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On a fractional Caputo–Hadamard problem with boundary value conditions via different orders of the Hadamard fractional operators

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Abstract

We investigate the existence of solutions for a Caputo–Hadamard fractional integro-differential equation with boundary value conditions involving the Hadamard fractional operators via different orders. By using the Krasnoselskii's fixed point theorem, the Leray–Schauder nonlinear alternative, and the Banach contraction principle, we prove our main results. Also, we provide three examples to illustrate our main results.

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1 Introduction

In recent decades, it has become clear to researchers that studying different types of fractional differential equations is of particular importance. This is a tool to complete our modeling information.

In fact, some practical instances done in the framework of the concepts and notions of the fractional calculus show us the power of this branch of mathematics in the modeling of different natural phenomena. In the meantime, fractional differential equations and inclusions of different types play an important role to reach desired practical goals. More precisely, in recent years, some researches invoked these fractional equations to model some processes and patterns via newly defined fractional operators (see, for example, [1–4]). The techniques used in these initial value problems are based on the analytical and the existence methods. In the following, some researchers designed new fractional models and investigated them via numerical techniques (see, for example, [5–14]). Therefore, the fractional calculus has been created a powerful tool for researchers to achieve more exact findings in other applied sciences. Also for further study, notice that a lot of works about different types of fractional integro-differential equations have been published (see, for example, [15–45]), q-difference equations (see, for example, [46–48]), integro-differential equations involving the Caputo–Fabrizio or the Caputo–Hadamard derivatives (see, for example, [49–51]), hybrid equations (see, for example, [52]), approximate solutions of different fractional equations (see, for example, [53, 54]), and modern models (see, for example, [55]).

In 2014, Ahmad et al. investigated the existence of solutions for the nonlinear fractional q-difference equation equipped with four-point nonlocal integral boundary conditions

$$\begin{cases} {}^{c}\mathcal{D}_q^{\beta}({}^{c}\mathcal{D}_q^{\gamma} + \lambda)u(t) = f(t, u(t)), \\ u(0) = a\mathcal{I}_q^{\alpha-1}u(\eta), \qquad u(1) = b\mathcal{I}_q^{\alpha-1}u(\sigma), \end{cases}$$

where $t \in [0,1]$, $q \in (0,1)$, $\lambda \in \mathbb{R}$, $0 < \eta, \sigma < 1$ and $\alpha > 2$; ${}^{c}\mathcal{D}_q^{\vartheta}$ denotes the Caputo q-fractional derivative of order $\vartheta \in \{\beta, \gamma : \beta, \gamma \in (0,1]\}$, $\mathcal{I}_q^{\alpha}$ denotes the Riemann–Liouville q-fractional integral of order α, and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function [46]. In 2016, Niyom et al. reviewed the problem

$$\begin{cases} (\lambda\mathcal{D}^{\alpha} + (1-\lambda)\mathcal{D}^{\beta})u(t) = f(t, u(t)), \\ u(0) = 0, \qquad \mu\mathcal{D}^{\gamma_1}u(T) + (1-\mu)\mathcal{D}^{\gamma_2}u(T) = \gamma_3, \end{cases}$$

where $T > 0$, $t \in [0, T]$, $1 < \alpha, \beta < 2$, $0 < \gamma_1$, $\gamma_2 < \alpha - \beta$, $\mathcal{D}^{\phi}$ is the Riemann–Liouville fractional derivative of order $\phi \in \{\alpha, \beta, \gamma_1, \gamma_2\}$, $0 < \lambda \leq 1$, $0 \leq \mu \leq 1$, $\gamma_3 \in \mathbb{R}$, and $f \in C([0, T] \times \mathbb{R}, \mathbb{R})$ [56]. In the same year, Ahmad et al. extended the boundary value problem presented by Niyom to the sequential fractional integro-differential equation of the form

$$\begin{cases} ({}^{c}\mathcal{D}^{q} + k{}^{c}\mathcal{D}^{q-1})x(t) = f(t, x(t), {}^{c}\mathcal{D}^{\beta}x(t), \mathcal{I}^{\gamma}x(t)), \\ x(0) = 0, \qquad x'(0) = 0, \qquad \sum_{i=1}^{m} a_i x(\xi_i) = \lambda\mathcal{I}^{\delta}x(\eta), \end{cases}$$

where $t \in [0,1]$, $\xi_i, \eta \in (0,1)$, $q \in (2,3]$, $\beta, \gamma \in (0,1)$, $k, \delta > 0$, λ, a_i $(i = 1, \dots, m)$ are real constants, ${}^{c}\mathcal{D}^{(\cdot)}$ denotes the Caputo derivative of the fractional order $(\cdot)$, and $f : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ is a continuous function [57].

By using main ideas of the aforementioned articles, we investigate the Caputo–Hadamard fractional integro-differential equation of different orders:

$$\bigl[\kappa\,{}^{\mathrm{CH}}\mathcal{D}_{1^+}^{\varrho} + (1-\kappa)\,{}^{\mathrm{CH}}\mathcal{D}_{1^+}^{\varpi}\bigr]w(t) = \alpha\psi\bigl(t, w(t)\bigr) + \beta\,{}^{H}\mathcal{I}_{1^+}^{\mu}\varphi\bigl(t, w(t)\bigr), \tag{1}$$

with mixed Hadamard and Caputo–Hadamard boundary value conditions

$$\begin{cases} w(1) = 0, \qquad {}^{\mathrm{CH}}\mathcal{D}_{1^+}^{\delta}w(e) = 0, \\ {}^{\mathrm{CH}}\mathcal{D}_{1^+}w(1) = 0, \qquad \frac{1}{\Gamma(\vartheta)}\int_1^e (\ln\frac{e}{s})^{\vartheta-1}w(s)\frac{\mathrm{d}s}{s} = 0, \end{cases} \tag{2}$$

where $t \in [1, e]$, $\varrho, \varpi \in (3,4]$, $\delta \in (1,2]$, $\kappa \in (0,1]$, $\mu, \vartheta > 0$ with $\delta + \vartheta \neq 0$ and also $\alpha, \beta \in \mathbb{R}^+$. The notation ${}^{\mathrm{CH}}\mathcal{D}_{1^+}^{\nu}$ denotes the Caputo–Hadamard fractional derivative of order $\nu \in \{\varrho, \varpi\}$ and ${}^{H}\mathcal{I}^{\mu}$ is the Hadamard fractional integral of order μ. Moreover, functions $\psi, \varphi : [1, e] \times \mathbb{R} \to \mathbb{R}$ are continuous. Note that the integro-differential equation (1) contains the Caputo–Hadamard derivatives of fractional orders ϱ and ϖ and a Hadamard integral of fractional order μ, while the Caputo–Hadamard derivative of order δ and Hadamard integral of order ϑ are involved in the boundary value conditions (2).

It should also be noted that boundary value conditions given in this paper are general and cover many different special cases. This new type of the modeling is an abstract idea and can include various existing natural processes in the future studies. Therefore, the main purpose of this manuscript is to focus on the existence results and provide some necessary conditions for the analytical investigation and so the practical aspects of boundary value problem (1)–(2) is not our main desire here. To reach our main aim, we apply three different fixed point theorems to establish the existence and uniqueness results. These analytical results guarantee the convergence of the numerical methods to desired solution with the least error, and so this can be a reliable criterion for modeling real processes.

The rest of the paper is arranged by follows. In the next section, we recall some basic notions and definitions which are necessary in the sequel. In Sect. 3, our main existence results are presented by three different analytical techniques such as the Krasnoselskii's fixed point theorem, the Leray–Schauder nonlinear alternative, and the Banach contraction principle. In Sect. 4, we examine the validity of our theoretical findings by providing three illustrative examples. In Sect. 5, the conclusion is stated.

2 Preliminaries

In this section, we recall some important and basic definitions on the fractional operators.

Definition 1 ([58, 59]) Let $\varrho \geq 0$. The Hadamard fractional integral of a continuous function $w : (a,b) \to \mathbb{R}$ of order ϱ is defined by $({}^{H}\mathcal{I}^{0}_{a^+} w)(t) = w(t)$ and

$$\left({}^{H}\mathcal{I}^{\varrho}_{a^+} w\right)(t) = \frac{1}{\Gamma(\varrho)} \int_a^t \left(\ln \frac{t}{s}\right)^{(\varrho-1)} w(s) \frac{\mathrm{d}s}{s}$$

provided that the right-hand side integral exists.

Note that the semigroup property is satisfied by the Hadamard fractional integral as follows: ${}^{H}\mathcal{I}^{\varpi}_{a^+} {}^{H}\mathcal{I}^{\varrho}_{a^+} w(t) = {}^{H}\mathcal{I}^{\varpi+\varrho}_{a^+} w(t)$ for $\varrho, \varpi \in \mathbb{R}^+$. Also, we have

$${}^{H}\mathcal{I}^{\varrho}_{a^+} \left(\ln \frac{t}{a}\right)^{\varpi} = \frac{\Gamma(\varpi+1)}{\Gamma(\varrho+\varpi+1)} \left(\ln \frac{t}{a}\right)^{\varrho+\varpi}$$

for $\varrho, \varpi \geq 0$ and $t > a$ [58, 59]. It is clear that ${}^{H}\mathcal{I}^{\varrho}_{a^+} 1 = \frac{1}{\Gamma(\varrho+1)} (\ln \frac{t}{a})^{\varrho}$ for all $t > a$ by putting $\varpi = 0$ [59].

Definition 2 ([58, 59]) Let $n = [\varrho] + 1$ and $n - 1 < \varrho \leq n$. The Hadamard fractional derivative of order ϱ for a continuous function $w : (a,b) \to \mathbb{R}$ is defined by

$$\left({}^{H}\mathcal{D}^{\varrho}_{a^+} w\right)(t) = \frac{1}{\Gamma(n-\varrho)} \left(t \frac{\mathrm{d}t}{t}\right)^{n} \int_a^t \left(\ln \frac{t}{s}\right)^{(n-\varrho-1)} w(s) \frac{\mathrm{d}s}{s}$$

provided that the right-hand side integral exists.

Definition 3 ([51, 58]) Let $\mathrm{AC}^{n}_{\theta}[a,b] = \{w : [a,b] \to \mathbb{R} : \theta^{n-1} w(t) \in \mathrm{AC}[a,b], \theta = t\frac{\mathrm{d}}{\mathrm{d}t}\}$. The Caputo–Hadamard fractional derivative of order ϱ for an absolutely continuous function

$w \in AC_\theta^n([a,b],\mathbb{R})$ is defined by

$$\left({}^{\mathrm{CH}}\mathcal{D}_{a^+}^{\varrho} w\right)(t) = \frac{1}{\Gamma(n-\varrho)} \int_a^t \left(\ln\frac{t}{s}\right)^{(n-\varrho-1)} \left(t\frac{\mathrm{d}t}{t}\right)^n w(s)\frac{\mathrm{d}s}{s}$$

whenever the right-hand side integral exists.

Assume that $w \in AC_\theta^n([a,b],\mathbb{R})$ and $n-1 < \varrho \leq n$. It has been proved that the solution of the Caputo–Hadamard fractional differential equation $({}^{\mathrm{CH}}\mathcal{D}_{a^+}^{\varrho} w)(t) = 0$ is in the form $w(t) = \sum_{k=0}^{n-1} c_k(\ln\frac{t}{a})^k$, and we have

$${}^{H}\mathcal{I}_{a^+}^{\varrho}\,{}^{\mathrm{CH}}\mathcal{D}_{a^+}^{\varrho} w(t) = w(t) + c_0 + c_1\left(\ln\frac{t}{a}\right) + c_2\left(\ln\frac{t}{a}\right)^2 + \cdots + c_{n-1}\left(\ln\frac{t}{a}\right)^{n-1}$$

for all $t > a$ [58, 59]. We need the following results.

Lemma 4 (Krasnoselskii's, [60]) *Let M be a closed, bounded, convex, and nonempty subset of a Banach space $\mathcal{E}$. Consider two operators Υ_1 and Υ_2 from M into $\mathcal{E}$ such that*
(i) *$\Upsilon_1 w_1 + \Upsilon_2 w_2 \in M$ for all $w_1, w_2 \in M$,*
(ii) *Υ_1 is compact and continuous,*
(iii) *Υ_2 is a contraction map.*
Then there exists $z \in M$ such that $z = \Upsilon_1 z + \Upsilon_2 z$.

Lemma 5 ([61]) *Let $\mathcal{E}$ be a Banach space, C a closed, convex subset of $\mathcal{E}$, $\mathcal{U}$ an open subset of C, and $0 \in \mathcal{U}$. Suppose that $\Upsilon : \overline{\mathcal{U}} \to C$ is a continuous and compact map (that is, $\Upsilon(\overline{\mathcal{U}})$ is a relatively compact subset of C). Then Υ has a fixed point in $\overline{\mathcal{U}}$ or there is a $w \in \partial\mathcal{U}$ (the boundary of $\mathcal{U}$ in C) and $\lambda \in (0,1)$ with $w = \lambda\Upsilon(w)$.*

Lemma 6 ([62]) *Let $\mathcal{E}$ be a Banach space and M a closed subset of $\mathcal{E}$. Suppose that $\Upsilon : M \to M$ is a contraction. Then Υ has a unique fixed point in M.*

3 Main results

Here, we are ready to prove our main results. We first characterize the structure of the solutions of the problem (1)–(2). Consider the Banach space $\mathcal{E} = \{w : w(t) \in C([1,e],\mathbb{R})\}$ with the norm $\|w\|_{\mathcal{E}} = \sup_{t\in[1,e]} |w(t)|$. We first provide our key lemma.

Lemma 7 *Let $\phi(t) \in \mathcal{E}$. Then w_0 is a solution for the Caputo–Hadamard problem*

$$\begin{cases} [\kappa\,{}^{\mathrm{CH}}\mathcal{D}_{1^+}^{\varrho} + (1-\kappa)\,{}^{\mathrm{CH}}\mathcal{D}_{1^+}^{\varpi}]w(t) = \phi(t) \quad (t \in [1,e]), \\ w(1) = 0, \qquad {}^{\mathrm{CH}}\mathcal{D}_{1^+}^{\delta} w(e) = 0, \\ {}^{\mathrm{CH}}\mathcal{D}_{1^+} w(1) = 0, \qquad \frac{1}{\Gamma(\vartheta)}\int_1^e (\ln\frac{e}{s})^{\vartheta-1} w(s)\frac{\mathrm{d}s}{s} = 0 \end{cases} \tag{3}$$

if and only if w_0 is a solution for the fractional integral equation

$$\begin{aligned} w(t) = {}& \frac{(\kappa-1)}{\kappa\Gamma(\varrho-\varpi)} \int_1^t \left(\ln\frac{t}{s}\right)^{\varrho-\varpi-1} w(s)\frac{\mathrm{d}s}{s} + \frac{1}{\kappa\Gamma(\varrho)} \int_1^t \left(\ln\frac{t}{s}\right)^{\varrho-1} \phi(s)\frac{\mathrm{d}s}{s} \\ & + \frac{(1-\kappa)[3\Gamma(4+\vartheta)(\ln t)^2 + (\delta-3)\Gamma(4-\vartheta)(\ln t)^3]}{6\kappa(\delta+\vartheta)\Gamma(\varrho-\varpi+\vartheta)} \end{aligned}$$

$$\begin{aligned}
&\times \int_1^e \left(\ln\frac{e}{s}\right)^{\varrho-\varpi+\vartheta-1} w(s)\frac{\mathrm{d}s}{s} \\
&+ \frac{(1-\kappa)\Gamma(4-\delta)[\Gamma(4-\vartheta)(\ln t)^3 - 3\Gamma(3+\vartheta)(\ln t)^2]}{6\kappa(\delta+\vartheta)\Gamma(3+\vartheta)\Gamma(\varrho-\varpi-\delta)} \\
&\times \int_1^e \left(\ln\frac{e}{s}\right)^{\varrho-\varpi-\delta-1} w(s)\frac{\mathrm{d}s}{s} \\
&+ \frac{(3-\delta)\Gamma(4-\vartheta)(\ln t)^3 - 3\Gamma(4+\vartheta)(\ln t)^2}{6\kappa(\delta+\vartheta)\Gamma(\varrho+\vartheta)} \int_1^e \left(\ln\frac{e}{s}\right)^{\varrho+\vartheta-1} \phi(s)\frac{\mathrm{d}s}{s} \\
&+ \frac{\Gamma(4-\delta)[3\Gamma(3+\vartheta)(\ln t)^2 - \Gamma(4-\vartheta)(\ln t)^3]}{6\kappa(\delta+\vartheta)\Gamma(3+\vartheta)\Gamma(\varrho-\delta)} \\
&\times \int_1^e \left(\ln\frac{e}{s}\right)^{\varrho-\delta-1} \phi(s)\frac{\mathrm{d}s}{s}. \qquad (4)
\end{aligned}$$

Proof Let w_0 be a solution for the Caputo–Hadamard problem (3). Then, we have

$$\kappa\,{}^{\mathrm{CH}}\mathcal{D}_{1^+}^{\varrho} w_0(t) + (1-\kappa)\,{}^{\mathrm{CH}}\mathcal{D}_{1^+}^{\varpi} w_0(t) = \phi(t)$$

and so ${}^{\mathrm{CH}}\mathcal{D}_{1^+}^{\varrho} w_0(t) = \frac{\kappa-1}{\kappa}\,{}^{\mathrm{CH}}\mathcal{D}_{1^+}^{\varpi} w_0(t) + \frac{1}{\kappa}\phi(t)$. By using the Hadamard fractional integral of order ϱ, we obtain

$$\begin{aligned}
w_0(t) = {}& \frac{\kappa-1}{\kappa}\,{}^{H}\mathcal{I}_{1^+}^{\varrho}\,{}^{\mathrm{CH}}\mathcal{D}_{1^+}^{\varpi} w_0(t) + \frac{1}{\kappa}\,{}^{H}\mathcal{I}_{1^+}^{\varrho}\phi(t) \\
&+ b_0 + b_1(\ln t) + b_2(\ln t)^2 + b_3(\ln t)^3,
\end{aligned}$$

where b_0, b_1, b_2, and b_3 are some real constants. Hence,

$$\begin{aligned}
w_0(t) = {}& \frac{\kappa-1}{\kappa\Gamma(\varrho-\varpi)} \int_1^t \left(\ln\frac{t}{s}\right)^{\varrho-\varpi-1} w_0(s)\frac{\mathrm{d}s}{s} \\
&+ \frac{1}{\kappa\Gamma(\varrho)} \int_1^t \left(\ln\frac{t}{s}\right)^{\varrho-1} \phi(s)\frac{\mathrm{d}s}{s} \\
&+ b_0 + b_1(\ln t) + b_2(\ln t)^2 + b_3(\ln t)^3. \qquad (5)
\end{aligned}$$

Now by using the boundary value conditions and properties of the Hadamard and Caputo–Hadamard fractional operators, we get

$$\begin{aligned}
{}^{\mathrm{CH}}\mathcal{D}_{1^+} w_0(t) = {}& \frac{\kappa-1}{\kappa\Gamma(\varrho-\varpi-1)} \int_1^t \left(\ln\frac{t}{s}\right)^{\varrho-\varpi-2} w_0(s)\frac{\mathrm{d}s}{s} \\
&+ \frac{1}{\kappa\Gamma(\varrho-1)} \int_1^t \left(\ln\frac{t}{s}\right)^{\varrho-2} \phi(s)\frac{\mathrm{d}s}{s} + b_1 + 2b_2(\ln t) + 3b_3(\ln t)^2, \\
{}^{\mathrm{CH}}\mathcal{D}_{1^+}^{\delta} w_0(t) = {}& \frac{\kappa-1}{\kappa\Gamma(\varrho-\varpi-\delta)} \int_1^t \left(\ln\frac{t}{s}\right)^{\varrho-\varpi-\delta-1} w_0(s)\frac{\mathrm{d}s}{s} \\
&+ \frac{1}{\kappa\Gamma(\varrho-\delta)} \int_1^t \left(\ln\frac{t}{s}\right)^{\varrho-\delta-1} \phi(s)\frac{\mathrm{d}s}{s} + b_2\frac{2}{\Gamma(3-\delta)}(\ln t)^{2-\delta} \\
&+ b_3\frac{6}{\Gamma(4-\delta)}(\ln t)^{3-\delta},
\end{aligned}$$

and

$$\begin{aligned}
{}^{H}\mathcal{I}_{1^+}^{\vartheta} w_0(t) &= \frac{\kappa - 1}{\kappa \Gamma(\varrho - \varpi + \vartheta)} \int_1^t \left(\ln \frac{t}{s}\right)^{\varrho - \varpi + \vartheta - 1} w_0(s) \frac{\mathrm{d}s}{s} \\
&\quad + \frac{1}{\kappa \Gamma(\varrho + \vartheta)} \int_1^t \left(\ln \frac{t}{s}\right)^{\varrho + \vartheta - 1} \phi(s) \frac{\mathrm{d}s}{s} + b_0 \frac{1}{\Gamma(1+\vartheta)} (\ln t)^{\vartheta} \\
&\quad + b_1 \frac{1}{\Gamma(2+\vartheta)} (\ln t)^{1+\vartheta} \\
&\quad + b_2 \frac{2}{\Gamma(3+\vartheta)} (\ln t)^{2+\vartheta} + b_3 \frac{6}{\Gamma(4+\vartheta)} (\ln t)^{3+\vartheta}.
\end{aligned}$$

By using two first boundary conditions, we obtain $b_0 = b_1 = 0$. By using two other boundary conditions, we obtain

$$\begin{aligned}
b_2 &= \frac{(1-\kappa)\Gamma(4+\vartheta)}{2\kappa(\delta+\vartheta)\Gamma(\varrho - \varpi + \vartheta)} \int_1^e \left(\ln \frac{e}{s}\right)^{\varrho - \varpi + \vartheta - 1} w_0(s) \frac{\mathrm{d}s}{s} \\
&\quad + \frac{(\kappa-1)\Gamma(4-\delta)}{2\kappa(\delta+\vartheta)\Gamma(\varrho - \varpi - \delta)} \int_1^e \left(\ln \frac{e}{s}\right)^{\varrho - \varpi - \delta - 1} w_0(s) \frac{\mathrm{d}s}{s} \\
&\quad - \frac{\Gamma(4+\vartheta)}{2\kappa(\delta+\vartheta)\Gamma(\varrho + \vartheta)} \int_1^e \left(\ln \frac{e}{s}\right)^{\varrho + \vartheta - 1} \phi(s) \frac{\mathrm{d}s}{s} \\
&\quad + \frac{\Gamma(4-\delta)}{2\kappa(\delta+\vartheta)\Gamma(\varrho - \delta)} \int_1^e \left(\ln \frac{e}{s}\right)^{\varrho - \delta - 1} \phi(s) \frac{\mathrm{d}s}{s}
\end{aligned}$$

and

$$\begin{aligned}
b_3 &= \frac{(1-\kappa)(\delta-3)\Gamma(4-\vartheta)}{6\kappa(\delta+\vartheta)\Gamma(\varrho - \varpi + \vartheta)} \int_1^e \left(\ln \frac{e}{s}\right)^{\varrho - \varpi + \vartheta - 1} w_0(s) \frac{\mathrm{d}s}{s} \\
&\quad + \frac{(1-\kappa)\Gamma(4-\delta)\Gamma(4-\vartheta)}{6\kappa(\delta+\vartheta)\Gamma(3+\vartheta)\Gamma(\varrho - \varpi - \delta)} \int_1^e \left(\ln \frac{e}{s}\right)^{\varrho - \varpi - \delta - 1} w_0(s) \frac{\mathrm{d}s}{s} \\
&\quad + \frac{(3-\delta)\Gamma(4-\vartheta)}{6\kappa(\delta+\vartheta)\Gamma(\varrho + \vartheta)} \int_1^e \left(\ln \frac{e}{s}\right)^{\varrho + \vartheta - 1} \phi(s) \frac{\mathrm{d}s}{s} \\
&\quad - \frac{\Gamma(4-\delta)\Gamma(4-\vartheta)}{6\kappa(\delta+\vartheta)\Gamma(3+\vartheta)\Gamma(\varrho - \delta)} \int_1^e \left(\ln \frac{e}{s}\right)^{\varrho - \delta - 1} \phi(s) \frac{\mathrm{d}s}{s}.
\end{aligned}$$

Now by substituting the values for b_0, b_1, b_2, b_3 in equation (5), we see that w_0 is a solution for the integral equation. For the converse part, by using some direct calculations, one can see that w_0 is a solution for the Caputo–Hadamard problem (3) whenever w_0 is a solution for the integral equation (4). This completes the proof. □

Now, consider the operator $\Upsilon : \mathcal{E} \to \mathcal{E}$ defined by

$$\begin{aligned}
(\Upsilon w)(t) &= \frac{(\kappa - 1)}{\kappa \Gamma(\varrho - \varpi)} \int_1^t \left(\ln \frac{t}{s}\right)^{\varrho - \varpi - 1} w(s) \frac{\mathrm{d}s}{s} \\
&\quad + \frac{\alpha}{\kappa \Gamma(\varrho)} \int_1^t \left(\ln \frac{t}{s}\right)^{\varrho - 1} \psi\bigl(s, w(s)\bigr) \frac{\mathrm{d}s}{s} + \frac{\beta}{\kappa \Gamma(\varrho + \mu)}
\end{aligned}$$

$$
\begin{aligned}
&\times \int_1^t \left(\ln \frac{t}{s}\right)^{\varrho+\mu-1} \varphi\bigl(s,w(s)\bigr)\frac{\mathrm{d}s}{s}\\
&+ \frac{(1-\kappa)[3\Gamma(4+\vartheta)(\ln t)^2 + (\delta-3)\Gamma(4-\vartheta)(\ln t)^3]}{6\kappa(\delta+\vartheta)\Gamma(\varrho-\varpi+\vartheta)}\\
&\times \int_1^e \left(\ln \frac{e}{s}\right)^{\varrho-\varpi+\vartheta-1} w(s)\frac{\mathrm{d}s}{s}\\
&+ \frac{(1-\kappa)\Gamma(4-\delta)[\Gamma(4-\vartheta)(\ln t)^3 - 3\Gamma(3+\vartheta)(\ln t)^2]}{6\kappa(\delta+\vartheta)\Gamma(3+\vartheta)\Gamma(\varrho-\varpi-\delta)}\\
&\times \int_1^e \left(\ln \frac{e}{s}\right)^{\varrho-\varpi-\delta-1} w(s)\frac{\mathrm{d}s}{s}\\
&+ \frac{\alpha[(3-\delta)\Gamma(4-\vartheta)(\ln t)^3 - 3\Gamma(4+\vartheta)(\ln t)^2]}{6\kappa(\delta+\vartheta)\Gamma(\varrho+\vartheta)}\\
&\times \int_1^e \left(\ln \frac{e}{s}\right)^{\varrho+\vartheta-1} \psi\bigl(s,w(s)\bigr)\frac{\mathrm{d}s}{s}\\
&+ \frac{\beta[(3-\delta)\Gamma(4-\vartheta)(\ln t)^3 - 3\Gamma(4+\vartheta)(\ln t)^2]}{6\kappa(\delta+\vartheta)\Gamma(\varrho+\vartheta+\mu)}\\
&\times \int_1^e \left(\ln \frac{e}{s}\right)^{\varrho+\vartheta+\mu-1} \varphi\bigl(s,w(s)\bigr)\frac{\mathrm{d}s}{s}\\
&+ \frac{\alpha\Gamma(4-\delta)[3\Gamma(3+\vartheta)(\ln t)^2 - \Gamma(4-\vartheta)(\ln t)^3]}{6\kappa(\delta+\vartheta)\Gamma(3+\vartheta)\Gamma(\varrho-\delta)}\\
&\times \int_1^e \left(\ln \frac{e}{s}\right)^{\varrho-\delta-1} \psi\bigl(s,w(s)\bigr)\frac{\mathrm{d}s}{s}\\
&+ \frac{\beta\Gamma(4-\delta)[3\Gamma(3+\vartheta)(\ln t)^2 - \Gamma(4-\vartheta)(\ln t)^3]}{6\kappa(\delta+\vartheta)\Gamma(3+\vartheta)\Gamma(\varrho+\mu-\delta)}\\
&\times \int_1^e \left(\ln \frac{e}{s}\right)^{\varrho+\mu-\delta-1} \varphi\bigl(s,w(s)\bigr)\frac{\mathrm{d}s}{s},
\end{aligned}
\tag{6}
$$

where $w \in \mathcal{E}$ and $t \in [1,e]$. Put

$$
\begin{aligned}
\mathcal{K}_0^* &:= \frac{|\kappa-1|}{\kappa\Gamma(\varrho-\varpi+1)} + \frac{(1-\kappa)[|3\Gamma(4+\vartheta)| + |(\delta-3)\Gamma(4-\vartheta)|]}{6\kappa(\delta+\vartheta)\Gamma(\varrho-\varpi+\vartheta+1)}\\
&\quad + \frac{(1-\kappa)|\Gamma(4-\delta)|[|\Gamma(4-\vartheta)| + 3|\Gamma(3+\vartheta)|]}{6\kappa(\delta+\vartheta)\Gamma(3+\vartheta)\Gamma(\varrho-\varpi-\delta+1)},\\
\mathcal{K}_1^* &:= \frac{\alpha}{\kappa\Gamma(\varrho+1)} + \frac{\alpha[|(3-\delta)\Gamma(4-\vartheta)| + 3|\Gamma(4+\vartheta)|]}{6\kappa(\delta+\vartheta)\Gamma(\varrho+\vartheta+1)}\\
&\quad + \frac{\alpha|\Gamma(4-\delta)|[3|\Gamma(3+\vartheta)| + |\Gamma(4-\vartheta)|]}{6\kappa(\delta+\vartheta)\Gamma(3+\vartheta)\Gamma(\varrho-\delta+1)},\\
\mathcal{K}_2^* &:= \frac{\beta}{\kappa\Gamma(\varrho+\mu+1)} + \frac{\beta[|(3-\delta)\Gamma(4-\vartheta)| + 3|\Gamma(4+\vartheta)|]}{6\kappa(\delta+\vartheta)\Gamma(\varrho+\vartheta+\mu+1)}\\
&\quad + \frac{\beta|\Gamma(4-\delta)|[3|\Gamma(3+\vartheta)| + |\Gamma(4-\vartheta)|]}{6\kappa(\delta+\vartheta)\Gamma(3+\vartheta)\Gamma(\varrho+\mu-\delta+1)}.
\end{aligned}
\tag{7}
$$

Theorem 8 *Suppose that $\psi, \varphi : [1,e] \times \mathbb{R} \to \mathbb{R}$ are continuous functions such that*

($\mathcal{N}_1$) *there is $L > 0$ so that $|\psi(t,w_1) - \psi(t,w_2)| \le L|w_1 - w_2|$ for all $w_1, w_2 \in \mathbb{R}$ and $t \in [1,e]$,*

($\mathcal{N}_2$) *there exists a real-valued continuous function σ on $[1,e]$ such that $|\varphi(t,w)| \leq \sigma(t)$ for all $w \in \mathbb{R}$ and $t \in [1,e]$.*

If $\mathcal{K}_0^* + L\mathcal{K}_1^* < 1$, then the Caputo–Hadamard boundary value problem (1)–(2) has at least one solution, where $\mathcal{K}_0^*$ and $\mathcal{K}_1^*$ are given by (7).

Proof Let $\|\sigma\| := \sup_{t\in[1,e]} |\sigma(t)|$ and $O := \sup_{t\in[1,e]} |\psi(t,0)|$. Consider the operator $\Upsilon : \mathcal{E} \to \mathcal{E}$ and the set $\mathcal{V}_r := \{w \in \mathcal{E} : \|w\|_{\mathcal{E}} \leq r\}$ which is a closed, convex, and bounded nonempty subset of Banach space $\mathcal{E}$, where $r \geq \frac{\|\sigma\|\mathcal{K}_2^* + O\mathcal{K}_1^*}{1-(\mathcal{K}_0^*+L\mathcal{K}_1^*)}$ and $\mathcal{K}_1^*$ and $\mathcal{K}_2^*$ are given by (7). Note that each fixed point of Υ is a solution for the Caputo–Hadamard problem (1)–(2). Let $t \in [1,e]$ be given. Then, we have

$$\begin{aligned}
(\Upsilon_1 w)(t) &= \frac{(\kappa-1)}{\kappa\Gamma(\varrho-\varpi)} \int_1^t \left(\ln\frac{t}{s}\right)^{\varrho-\varpi-1} w(s)\frac{\mathrm{d}s}{s} \\
&\quad + \frac{\alpha}{\kappa\Gamma(\varrho)} \int_1^t \left(\ln\frac{t}{s}\right)^{\varrho-1} \psi\bigl(s,w(s)\bigr)\frac{\mathrm{d}s}{s} \\
&\quad + \frac{(1-\kappa)[3\Gamma(4+\vartheta)(\ln t)^2 + (\delta-3)\Gamma(4-\vartheta)(\ln t)^3]}{6\kappa(\delta+\vartheta)\Gamma(\varrho-\varpi+\vartheta)} \\
&\quad \times \int_1^e \left(\ln\frac{e}{s}\right)^{\varrho-\varpi+\vartheta-1} w(s)\frac{\mathrm{d}s}{s} \\
&\quad + \frac{(1-\kappa)\Gamma(4-\delta)[\Gamma(4-\vartheta)(\ln t)^3 - 3\Gamma(3+\vartheta)(\ln t)^2]}{6\kappa(\delta+\vartheta)\Gamma(3+\vartheta)\Gamma(\varrho-\varpi-\delta)} \\
&\quad \times \int_1^e \left(\ln\frac{e}{s}\right)^{\varrho-\varpi-\delta-1} w(s)\frac{\mathrm{d}s}{s} \\
&\quad + \frac{\alpha[(3-\delta)\Gamma(4-\vartheta)(\ln t)^3 - 3\Gamma(4+\vartheta)(\ln t)^2]}{6\kappa(\delta+\vartheta)\Gamma(\varrho+\vartheta)} \\
&\quad \times \int_1^e \left(\ln\frac{e}{s}\right)^{\varrho+\vartheta-1} \psi\bigl(s,w(s)\bigr)\frac{\mathrm{d}s}{s} \\
&\quad + \frac{\alpha\Gamma(4-\delta)[3\Gamma(3+\vartheta)(\ln t)^2 - \Gamma(4-\vartheta)(\ln t)^3]}{6\kappa(\delta+\vartheta)\Gamma(3+\vartheta)\Gamma(\varrho-\delta)} \\
&\quad \times \int_1^e \left(\ln\frac{e}{s}\right)^{\varrho-\delta-1} \psi\bigl(s,w(s)\bigr)\frac{\mathrm{d}s}{s}
\end{aligned}$$

and

$$\begin{aligned}
(\Upsilon_2 w)(t) &= \frac{\beta}{\kappa\Gamma(\varrho+\mu)} \int_1^t \left(\ln\frac{t}{s}\right)^{\varrho+\mu-1} \varphi\bigl(s,w(s)\bigr)\frac{\mathrm{d}s}{s} \\
&\quad + \frac{\beta[(3-\delta)\Gamma(4-\vartheta)(\ln t)^3 - 3\Gamma(4+\vartheta)(\ln t)^2]}{6\kappa(\delta+\vartheta)\Gamma(\varrho+\vartheta+\mu)} \\
&\quad \times \int_1^e \left(\ln\frac{e}{s}\right)^{\varrho+\vartheta+\mu-1} \varphi\bigl(s,w(s)\bigr)\frac{\mathrm{d}s}{s} \\
&\quad + \frac{\beta\Gamma(4-\delta)[3\Gamma(3+\vartheta)(\ln t)^2 - \Gamma(4-\vartheta)(\ln t)^3]}{6\kappa(\delta+\vartheta)\Gamma(3+\vartheta)\Gamma(\varrho+\mu-\delta)} \\
&\quad \times \int_1^e \left(\ln\frac{e}{s}\right)^{\varrho+\mu-\delta-1} \varphi\bigl(s,w(s)\bigr)\frac{\mathrm{d}s}{s}.
\end{aligned}$$

Thus, we have

$$
\begin{aligned}
&\big|(\Upsilon_1 w_1)(t) + (\Upsilon_2 w_2)(t)\big| \\
&\quad\leq \frac{|\kappa-1|}{\kappa\Gamma(\varrho-\varpi)}\int_1^t\left(\ln\frac{t}{s}\right)^{\varrho-\varpi-1}\big|w_1(s)\big|\frac{\mathrm{d}s}{s} \\
&\qquad+\frac{\alpha}{\kappa\Gamma(\varrho)}\int_1^t\left(\ln\frac{t}{s}\right)^{\varrho-1}\big(\big|\psi\big(s,w_1(s)\big)-\psi(s,0)\big|+\big|\psi(s,0)\big|\big)\frac{\mathrm{d}s}{s} \\
&\qquad+\frac{\beta}{\kappa\Gamma(\varrho+\mu)}\int_1^t\left(\ln\frac{t}{s}\right)^{\varrho+\mu-1}\big|\varphi\big(s,w_2(s)\big)\big|\frac{\mathrm{d}s}{s} \\
&\qquad+\frac{(1-\kappa)[|3\Gamma(4+\vartheta)|(\ln t)^2+|(\delta-3)\Gamma(4-\vartheta)|(\ln t)^3]}{6\kappa(\delta+\vartheta)\Gamma(\varrho-\varpi+\vartheta)} \\
&\qquad\times\int_1^e\left(\ln\frac{e}{s}\right)^{\varrho-\varpi+\vartheta-1}\big|w_1(s)\big|\frac{\mathrm{d}s}{s} \\
&\qquad+\frac{(1-\kappa)|\Gamma(4-\delta)|[|\Gamma(4-\vartheta)|(\ln t)^3+3|\Gamma(3+\vartheta)|(\ln t)^2]}{6\kappa(\delta+\vartheta)\Gamma(3+\vartheta)\Gamma(\varrho-\varpi-\delta)} \\
&\qquad\times\int_1^e\left(\ln\frac{e}{s}\right)^{\varrho-\varpi-\delta-1}\big|w_1(s)\big|\frac{\mathrm{d}s}{s} \\
&\qquad+\frac{\alpha[|(3-\delta)\Gamma(4-\vartheta)|(\ln t)^3+3|\Gamma(4+\vartheta)|(\ln t)^2]}{6\kappa(\delta+\vartheta)\Gamma(\varrho+\vartheta)} \\
&\qquad\times\int_1^e\left(\ln\frac{e}{s}\right)^{\varrho+\vartheta-1}\big(\big|\psi\big(s,w_1(s)\big)-\psi(s,0)\big|+\big|\psi(s,0)\big|\big)\frac{\mathrm{d}s}{s} \\
&\qquad+\frac{\beta[|(3-\delta)\Gamma(4-\vartheta)|(\ln t)^3+3|\Gamma(4+\vartheta)|(\ln t)^2]}{6\kappa(\delta+\vartheta)\Gamma(\varrho+\vartheta+\mu)} \\
&\qquad\times\int_1^e\left(\ln\frac{e}{s}\right)^{\varrho+\vartheta+\mu-1}\big|\varphi\big(s,w_2(s)\big)\big|\frac{\mathrm{d}s}{s} \\
&\qquad+\frac{\alpha|\Gamma(4-\delta)|[3|\Gamma(3+\vartheta)|(\ln t)^2+|\Gamma(4-\vartheta)|(\ln t)^3]}{6\kappa(\delta+\vartheta)\Gamma(3+\vartheta)\Gamma(\varrho-\delta)} \\
&\qquad\times\int_1^e\left(\ln\frac{e}{s}\right)^{\varrho-\delta-1}\big(\big|\psi\big(s,w_1(s)\big)-\psi(s,0)\big|+\big|\psi(s,0)\big|\big)\frac{\mathrm{d}s}{s} \\
&\qquad+\frac{\beta|\Gamma(4-\delta)|[3|\Gamma(3+\vartheta)|(\ln t)^2+|\Gamma(4-\vartheta)|(\ln t)^3]}{6\kappa(\delta+\vartheta)\Gamma(3+\vartheta)\Gamma(\varrho+\mu-\delta)} \\
&\qquad\times\int_1^e\left(\ln\frac{e}{s}\right)^{\varrho+\mu-\delta-1}\big|\varphi\big(s,w_2(s)\big)\big|\frac{\mathrm{d}s}{s} \\
&\quad\leq\frac{|\kappa-1|}{\kappa\Gamma(\varrho-\varpi+1)}\|w_1\|+\frac{\alpha}{\kappa\Gamma(\varrho+1)}\big(L\|w_1\|+O\big)+\frac{\beta}{\kappa\Gamma(\varrho+\mu+1)}\|\sigma\| \\
&\qquad+\frac{(1-\kappa)[|3\Gamma(4+\vartheta)|+|(\delta-3)\Gamma(4-\vartheta)|]}{6\kappa(\delta+\vartheta)\Gamma(\varrho-\varpi+\vartheta+1)}\|w_1\| \\
&\qquad+\frac{(1-\kappa)|\Gamma(4-\delta)|[|\Gamma(4-\vartheta)|+3|\Gamma(3+\vartheta)|]}{6\kappa(\delta+\vartheta)\Gamma(3+\vartheta)\Gamma(\varrho-\varpi-\delta+1)}\|w_1\| \\
&\qquad+\frac{\alpha[|(3-\delta)\Gamma(4-\vartheta)|+3|\Gamma(4+\vartheta)|]}{6\kappa(\delta+\vartheta)\Gamma(\varrho+\vartheta+1)}\big(L\|w_1\|+O\big)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\beta[|(3-\delta)\Gamma(4-\vartheta)|+3|\Gamma(4+\vartheta)|]}{6\kappa(\delta+\vartheta)\Gamma(\varrho+\vartheta+\mu+1)}\|\sigma\| \\
&+\frac{\alpha|\Gamma(4-\delta)|[3|\Gamma(3+\vartheta)|+|\Gamma(4-\vartheta)|]}{6\kappa(\delta+\vartheta)\Gamma(3+\vartheta)\Gamma(\varrho-\delta+1)}\big(L\|w_1\|+O\big) \\
&+\frac{\beta|\Gamma(4-\delta)|[3|\Gamma(3+\vartheta)|+|\Gamma(4-\vartheta)|]}{6\kappa(\delta+\vartheta)\Gamma(3+\vartheta)\Gamma(\varrho+\mu-\delta+1)}\|\sigma\| \\
&=\big(\mathcal{K}_0^*+L\mathcal{K}_1^*\big)\|w_1\|+\mathcal{K}_2^*\|\sigma\|+\mathcal{K}_1^*O \\
&\le\big(\mathcal{K}_0^*+L\mathcal{K}_1^*\big)r+\mathcal{K}_2^*\|\sigma\|+\mathcal{K}_1^*O\le r
\end{aligned}
$$

for all $w_1, w_2 \in \mathcal{V}_r$. Hence, $\|\Upsilon_1 w_1+\Upsilon_2 w_2\|\le r$ and so $\Upsilon_1 w_1+\Upsilon_2 w_2\in\mathcal{V}_r$ for all $w_1, w_2\in\mathcal{V}_r$. Now let $\{w_n\}_{n\ge 1}$ be a sequence in $\mathcal{V}_r$ with $w_n\to w$ and $t\in[1,e]$. Then, we have

$$
\begin{aligned}
&\big|(\Upsilon_2 w_n)(t)-(\Upsilon_2 w)(t)\big| \\
&\le\frac{\beta}{\kappa\Gamma(\varrho+\mu)}\int_1^t\left(\ln\frac{t}{s}\right)^{\varrho+\mu-1}\big|\varphi\big(s,w_n(s)\big)-\varphi\big(s,w(s)\big)\big|\frac{\mathrm{d}s}{s} \\
&\quad+\frac{\beta[|(3-\delta)\Gamma(4-\vartheta)|(\ln t)^3+3|\Gamma(4+\vartheta)|(\ln t)^2]}{6\kappa(\delta+\vartheta)\Gamma(\varrho+\vartheta+\mu)} \\
&\quad\times\int_1^e\left(\ln\frac{e}{s}\right)^{\varrho+\vartheta+\mu-1}\big|\varphi\big(s,w_n(s)\big)-\varphi\big(s,w(s)\big)\big|\frac{\mathrm{d}s}{s} \\
&\quad+\frac{\beta|\Gamma(4-\delta)|[3|\Gamma(3+\vartheta)|(\ln t)^2+|\Gamma(4-\vartheta)|(\ln t)^3]}{6\kappa(\delta+\vartheta)\Gamma(3+\vartheta)\Gamma(\varrho+\mu-\delta)} \\
&\quad\times\int_1^e\left(\ln\frac{e}{s}\right)^{\varrho+\mu-\delta-1}\big|\varphi\big(s,w_n(s)\big)-\varphi\big(s,w(s)\big)\big|\frac{\mathrm{d}s}{s} \\
&\le\frac{\beta}{\kappa\Gamma(\varrho+\mu+1)}\big|\varphi\big(s,w_n(s)\big)-\varphi\big(s,w(s)\big)\big| \\
&\quad+\frac{\beta[|(3-\delta)\Gamma(4-\vartheta)|+3|\Gamma(4+\vartheta)|]}{6\kappa(\delta+\vartheta)\Gamma(\varrho+\vartheta+\mu+1)}\big|\varphi\big(s,w_n(s)\big)-\varphi\big(s,w(s)\big)\big| \\
&\quad+\frac{\beta|\Gamma(4-\delta)|[3|\Gamma(3+\vartheta)|+|\Gamma(4-\vartheta)|]}{6\kappa(\delta+\vartheta)\Gamma(3+\vartheta)\Gamma(\varrho+\mu-\delta+1)}\big|\varphi\big(s,w_n(s)\big)-\varphi\big(s,w(s)\big)\big|.
\end{aligned}
$$

Since φ is continuous, $\|\Upsilon_2 w_n-\Upsilon_2 w\|\to 0$, and so the operator Υ_2 is continuous on the open ball $\mathcal{V}_r$. Now, we show that Υ_2 is uniformly bounded. Let $w\in\mathcal{V}_r$ and $t\in[1,e]$. Then, we get

$$
\begin{aligned}
\big|(\Upsilon_2 w)(t)\big|\le{}&\frac{\beta}{\kappa\Gamma(\varrho+\mu)}\int_1^t\left(\ln\frac{t}{s}\right)^{\varrho+\mu-1}\big|\varphi\big(s,w(s)\big)\big|\frac{\mathrm{d}s}{s} \\
&+\frac{\beta[|(3-\delta)\Gamma(4-\vartheta)|(\ln t)^3+3|\Gamma(4+\vartheta)|(\ln t)^2]}{6\kappa(\delta+\vartheta)\Gamma(\varrho+\vartheta+\mu)} \\
&\times\int_1^e\left(\ln\frac{e}{s}\right)^{\varrho+\vartheta+\mu-1}\big|\varphi\big(s,w(s)\big)\big|\frac{\mathrm{d}s}{s} \\
&+\frac{\beta|\Gamma(4-\delta)|[3|\Gamma(3+\vartheta)|(\ln t)^2+|\Gamma(4-\vartheta)|(\ln t)^3]}{6\kappa(\delta+\vartheta)\Gamma(3+\vartheta)\Gamma(\varrho+\mu-\delta)} \\
&\times\int_1^e\left(\ln\frac{e}{s}\right)^{\varrho+\mu-\delta-1}\big|\varphi\big(s,w(s)\big)\big|\frac{\mathrm{d}s}{s}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{\beta}{\kappa\Gamma(\varrho+\mu+1)}\|\sigma\| + \frac{\beta[|(3-\delta)\Gamma(4-\vartheta)|+3|\Gamma(4+\vartheta)|]}{6\kappa(\delta+\vartheta)\Gamma(\varrho+\vartheta+\mu+1)}\|\sigma\| \\
&\quad + \frac{\beta|\Gamma(4-\delta)|[3|\Gamma(3+\vartheta)|+|\Gamma(4-\vartheta)|]}{6\kappa(\delta+\vartheta)\Gamma(3+\vartheta)\Gamma(\varrho+\mu-\delta+1)}\|\sigma\| \\
&\leq \|\sigma\|\biggl[\frac{\beta}{\kappa\Gamma(\varrho+\mu+1)} + \frac{\beta[|(3-\delta)\Gamma(4-\vartheta)|+3|\Gamma(4+\vartheta)|]}{6\kappa(\delta+\vartheta)\Gamma(\varrho+\vartheta+\mu+1)} \\
&\quad + \frac{\beta|\Gamma(4-\delta)|[3|\Gamma(3+\vartheta)|+|\Gamma(4-\vartheta)|]}{6\kappa(\delta+\vartheta)\Gamma(3+\vartheta)\Gamma(\varrho+\mu-\delta+1)}\biggr] \\
&= \mathcal{K}_2^*\|\sigma\|
\end{aligned}
$$

which implies that $\|\Upsilon_2 w\| \leq \mathcal{K}_2^*\|\sigma\|$. This shows that Υ_2 is uniformly bounded. Here, we prove that Υ_2 is equicontinuous. Let $t_1, t_2 \in [1,e]$ with $t_1 < t_2$. We show that Υ_2 maps bounded sets into equicontinuous sets. For each $w \in \mathcal{V}_r$, we have

$$
\begin{aligned}
&\bigl|(\Upsilon_2 w)(t_2) - (\Upsilon_2 w)(t_1)\bigr| \\
&\quad\leq \frac{\beta}{\kappa\Gamma(\varrho+\mu)}\int_1^{t_1}\biggl[\biggl(\ln\frac{t_2}{s}\biggr)^{\varrho+\mu-1} - \biggl(\ln\frac{t_1}{s}\biggr)^{\varrho+\mu-1}\biggr]\bigl|\varphi\bigl(s,w(s)\bigr)\bigr|\frac{\mathrm{d}s}{s} \\
&\qquad + \frac{\beta}{\kappa\Gamma(\varrho+\mu)}\int_{t_1}^{t_2}\biggl(\ln\frac{t_2}{s}\biggr)^{\varrho+\mu-1}\bigl|\varphi\bigl(s,w(s)\bigr)\bigr|\frac{\mathrm{d}s}{s} \\
&\qquad + \frac{\beta[|(3-\delta)\Gamma(4-\vartheta)|[(\ln t_2)^3-(\ln t_1)^3]+3|\Gamma(4+\vartheta)|[(\ln t_2)^2-(\ln t_1)^2]]}{6\kappa(\delta+\vartheta)\Gamma(\varrho+\vartheta+\mu)} \\
&\qquad \times \int_1^e\biggl(\ln\frac{e}{s}\biggr)^{\varrho+\vartheta+\mu-1}\bigl|\varphi\bigl(s,w(s)\bigr)\bigr|\frac{\mathrm{d}s}{s} \\
&\qquad + \frac{\beta|\Gamma(4-\delta)|[3|\Gamma(3+\vartheta)|[(\ln t_2)^2-(\ln t_1)^2]+|\Gamma(4-\vartheta)|[(\ln t_2)^3-(\ln t_1)^3]]}{6\kappa(\delta+\vartheta)\Gamma(3+\vartheta)\Gamma(\varrho+\mu-\delta)} \\
&\qquad \times \int_1^e\biggl(\ln\frac{e}{s}\biggr)^{\varrho+\mu-\delta-1}\bigl|\varphi\bigl(s,w(s)\bigr)\bigr|\frac{\mathrm{d}s}{s} \\
&\quad\leq \|\sigma\|\biggl(\frac{2\beta}{\kappa\Gamma(\varrho+\mu+1)}\biggl(\ln\frac{t_2}{t_1}\biggr)^{\varrho+\mu} + \frac{\beta}{\kappa\Gamma(\varrho+\mu+1)}\bigl|(\ln t_2)^{\varrho+\mu}-(\ln t_1)^{\varrho+\mu}\bigr| \\
&\qquad + \frac{\beta[|(3-\delta)\Gamma(4-\vartheta)|[(\ln t_2)^3-(\ln t_1)^3]+3|\Gamma(4+\vartheta)|[(\ln t_2)^2-(\ln t_1)^2]]}{6\kappa(\delta+\vartheta)\Gamma(\varrho+\vartheta+\mu+1)} \\
&\qquad + \frac{\beta|\Gamma(4-\delta)|[3|\Gamma(3+\vartheta)|[(\ln t_2)^2-(\ln t_1)^2]+|\Gamma(4-\vartheta)|[(\ln t_2)^3-(\ln t_1)^3]]}{6\kappa(\delta+\vartheta)\Gamma(3+\vartheta)\Gamma(\varrho+\mu-\delta+1)}\biggr).
\end{aligned}
$$

Note that the right-hand side is independent of $w \in \mathcal{V}_r$ and converges to zero as $t_1 \to t_2$. This means that Υ_2 is equicontinuous. Consequently, the operator Υ_2 is relatively compact on $\mathcal{V}_r$ and, by using the Arzela–Ascoli theorem, we conclude that Υ_2 is completely continuous. Hence, Υ_2 is compact on the open ball $\mathcal{V}_r$. Now, we show that Υ_1 is a contraction. Let $w_1, w_2 \in \mathcal{V}_r$ and $t \in [1,e]$. Then, we have

$$
\begin{aligned}
&\bigl|(\Upsilon_1 w_1)(t) - (\Upsilon_1 w_2)(t)\bigr| \\
&\quad\leq \frac{|\kappa-1|}{\kappa\Gamma(\varrho-\varpi)}\int_1^t\biggl(\ln\frac{t}{s}\biggr)^{\varrho-\varpi-1}\bigl|w_1(s)-w_2(s)\bigr|\frac{\mathrm{d}s}{s}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\alpha}{\kappa\Gamma(\varrho)}\int_1^t\left(\ln\frac{t}{s}\right)^{\varrho-1}\left|\psi\left(s,w_1(s)\right)-\psi\left(s,w_2(s)\right)\right|\frac{\mathrm{d}s}{s}\\
&+\frac{(1-\kappa)[|3\Gamma(4+\vartheta)|(\ln t)^2+|(\delta-3)\Gamma(4-\vartheta)|(\ln t)^3]}{6\kappa(\delta+\vartheta)\Gamma(\varrho-\varpi+\vartheta)}\\
&\times\int_1^e\left(\ln\frac{e}{s}\right)^{\varrho-\varpi+\vartheta-1}\left|w_1(s)-w_2(s)\right|\frac{\mathrm{d}s}{s}\\
&+\frac{(1-\kappa)|\Gamma(4-\delta)|[|\Gamma(4-\vartheta)|(\ln t)^3+3|\Gamma(3+\vartheta)|(\ln t)^2]}{6\kappa(\delta+\vartheta)\Gamma(3+\vartheta)\Gamma(\varrho-\varpi-\delta)}\\
&\times\int_1^e\left(\ln\frac{e}{s}\right)^{\varrho-\varpi-\delta-1}\left|w_1(s)-w_2(s)\right|\frac{\mathrm{d}s}{s}\\
&+\frac{\alpha[|(3-\delta)\Gamma(4-\vartheta)|(\ln t)^3+3|\Gamma(4+\vartheta)|(\ln t)^2]}{6\kappa(\delta+\vartheta)\Gamma(\varrho+\vartheta)}\\
&\times\int_1^e\left(\ln\frac{e}{s}\right)^{\varrho+\vartheta-1}\left|\psi\left(s,w_1(s)\right)-\psi\left(s,w_2(s)\right)\right|\frac{\mathrm{d}s}{s}\\
&+\frac{\alpha|\Gamma(4-\delta)|[3|\Gamma(3+\vartheta)|(\ln t)^2+|\Gamma(4-\vartheta)|(\ln t)^3]}{6\kappa(\delta+\vartheta)\Gamma(3+\vartheta)\Gamma(\varrho-\delta)}\\
&\times\int_1^e\left(\ln\frac{e}{s}\right)^{\varrho-\delta-1}\left|\psi\left(s,w_1(s)\right)-\psi\left(s,w_2(s)\right)\right|\frac{\mathrm{d}s}{s}\\
\le{}&\left[\frac{|\kappa-1|}{\kappa\Gamma(\varrho-\varpi+1)}+\frac{(1-\kappa)[|3\Gamma(4+\vartheta)|+|(\delta-3)\Gamma(4-\vartheta)|]}{6\kappa(\delta+\vartheta)\Gamma(\varrho-\varpi+\vartheta+1)}\right.\\
&+\frac{(1-\kappa)|\Gamma(4-\delta)|[|\Gamma(4-\vartheta)|+3|\Gamma(3+\vartheta)|]}{6\kappa(\delta+\vartheta)\Gamma(3+\vartheta)\Gamma(\varrho-\varpi-\delta+1)}+\frac{L\alpha}{\kappa\Gamma(\varrho+1)}\\
&+\frac{L\alpha[|(3-\delta)\Gamma(4-\vartheta)|+3|\Gamma(4+\vartheta)|]}{6\kappa(\delta+\vartheta)\Gamma(\varrho+\vartheta+1)}\\
&\left.+\frac{L\alpha|\Gamma(4-\delta)|[3|\Gamma(3+\vartheta)|+|\Gamma(4-\vartheta)|]}{6\kappa(\delta+\vartheta)\Gamma(3+\vartheta)\Gamma(\varrho-\delta+1)}\right]\|w_1-w_2\|\\
={}&\left(\mathcal{K}_0^*+L\mathcal{K}_1^*\right)\|w_1-w_2\|.
\end{aligned}
$$

Since $\mathcal{K}_0^*+L\mathcal{K}_1^*<1$, Υ_1 is a contraction. Note that $\Upsilon=\Upsilon_1+\Upsilon_2$. Now, by using Lemma 4, the operator Υ has a fixed point which is a solution for the Caputo–Hadamard boundary value problem (1)–(2). □

Here, we are going to investigate the existence of solutions for the Caputo–Hadamard problem (1)–(2) by considering different conditions.

Theorem 9 *Suppose that $\psi,\varphi:[1,e]\times\mathbb{R}\to\mathbb{R}$ are continuous functions such that*

($\mathcal{N}_3$) *there are continuous nondecreasing functions $\xi_1,\xi_2:[0,\infty)\to(0,\infty)$ and two maps $\theta_1,\theta_2\in C([0,1],\mathbb{R}^+)$ such that $|\psi(t,w)|\le\theta_1(t)\xi_1(|w|)$ and $|\varphi(t,w)|\le\theta_2(t)\xi_2(|w|)$ for all $(t,w)\in[1,e]\times\mathbb{R}$,*

($\mathcal{N}_4$) *$\mathcal{K}_0^*<1$ and there is a constant $\Xi>0$ such that $\frac{(1-\mathcal{K}_0^*)\Xi}{\mathcal{K}_1^*\|\theta_1\|\xi_1(\Xi)+\mathcal{K}_2^*\|\theta_2\|\xi_2(\Xi)}>1$, where $\mathcal{K}_0^*$, $\mathcal{K}_1^*$, $\mathcal{K}_2^*$ are defined by (7).*

Then the Caputo–Hadamard problem (1)–(2) has at least one solution.

Proof We first show that the operator Υ maps bounded sets of $\mathcal{E}$ into bounded sets. Let $\epsilon > 0$, $\mathcal{B}_\epsilon = \{w \in \mathcal{E} : \|w\| \leq \epsilon\}$ and $t \in [1, e]$. Then, we have

$$\begin{aligned}
\bigl|(\Upsilon w)(t)\bigr| &\leq \frac{|\kappa - 1|}{\kappa\Gamma(\varrho - \varpi)} \int_1^t \left(\ln\frac{t}{s}\right)^{\varrho-\varpi-1} \|w\| \frac{\mathrm{d}s}{s} \\
&\quad + \frac{\alpha}{\kappa\Gamma(\varrho)} \int_1^t \left(\ln\frac{t}{s}\right)^{\varrho-1} \|\theta_1\| \xi_1\bigl(\|w\|\bigr) \frac{\mathrm{d}s}{s} \\
&\quad + \frac{\beta}{\kappa\Gamma(\varrho + \mu)} \int_1^t \left(\ln\frac{t}{s}\right)^{\varrho+\mu-1} \|\theta_2\| \xi_2\bigl(\|w\|\bigr) \frac{\mathrm{d}s}{s} \\
&\quad + \frac{(1-\kappa)[|3\Gamma(4+\vartheta)|(\ln t)^2 + |(\delta - 3)\Gamma(4-\vartheta)|(\ln t)^3]}{6\kappa(\delta+\vartheta)\Gamma(\varrho - \varpi + \vartheta)} \\
&\quad \times \int_1^e \left(\ln\frac{e}{s}\right)^{\varrho-\varpi+\vartheta-1} \|w\| \frac{\mathrm{d}s}{s} \\
&\quad + \frac{(1-\kappa)|\Gamma(4-\delta)|[|\Gamma(4-\vartheta)|(\ln t)^3 + 3|\Gamma(3+\vartheta)|(\ln t)^2]}{6\kappa(\delta+\vartheta)\Gamma(3+\vartheta)\Gamma(\varrho - \varpi - \delta)} \\
&\quad \times \int_1^e \left(\ln\frac{e}{s}\right)^{\varrho-\varpi-\delta-1} \|w\| \frac{\mathrm{d}s}{s} \\
&\quad + \frac{\alpha[|(3-\delta)\Gamma(4-\vartheta)|(\ln t)^3 + 3|\Gamma(4+\vartheta)|(\ln t)^2]}{6\kappa(\delta+\vartheta)\Gamma(\varrho+\vartheta)} \\
&\quad \times \int_1^e \left(\ln\frac{e}{s}\right)^{\varrho+\vartheta-1} \|\theta_1\| \xi_1\bigl(\|w\|\bigr) \frac{\mathrm{d}s}{s} \\
&\quad + \frac{\beta[|(3-\delta)\Gamma(4-\vartheta)|(\ln t)^3 + 3|\Gamma(4+\vartheta)|(\ln t)^2]}{6\kappa(\delta+\vartheta)\Gamma(\varrho+\vartheta+\mu)} \\
&\quad \times \int_1^e \left(\ln\frac{e}{s}\right)^{\varrho+\vartheta+\mu-1} \|\theta_2\| \xi_2\bigl(\|w\|\bigr) \frac{\mathrm{d}s}{s} \\
&\quad + \frac{\alpha|\Gamma(4-\delta)|[3|\Gamma(3+\vartheta)|(\ln t)^2 + |\Gamma(4-\vartheta)|(\ln t)^3]}{6\kappa(\delta+\vartheta)\Gamma(3+\vartheta)\Gamma(\varrho-\delta)} \\
&\quad \times \int_1^e \left(\ln\frac{e}{s}\right)^{\varrho-\delta-1} \|\theta_1\| \xi_1\bigl(\|w\|\bigr) \frac{\mathrm{d}s}{s} \\
&\quad + \frac{\beta|\Gamma(4-\delta)|[3|\Gamma(3+\vartheta)|(\ln t)^2 + |\Gamma(4-\vartheta)|(\ln t)^3]}{6\kappa(\delta+\vartheta)\Gamma(3+\vartheta)\Gamma(\varrho+\mu-\delta)} \\
&\quad \times \int_1^e \left(\ln\frac{e}{s}\right)^{\varrho+\mu-\delta-1} \|\theta_2\| \xi_2\bigl(\|w\|\bigr) \frac{\mathrm{d}s}{s} \\
&\leq \mathcal{K}_0^* \|w\| + \mathcal{K}_1^* \|\theta_1\| \xi_1\bigl(\|w\|\bigr) + \mathcal{K}_2^* \|\theta_2\| \xi_2\bigl(\|w\|\bigr).
\end{aligned}$$

Hence, $\|\Upsilon w\| \leq \mathcal{K}_0^* \epsilon + \mathcal{K}_1^* \|\theta_1\| \xi_1(\epsilon) + \mathcal{K}_2^* \|\theta_2\| \xi_1(\epsilon)$. Now, we prove that Υ maps bounded sets into equicontinuous sets of $\mathcal{E}$. Let $t_1, t_2 \in [1, e]$ with $t_1 < t_2$ and $w \in \mathcal{B}_\epsilon$. Then, we get

$$\begin{aligned}
&\bigl|(\Upsilon w)(t_2) - (\Upsilon w)(t_1)\bigr| \\
&\quad \leq \frac{|\kappa - 1|\epsilon}{\kappa\Gamma(\varrho - \varpi)} \left[\int_1^{t_1} \left[\left(\ln\frac{t_2}{s}\right)^{\varrho+\varpi-1} - \left(\ln\frac{t_1}{s}\right)^{\varrho+\varpi-1}\right] \frac{\mathrm{d}s}{s} + \int_{t_1}^{t_2} \left(\ln\frac{t_2}{s}\right)^{\varrho+\varpi-1} \frac{\mathrm{d}s}{s}\right]
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\alpha\|\theta_1\|\xi_1(\epsilon)}{\kappa\Gamma(\varrho)}\left[\int_1^{t_1}\left[\left(\ln\frac{t_2}{s}\right)^{\varrho-1}-\left(\ln\frac{t_1}{s}\right)^{\varrho-1}\right]\frac{ds}{s}+\int_{t_1}^{t_2}\left(\ln\frac{t_2}{s}\right)^{\varrho-1}\frac{ds}{s}\right]\\
&+\frac{\beta\|\theta_2\|\xi_2(\epsilon)}{\kappa\Gamma(\varrho+\mu)}\left[\int_1^{t_1}\left[\left(\ln\frac{t_2}{s}\right)^{\varrho+\mu-1}-\left(\ln\frac{t_1}{s}\right)^{\varrho+\mu-1}\right]\frac{ds}{s}+\int_{t_1}^{t_2}\left(\ln\frac{t_2}{s}\right)^{\varrho+\mu-1}\frac{ds}{s}\right]\\
&+\frac{(1-\kappa)[|3\Gamma(4+\vartheta)|[(\ln t_2)^2-(\ln t_1)^2]+|(\delta-3)\Gamma(4-\vartheta)|[(\ln t_2)^3-(\ln t_1)^3]]\epsilon}{6\kappa(\delta+\vartheta)\Gamma(\varrho-\varpi+\vartheta+1)}\\
&+\Big((1-\kappa)\big|\Gamma(4-\delta)\big|\Big[\big|\Gamma(4-\vartheta)\big|\big[(\ln t_2)^3-(\ln t_1)^3\big]\\
&+3\big|\Gamma(3+\vartheta)\big|\big[(\ln t_2)^2-(\ln t_1)^2\big]\Big]\epsilon\Big)\\
&/\big(6\kappa(\delta+\vartheta)\Gamma(3+\vartheta)\Gamma(\varrho-\varpi-\delta+1)\big)\\
&+\Big(\alpha\Big[\big|(3-\delta)\Gamma(4-\vartheta)\big|\big[(\ln t_2)^3-(\ln t_1)^3\big]\\
&+3\big|\Gamma(4+\vartheta)\big|\big[(\ln t_2)^2-(\ln t_1)^2\big]\Big]\|\theta_1\|\xi_1(\epsilon)\Big)\\
&/\big(6\kappa(\delta+\vartheta)\Gamma(\varrho+\vartheta+1)\big)\\
&+\Big(\beta\Big[\big|(3-\delta)\Gamma(4-\vartheta)\big|\big[(\ln t_2)^3-(\ln t_1)^3\big]\\
&+3\big|\Gamma(4+\vartheta)\big|\big[(\ln t_2)^2-(\ln t_1)^2\big]\Big]\|\theta_2\|\xi_2(\epsilon)\Big)\\
&/\big(6\kappa(\delta+\vartheta)\Gamma(\varrho+\vartheta+\mu+1)\big)\\
&+\Big(\alpha\big|\Gamma(4-\delta)\big|\Big[3\big|\Gamma(3+\vartheta)\big|\big[(\ln t_2)^2-(\ln t_1)^2\big]\\
&+\big|\Gamma(4-\vartheta)\big|\big[(\ln t_2)^3-(\ln t_1)^3\big]\Big]\|\theta_1\|\xi_1(\epsilon)\Big)\\
&/\big(6\kappa(\delta+\vartheta)\Gamma(3+\vartheta)\Gamma(\varrho-\delta+1)\big)\\
&+\Big(\beta\big|\Gamma(4-\delta)\big|\Big[3\big|\Gamma(3+\vartheta)\big|\big[(\ln t_2)^2-(\ln t_1)^2\big]\\
&+\big|\Gamma(4-\vartheta)\big|\big[(\ln t_2)^3-(\ln t_1)^3\big]\Big]\|\theta_2\|\xi_2(\epsilon)\Big)\\
&/\big(6\kappa(\delta+\vartheta)\Gamma(3+\vartheta)\Gamma(\varrho+\mu-\delta+1)\big).
\end{aligned}
$$

Note that the right-hand side tends to zero independently of $w\in\mathcal{B}_\epsilon$ as $t_2\to t_1$. By using the Arzela–Ascoli theorem, we deduce that $\Upsilon:\mathcal{E}\to\mathcal{E}$ is completely continuous. Here, we prove that the set of all solutions of the equation $w=\lambda(\Upsilon w)$ is bounded for each $\lambda\in[0,1]$. Let $\lambda\in[0,1]$, w be such that $w=\lambda(\Upsilon w)$ and $t\in[1,e]$. Then by using computations used in the first step, we obtain $\|w\|\le\mathcal{K}_0^*\|w\|+\mathcal{K}_1^*\|\theta_1\|\xi_1(\|w\|)+\mathcal{K}_2^*\|\theta_2\|\xi_2(\|w\|)$. Thus, we conclude that $\frac{(1-\mathcal{K}_0^*)\|w\|}{\mathcal{K}_1^*\|\theta_1\|\xi_1(\|w\|)+\mathcal{K}_2^*\|\theta_2\|\xi_2(\|w\|)}\le 1$. By using the assumption $(\mathcal{N}_4)$, we can choose a number $\Xi>0$ such that $\|w\|\ne\Xi$ and $\frac{(1-\mathcal{K}_0^*)\Xi}{\mathcal{K}_1^*\|\theta_1\|\xi_1(\Xi)+\mathcal{K}_2^*\|\theta_2\|\xi_2(\Xi)}>1$. Consider the set $\mathcal{U}=\{w\in\mathcal{E}:\|w\|<\Xi\}$. Note that the operator $\Upsilon:\overline{\mathcal{U}}\to\mathcal{E}$ is continuous and completely continuous and also we can not find $w\in\partial\mathcal{U}$ such that $w=\lambda(\Upsilon w)$ holds for some $\lambda\in(0,1)$. Now, by using Lemma 5, the operator Υ has a fixed point in $\overline{\mathcal{U}}$ which is a solution for the Caputo–Hadamard fractional integro-differential boundary value problem (1)–(2). □

Now by using the Banach contraction principle, we review the Caputo–Hadamard problem (1)–(2) under some different conditions.

Theorem 10 *Suppose that $\psi:[1,e]\times\mathbb{R}\to\mathbb{R}$ is a continuous function satisfying assumption $(\mathcal{N}_1)$. Assume that the function $\varphi:[1,e]\times\mathbb{R}\to\mathbb{R}$ satisfies the following condition*:

($\mathcal{N}_5$) *there is a positive constant $\tilde{L}$ such that for each $|\varphi(t,w_1)-\varphi(t,w_2)|\leq\tilde{L}|w_1-w_2|$ for all $w_1,w_2\in\mathbb{R}$ and $t\in[1,e]$.*

If $\mathcal{K}_0^*+L\mathcal{K}_1^*+\tilde{L}\mathcal{K}_2^*<1$, then the Caputo–Hadamard problem (1)–(2) has a unique solution, where $\mathcal{K}_0^*$, $\mathcal{K}_1^*$, and $\mathcal{K}_2^*$ are given by (7).

Proof Put $K^*=\sup_{t\in[1,e]}|\psi(t,0)|<\infty$ and $N^*=\sup_{t\in[1,e]}|\varphi(t,0)|<\infty$. Choose $r>0$ such that $r\geq\frac{N^*\mathcal{K}_2^*+K^*\mathcal{K}_1^*}{1-(\mathcal{K}_0^*+L\mathcal{K}_1^*+\tilde{L}\mathcal{K}_2^*)}$. Let $\mathcal{B}_r=\{w\in\mathcal{E}:\|w\|\leq r\}$. We show that $\Upsilon\mathcal{B}_r\subset\mathcal{B}_r$. Let $w\in\mathcal{B}_r$. By using assumptions ($\mathcal{N}_1$) and ($\mathcal{N}_5$), we have

$$\begin{aligned}
\|\Upsilon w\| &\leq \frac{|\kappa-1|}{\kappa\Gamma(\varrho-\varpi)}\int_1^t\left(\ln\frac{t}{s}\right)^{\varrho-\varpi-1}\|w\|\frac{\mathrm{d}s}{s}\\
&\quad+\frac{\alpha}{\kappa\Gamma(\varrho)}\int_1^t\left(\ln\frac{t}{s}\right)^{\varrho-1}\big(L\|w\|+K^*\big)\frac{\mathrm{d}s}{s}\\
&\quad+\frac{\beta}{\kappa\Gamma(\varrho+\mu)}\int_1^t\left(\ln\frac{t}{s}\right)^{\varrho+\mu-1}\big(\tilde{L}\|w\|+N^*\big)\frac{\mathrm{d}s}{s}\\
&\quad+\frac{(1-\kappa)[|3\Gamma(4+\vartheta)|(\ln t)^2+|(\delta-3)\Gamma(4-\vartheta)|(\ln t)^3]}{6\kappa(\delta+\vartheta)\Gamma(\varrho-\varpi+\vartheta)}\\
&\quad\times\int_1^e\left(\ln\frac{e}{s}\right)^{\varrho-\varpi+\vartheta-1}\|w\|\frac{\mathrm{d}s}{s}\\
&\quad+\frac{(1-\kappa)|\Gamma(4-\delta)|[|\Gamma(4-\vartheta)|(\ln t)^3+3|\Gamma(3+\vartheta)|(\ln t)^2]}{6\kappa(\delta+\vartheta)\Gamma(3+\vartheta)\Gamma(\varrho-\varpi-\delta)}\\
&\quad\times\int_1^e\left(\ln\frac{e}{s}\right)^{\varrho-\varpi-\delta-1}\|w\|\frac{\mathrm{d}s}{s}\\
&\quad+\frac{\alpha[|(3-\delta)\Gamma(4-\vartheta)|(\ln t)^3+3|\Gamma(4+\vartheta)|(\ln t)^2]}{6\kappa(\delta+\vartheta)\Gamma(\varrho+\vartheta)}\\
&\quad\times\int_1^e\left(\ln\frac{e}{s}\right)^{\varrho+\vartheta-1}\big(L\|w\|+K^*\big)\frac{\mathrm{d}s}{s}\\
&\quad+\frac{\beta[|(3-\delta)\Gamma(4-\vartheta)|(\ln t)^3+3|\Gamma(4+\vartheta)|(\ln t)^2]}{6\kappa(\delta+\vartheta)\Gamma(\varrho+\vartheta+\mu)}\\
&\quad\times\int_1^e\left(\ln\frac{e}{s}\right)^{\varrho+\vartheta+\mu-1}\big(\tilde{L}\|w\|+N^*\big)\frac{\mathrm{d}s}{s}\\
&\quad+\frac{\alpha|\Gamma(4-\delta)|[3|\Gamma(3+\vartheta)|(\ln t)^2+|\Gamma(4-\vartheta)|(\ln t)^3]}{6\kappa(\delta+\vartheta)\Gamma(3+\vartheta)\Gamma(\varrho-\delta)}\\
&\quad\times\int_1^e\left(\ln\frac{e}{s}\right)^{\varrho-\delta-1}\big(L\|w\|+K^*\big)\frac{\mathrm{d}s}{s}\\
&\quad+\frac{\beta|\Gamma(4-\delta)|[3|\Gamma(3+\vartheta)|(\ln t)^2+|\Gamma(4-\vartheta)|(\ln t)^3]}{6\kappa(\delta+\vartheta)\Gamma(3+\vartheta)\Gamma(\varrho+\mu-\delta)}\\
&\quad\times\int_1^e\left(\ln\frac{e}{s}\right)^{\varrho+\mu-\delta-1}\big(\tilde{L}\|w\|+N^*\big)\frac{\mathrm{d}s}{s}\\
&\leq\big(\mathcal{K}_0^*+L\mathcal{K}_1^*+\tilde{L}\mathcal{K}_2^*\big)r+\mathcal{K}_2^*N^*+\mathcal{K}_1^*K^*<r.
\end{aligned}$$

Hence, $\Upsilon\mathcal{B}_r \subset \mathcal{B}_r$. Let $t \in [1,e]$ and $w_1, w_2 \in \mathbb{R}$. Then, we have

$$\begin{aligned}
\big\|(\Upsilon w_1)(t) - (\Upsilon w_2)(t)\big\| \leq{}& \frac{|\kappa - 1|}{\kappa\Gamma(\varrho - \varpi)} \int_1^t \left(\ln\frac{t}{s}\right)^{\varrho-\varpi-1} \big|w_1(s) - w_2(s)\big| \frac{\mathrm{d}s}{s} \\
&+ \frac{\alpha}{\kappa\Gamma(\varrho)} \int_1^t \left(\ln\frac{t}{s}\right)^{\varrho-1} \big|\psi\big(s, w_1(s)\big) - \psi\big(s, w_2(s)\big)\big| \frac{\mathrm{d}s}{s} \\
&+ \frac{\beta}{\kappa\Gamma(\varrho + \mu)} \int_1^t \left(\ln\frac{t}{s}\right)^{\varrho+\mu-1} \big|\varphi\big(s, w_1(s)\big) - \varphi\big(s, w_2(s)\big)\big| \frac{\mathrm{d}s}{s} \\
&+ \frac{(1-\kappa)[|3\Gamma(4+\vartheta)|(\ln t)^2 + |(\delta - 3)\Gamma(4-\vartheta)|(\ln t)^3]}{6\kappa(\delta+\vartheta)\Gamma(\varrho - \varpi + \vartheta)} \\
&\times \int_1^e \left(\ln\frac{e}{s}\right)^{\varrho-\varpi+\vartheta-1} \big|w_1(s) - w_2(s)\big| \frac{\mathrm{d}s}{s} \\
&+ \frac{(1-\kappa)|\Gamma(4-\delta)|[|\Gamma(4-\vartheta)|(\ln t)^3 + 3|\Gamma(3+\vartheta)|(\ln t)^2]}{6\kappa(\delta+\vartheta)\Gamma(3+\vartheta)\Gamma(\varrho - \varpi - \delta)} \\
&\times \int_1^e \left(\ln\frac{e}{s}\right)^{\varrho-\varpi-\delta-1} \big|w_1(s) - w_2(s)\big| \frac{\mathrm{d}s}{s} \\
&+ \frac{\alpha[|(3-\delta)\Gamma(4-\vartheta)|(\ln t)^3 + 3|\Gamma(4+\vartheta)|(\ln t)^2]}{6\kappa(\delta+\vartheta)\Gamma(\varrho + \vartheta)} \\
&\times \int_1^e \left(\ln\frac{e}{s}\right)^{\varrho+\vartheta-1} \big|\psi\big(s, w_1(s)\big) - \psi\big(s, w_2(s)\big)\big| \frac{\mathrm{d}s}{s} \\
&+ \frac{\beta[|(3-\delta)\Gamma(4-\vartheta)|(\ln t)^3 + 3|\Gamma(4+\vartheta)|(\ln t)^2]}{6\kappa(\delta+\vartheta)\Gamma(\varrho + \vartheta + \mu)} \\
&\times \int_1^e \left(\ln\frac{e}{s}\right)^{\varrho+\vartheta+\mu-1} \big|\varphi\big(s, w_1(s)\big) - \varphi\big(s, w_2(s)\big)\big| \frac{\mathrm{d}s}{s} \\
&+ \frac{\alpha|\Gamma(4-\delta)|[3|\Gamma(3+\vartheta)|(\ln t)^2 + |\Gamma(4-\vartheta)|(\ln t)^3]}{6\kappa(\delta+\vartheta)\Gamma(3+\vartheta)\Gamma(\varrho - \delta)} \\
&\times \int_1^e \left(\ln\frac{e}{s}\right)^{\varrho-\delta-1} \big|\psi\big(s, w_1(s)\big) - \psi\big(s, w_2(s)\big)\big| \frac{\mathrm{d}s}{s} \\
&+ \frac{\beta|\Gamma(4-\delta)|[3|\Gamma(3+\vartheta)|(\ln t)^2 + |\Gamma(4-\vartheta)|(\ln t)^3]}{6\kappa(\delta+\vartheta)\Gamma(3+\vartheta)\Gamma(\varrho + \mu - \delta)} \\
&\times \int_1^e \left(\ln\frac{e}{s}\right)^{\varrho+\mu-\delta-1} \big|\varphi\big(s, w_1(s)\big) - \varphi\big(s, w_2(s)\big)\big| \frac{\mathrm{d}s}{s} \\
\leq{}& \big(\mathcal{K}_0^* + L\mathcal{K}_1^* + \tilde{L}\mathcal{K}_2^*\big)\|w_1 - w_2\|.
\end{aligned}$$

Since we have $\mathcal{K}_0^* + L\mathcal{K}_1^* + \tilde{L}\mathcal{K}_2^* < 1$, Υ is a contraction. By using the Banach contraction principle, Υ has a unique fixed point which is the unique solution of the Caputo–Hadamard problem (1)–(2). This completes the proof. □

4 Examples

In this section, we provide three numerical examples to examine the validity of our theoretical findings. To do this, we consider constants $\kappa = 0.78$, $\alpha = 0.69$, $\beta = 0.73$, $\varrho = 3.95$, $\varpi = 3.87$, $\mu = 1.3$, $\delta = 1.92$, and $\vartheta = 0.001$ with $\delta + \vartheta = 1.921 \neq 0$ for our examples. The next example illustrates Theorem 8.

Example 1 Consider the Caputo–Hadamard fractional integro-differential equation

$$\left[0.78^{\mathrm{CH}}\mathcal{D}_{1^+}^{3.95} + 0.22^{\mathrm{CH}}\mathcal{D}_{1^+}^{3.87}\right]w(t) = 0.69\frac{0.01t|w(t)|}{7+|w(t)|} + 0.73^{H}\mathcal{I}_{1^+}^{1.3}\ln t\bigl(\sin w(t)\bigr) \tag{8}$$

with boundary value conditions

$$\begin{cases} w(1)=0, & {}^{\mathrm{CH}}\mathcal{D}_{1^+}^{1.92}w(e)=0,\\ {}^{\mathrm{CH}}\mathcal{D}_{1^+}w(1)=0, & \frac{1}{\Gamma(0.001)}\int_1^e(\ln\frac{e}{s})^{0.001-1}w(s)\frac{ds}{s}=0, \end{cases} \tag{9}$$

where $t\in[1,e]$. Define the continuous functions $\psi,\varphi:[1,e]\times\mathbb{R}\to\mathbb{R}$ by $\psi(t,w)=\frac{0.99t|w|}{7+|w|}$ and $\varphi(t,w)=\ln t(\sin w)$. Note that $|\psi(t,w_1)-\psi(t,w_2)|\le L|w_1-w_2|$ holds for all $w_1,w_2\in\mathbb{R}$, where $L=0.01e>0$. Also, the continuous function $\sigma(t)=\ln t$ on $[1,e]$ is such that $|\varphi(t,w)|\le\sigma(t)=\ln t$ for all $w\in\mathbb{R}$. In this case, we have $\|\sigma\|=\sup_{t\in[1,e]}\sigma(t)=1$. Some calculations show that $\mathcal{K}_0^*=0.8968$ and $\mathcal{K}_1^*=0.3559$. Hence, $\mathcal{K}_0^*+L\mathcal{K}_1^*=0.9064<1$. By using Theorem 8, the Caputo–Hadamard problem (8)–(9) has a solution.

Next example illustrates Theorem 9.

Example 2 Consider the Caputo–Hadamard fractional integro-differential equation

$$\begin{aligned}\left[0.78^{\mathrm{CH}}\mathcal{D}_{1^+}^{3.95} + 0.22^{\mathrm{CH}}\mathcal{D}_{1^+}^{3.87}\right]w(t) &= 0.69\frac{1}{16+t}\left(\frac{3}{4}+\frac{|w(t)|}{2+|w(t)|}\right)\\ &\quad + 0.73^{H}\mathcal{I}_{1^+}^{1.3}\frac{1}{3+\sin\frac{\pi t}{2}}\left(\frac{4}{5}+\frac{|w(t)|}{3+|w(t)|}\right)\end{aligned} \tag{10}$$

with boundary value conditions

$$\begin{cases} w(1)=0, & {}^{\mathrm{CH}}\mathcal{D}_{1^+}^{1.92}w(e)=0,\\ {}^{\mathrm{CH}}\mathcal{D}_{1^+}w(1)=0, & \frac{1}{\Gamma(0.001)}\int_1^e(\ln\frac{e}{s})^{0.001-1}w(s)\frac{ds}{s}=0, \end{cases} \tag{11}$$

where $t\in[1,e]$. Define continuous maps $\psi,\varphi:[1,e]\times\mathbb{R}\to\mathbb{R}$ by $\psi(t,w)=\frac{1}{16+t}(\frac{3}{4}+\frac{|w|}{2+|w|})$ and $\varphi(t,w)=\frac{1}{3+\sin\frac{\pi t}{2}}(\frac{4}{5}+\frac{|w|}{3+|w|})$. Note that $|\psi(t,w(t))|\le\frac{1}{16+t}(1+\|w\|)$ and $|\varphi(t,w(t))|\le\frac{1}{3+\sin\frac{\pi t}{2}}(1+\|w\|)$ for all $w\in\mathbb{R}$ and $t\in[1,e]$. Put $\theta_1(t)=\frac{1}{16+t}$, $\theta_2(t)=\frac{1}{3+\sin\frac{\pi t}{2}}$, and $\xi_1(\|w\|)=\xi_2(\|w\|)=1+\|w\|$. Then, we have $\psi(t,w)\le\theta_1(t)\xi_1(|w|)$ and $\varphi(t,w)\le\theta_2(t)\xi_2(|w|)$. Note that $\|\theta_1\|=\frac{1}{17}=0.0588$, $\|\theta_2\|=\frac{1}{4}=0.25$, and $\xi_1(\Xi)=\xi_2(\Xi)=1+\Xi$. Also, $\mathcal{K}_0^*=0.8968<1$, $\mathcal{K}_1^*=0.3559$, and $\mathcal{K}_2^*=0.2995$. By considering assumption $(\mathcal{N}_4)$, choose $\Xi>12.76$. Now by using Theorem 9, the Caputo–Hadamard problem (10)–(11) has a solution.

Next example illustrates Theorem 10.

Example 3 Consider the Caputo–Hadamard fractional integro-differential equation

$$\begin{aligned}&\left[0.78^{\mathrm{CH}}\mathcal{D}_{1^+}^{3.95} + 0.22^{\mathrm{CH}}\mathcal{D}_{1^+}^{3.87}\right]w(t)\\ &\quad = 0.69\frac{\cos t|w(t)|}{1+|w(t)|} + 0.73^{H}\mathcal{I}_{1^+}^{1.3}\frac{2}{5+t}\left(\frac{|\arctan w(t)|}{|\arctan w(t)|+1}\right)\end{aligned} \tag{12}$$

with boundary value conditions

$$\begin{cases} w(1) = 0, & {}^{\mathrm{CH}}\mathcal{D}_{1^+}^{1.92} w(e) = 0, \\ {}^{\mathrm{CH}}\mathcal{D}_{1^+} w(1) = 0, & \frac{1}{\Gamma(0.001)} \int_1^e (\ln \frac{e}{s})^{0.001-1} w(s) \frac{ds}{s} = 0, \end{cases} \tag{13}$$

where $t \in [1,e]$. Define continuous maps $\psi, \varphi : [1,e] \times \mathbb{R} \to \mathbb{R}$ by $\psi(t,w) = \frac{\cos t|w|}{1+|w|}$ and $\varphi(t,w) = \frac{2}{7+t}(\frac{|\arctan w(t)|}{|\arctan w(t)|+1})$. Note that $|\psi(t,w_1(t)) - \psi(t,w_2(t))| \leq \cos t(|w_1(t) - w_2(t)|)$ and $|\varphi(t,w_1(t)) - \varphi(t,w_2(t))| \leq \frac{2}{7+t}(|w_1(t) - w_2(t)|)$. Put $L = |\cos(e)| = 0.9117$ and $\tilde{L} = 0.25$. Some calculations show that $\mathcal{K}_0^* + L\mathcal{K}_1^* + \tilde{L}\mathcal{K}_2^* = 0.98127 < 1$. Now by using Theorem 10, the Caputo–Hadamard problem (12)–(13) has a unique solution.

5 Conclusions

It is known that we should increase our ability for studying of different types of fractional integro-differential equations. In this case, we could create modern software in the future by using advanced modelings of distinct phenomena. In this way, we should try to review different types of fractional integro-differential equations. In this work, we study the existence of solutions for a Caputo–Hadamard fractional integro-differential equation with boundary value conditions involving the Hadamard fractional operators via different orders. Also, we provide three examples to illustrate our main results.

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References

1. Alizadeh, Sh., Baleanu, D., Rezapour, Sh.: Analyzing transient response of the parallel RCL circuit by using the Caputo–Fabrizio fractional derivative. Adv. Differ. Equ. **2020**, 55 (2020). https://doi.org/10.1186/s13662-020-2527-0
2. Baleanu, D., Etemad, S., Rezapour, Sh.: A hybrid Caputo fractional modeling for thermostat with hybrid boundary value conditions. Bound. Value Probl. **2020**, 64 (2020). https://doi.org/10.1186/s13661-020-01361-0

3. Baleanu, D., Jajarmi, A., Mohammadi, H., Rezapour, Sh.: A new study on the mathematical modeling of human liver with Caputo–Fabrizio fractional derivative. Chaos Solitons Fractals **134**, 109705 (2020). https://doi.org/10.1016/j.chaos.2020.109705
4. Baleanu, D., Mohammadi, H., Rezapour, Sh.: Analysis of the model of HIV-1 infection of CD4+ T-cell with a new approach of fractional derivative. Adv. Differ. Equ. **2020**, 71 (2020). https://doi.org/10.1186/s13662-020-02544-w
5. Alijani, Z., Baleanu, D., Shiri, B., Wu, G.C.: Spline collocation methods for systems of fuzzy fractional differential equations. Chaos Solitons Fractals **131**, 109510 (2020)
6. Baleanu, D., Shiri, B.: Collocation methods for fractional differential equations involving non-singular kernel. Chaos Solitons Fractals **116**, 136–145 (2018)
7. Baleanu, D., Shiri, B., Srivastava, H.M., Qurashi, M.A.: A Chebyshev spectral method based on operational matrix for fractional differential equations involving non-singular Mittag-Leffler kernel. Adv. Differ. Equ. **2018**, 353 (2018)
8. Bhatter, S., Mathur, A., Kumar, D., Singh, J.: A new analysis of fractional Drinfeld–Sokolov–Wilson model with exponential memory. Phys. A, Stat. Mech. Appl. **537**, 122578 (2020)
9. Khiabani, E.D., Ghaffarzadeh, H., Shiri, B., Katebi, J.: Spline collocation methods for seismic analysis of multiple degree of freedom systems with visco-elastic dampers using fractional models. J. Vib. Control (2020, in press). https://doi.org/10.1177/1077546319898570
10. Goswami, A., Rathore, S., Singh, J., Kumar, D.: Numerical computation of fractional Kersten–Krasil'shchik coupled KdV-mKdV system arising in multi-component plasmas. AIMS Math. **5**(3), 2346–2368 (2020)
11. Ma, C.Y., Shiri, B., Wu, G.C., Baleanu, D.: New signal smoothing equations with short memory and variable order. Optik **2020**, 164507 (2020). https://doi.org/10.1016/j.ijleo.2020.164507
12. Shiri, B., Baleanu, D.: System of fractional differential algebraic equations with applications. Chaos Solitons Fractals **120**, 203–212 (2019)
13. Singh, J., Kumar, D., Baleanu, D.: A new analysis of fractional fish farm model associated with Mittag-Leffler type kernel. Int. J. Biomath. **13**(2), 2050010 (2020)
14. Veeresha, P., Prakasha, D.G., Kumar, D., Baleanu, D., Singh, J.: An efficient computational technique for fractional model of generalized Hirota–Satsuma coupled KdV and coupled mKdV equations. J. Comput. Nonlinear Dyn. **15**(7), 071003 (2020)
15. Baleanu, D., Rezapour, Sh., Mohammadi, H.: Some existence results on nonlinear fractional differential equations. Philos. Trans. R. Soc. Lond. A **371**, 20120144 (2013). https://doi.org/10.1098/rsta.2012.0144
16. Delbosco, D.: Fractional calculus and function spaces. J. Fract. Calc. **6**, 45–53 (1994)
17. Zhang, S.: The existence of a positive solution for a nonlinear fractional differential equation. J. Math. Anal. Appl. **252**(2), 804–812 (2000). https://doi.org/10.1006/jmaa.2000.7123
18. Bai, Z., Lü, H.: Positive solutions for boundary value problem of nonlinear fractional differential equation. J. Math. Anal. Appl. **311**(2), 495–505 (2005). https://doi.org/10.1016/j.jmaa.2005.02.052
19. Kochubei, A.N.: Distributed order calculus and equations of ultraslow diffusion. J. Math. Anal. Appl. **340**(1), 252–281 (2008). https://doi.org/10.1016/j.jmaa.2007.08.024
20. Leggett, R.W., Williams, L.R.: Multiple positive fixed points of nonlinera operators on ordered Banach spaces. Indiana Univ. Math. J. **28**(4), 673–688 (1979) http://www.jstor.org/stable/24892256
21. Agarwal, R.P., O'Regan, D., Staněk, S.: The existence of solutions for a nonlinear mixed problem of singular fractional differential equations. Math. Nachr. **285**(1), 27–41 (2012). https://doi.org/10.1002/mana.201000043
22. Agarwal, R.P., O'Regan, D., Staněk, S.: Positive solutions for Dirichlet problems of singular nonlinear fractional differential equations. J. Math. Anal. Appl. **371**(1), 57–68 (2010). https://doi.org/10.1016/j.jmaa.2010.04.034
23. Jiang, M., Zhong, Sh.: Existence of solutions for nonlinear fractional q-difference equations with Riemann–Liouville type q-derivatives. J. Appl. Math. Comput. **47**(1–2), 429–459 (2015). https://doi.org/10.1007/s12190-014-0784-3
24. Zhang, X., Zhong, Q.: Multiple positive solutions for nonlocal boundary value problems of singular fractional differential equations. Bound. Value Probl. **2016**, 65 (2016). https://doi.org/10.1186/s13661-016-0572-0
25. Zhou, H., Alzabut, J., Yang, L.: On fractional Langevin differential equations with anti-periodic boundary conditions. Eur. Phys. J. Spec. Top. **226**, 3577–3590 (2017). https://doi.org/10.1140/epjst/e2018-00082-0
26. Xu, X., Jiang, D., Yuan, Ch.: Multiple positive solutions for the boundary value problem of a nonlinear fractional differential equation. Nonlinear Anal. **71**, 4676–4688 (2009). https://doi.org/10.1016/j.na.2009.03.030
27. Ahmad, B., Nieto, J.J.: Riemann–Liouville fractional integro-differential equations with fractional nonlocal integral boundary conditions. Bound. Value Probl. **2011**, 36 (2011). https://doi.org/10.1186/1687-2770-2011-36
28. Su, X., Zhang, S.: Solutions to boundary value problems for nonlinear differential equations of fractional order. Electron. J. Differ. Equ. **2009**(26), 1 (2009) http://ejde.math.txstate.edu
29. Ragusa, M.A.: Cauchy–Dirichlet problem associated to divergence form parabolic equations. Commun. Contemp. Math. **6**(3), 377–393 (2004). https://doi.org/10.1142/S0219199704001392
30. Chidouh, A., Torres, D.: Existence of positive solutions to a discrete fractional boundary value problem and corresponding Lyapunov-type inequalities. Opusc. Math. **38**(1), 31–40 (2018). https://doi.org/10.7494/OpMath.2018.38.1.31
31. Denton, Z., Ramirez, J.D.: Existence of minimal and maximal solutions to RL fractional integro-differential initial value problems. Opusc. Math. **37**(5), 705–724 (2017). https://doi.org/10.7494/OpMath.2017.37.5.705
32. Liu, Y.: A new method for converting boundary value problems for impulsive fractional differential equations to integral equations and its applications. Adv. Nonlinear Anal. **8**(1), 386–454 (2019). https://doi.org/10.1515/anona-2016-0064
33. Wang, Y., Liu, L.: Necessary and sufficient condition for the existence of positive solution to singular fractional differential equations. Adv. Differ. Equ. **2015**, 207 (2015)
34. Wang, Y.: Positive solutions for a class of two-term fractional differential equations with multipoint boundary value conditions. Adv. Differ. Equ. **2019**, 304 (2019). https://doi.org/10.1186/s13662-019-2250-x
35. Wang, Y.: Necessary conditions for the existence of positive solutions to fractional boundary value problems at resonance. Appl. Math. Lett. **97**, 34–40 (2019). https://doi.org/10.1016/j.aml.2019.05.007
36. Bungardi, S., Cardinali, T., Rubbioni, P.: Nonlocal semi-linear integro-differential inclusions via vectorial measures of non-compactness. Appl. Anal. **96**(15), 2526–2544 (2015)

37. Ndaírou, F., Area, I., Nieto, J.J., Torres, D.F.M.: Mathematical modeling of COVID-19 transmission dynamics with a case study of Wuhan. Chaos Solitons Fractals **135**, 109846 (2020)
38. Kucche, K.D., Nieto, J.J., Venktesh, V.: Theory of nonlinear implicit fractional differential equations. Differ. Equ. Dyn. Syst. **28**(1), 1–17 (2020)
39. Ahmad, B., Alruwaily, Y., Alsaedi, A., Nieto, J.J.: Fractional integro-differential equations with dual anti-periodic boundary conditions. Differ. Integral Equ. **33**(3–4), 181–206 (2020)
40. Nisar, K.S., Suthar, D.L., Agarwal, R., Purohit, S.D.: Fractional calculus operators with Appell function kernels applied to Srivastava polynomials and extended Mittag-Leffler function. Adv. Differ. Equ. **2020**, 148 (2020)
41. Agarwal, R., Golev, A., Hristova, S., O'Regan, D., Stefanova, K.: Iterative techniques with computer realization for the initial value problem for Caputo fractional differential equations. J. Appl. Math. Comput. **58**(1–2), 433–467 (2018)
42. Hristova, S., Agarwal, R., O'Regan, D.: Explicit solutions of initial value problems for systems of linear Riemann–Liouville fractional differential equations with constant delay. Adv. Differ. Equ. **2020**, 180 (2020)
43. Wang, X., Li, X., Huang, N., O'Regan, D.: Asymptotical consensus of fractional-order multi-agent systems with current and delay states. Appl. Math. Mech. **40**(11), 1677–1694 (2019)
44. Song, J., Xia, Y., Bai, Y., Cai, Y., O'Regan, D.: A non-autonomous Leslie–Gower model with Holling type IV functional response and harvesting complexity. Adv. Differ. Equ. **2019**, 299 (2019)
45. Agarwal, P., Chand, M., Baleanu, D., O'Regan, D., Jain, S.: On the solutions of certain fractional kinetic equations involving k-Mittag-Leffler function. Adv. Differ. Equ. **2018**, 249 (2018)
46. Ahmad, B., Nieto, J.J., Alsaedi, A., Al-Hutami, H.: Boundary value problems of nonlinear fractional q-difference (integral) equations with two fractional orders and four-point nonlocal integral boundary conditions. Filomat **28**(8), 1719–1736 (2014)
47. Zhai, C.B., Ren, J.: Positive and negative solutions of a boundary value problem for a fractional q-difference equation. Adv. Differ. Equ. **2017**, 82 (2017)
48. Zhao, Y., Chen, H., Zhang, Q.: Existence results for fractional q-difference equations with nonlocal q-integral boundary conditions. Adv. Differ. Equ. **2013**, 48 (2013). https://doi.org/10.1186/1687-1847-2013-48
49. Aydogan, M.S., Baleanu, D., Mousalou, A., Rezapour, Sh.: On high order fractional integro-differential equations including the Caputo–Fabrizio derivative. Bound. Value Probl. **2018**(1), 90 (2018). https://doi.org/10.1186/s13661-018-1008-9
50. Baleanu, D., Rezapour, Sh., Saberpour, Z.: On fractional integro-differential inclusions via the extended fractional Caputo–Fabrizio derivation. Bound. Value Probl. **2019**, 79 (2019). https://doi.org/10.1186/s13661-019-1194-0
51. Baleanu, F.J.D., Abdeljawad, A.: Caputo-type modification of the Hadamard fractional derivatives. Adv. Differ. Equ. **2020**, 142 (2020)
52. Baleanu, D., Khan, H., Jafari, H., Khan, R.A., Alipour, M.: On existence results for solutions of a coupled system of hybrid boundary value problems with hybrid conditions. Adv. Differ. Equ. **2015**, 318 (2015). https://doi.org/10.1186/s13662-015-0651-z
53. Akbari Kojabad, E., Rezapour, Sh.: Approximate solutions of a sum-type fractional integro-differential equation by using Chebyshev and Legendre polynomials. Adv. Differ. Equ. **2017**, 351 (2017)
54. Aydogan, S.M., Baleanu, D., Mousalou, A., Rezapour, Sh.: On approximate solutions for two higher-order Caputo–Fabrizio fractional integro-differential equations. Adv. Differ. Equ. **2017**(1), 221 (2017). https://doi.org/10.1186/s13662-017-1258-3
55. Baleanu, D., Ghafarnezhad, K., Rezapour, Sh.: On a three steps crisis integro-differential equation. Adv. Differ. Equ. **2019**, 153 (2019)
56. Niyom, S., Ntouyas, S.K., Laoprasittichok, S., Tariboon, J.: Boundary value problems with four orders of Riemann–Liouville fractional derivatives. Adv. Differ. Equ. **2016**, 165 (2016)
57. Ahmad, B., Ntouyas, S.K., Agarwal, R.P., Alsaedi, A.: Existence results for sequential fractional integro-differential equations with nonlocal multi-point and strip conditions. Bound. Value Probl. **2016**, 205 (2016)
58. Aljoudi, Sh., Ahmad, B., Nieto, J.J., Alsaedi, A.: A coupled system of Hadamard type sequential fractional differential equations with coupled strip conditions. Chaos Solitons Fractals **91**, 39–46 (2016)
59. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Studies. Elsevier Science, Amsterdam (2006)
60. Krasnoselskii, M.A.: Two remarks on the method of successive approximations. Usp. Mat. Nauk **10**, 123–127 (1955)
61. Granas, A., Dugundji, J.: Fixed Point Theory. Springer, New York (2003)
62. Deimling, K.: Nonlinear Functional Analysis. Springer, New York (1985)