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Time-space variable-order fractional nonlinear system of thermoelasticity: numerical treatment

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Abstract

This paper focuses on a numerical study of the general time-space variable-order fractional nonlinear problem of thermoelasticity in one dimension using the weighted average nonstandard finite difference (WANSFD). By replacing the second order space derivative with a Riesz fractional variable-order derivative and the time derivative by Caputo fractional variable-order operator in the standard system which arises in thermoelasticity, we obtain this general system. Using a kind of John von Neumann technique, we study the stability of the designed schemes. Also, the truncation error of the introduced schemes is studied. Our numerical treatment is shown graphically. These results expose that WANSFD approach is suitable and effective for solving the proposed system; moreover, it is easy to implement.

Keywords: Time-space variable-order fractional nonlinear thermoelasticity; Riesz fractional fractional variable-order derivative; John von Neumann stability method; Weighted average nonstandard finite difference methods

1 Introduction

Today scientists in different fields such as plasma waves, fluid mechanics systems and solid state physics use coupled partial differential equations to describe many phenomena. A system of nonlinear coupled hyperbolic and parabolic equations is always used in studies of circled fuel reactor, radiation hydrodynamics, magnetoelasticity, thermoelasticity, and in biology [1–5]. It is known that studying the behavior of solutions of these systems is a very important and difficult area of research. A lot of authors [4–6] have been studying the existence of solutions for such models, as well as their uniqueness and stability.

It is very difficult to obtain the analytic exact solutions of such nonlinear problems and, unfortunately, only in simple cases one can find these solutions. Therefore, it is very important to approximate them numerically, which requires great potential from the researchers. The authors of [7] and of the references therein studied a system of coupled parabolic equations. Some of these studies depended on the finite element technique [8, 9] and on the finite difference technique [1, 2, 10] for a nonlinear system of coupled hyperbolic and parabolic equations. In [11, 12], the authors apply the Adomian decomposition approach and the variational iteration approach for solving the model. In [3], the authors

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found the solution of a nonlinear system of coupled hyperbolic and parabolic equations with specified harmonic displacement at the border depending on the Poincaré extension with a small parameter.

It is well known, since the last few years, that the fractional derivative operators are more compatible to describe the properties of many real-life problems and more helpful than those of integer-order to understand such problems [13]. In fact, the next state in the model with a fractional-order derivative is based on all of its prior states and not only on its present state as in the case with a derivative of integer order. And therefore, the derivative of fractional order will be more authentic than that of integer order. Inclusion of the hereditary properties and memory effect of processes in numerous real-life problems is considered as one of the most important advantages of the fractional derivative [14].

The fractional differential equations is the best way to describe the fractional models. These equations are solved by different methods. Some of the introduced numerical methods are: Adomian decomposition technique [15], finite element technique [16], finite difference technique [17, 18], homotopy perturbation technique [19], spectral methods [20, 21], Taylor collocation methods [22], and variational iteration methods [23].

Recently, many physical, mathematical, financial, viscoelasticity, mechanical, and engineering problems have been described by the fractional variable-order pseudodifferential operators, see [24–32]. Also, the underlying differential operators in many dynamic processes appear as fractional and are dynamic in a sense that their derivative order is field-changing, i.e., it may change with space and/or time. Subsequently, a lot of scientists have been studying the characteristic of the variable-order fractional derivatives. In 1993, Samko with others [31] introduced this motivating expansion of the standard fractional derivatives by letting the order of the fractional derivative be not a constant but a function of an independent variable. After that, many authors have proposed various definitions of variable-order fractional derivative to fit coveted aims. Lorenzo et al. in [33] introduced in details the connotation of variable-order derivative and inspected some variable-order definitions. Coimbra [34] suggested a new definition for the variable-order fractional operator by using the Laplace transform of the Caputo fractional derivative.

The nonstandard finite difference methods (NSFDMs) were suggested by Mickens [35–37] to improve the discretization of some terms in the studied differential equation. Using a specific discretization and the denominator function, this approach will be more stable and more accurate than the standard approach [38, 39], such that the proposed approach is not difficult to build [40]. The workable implementation of the NSFDMs exists in the research areas of chemistry, physics, and engineering [41–43]. In particular, the most fascinating usage is in mathematical ecology and mathematical biology [44, 45], where the advantages of the NSFDMs have been shown markedly. Also, the active keeping property of the NSFDMs is well behaved in solving systems of fractional-order differential equations, like the fractional-order neuron model [46], the fractional-order Rössler model [47], and the Hodgkin–Huxley system with fractional order derivative [48].

In addition, the weighted average finite difference method (WAFDM) [49–51] can depend on the weight factor, and the implicit or explicit method (more stable or easy for coding, respectively).

In this work, we merge between those two techniques to find numerical approximating scheme for a time-space variable-order fractional nonlinear system of coupled hyperbolic and parabolic differential equations. This scheme has the advantages of these two

approaches and therefore will have a larger stability region and will be more accurate. The introduced method is called the weighted average nonstandard finite difference method (WANSFDM) (see [18, 50], and [52]).

Consider the system of general time-space variable-order fractional nonlinear, coupled, hyperbolic and parabolic partial differential equations in one-dimension with additional terms of a heat supply and a force term in the equation of heat conduction and in the equation of the motion, respectively:

$$\begin{aligned} {}^c_0D_t^{\mu(x,t)}u - a_1(u_x, \theta)R^{\alpha(x,t)}u + b_1(u_x, \theta)R^{\alpha(x,t)-1}\theta &= f(x, t), \quad x \in \mathbb{R}, \\ {}^c_0D_t^{\mu(x,t)-1}\theta - d_1(u_x, \theta)R^{\alpha(x,t)}\theta - h_1(u_{xx}, \theta)R^{\alpha(x,t)-1}\theta + c_1(u_x, \theta)u_{xt} &= l(x, t), \quad t > 0, \end{aligned} \quad (1)$$

with initial conditions:

$$\theta(x, 0) = \theta^0, \quad u(x, 0) = u^0(x), \quad u_t(x, 0) = u^1(x),$$

such that $\theta = \theta(x, t)$ is the difference of the temperature of the body from $t_0 = 0$ and $u = u(x, t)$ is the displacement of body from balance, ${}^c_0D_t^{\mu(x,t)}$ is the variable-order Caputo time fractional derivative with $1 < \mu(x, t) \leq 2$, $R^{\alpha(x,t)}$ is the Riesz variable-order fractional space derivative with $1 < \alpha(x, t) \leq 2$, a_1 , b_1 , c_1 , d_1 , and h_1 , are given smooth functions and the subscripts denote partial derivatives.

Many articles like [1, 53–55] have widely studied this system (1). When $\mu(x, t) = 2$ and $\alpha(x, t) = 2$, in [3, 53] we found specific details of the meaning and application of such this system in physics. Also, in [50] the authors studied this system in case of time fractional derivative only. These works depend on the fundamental researches in which the theory of fractional thermoelasticity was introduced (see [56–58], and [59]).

In reality, the Caputo variable-order fractional derivative operator has many advantages when used to describe the derivative in initial value differential equations. The most important feature of variable-order Caputo's definition is that the conditions of the initial value for the variable-order fractional differential equations with the variable-order Caputo derivatives will be taken as in the case of differential equations of integer order, therefore, the time variable-order fractional derivatives are often taken in the Caputo sense. Also, it is usually to define the space variable-order fractional derivative as the Riesz variable-order fractional operator. The author in [13] concluded that “the complete theory of fractional differential equations, especially the theory of boundary value problems for fractional differential equations, can be developed only with the use of both left and right derivatives.” So, in this paper all the spatial derivatives are Riesz variable-order fractional derivatives, which include the right and left variable-order Riemann–Liouville fractional operators.

The fundamental aim of the present article is to carry out numerical treatment of the time-space variable-order fractional differential model (1) which arises in fractional variable-order thermoelasticity using the WANSFD technique. In reality, there are no studies in the literature using WANSFD technique to solve similar systems with variable-order fractional derivatives, so our attention to such an approach is motivated.

The article is arranged as follows: In the next section, we remind the elementary facts of the used methods (NSFDMs) and present many definitions of the derivatives of fractional variable order. In Sect. 3, in three different cases, we propose WANSFD schemes

for the time-space variable order fractional model (1). We analyze, in Sect. 4, the stability of the suggested schemes and discuss their truncation error. Section 5 is devoted to some numerical consequences which are reported to exhibit the accuracy and the efficiency of the offered technique. Finally, in Sect. 6, we give conclusions of this work.

2 Preliminaries and definitions

2.1 The nonstandard finite difference technique

The NSFDMs were suggested firstly by Mickens [35–37]. These techniques construct an approximating discrete schemes for partial differential equations (PDEs) or ordinary differential equations (ODEs). NSFDMs can preserve the properties of the exact solution of the studied PDEs or ODEs depending on the following principles [36]:

1. In general, the denominator functions for the discrete derivatives must be expressed in terms of the step sizes.
2. The orders of the discrete derivatives should be equal to the orders of the corresponding derivatives of the differential equations.
3. The nonlinear terms should be approximated in a nonlocal approximation manner.
4. The schemes must not have solutions which do not coincide with solutions of the studied differential models.
5. Specific conditions which keep the analytic solutions of the studied differential equations must also be specific discrete conditions for a scheme of the finite difference.

In brief, for approximating $\frac{dy}{dt}$ using the Euler approach, we usually depend on $\frac{y(t+h)-y(t)}{h}$ but here we will use $\frac{y(t+h)-y(t)}{\phi(h)}$ instead, where $\phi(h)$ is a continuous function of step size h which satisfies the following conditions:

$$\phi(h) = h + O(h^2), \quad 0 < \phi(h) < 1, \quad h \rightarrow 0.$$

Furthermore, if there is a nonlinear term in the studied differential equation, we replace it in a nonlocal manner as, for example,

$$xy \rightarrow \begin{cases} x_{n-1}y_n, \\ x_{n+1}y_n, \\ x_ny_{n+1}. \end{cases}$$

2.2 Fractional variable-order calculus definitions

In the literature, there are a lot of definitions of the operators of fractional variable-order derivatives (see, e.g., [60–63]). The space-fractional variable-order derivatives are usually given by the Riesz fractional variable-order derivatives. As for the time-fractional derivatives, they are usually defined in the Grünwald–Letnikov, Riemann–Liouville, or Caputo sense.

Definition 2.1 ([28]) Let $\alpha(t) \in \mathbb{R}^+$. The variable-order Caputo fractional derivative of order $\alpha(t)$ is defined by

$$({}^c D_t^{\alpha(t)} f)(t) = \frac{1}{\Gamma(n - \alpha(t))} \int_0^t \frac{f^{(n)}(\xi)}{(t - \xi)^{1 - n + \alpha(t)}} d\xi, \quad t > 0, \quad (2)$$

where $f(t) \in C^n[0, \infty[$, $n = [\alpha(t)] + 1$.

Definition 2.2 ([28]) The right- and left-sided variable-order Riemann–Liouville operators of fractional derivatives where $t \in \mathbb{R}$ and $\alpha(t) \in \mathbb{R}^+$, $n = [\alpha(t)] + 1$ are

$$\begin{aligned}({}_t D_b^{\alpha(t)} f)(t) &= \frac{(-1)^n}{\Gamma(n - \alpha(t))} \left(\frac{d}{dt} \right)^n \int_t^b \frac{f(\xi)}{(\xi - t)^{1 + \alpha(t) - n}} d\xi, \\({}_a D_t^{\alpha(t)} f)(t) &= \frac{1}{\Gamma(n - \alpha(t))} \left(\frac{d}{dt} \right)^n \int_a^t \frac{f(\xi)}{(t - \xi)^{1 + \alpha(t) - n}} d\xi.\end{aligned}\quad (3)$$

Definition 2.3 ([28]) The right- and left-sided variable-order Grünwald–Letnikov fractional derivatives are given by

$$D_{b-}^{\alpha(t)} f(t) = \lim_{h \rightarrow 0, n\tau = b-t} h^{-\alpha(t)} \sum_{j=0}^n (-1)^j \binom{\alpha(t)}{j} f(t + jh), \quad (4)$$

$$D_{a+}^{\alpha(t)} f(t) = \lim_{h \rightarrow 0, n\tau = t-a} h^{-\alpha(t)} \sum_{j=0}^n (-1)^j \binom{\alpha(t)}{j} f(t - jh), \quad (5)$$

where $[x]$ denotes the integer part of x and h is the step size.

Definition 2.4 ([30]) The variable order Riesz fractional derivative is given by

$$R^{\alpha(t)} f(t) = -\frac{1}{2 \cos(\frac{\pi \alpha(t)}{2})} ({}_a D_t^{\alpha(t)} + {}_t D_b^{\alpha(t)}) f(t). \quad (6)$$

3 Structure of the WANSFD for the proposed system

In the current part of this paper, we introduce WANSFDM for obtaining a discrete scheme for the system (1). Here, it is valuable to remind that the distinction between “the weighted average nonstandard finite difference technique” and “the weighted average standard finite difference technique” is analogous to the distinction between the nonstandard finite difference technique and the finite difference technique. The problem of finding the numerical solution of the system (1) is in approximating the fractional variable-order Caputo and Riemann–Liouville variable-order derivatives by WANSFD scheme when $\sigma \in [0, 1]$ is the weight factor.

Let $N, M \in \mathbb{N}$ and consider

$$x_n = nh, \quad n = 0, 1, 2, \dots, N, \quad t_m = m\Delta t, \quad m = 0, 1, 2, \dots, M,$$

the coordinates of the mesh points, where

$$\Delta t = T/M, \quad h = (b - a)/N.$$

We will denote, at a grid point $(x_n, t_m) = (nh, m\Delta t)$, the approximated values of u , θ by u_n^m , θ_n^m , respectively. The nonstandard difference approximation of variable-order Caputo derivative is introduced in the form:

$$\begin{aligned}& {}^c_0 D_t^{\mu(x,t)} u(x, t) \\&= \frac{1}{\Gamma(2 - \mu(x, t))} \int_0^t \frac{1}{(t - \tau)^{(\mu(x,t)-1)}} \frac{\partial^2 u(x, \tau)}{\partial \tau^2} d\tau\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\Gamma(2-\mu(x,t))} \sum_{j=0}^{m-1} \int_{j\Delta t}^{(j+1)\Delta t} z^{(1-\mu(x,t))} \frac{\partial^2 u(x,t-z)}{\partial z^2} dz \\
&= \frac{1}{\Gamma(2-\mu(x,t))} \sum_{j=0}^{m-1} \frac{u(x,t-(j-1)\Delta t) - 2u(x,t-j\Delta t) + u(x,t-(j+1)\Delta t)}{(\varphi(\Delta t))^2} \\
&\quad \times \int_{j\Delta t}^{(j+1)\Delta t} z^{(1-\mu(x,t))} dz \\
&= \frac{(\Delta t)^{2-\mu(x,t)}}{\Gamma(3-\mu(x,t))(\varphi(\Delta t))^2} \\
&\quad \times \sum_{j=0}^{m-1} [u(x,t-(j-1)\Delta t) - 2u(x,t-j\Delta t) + u(x,t-(j+1)\Delta t)] \\
&\quad \times [(j+1)^{2-\mu(x,t)} - j^{2-\mu(x,t)}]. \tag{7}
\end{aligned}$$

Similarly,

$$\begin{aligned}
{}_0^c D_t^{\mu(x,t)-1} \theta(x,t) &= \frac{(\Delta t)^{2-\mu(x,t)}}{2\Gamma(3-\mu(x,t))(\varphi(\Delta t))} \sum_{j=0}^{m-1} [\theta(x,t-(j-1)\Delta t) - \theta(x,t-(j+1)\Delta t)] \\
&\quad \times [(j+1)^{2-\mu(x,t)} - j^{2-\mu(x,t)}]. \tag{8}
\end{aligned}$$

Also, the approximations of variable-order Riemann–Liouville derivative using shifted Grünwald–Letnikov variable-order fractional derivative [29] are given as follows:

$$\begin{aligned}
({}_a D_x^{\alpha(x,t)} u(x,t))_n^m &= h^{-\alpha_{n+1}^m} \sum_{j=0}^{n+1} g_j^{\alpha_{n+1}^m} u_{n+1-j}^m, \\
({}_x D_b^{\alpha(x,t)} u)(x,t)_n^m &= h^{-\alpha_{n+1}^m} \sum_{j=0}^{N-n+1} g_j^{\alpha_{n+1}^m} u_{n-1+j}^m, \tag{9}
\end{aligned}$$

where

$$g_j^{\alpha_n^m} = (-1)^j \binom{\alpha_n^m}{j}.$$

In this paper we study the proposed model in three different cases:

1. $a_1(x,t) = 1 + 2\gamma u_x + 3\delta u_x^2 - \beta_2 \theta$, $b_1(x,t) = -(\beta_1 + \beta_2 u_x)$, $c_1(x,t) = -a - bu_x$, $d_1(x,t) = 1 + \alpha^* u_x$, and $h_1(x,t) = \alpha^* u_{xx}$ (as in [1]) such that the constants have specific orders as follows:

$$\begin{aligned}
\delta &= O(1 \text{ to } 10^{-1}), & \gamma &= O(1), & \beta_1 &= O(10^{-3}), & \beta_2 &= O(10^{-3}), \\
\alpha^* &= O(1), & a &= O(10^{-1}), & b &= O(10^{-1}),
\end{aligned}$$

therefore system(1) can be written as follows:

$$\begin{aligned}
&{}_0^c D_t^{\mu(x,t)} u - (1 + 2\gamma u_x + 3\delta u_x^2 - \beta_2 \theta) R^{\alpha(x,t)} u - (\beta_1 + \beta_2 u_x) R^{\alpha(x,t)-1} \theta \\
&= f(x,t), \quad x \in \mathbb{R}, \tag{10} \\
&{}_0^c D_t^{\mu(x,t)-1} \theta - (1 + \alpha^* u_x) R^{\alpha(x,t)} \theta - \alpha^* u_{xx} R^{\alpha(x,t)-1} \theta + (-a - bu_x) u_{xt} = l(x,t), \quad t > 0,
\end{aligned}$$

depending on the following difference approximations

$$\begin{aligned}({}_0^c D_t^{\mu(x,t)} u(x,t))_n^m &= \frac{(\Delta t)^{2-\mu_n^m}}{\Gamma(3-\mu_n^m)(\varphi(\Delta t))^2} \sum_{j=0}^{m-1} [u_n^{m-j+1} - 2u_n^{m-j} + u_n^{m-j-1}] \\ &\quad \times [(j+1)^{2-\mu_n^m} - j^{2-\mu_n^m}] + O((\varphi(\Delta t))^2),\end{aligned}\quad (11a)$$

$$\begin{aligned}({}_0^c D_t^{\mu(x,t)-1} \theta(x,t))_n^m &= \frac{(\Delta t)^{2-\mu_n^m}}{2\Gamma(3-\mu_n^m)(\varphi(\Delta t))} \sum_{j=0}^{m-1} [\theta_n^{m-j+1} - \theta_n^{m-j-1}] [(j+1)^{2-\mu_n^m} - j^{2-\mu_n^m}] \\ &\quad + O((\varphi(\Delta t))^2),\end{aligned}\quad (11b)$$

$$(R^{\alpha(x,t)} u(x,t))_n^m = -\frac{(\phi(h))^{-\alpha_{n+1}^m}}{2\cos(\frac{\pi\alpha_n^m}{2})} \left(\sum_{j=0}^{n+1} g_j^{\alpha_{n+1}^m} u_{n+1-j}^m + \sum_{j=0}^{N-n+1} g_j^{\alpha_{n-1}^m} u_{n-1+j}^m \right), \quad (11c)$$

$$(R^{\alpha(x,t)-1} u(x,t))_n^m = -\frac{(\phi(h))^{-\alpha_{n+1}^m-1}}{2\cos(\frac{\pi\alpha_n^m-1}{2})} \left(\sum_{j=0}^{n+1} g_j^{\alpha_{n+1}^m-1} u_{n+1-j}^m + \sum_{j=0}^{N-n+1} g_j^{\alpha_{n-1}^m-1} u_{n-1+j}^m \right), \quad (11d)$$

$$(u_{xt})_n^m = \frac{u_{n+1}^{m+1} - u_{n+1}^{m-1} - u_{n-1}^{m+1} + u_{n-1}^{m-1}}{4(\phi(h)) \cdot \varphi(\Delta t)} + O((\varphi(\Delta t))^2 + (\phi(h))^2), \quad (11e)$$

$$(u_x)_n^m = \frac{u_{n+1}^m - u_{n-1}^m}{2(\phi(h))} + O((\phi(h))^2), \quad (11f)$$

then system (10) takes the following form:

$$\begin{aligned}&\frac{(\Delta t)^{2-\mu_n^m}}{\Gamma(3-\mu_n^m)(\varphi(\Delta t))^2} \sum_{j=0}^{m-1} [u_n^{m-j+1} - 2u_n^{m-j} + u_n^{m-j-1}] [(j+1)^{2-\mu_n^m} - j^{2-\mu_n^m}] \\ &\quad + \sigma \left(1 + 2\gamma \frac{u_{n+1}^m - u_{n-1}^m}{2(\phi(h))} + 3\delta \left(\frac{u_{n+1}^m - u_{n-1}^m}{2(\phi(h))} \right)^2 - \beta_2 \theta_n^m \right) \\ &\quad \times \frac{(\phi(h))^{-\alpha_{n+1}^m}}{2\cos(\frac{\pi\alpha_n^m}{2})} \left(\sum_{j=0}^{n+1} g_j^{\alpha_{n+1}^m} u_{n+1-j}^m + \sum_{j=0}^{N-n+1} g_j^{\alpha_{n-1}^m} u_{n-1+j}^m \right) \\ &\quad + (1-\sigma) \left(1 + 2\gamma \frac{u_{n+1}^{m+1} - u_{n-1}^{m+1}}{2(\phi(h))} + 3\delta \left(\frac{u_{n+1}^{m+1} - u_{n-1}^{m+1}}{2(\phi(h))} \right)^2 - \beta_2 \theta_n^{m+1} \right) \\ &\quad \times \frac{(\phi(h))^{-\alpha_{n+1}^m}}{2\cos(\frac{\pi\alpha_n^m}{2})} \left(\sum_{j=0}^{n+1} g_j^{\alpha_{n+1}^m} u_{n+1-j}^{m+1} + \sum_{j=0}^{N-n+1} g_j^{\alpha_{n-1}^m} u_{n-1+j}^{m+1} \right) \\ &\quad + \sigma \left(\beta_1 + \beta_2 \frac{u_{n+1}^m - u_{n-1}^m}{2(\phi(h))} \right) \\ &\quad \times \frac{(\phi(h))^{-\alpha_{n+1}^m-1}}{2\cos(\frac{\pi\alpha_n^m-1}{2})} \left(\sum_{j=0}^{n+1} g_j^{\alpha_{n+1}^m-1} \theta_{n+1-j}^m + \sum_{j=0}^{N-n+1} g_j^{\alpha_{n-1}^m-1} \theta_{n-1+j}^m \right) \\ &\quad + (1-\sigma) \left(\beta_1 + \beta_2 \frac{u_{n+1}^{m+1} - u_{n-1}^{m+1}}{2(\phi(h))} \right) \\ &\quad \times \frac{(\phi(h))^{-\alpha_{n+1}^m-1}}{2\cos(\frac{\pi\alpha_n^m-1}{2})} \left(\sum_{j=0}^{n+1} g_j^{\alpha_{n+1}^m-1} \theta_{n+1-j}^{m+1} + \sum_{j=0}^{N-n+1} g_j^{\alpha_{n-1}^m-1} \theta_{n-1+j}^{m+1} \right) \\ &\quad - \sigma f_n^m + (1-\sigma) f_n^{m+1} = T_n^m,\end{aligned}$$

$$\begin{aligned}
& \frac{(\Delta t)^{2-\mu_n^m}}{2\Gamma(3-\mu_n^m)(\varphi(\Delta t))} \sum_0^{m-1} [\theta_n^{m-j+1} - \theta_n^{m-j-1}] [(j+1)^{2-\mu_n^m} - j^{2-\mu_n^m}] \\
& + \sigma \left(1 + \alpha^* \frac{u_{n+1}^m - u_{n-1}^m}{2(\phi(h))} \right) \\
& \times \frac{(\phi(h))^{-\alpha_{n+1}^m}}{2 \cos(\frac{\pi \alpha_n^m}{2})} \left(\sum_{j=0}^{n+1} g_j^{\alpha_{n+1}^m} \theta_{n+1-j}^m + \sum_{j=0}^{N-n+1} g_j^{\alpha_{n-1}^m} \theta_{n-1+j}^m \right) \\
& + (1-\sigma) \left(1 + \alpha^* \frac{u_{n+1}^{m+1} - u_{n-1}^{m+1}}{2(\phi(h))} \right) \\
& \times \frac{(\phi(h))^{-\alpha_{n+1}^{m+1}}}{2 \cos(\frac{\pi \alpha_n^m}{2})} \left(\sum_{j=0}^{n+1} g_j^{\alpha_{n+1}^{m+1}} \theta_{n+1-j}^{m+1} + \sum_{j=0}^{N-n+1} g_j^{\alpha_{n-1}^{m+1}} \theta_{n-1+j}^{m+1} \right) \\
& + \sigma \alpha^* \frac{u_{n+1}^m - 2u_n^m + u_{n-1}^m}{(\phi(h))^2} \\
& \times \frac{(\phi(h))^{-\alpha_{n+1}^{m-1}}}{2 \cos(\frac{\pi \alpha_n^{m-1}}{2})} \left(\sum_{j=0}^{n+1} g_j^{\alpha_{n+1}^{m-1}} \theta_{n+1-j}^m + \sum_{j=0}^{N-n+1} g_j^{\alpha_{n-1}^{m-1}} \theta_{n-1+j}^m \right) \\
& + (1-\sigma) \alpha^* \frac{u_{n+1}^{m+1} - 2u_n^{m+1} + u_{n-1}^{m+1}}{(\phi(h))^2} \\
& \times \frac{(\phi(h))^{-\alpha_{n+1}^{m+1-1}}}{2 \cos(\frac{\pi \alpha_n^{m+1-1}}{2})} \left(\sum_{j=0}^{n+1} g_j^{\alpha_{n+1}^{m+1-1}} \theta_{n+1-j}^{m+1} + \sum_{j=0}^{N-n+1} g_j^{\alpha_{n-1}^{m+1-1}} \theta_{n-1+j}^{m+1} \right) \\
& + \left(-a - b \frac{u_{n+1}^m - u_{n-1}^m}{2(\phi(h))} \right) \frac{u_{n+1}^{m+1} - u_{n+1}^{m-1} - u_{n-1}^{m+1} + u_{n-1}^{m-1}}{4\varphi(\Delta t) \cdot (\phi(h))} \\
& - \sigma l_n^m + (1-\sigma) l_n^{m+1} = T_n^m.
\end{aligned} \tag{12}$$

The previous substitution gives an error, truncating error, symbolized here by T_n^m . Its estimation will be discussed in Sect. 4.2. After dropping this truncating error, we have the following computable difference scheme:

$$\begin{aligned}
& \frac{(\Delta t)^{2-\mu_n^m}}{\Gamma(3-\mu_n^m)(\varphi(\Delta t))^2} \sum_0^{m-1} [u_n^{m-j+1} - 2u_n^{m-j} + u_n^{m-j-1}] [(j+1)^{2-\mu_n^m} - j^{2-\mu_n^m}] \\
& + \sigma \left(1 + 2\gamma \frac{u_{n+1}^m - u_{n-1}^m}{2(\phi(h))} + 3\delta \left(\frac{u_{n+1}^m - u_{n-1}^m}{2(\phi(h))} \right)^2 - \beta_2 \theta_n^m \right) \\
& \times \frac{(\phi(h))^{-\alpha_{n+1}^m}}{2 \cos(\frac{\pi \alpha_n^m}{2})} \left(\sum_{j=0}^{n+1} g_j^{\alpha_{n+1}^m} u_{n+1-j}^m + \sum_{j=0}^{N-n+1} g_j^{\alpha_{n-1}^m} u_{n-1+j}^m \right) \\
& + (1-\sigma) \left(1 + 2\gamma \frac{u_{n+1}^{m+1} - u_{n-1}^{m+1}}{2(\phi(h))} + 3\delta \left(\frac{u_{n+1}^{m+1} - u_{n-1}^{m+1}}{2(\phi(h))} \right)^2 - \beta_2 \theta_n^{m+1} \right) \\
& \times \frac{(\phi(h))^{-\alpha_{n+1}^{m+1}}}{2 \cos(\frac{\pi \alpha_n^m}{2})} \left(\sum_{j=0}^{n+1} g_j^{\alpha_{n+1}^{m+1}} u_{n+1-j}^{m+1} + \sum_{j=0}^{N-n+1} g_j^{\alpha_{n-1}^{m+1}} u_{n-1+j}^{m+1} \right) \\
& + \sigma \left(\beta_1 + \beta_2 \frac{u_{n+1}^m - u_{n-1}^m}{2(\phi(h))} \right)
\end{aligned}$$

$$\begin{aligned}
& \times \frac{(\phi(h))^{-\alpha_{n+1}^m-1}}{2 \cos(\frac{\pi \alpha_n^m-1}{2})} \left(\sum_{j=0}^{n+1} g_j^{\alpha_{n+1}^m-1} \theta_{n+1-j}^m + \sum_{j=0}^{N-n+1} g_j^{\alpha_{n-1}^m-1} \theta_{n-1+j}^m \right) \\
& + (1-\sigma) \left(\beta_1 + \beta_2 \frac{u_{n+1}^m - u_{n-1}^m}{2(\phi(h))} \right) \\
& \times \frac{(\phi(h))^{-\alpha_{n+1}^{m+1}-1}}{2 \cos(\frac{\pi \alpha_n^{m+1}-1}{2})} \left(\sum_{j=0}^{n+1} g_j^{\alpha_{n+1}^{m+1}-1} \theta_{n+1-j}^{m+1} + \sum_{j=0}^{N-n+1} g_j^{\alpha_{n-1}^{m+1}-1} \theta_{n-1+j}^{m+1} \right) \\
& - \sigma f_n^m + (1-\sigma) f_n^{m+1} = 0, \\
& \frac{(\Delta t)^{2-\mu_n^m}}{2\Gamma(3-\mu_n^m)(\varphi(\Delta t))} \sum_0^{m-1} [\theta_n^{m-j+1} - \theta_n^{m-j-1}] [(j+1)^{2-\mu_n^m} - j^{2-\mu_n^m}] \\
& + \sigma \left(1 + \alpha^* \frac{u_{n+1}^m - u_{n-1}^m}{2(\phi(h))} \right) \\
& \times \frac{(\phi(h))^{-\alpha_{n+1}^m}}{2 \cos(\frac{\pi \alpha_n^m}{2})} \left(\sum_{j=0}^{n+1} g_j^{\alpha_{n+1}^m} \theta_{n+1-j}^m + \sum_{j=0}^{N-n+1} g_j^{\alpha_{n-1}^m} \theta_{n-1+j}^m \right) \\
& + (1-\sigma) \left(1 + \alpha^* \frac{u_{n+1}^{m+1} - u_{n-1}^{m+1}}{2(\phi(h))} \right) \\
& \times \frac{(\phi(h))^{-\alpha_{n+1}^{m+1}}}{2 \cos(\frac{\pi \alpha_n^{m+1}}{2})} \left(\sum_{j=0}^{n+1} g_j^{\alpha_{n+1}^{m+1}} \theta_{n+1-j}^{m+1} + \sum_{j=0}^{N-n+1} g_j^{\alpha_{n-1}^{m+1}} \theta_{n-1+j}^{m+1} \right) \\
& + \sigma \alpha^* \frac{u_{n+1}^m - 2u_n^m + u_{n-1}^m}{(\phi(h))^2} \\
& \times \frac{(\phi(h))^{-\alpha_{n+1}^m-1}}{2 \cos(\frac{\pi \alpha_n^m-1}{2})} \left(\sum_{j=0}^{n+1} g_j^{\alpha_{n+1}^m-1} \theta_{n+1-j}^m + \sum_{j=0}^{N-n+1} g_j^{\alpha_{n-1}^m-1} \theta_{n-1+j}^m \right) \\
& + (1-\sigma) \alpha^* \frac{u_{n+1}^{m+1} - 2u_n^{m+1} + u_{n-1}^{m+1}}{(\phi(h))^2} \\
& \times \frac{(\phi(h))^{-\alpha_{n+1}^{m+1}-1}}{2 \cos(\frac{\pi \alpha_n^{m+1}-1}{2})} \left(\sum_{j=0}^{n+1} g_j^{\alpha_{n+1}^{m+1}-1} \theta_{n+1-j}^{m+1} + \sum_{j=0}^{N-n+1} g_j^{\alpha_{n-1}^{m+1}-1} \theta_{n-1+j}^{m+1} \right) \\
& + \left(-a - b \frac{u_{n+1}^m - u_{n-1}^m}{2(\phi(h))} \right) \frac{u_{n+1}^{m+1} - u_{n+1}^m - u_{n-1}^{m+1} + u_{n-1}^m}{4\varphi(\Delta t) \cdot (\phi(h))} \\
& - \sigma l_n^m + (1-\sigma) l_n^{m+1} = 0.
\end{aligned} \tag{13}$$

2. $a_1(x, t) = d_1(x, t) = 1$, $b_1(x, t) = c_1(x, t) = \theta u_x$, and $h_1(x, t) = 0$ (as in [55]) so model (1) has the following form:

$$\begin{aligned}
& {}_0^c D_t^{\mu(x,t)} u - R^{\alpha(x,t)} u + \theta u_x R^{\alpha(x,t)-1} \theta = f(x, t), \\
& {}_0^c D_t^{\mu(x,t)-1} \theta - R^{\alpha(x,t)} \theta + \theta u_x u_{xt} = l(x, t),
\end{aligned} \tag{14}$$

and then, using the difference approximations (11a)–(11f), system (14) can be written as follows:

$$\frac{(\Delta t)^{2-\mu_n^m}}{\Gamma(3-\mu_n^m)(\varphi(\Delta t))^2} \sum_0^{m-1} [u_n^{m-j+1} - 2u_n^{m-j} + u_n^{m-j-1}] [(j+1)^{2-\mu_n^m} - j^{2-\mu_n^m}]$$

$$\begin{aligned}
& + \sigma \frac{(\phi(h))^{-\alpha_{n+1}^m}}{2 \cos(\frac{\pi \alpha_n^m}{2})} \left(\sum_{j=0}^{n+1} g_j^{\alpha_{n+1}^m} u_{n+1-j}^m + \sum_{j=0}^{N-n+1} g_j^{\alpha_{n-1}^m} u_{n-1+j}^m \right) \\
& + (1-\sigma) \frac{(\phi(h))^{-\alpha_{n+1}^{m+1}}}{2 \cos(\frac{\pi \alpha_n^{m+1}}{2})} \left(\sum_{j=0}^{n+1} g_j^{\alpha_{n+1}^{m+1}} u_{n+1-j}^{m+1} + \sum_{j=0}^{N-n+1} g_j^{\alpha_{n-1}^{m+1}} u_{n-1+j}^{m+1} \right) \\
& - \sigma \theta_n^m \frac{u_{n+1}^m - u_{n-1}^m}{2(\phi(h))} \frac{(\phi(h))^{-\alpha_{n+1}^m - 1}}{2 \cos(\frac{\pi \alpha_n^m - 1}{2})} \left(\sum_{j=0}^{n+1} g_j^{\alpha_{n+1}^m - 1} \theta_{n+1-j}^m + \sum_{j=0}^{N-n+1} g_j^{\alpha_{n-1}^m - 1} \theta_{n-1+j}^m \right) \\
& - (1-\sigma) \theta_n^m \frac{u_{n+1}^{m+1} - u_{n-1}^{m+1}}{2(\phi(h))} \frac{(\phi(h))^{-\alpha_{n+1}^{m+1} - 1}}{2 \cos(\frac{\pi \alpha_n^{m+1} - 1}{2})} \left(\sum_{j=0}^{n+1} g_j^{\alpha_{n+1}^{m+1} - 1} \theta_{n+1-j}^{m+1} + \sum_{j=0}^{N-n+1} g_j^{\alpha_{n-1}^{m+1} - 1} \theta_{n-1+j}^{m+1} \right) \\
& - \sigma f_n^m + (1-\sigma) f_n^{m+1} = T_n^m, \tag{15} \\
& \frac{(\Delta t)^{2-\mu_n^m}}{2\Gamma(3-\mu_n^m)(\varphi(\Delta t))} \sum_0^{m-1} [\theta_n^{m-j+1} - \theta_n^{m-j-1}] [(j+1)^{2-\mu_n^m} - j^{2-\mu_n^m}] \\
& + \sigma \frac{(\phi(h))^{-\alpha_{n+1}^m}}{2 \cos(\frac{\pi \alpha_n^m}{2})} \left(\sum_{j=0}^{n+1} g_j^{\alpha_{n+1}^m} \theta_{n+1-j}^m + \sum_{j=0}^{N-n+1} g_j^{\alpha_{n-1}^m} \theta_{n-1+j}^m \right) \\
& + (1-\sigma) \frac{(\phi(h))^{-\alpha_{n+1}^{m+1}}}{2 \cos(\frac{\pi \alpha_n^{m+1}}{2})} \left(\sum_{j=0}^{n+1} g_j^{\alpha_{n+1}^{m+1}} \theta_{n+1-j}^{m+1} + \sum_{j=0}^{N-n+1} g_j^{\alpha_{n-1}^{m+1}} \theta_{n-1+j}^{m+1} \right) \\
& + \theta_n^m \frac{u_{n+1}^m - u_{n-1}^m}{2(\phi(h))} \frac{u_{n+1}^{m+1} - u_{n-1}^{m+1} - u_{n-1}^{m+1} + u_{n-1}^{m-1}}{4\varphi(\Delta t) \cdot (\phi(h))} \\
& - \sigma l_n^m + (1-\sigma) l_n^{m+1} = T_n^m.
\end{aligned}$$

Neglecting the truncation error, then we have

$$\begin{aligned}
& \frac{(\Delta t)^{2-\mu_n^m}}{\Gamma(3-\mu_n^m)(\varphi(\Delta t))^2} \sum_0^{m-1} [u_n^{m-j+1} - 2u_n^{m-j} + u_n^{m-j-1}] [(j+1)^{2-\mu_n^m} - j^{2-\mu_n^m}] \\
& + \sigma \frac{(\phi(h))^{-\alpha_{n+1}^m}}{2 \cos(\frac{\pi \alpha_n^m}{2})} \left(\sum_{j=0}^{n+1} g_j^{\alpha_{n+1}^m} u_{n+1-j}^m + \sum_{j=0}^{N-n+1} g_j^{\alpha_{n-1}^m} u_{n-1+j}^m \right) \\
& + (1-\sigma) \frac{(\phi(h))^{-\alpha_{n+1}^{m+1}}}{2 \cos(\frac{\pi \alpha_n^{m+1}}{2})} \left(\sum_{j=0}^{n+1} g_j^{\alpha_{n+1}^{m+1}} u_{n+1-j}^{m+1} + \sum_{j=0}^{N-n+1} g_j^{\alpha_{n-1}^{m+1}} u_{n-1+j}^{m+1} \right) \\
& - \sigma \theta_n^m \frac{u_{n+1}^m - u_{n-1}^m}{2(\phi(h))} \frac{(\phi(h))^{-\alpha_{n+1}^m - 1}}{2 \cos(\frac{\pi \alpha_n^m - 1}{2})} \left(\sum_{j=0}^{n+1} g_j^{\alpha_{n+1}^m - 1} \theta_{n+1-j}^m + \sum_{j=0}^{N-n+1} g_j^{\alpha_{n-1}^m - 1} \theta_{n-1+j}^m \right) \\
& - (1-\sigma) \theta_n^m \frac{u_{n+1}^{m+1} - u_{n-1}^{m+1}}{2(\phi(h))} \frac{(\phi(h))^{-\alpha_{n+1}^{m+1} - 1}}{2 \cos(\frac{\pi \alpha_n^{m+1} - 1}{2})} \left(\sum_{j=0}^{n+1} g_j^{\alpha_{n+1}^{m+1} - 1} \theta_{n+1-j}^{m+1} + \sum_{j=0}^{N-n+1} g_j^{\alpha_{n-1}^{m+1} - 1} \theta_{n-1+j}^{m+1} \right) \\
& - \sigma f_n^m + (1-\sigma) f_n^{m+1} = 0, \tag{16} \\
& \frac{(\Delta t)^{2-\mu_n^m}}{2\Gamma(3-\mu_n^m)(\varphi(\Delta t))} \sum_0^{m-1} [\theta_n^{m-j+1} - \theta_n^{m-j-1}] [(j+1)^{2-\mu_n^m} - j^{2-\mu_n^m}] \\
& + \sigma \frac{(\phi(h))^{-\alpha_{n+1}^m}}{2 \cos(\frac{\pi \alpha_n^m}{2})} \left(\sum_{j=0}^{n+1} g_j^{\alpha_{n+1}^m} \theta_{n+1-j}^m + \sum_{j=0}^{N-n+1} g_j^{\alpha_{n-1}^m} \theta_{n-1+j}^m \right)
\end{aligned}$$

$$\begin{aligned}
& + (1 - \sigma) \frac{(\phi(h))^{-\alpha_{n+1}^{m+1}}}{2 \cos(\frac{\pi \alpha_{n+1}^{m+1}}{2})} \left(\sum_{j=0}^{n+1} g_j^{\alpha_{n+1}^{m+1}} \theta_{n+1-j}^{m+1} + \sum_{j=0}^{N-n+1} g_j^{\alpha_{n-1}^{m+1}} \theta_{n-1+j}^{m+1} \right) \\
& + \theta_n^m \frac{u_{n+1}^m - u_{n-1}^m}{2(\phi(h))} \frac{u_{n+1}^{m+1} - u_{n+1}^{m-1} - u_{n-1}^{m+1} + u_{n-1}^{m-1}}{4\varphi(\Delta t) \cdot (\phi(h))} \\
& - \sigma l_n^m + (1 - \sigma) l_n^{m+1} = 0.
\end{aligned}$$

3. $a_1(x, t) = 2 - \theta u_x$, $b_1(x, t) = c_1(x, t) = 2 + \theta u_x$, $d_1(x, t) = \theta$, and $h_1(x, t) = 0$ (as in [12]) so model (1) has the following form:

$$\begin{aligned}
& {}^c D_t^{\mu(x,t)} u - (2 - \theta u_x) R^{\alpha(x,t)} u + (2 + \theta u_x) R^{\alpha(x,t)-1} \theta = f(x, t), \quad x \in \mathbb{R}, \\
& {}^c D_t^{\mu(x,t)-1} \theta - \theta R^{\alpha(x,t)} \theta + (2 + \theta u_x) u_{xt} = l(x, t), \quad t > 0,
\end{aligned} \tag{17}$$

where, depending on the difference approximations (11a)–(11f), model (17) then takes the form:

$$\begin{aligned}
& \frac{(\Delta t)^{2-\mu_n^m}}{\Gamma(3-\mu_n^m)(\varphi(\Delta t))^2} \sum_0^{m-1} [u_n^{m-j+1} - 2u_n^{m-j} + u_n^{m-j-1}] [(j+1)^{2-\mu_n^m} - j^{2-\mu_n^m}] \\
& + \sigma \left(2 - \theta_n^m \frac{u_{n+1}^m - u_{n-1}^m}{2(\phi(h))} \right) \frac{(\phi(h))^{-\alpha_{n+1}^m}}{2 \cos(\frac{\pi \alpha_{n+1}^m}{2})} \left(\sum_{j=0}^{n+1} g_j^{\alpha_{n+1}^m} u_{n+1-j}^m + \sum_{j=0}^{N-n+1} g_j^{\alpha_{n-1}^m} u_{n-1+j}^m \right) \\
& + (1 - \sigma) \left(2 - \theta_n^{m+1} \frac{u_{n+1}^{m+1} - u_{n-1}^{m+1}}{2(\phi(h))} \right) \\
& \times \frac{h^{-\alpha_{n+1}^{m+1}}}{2 \cos(\frac{\pi \alpha_{n+1}^{m+1}}{2})} \left(\sum_{j=0}^{n+1} g_j^{\alpha_{n+1}^{m+1}} u_{n+1-j}^{m+1} + \sum_{j=0}^{N-n+1} g_j^{\alpha_{n-1}^{m+1}} u_{n-1+j}^{m+1} \right) \\
& - \sigma \left(2 + \theta_n^m \frac{u_{n+1}^m - u_{n-1}^m}{2(\phi(h))} \right) \frac{(\phi(h))^{-\alpha_{n+1}^m-1}}{2 \cos(\frac{\pi \alpha_{n+1}^m-1}{2})} \left(\sum_{j=0}^{n+1} g_j^{\alpha_{n+1}^m-1} \theta_{n+1-j}^m + \sum_{j=0}^{N-n+1} g_j^{\alpha_{n-1}^m-1} \theta_{n-1+j}^m \right) \\
& - (1 - \sigma) \left(2 + \theta_n^{m+1} \frac{u_{n+1}^{m+1} - u_{n-1}^{m+1}}{2(\phi(h))} \right) \\
& \times \frac{(\phi(h))^{-\alpha_{n+1}^{m+1}-1}}{2 \cos(\frac{\pi \alpha_{n+1}^{m+1}-1}{2})} \left(\sum_{j=0}^{n+1} g_j^{\alpha_{n+1}^{m+1}-1} \theta_{n+1-j}^{m+1} + \sum_{j=0}^{N-n+1} g_j^{\alpha_{n-1}^{m+1}-1} \theta_{n-1+j}^{m+1} \right) \\
& - \sigma f_n^m + (1 - \sigma) f_n^{m+1} = T_n^m, \\
& \frac{(\Delta t)^{2-\mu_n^m}}{2\Gamma(3-\mu_n^m)(\varphi(\Delta t))} \sum_0^{m-1} [\theta_n^{m-j+1} - \theta_n^{m-j-1}] [(j+1)^{2-\mu_n^m} - j^{2-\mu_n^m}] \\
& + \sigma \theta_n^m \frac{(\phi(h))^{-\alpha_{n+1}^m}}{2 \cos(\frac{\pi \alpha_{n+1}^m}{2})} \left(\sum_{j=0}^{n+1} g_j^{\alpha_{n+1}^m} \theta_{n+1-j}^m + \sum_{j=0}^{N-n+1} g_j^{\alpha_{n-1}^m} \theta_{n-1+j}^m \right) \\
& + (1 - \sigma) \theta_n^{m+1} \frac{(\phi(h))^{-\alpha_{n+1}^{m+1}}}{2 \cos(\frac{\pi \alpha_{n+1}^{m+1}}{2})} \left(\sum_{j=0}^{n+1} g_j^{\alpha_{n+1}^{m+1}} \theta_{n+1-j}^{m+1} + \sum_{j=0}^{N-n+1} g_j^{\alpha_{n-1}^{m+1}} \theta_{n-1+j}^{m+1} \right) \\
& + \left(2 + \theta_n^m \frac{u_{n+1}^m - u_{n-1}^m}{2(\phi(h))} \right) \frac{u_{n+1}^{m+1} - u_{n+1}^{m-1} - u_{n-1}^{m+1} + u_{n-1}^{m-1}}{4\varphi(\Delta t) \cdot (\phi(h))} \\
& - \sigma l_n^m + (1 - \sigma) l_n^{m+1} = T_n^m.
\end{aligned} \tag{18}$$

Neglecting the truncation error, then we have

$$\begin{aligned}
& \frac{(\Delta t)^{2-\mu_n^m}}{\Gamma(3-\mu_n^m)(\varphi(\Delta t))^2} \sum_0^{m-1} [u_n^{m-j+1} - 2u_n^{m-j} + u_n^{m-j-1}] [(j+1)^{2-\mu_n^m} - j^{2-\mu_n^m}] \\
& + \sigma \left(2 - \theta_n^m \frac{u_{n+1}^m - u_{n-1}^m}{2(\phi(h))} \right) \frac{(\phi(h))^{-\alpha_{n+1}^m}}{2 \cos(\frac{\pi \alpha_n^m}{2})} \left(\sum_{j=0}^{n+1} g_j^{\alpha_{n+1}^m} u_{n+1-j}^m + \sum_{j=0}^{N-n+1} g_j^{\alpha_{n-1}^m} u_{n-1+j}^m \right) \\
& + (1-\sigma) \left(2 - \theta_n^{m+1} \frac{u_{n+1}^{m+1} - u_{n-1}^{m+1}}{2(\phi(h))} \right) \\
& \times \frac{h^{-\alpha_{n+1}^{m+1}}}{2 \cos(\frac{\pi \alpha_n^{m+1}}{2})} \left(\sum_{j=0}^{n+1} g_j^{\alpha_{n+1}^{m+1}} u_{n+1-j}^{m+1} + \sum_{j=0}^{N-n+1} g_j^{\alpha_{n-1}^{m+1}} u_{n-1+j}^{m+1} \right) \\
& - \sigma \left(2 + \theta_n^m \frac{u_{n+1}^m - u_{n-1}^m}{2(\phi(h))} \right) \frac{(\phi(h))^{-\alpha_{n+1}^m-1}}{2 \cos(\frac{\pi \alpha_n^m-1}{2})} \left(\sum_{j=0}^{n+1} g_j^{\alpha_{n+1}^m-1} \theta_{n+1-j}^m + \sum_{j=0}^{N-n+1} g_j^{\alpha_{n-1}^m-1} \theta_{n-1+j}^m \right) \\
& - (1-\sigma) \left(2 + \theta_n^{m+1} \frac{u_{n+1}^{m+1} - u_{n-1}^{m+1}}{2(\phi(h))} \right) \\
& \times \frac{(\phi(h))^{-\alpha_{n+1}^{m+1}-1}}{2 \cos(\frac{\pi \alpha_n^{m+1}-1}{2})} \left(\sum_{j=0}^{n+1} g_j^{\alpha_{n+1}^{m+1}-1} \theta_{n+1-j}^{m+1} + \sum_{j=0}^{N-n+1} g_j^{\alpha_{n-1}^{m+1}-1} \theta_{n-1+j}^{m+1} \right) \\
& - \sigma f_n^m + (1-\sigma) f_n^{m+1} = 0, \\
& \frac{(\Delta t)^{2-\mu_n^m}}{2\Gamma(3-\mu_n^m)(\varphi(\Delta t))} \sum_0^{m-1} [\theta_n^{m-j+1} - \theta_n^{m-j-1}] [(j+1)^{2-\mu_n^m} - j^{2-\mu_n^m}] \\
& + \sigma \theta_n^m \sigma \frac{(\phi(h))^{-\alpha_{n+1}^m}}{2 \cos(\frac{\pi \alpha_n^m}{2})} \left(\sum_{j=0}^{n+1} g_j^{\alpha_{n+1}^m} \theta_{n+1-j}^m + \sum_{j=0}^{N-n+1} g_j^{\alpha_{n-1}^m} \theta_{n-1+j}^m \right) \\
& + (1-\sigma) \theta_n^{m+1} \frac{(\phi(h))^{-\alpha_{n+1}^{m+1}}}{2 \cos(\frac{\pi \alpha_n^{m+1}}{2})} \left(\sum_{j=0}^{n+1} g_j^{\alpha_{n+1}^{m+1}} \theta_{n+1-j}^{m+1} + \sum_{j=0}^{N-n+1} g_j^{\alpha_{n-1}^{m+1}} \theta_{n-1+j}^{m+1} \right) \\
& + \left(2 + \theta_n^m \frac{u_{n+1}^m - u_{n-1}^m}{2(\phi(h))} \right) \frac{u_{n+1}^{m+1} - u_{n+1}^m - u_{n-1}^{m+1} + u_{n-1}^m}{4\varphi(\Delta t) \cdot (\phi(h))} \\
& - \sigma l_n^m + (1-\sigma) l_n^{m+1} = 0.
\end{aligned} \tag{19}$$

All of the above schemes (13), (16), and (19) with their particular initial conditions and boundary conditions form an algebraic system of $2(N+1)(M+1)$ nonlinear equations with u_n^m, θ_n^m ($n = 0, 1, 2, \dots, N, m = 0, 1, 2, \dots, M$) as unknowns. The algebraic system can be solved using Newton's iteration method [64].

The schemes (13), (16), and (19) are explicit for $\sigma = 1$, partially implicit for $0 < \sigma < 1$, fully implicit for $\sigma = 0$, and for the special case $\sigma = 1/2$ therefore we use the Crank–Nicholson scheme [51].

4 Analysis of the stability and truncation error

4.1 Stability analysis

To discuss the stability of the proposed schemes, we use the von Neumann-type stability technique and, because this technique deals only with linear problems, we have to linearize

these schemes by supposing that the terms a_1, b_1, c_1, d_1, h_1 are constants and considering $f = l = 0$ (free source term) (see [1] and [65] for more details). To start our study analysis, we write system (1) in another form as follows:

$$\begin{aligned} {}^c D_t^{\mu(x,t)} u - a_1 R^{\alpha(x,t)} u + b_1 R^{\alpha(x,t)-1} \theta &= 0, \\ {}^c D_t^{\mu(x,t)-1} \theta - d_1 R^{\alpha(x,t)} \theta - h_1 R^{\alpha(x,t)-1} \theta + c_1 u_{xt} &= 0. \end{aligned} \quad (20)$$

Letting $u_t = v$, then we can express:

$$\begin{aligned} {}^c D_t^{\mu(x,t)-1} v - a_1 R^{\alpha(x,t)} u + b_1 R^{\alpha(x,t)-1} \theta &= 0, \\ u_t &= v, \\ {}^c D_t^{\mu(x,t)-1} \theta - d_1 R^{\alpha(x,t)} \theta - h_1 R^{\alpha(x,t)-1} \theta + c_1 v_x &= 0. \end{aligned} \quad (21)$$

Firstly, we write down system (21) in matrix form as follows:

$$\mathbf{Y}_1 {}^c D_t^{\mu-1} \mathbf{X} + \mathbf{Y}_2 \mathbf{R}^{\alpha(x,t)} \mathbf{X} + \mathbf{Y}_3 \mathbf{R}^{\alpha(x,t)-1} \mathbf{X} + \mathbf{Y}_4 \mathbf{X}_t + \mathbf{Y}_5 \mathbf{X}_x + \mathbf{Y}_6 \mathbf{X} = 0, \quad (22)$$

where

$$\begin{aligned} \mathbf{X} &= \begin{pmatrix} u \\ v \\ \theta \end{pmatrix}, \quad \mathbf{Y}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ \mathbf{Y}_2 &= \begin{pmatrix} -a_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -d_1 \end{pmatrix}, \quad \mathbf{Y}_3 = \begin{pmatrix} 0 & 0 & b_1 \\ 0 & 0 & 0 \\ 0 & 0 & -h_1 \end{pmatrix}, \\ \mathbf{Y}_4 &= \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{Y}_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & c_1 & 0 \end{pmatrix}, \quad \mathbf{Y}_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

Now, we write system (22) using WANSFDM as follows:

$$\begin{aligned} &\mathbf{Y}_1 \frac{(\Delta t)^{2-\mu_n^m}}{2\Gamma(3-\mu_n^m)(\varphi(\Delta t))} \sum_{j=0}^{m-1} [\mathbf{X}_n^{m-j+1} - \mathbf{X}_n^{m-j-1}] [(j+1)^{2-\mu_n^m} - j^{2-\mu_n^m}] \\ &- \sigma \mathbf{Y}_2 \frac{(\phi(h))^{-\alpha_{n+1}^m}}{2 \cos(\frac{\pi \alpha_n^m}{2})} \left(\sum_{j=0}^{n+1} g_j^{\alpha_{n+1}^m} \mathbf{X}_{n+1-j}^m + \sum_{j=0}^{N-n+1} g_j^{\alpha_{n-1}^m} \mathbf{X}_{n-1+j}^m \right) \\ &- (1-\sigma) \mathbf{Y}_2 \frac{(\phi(h))^{-\alpha_{n+1}^{m+1}}}{2 \cos(\frac{\pi \alpha_n^m}{2})} \left(\sum_{j=0}^{n+1} g_j^{\alpha_{n+1}^{m+1}} \mathbf{X}_{n+1-j}^{m+1} + \sum_{j=0}^{N-n+1} g_j^{\alpha_{n-1}^{m+1}} \mathbf{X}_{n-1+j}^{m+1} \right) \\ &- \sigma \mathbf{Y}_3 \frac{(\phi(h))^{-\alpha_{n+1}^{m-1}}}{2 \cos(\frac{\pi \alpha_n^{m-1}}{2})} \left(\sum_{j=0}^{n+1} g_j^{\alpha_{n+1}^{m-1}} \mathbf{X}_{n+1-j}^m + \sum_{j=0}^{N-n+1} g_j^{\alpha_{n-1}^{m-1}} \mathbf{X}_{n-1+j}^m \right) \end{aligned}$$

$$\begin{aligned}
& - (1 - \sigma) Y_3 \frac{(\phi(h))^{-\alpha_{n+1}^{m+1}-1}}{2 \cos(\frac{\pi \alpha_n^m - 1}{2})} \left(\sum_{j=0}^{n+1} g_j^{\alpha_{n+1}^{m+1}-1} X_{n+1-j}^{m+1} + \sum_{j=0}^{N-n+1} g_j^{\alpha_{n-1}^{m+1}-1} X_{n-1+j}^{m+1} \right) \\
& + Y_4 \frac{X_n^{m+1} - X_n^m}{\varphi(\Delta t)} \\
& + \sigma Y_5 \frac{X_{n+1}^m - X_{n-1}^m}{2\phi(h)} + (1 - \sigma) Y_5 \frac{X_{n+1}^{m+1} - X_{n-1}^{m+1}}{2\phi(h)} \\
& + \sigma Y_6 X_n^m + (1 - \sigma) Y_6 X_n^{m+1} = 0,
\end{aligned} \tag{23}$$

which takes the following form:

$$\begin{aligned}
& Y_1 \frac{(\Delta t)^{2-\mu_n^m}}{2\Gamma(3-\mu_n^m)(\varphi(\Delta t))} X_n^{m+1} - (1 - \sigma) Y_2 \frac{(\phi(h))^{-\alpha_{n+1}^{m+1}}}{2 \cos(\frac{\pi \alpha_n^m}{2})} \left(\sum_{j=0}^N \hat{g}_{n,j}^{\alpha_{n+1}^{m+1}} X_j^{m+1} \right) \\
& - (1 - \sigma) Y_3 \frac{(\phi(h))^{-\alpha_{n+1}^{m+1}-1}}{2 \cos(\frac{\pi \alpha_n^m - 1}{2})} \left(\sum_{j=0}^N \hat{g}_{n,j}^{\alpha_{n+1}^{m+1}-1} X_j^{m+1} \right) \\
& + Y_4 \frac{X_n^{m+1}}{\varphi(\Delta t)} + (1 - \sigma) Y_5 \frac{X_{n+1}^{m+1} - X_{n-1}^{m+1}}{2\phi(h)} + (1 - \sigma) Y_6 X_n^{m+1} \\
& = \sigma Y_2 \frac{(\phi(h))^{-\alpha_{n+1}^m}}{2 \cos(\frac{\pi \alpha_n^m}{2})} \left(\sum_{j=0}^{n+1} \hat{g}_{n,j}^{\alpha_{n+1}^m} X_j^m \right) \\
& + \sigma Y_3 \frac{(\phi(h))^{-\alpha_{n+1}^m-1}}{2 \cos(\frac{\pi \alpha_n^m - 1}{2})} \left(\sum_{j=0}^{n+1} \hat{g}_{n,j}^{\alpha_{n+1}^m-1} X_j^m \right) \\
& + Y_4 \frac{X_n^m}{\varphi(\Delta t)} - \sigma Y_5 \frac{X_{n+1}^m - X_{n-1}^m}{2\phi(h)} - \sigma Y_6 X_n^m - \sum_{j=1}^{m-1} \bar{w}_j^{\mu_n^m} X_n^{m-j},
\end{aligned} \tag{24}$$

where

$$\bar{w}_j = w_{j+1} - w_j, \quad j = 1, 2, \dots, m-2, \quad \bar{w}_{m-1} = -w_{m-1},$$

and

$$w_j = [(j+1)^{2-\mu} - j^{2-\mu}], \quad j = 1, 2, \dots, m-1.$$

and

$$\hat{g}_{n,j}^{\alpha_n^m} = \begin{cases} g_{n-j+1}^{\alpha_n^m} & \text{if } j < n-1, \\ g_0^{\alpha_n^m} + g_2^{\alpha_n^m} & \text{if } j = n-1, \\ 2g_1^{\alpha_n^m} & \text{if } j = n, \\ g_0^{\alpha_n^m} + g_2^{\alpha_n^m} & \text{if } j = n+1, \\ g_{j-n+1}^{\alpha_n^m} & \text{if } j > n+1. \end{cases} \tag{25}$$

After some simplification we have:

$$\begin{aligned}
& \left(- (g_0^{\alpha_{n+1}^{m+1}} + g_2^{\alpha_{n+1}^{m+1}}) (1 - \sigma) \mathbf{Y}_2 \frac{(\phi(h))^{-\alpha_{n+1}^{m+1}}}{2 \cos(\frac{\pi \alpha_n^m}{2})} \right. \\
& \quad \left. - (g_0^{\alpha_{n+1}^{m+1}-1} + g_2^{\alpha_{n+1}^{m+1}-1}) (1 - \sigma) \mathbf{Y}_3 \frac{(\phi(h))^{-\alpha_{n+1}^{m+1}-1}}{2 \cos(\frac{\pi \alpha_n^m-1}{2})} + (1 - \sigma) \mathbf{Y}_5 \frac{1}{2 \phi(h)} \right) \mathbf{X}_{n+1}^{m+1} \\
& \quad + \left(\mathbf{Y}_1 \frac{(\Delta t)^{2-\mu_n^m}}{2 \Gamma(3 - \mu_n^m) (\varphi(\Delta t))} - 2 g_1^{\alpha_{n+1}^{m+1}} (1 - \sigma) \mathbf{Y}_2 \frac{(\phi(h))^{-\alpha_{n+1}^{m+1}}}{2 \cos(\frac{\pi \alpha_n^m}{2})} \right. \\
& \quad \left. - 2 g_1^{\alpha_{n+1}^{m+1}-1} (1 - \sigma) \mathbf{Y}_3 \frac{(\phi(h))^{-\alpha_{n+1}^{m+1}-1}}{2 \cos(\frac{\pi \alpha_n^m-1}{2})} + \mathbf{Y}_4 \frac{1}{\varphi(\Delta t)} + (1 - \sigma) \mathbf{Y}_6 \right) \mathbf{X}_n^{m+1} \\
& \quad + \left(- (g_0^{\alpha_{n+1}^{m+1}} + g_2^{\alpha_{n+1}^{m+1}}) (1 - \sigma) \mathbf{Y}_2 \frac{(\phi(h))^{-\alpha_{n+1}^{m+1}}}{2 \cos(\frac{\pi \alpha_n^m}{2})} \right. \\
& \quad \left. - (g_0^{\alpha_{n+1}^{m+1}-1} + g_2^{\alpha_{n+1}^{m+1}-1}) (1 - \sigma) \mathbf{Y}_3 \frac{(\phi(h))^{-\alpha_{n+1}^{m+1}-1}}{2 \cos(\frac{\pi \alpha_n^m-1}{2})} - (1 - \sigma) \mathbf{Y}_5 \frac{1}{2 \phi(h)} \right) \mathbf{X}_{n-1}^{m+1} \\
& \quad - (1 - \sigma) \mathbf{Y}_2 \frac{(\phi(h))^{-\alpha_{n+1}^{m+1}}}{2 \cos(\frac{\pi \alpha_n^m}{2})} \left(\sum_{j=0, j \neq \{n-1, n, n+1\}}^N \widehat{g}_{n,j}^{\alpha_{n+1}^{m+1}} \mathbf{X}_j^{m+1} \right) \\
& \quad - (1 - \sigma) \mathbf{Y}_3 \frac{(\phi(h))^{-\alpha_{n+1}^{m+1}-1}}{2 \cos(\frac{\pi \alpha_n^m-1}{2})} \left(\sum_{j=0, j \neq \{n-1, n, n+1\}}^N \widehat{g}_{n,j}^{\alpha_{n+1}^{m+1}-1} \mathbf{X}_j^{m+1} \right) \\
& = \left((g_0^{\alpha_{n+1}^m} + g_2^{\alpha_{n+1}^m}) \sigma \mathbf{Y}_2 \frac{(\phi(h))^{-\alpha_{n+1}^m}}{2 \cos(\frac{\pi \alpha_n^m}{2})} \right. \\
& \quad + (g_0^{\alpha_{n+1}^m-1} + g_2^{\alpha_{n+1}^m-1}) \sigma \mathbf{Y}_3 \frac{(\phi(h))^{-\alpha_{n+1}^m-1}}{2 \cos(\frac{\pi \alpha_n^m-1}{2})} - \sigma \mathbf{Y}_5 \frac{1}{2 \phi(h)} \Big) \mathbf{X}_{n+1}^m \\
& \quad + \left(2 g_1^{\alpha_{n+1}^{m+1}} \sigma \mathbf{Y}_2 \frac{(\phi(h))^{-\alpha_{n+1}^{m+1}}}{2 \cos(\frac{\pi \alpha_n^m}{2})} \right. \\
& \quad + 2 g_1^{\alpha_{n+1}^{m+1}-1} \sigma \mathbf{Y}_3 \frac{(\phi(h))^{-\alpha_{n+1}^{m+1}-1}}{2 \cos(\frac{\pi \alpha_n^m-1}{2})} + \mathbf{Y}_4 \frac{1}{\varphi(\Delta t)} - \sigma \mathbf{Y}_6 \Big) \mathbf{X}_n^m \\
& \quad + \left((g_0^{\alpha_{n+1}^m} + g_2^{\alpha_{n+1}^m}) \sigma \mathbf{Y}_2 \frac{(\phi(h))^{-\alpha_{n+1}^m}}{2 \cos(\frac{\pi \alpha_n^m}{2})} \right. \\
& \quad + (g_0^{\alpha_{n+1}^m-1} + g_2^{\alpha_{n+1}^m-1}) \sigma \mathbf{Y}_3 \frac{(\phi(h))^{-\alpha_{n+1}^m-1}}{2 \cos(\frac{\pi \alpha_n^m-1}{2})} - \sigma \mathbf{Y}_5 \frac{1}{2 \phi(h)} \Big) \mathbf{X}_{n-1}^m \\
& \quad + \sigma \mathbf{Y}_2 \frac{(\phi(h))^{-\alpha_{n+1}^m}}{2 \cos(\frac{\pi \alpha_n^m}{2})} \left(\sum_{j=0, j \neq \{n-1, n, n+1\}}^N \widehat{g}_{n,j}^{\alpha_{n+1}^m} \mathbf{X}_j^m \right) \\
& \quad + \sigma \mathbf{Y}_3 \frac{(\phi(h))^{-\alpha_{n+1}^m-1}}{2 \cos(\frac{\pi \alpha_n^m-1}{2})} \left(\sum_{j=0, j \neq \{n-1, n, n+1\}}^N \widehat{g}_{n,j}^{\alpha_{n+1}^m-1} \mathbf{X}_j^m \right) \\
& \quad - \sum_{j=1}^{m-1} \bar{w}_j^{\mu_n^m} \mathbf{X}_n^{m-j},
\end{aligned} \tag{26}$$

this scheme takes the following composite form:

$$\begin{aligned} & \mathbf{A}_1 \mathbf{X}_{n+1}^{m+1} + \mathbf{A}_2 \mathbf{X}_n^{m+1} + \mathbf{A}_3 \mathbf{X}_{n-1}^{m+1} - (1 - \sigma) \sum_{j=0, j \neq \{n-1, n, n+1\}}^N (\mathbf{A}_{4,j} + \mathbf{A}_{5,j}) \mathbf{X}_j^{m+1} \\ &= \mathbf{B}_1 \mathbf{X}_{n+1}^m + \mathbf{B}_2 \mathbf{X}_n^m + \mathbf{B}_3 \mathbf{X}_{n-1}^m + \sigma \sum_{j=0, j \neq \{n-1, n, n+1\}}^N (\mathbf{B}_{4,j} + \mathbf{B}_{5,j}) \mathbf{X}_j^m \\ &\quad - \sum_{j=1}^{m-1} \bar{w}_j^{\mu_n^m} \mathbf{X}_n^{m-j}. \end{aligned} \quad (27)$$

Depending on the von Neumann stability method, we presume

$$\mathbf{X}_n^m = \boldsymbol{\xi}^m \boldsymbol{\Upsilon} e^{in\beta h}, \quad (28)$$

such that $\beta \in \mathbb{R}$, $i = \sqrt{-1}$, $\boldsymbol{\Upsilon} \in \mathbb{R}^{4 \times 1}$, and $\boldsymbol{\xi} \in \mathbb{R}^{4 \times 4}$ is an amplification element. Putting Eq. (28) into Eq. (27), after some manipulation we obtain the following form of the amplification matrix:

$$\begin{aligned} \boldsymbol{\xi} &= \left[\mathbf{A}_1 e^{i\beta h} + \mathbf{A}_2 + \mathbf{A}_3 e^{-i\beta h} - (1 - \sigma) \sum_{j=0, j \neq \{n-1, n, n+1\}}^N (\mathbf{A}_{4,j} + \mathbf{A}_{5,j}) e^{i(j-n)\beta h} \right]^{-1} \\ &\quad \times \left[\mathbf{B}_1 e^{i\beta h} + \mathbf{B}_2 + \mathbf{B}_3 e^{-i\beta h} + \sigma \sum_{j=0, j \neq \{n-1, n, n+1\}}^N (\mathbf{B}_{4,j} + \mathbf{B}_{5,j}) e^{i(j-n)\beta h} + \sum_1^{m-1} \bar{w}_j^{\mu_n^m} \boldsymbol{\xi}^{-j} \right]. \end{aligned} \quad (29)$$

The scheme will be stable when $\|\boldsymbol{\xi}\| \leq 1$. As in [50, 51, 66], we consider on the right side the time-independent $\boldsymbol{\xi} = \mathbf{I}$ so:

$$\sum_1^{m-1} \bar{w}_j^{\mu_n^m} \boldsymbol{\xi}^{-j} = (2^{2-\mu_n^m} - 1) \mathbf{I},$$

therefore,

$$\begin{aligned} \boldsymbol{\xi} &= \left[\mathbf{A}_1 e^{i\beta h} + \mathbf{A}_2 + \mathbf{A}_3 e^{-i\beta h} - (1 - \sigma) \sum_{j=0, j \neq \{n-1, n, n+1\}}^N (\mathbf{A}_{4,j} + \mathbf{A}_{5,j}) e^{i(j-n)\beta h} \right]^{-1} \\ &\quad \times \left[\mathbf{B}_1 e^{i\beta h} + \mathbf{B}_2 + \mathbf{B}_3 e^{-i\beta h} + \sigma \sum_{j=0, j \neq \{n-1, n, n+1\}}^N (\mathbf{B}_{4,j} + \mathbf{B}_{5,j}) e^{i(j-n)\beta h} + (2^{2-\mu_n^m} - 1) \mathbf{I} \right]. \end{aligned} \quad (30)$$

Here $\boldsymbol{\xi}$ depends on μ_n^m , σ , h , and Δt , as we see from Eq. (30).

Scheme (27) will be stable as soon as the following relation is satisfied:

$$\begin{aligned} & \left\| \left[\mathbf{A}_1 e^{i\beta h} + \mathbf{A}_2 + \mathbf{A}_3 e^{-i\beta h} - (1 - \sigma) \sum_{j=0, j \neq \{n-1, n, n+1\}}^N (\mathbf{A}_{4,j} + \mathbf{A}_{5,j}) e^{i(j-n)\beta h} \right]^{-1} \right. \\ & \quad \times \left. \left[\mathbf{B}_1 e^{i\beta h} + \mathbf{B}_2 + \mathbf{B}_3 e^{-i\beta h} + \sigma \sum_{j=0, j \neq \{n-1, n, n+1\}}^N (\mathbf{B}_{4,j} + \mathbf{B}_{5,j}) e^{i(j-n)\beta h} + (2^{2-\mu_n^m} - 1) \mathbf{I} \right] \right\| \\ & \leq 1. \end{aligned} \quad (31)$$

4.2 Truncation error

Depending on definition of truncation error (15), (18), and (12), and from relations (11a)–(11f), we have

$$T_n^m = O((\phi(h))^2 + (\varphi(\Delta t))^2),$$

but $\phi(h) = h + O(h)^2$ and $\varphi(\Delta t) = \Delta t + O(\Delta t)^2$, so

$$T_n^m = O(h^2 + (\Delta t)^2).$$

Remark 4.1 It is well known that the Lax–Richtmyer equivalence theorem says: for consistent numerical approximations, stability and convergence are equivalent in the integer order differential equations. Yuste and Murillo in 2012 generalized this theorem to the fractional derivative case. They proved in [67, Sect. 4] that for a whole class of the fractional difference algorithms the consistency and stability of the difference scheme implies its convergence. Therefore our scheme is convergent under condition (31).

5 Numerical simulations

In the present section, three test examples are introduced to show the application of the used technique to solve the proposed nonlinear coupled model of a time-space fractional variable-order hyperbolic and parabolic equations. To check the accuracy of the proposed scheme, we estimate the L_∞ -error (when $\mu(x, t) = \alpha(x, t) = 2$, the derivatives of integer order) using formulas

$$E_1 = \max |u_{\text{exact}}(x_n, t_m) - u_{\text{approx}}(x_n, t_m)|, \quad E_2 = \max |\theta_{\text{exact}}(x_n, t_m) - \theta_{\text{approx}}(x_n, t_m)|$$

for all n, m . Let $\varphi(\Delta t) = 1 - e^{-\Delta t}$ and $\phi(h) = \sinh(\Delta x)$.

Example 1 ([1]) We deal with the system (10) for:

$$\begin{aligned} f(x, t) &= e^{-x} \cos t + (\beta_1 - 1)e^{-x}(1 - \cos t) \\ &\quad + 2\gamma e^{-2x}(1 - \cos t)^2 - 3\delta e^{-3x}(1 - \cos t)^3, \\ l(x, t) &= (1 + a)e^{-x} \sin t - (1 + be^{-x} \sin t)e^{-x}(1 - \cos t) + 2\alpha^* e^{-2x}(1 - \cos t)^2, \end{aligned} \quad (32)$$

and the exact solutions when $\mu(x, t) = \alpha(x, t) = 2$ (derivatives of integer order) are

$$u(x, t) = e^{-x}(1 - \cos t) = \theta(x, t), \quad 0 \leq x \leq 1.$$

The boundary conditions are

$$u(0, t) = 1 - \cos t, \quad u(1, t) = \frac{1 - \cos t}{e}, \quad \theta(0, t) = 1 - \cos t, \quad \theta(1, t) = \frac{1 - \cos t}{e},$$

and the initial conditions are

$$u(x, 0) = 0, \quad u_t(x, 0) = 0, \quad \theta(x, 0) = 0,$$

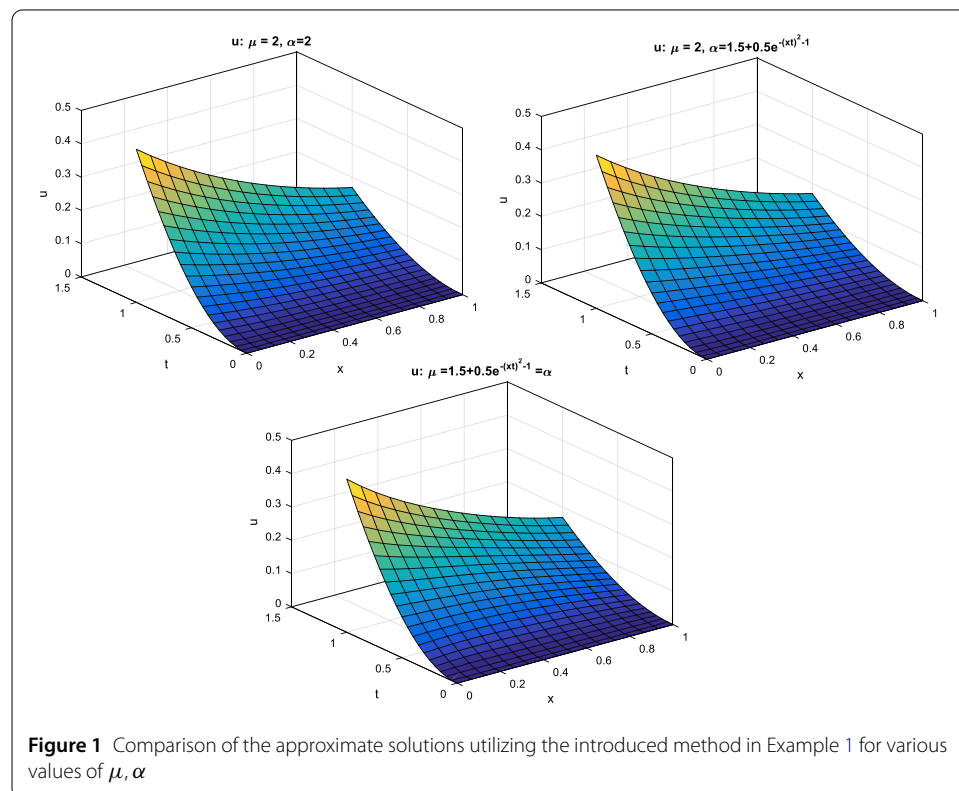
and $a = 0.1$, $b = 0.1$, $\alpha^* = 1$, $\beta_1 = 0.5$, $\beta_2 = 0.5$, $\gamma = 1$, $\delta = 0.8$.

Table 1 Comparison of E_1 and E_2 , the maximum absolute errors, for different values of N, M in Example 1, when $0 < t \leq 1$

N	M	E_1			E_2		
		$\sigma = 1$	$\sigma = 0.5$	$\sigma = 0$	$\sigma = 1$	$\sigma = 0.5$	$\sigma = 0$
10	10	3.5478e-01	1.1329e-02	4.4571e-03	4.3610e-01	5.8361e-02	2.2423e-02
25	25	4.1842e+00	4.6783e-03	1.9563e-03	divergent	4.3784e-03	5.6345e-03
40	40	9.8712e+00	7.7532e-04	8.3712e-04	divergent	1.3741e-03	1.1524e-03
50	50	divergent	1.0763e-04	4.5137e-04	divergent	7.4723e-04	7.7859e-04

Table 2 Comparison of E_1 and E_2 , the maximum absolute errors, for different values of N, M in Example 1, when $0 < t \leq 5$

N	M	E_1			E_2		
		$\sigma = 1$	$\sigma = 0.5$	$\sigma = 0$	$\sigma = 1$	$\sigma = 0.5$	$\sigma = 0$
40	40	divergent	3.4226e-01	2.1399e-01	divergent	8.0341e-01	1.8633e-01
60	60	divergent	6.8113e-02	8.3989e-02	divergent	5.65394e-02	2.9847e-02
60	80	divergent	9.9546e-03	1.0302e-02	divergent	7.4853e-03	1.0301e-02



From Table 1 we see that E_1 and E_2 are reduced when N and M grow with $\sigma = 0, 0.5$ and $0 < t \leq 1$ and deduce that the approximations obtained by the explicit NSFDM do not converge because the condition of stability is not satisfied in this case.

Also, when $0 < t \leq 5$, we deduce the same conclusion from Table 2.

In Fig. 1 we see the approximation of u which was achieved by the introduced WANSFD scheme for various values of α, μ where $\sigma = 0.5$ and $0 < t \leq 1$.

In Fig. 2 we see the approximation of u which was achieved by the introduced WANSFD scheme for various values of α, μ where $\sigma = 0.5$ and $0 < t \leq 5$.

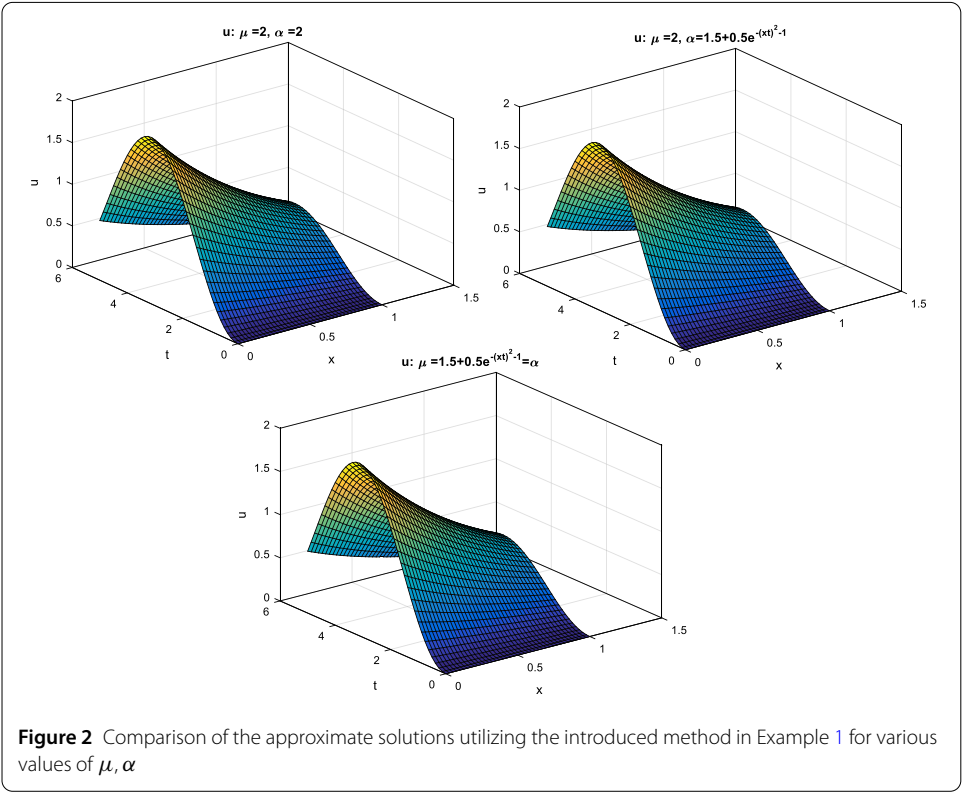


Figure 2 Comparison of the approximate solutions utilizing the introduced method in Example 1 for various values of μ, α

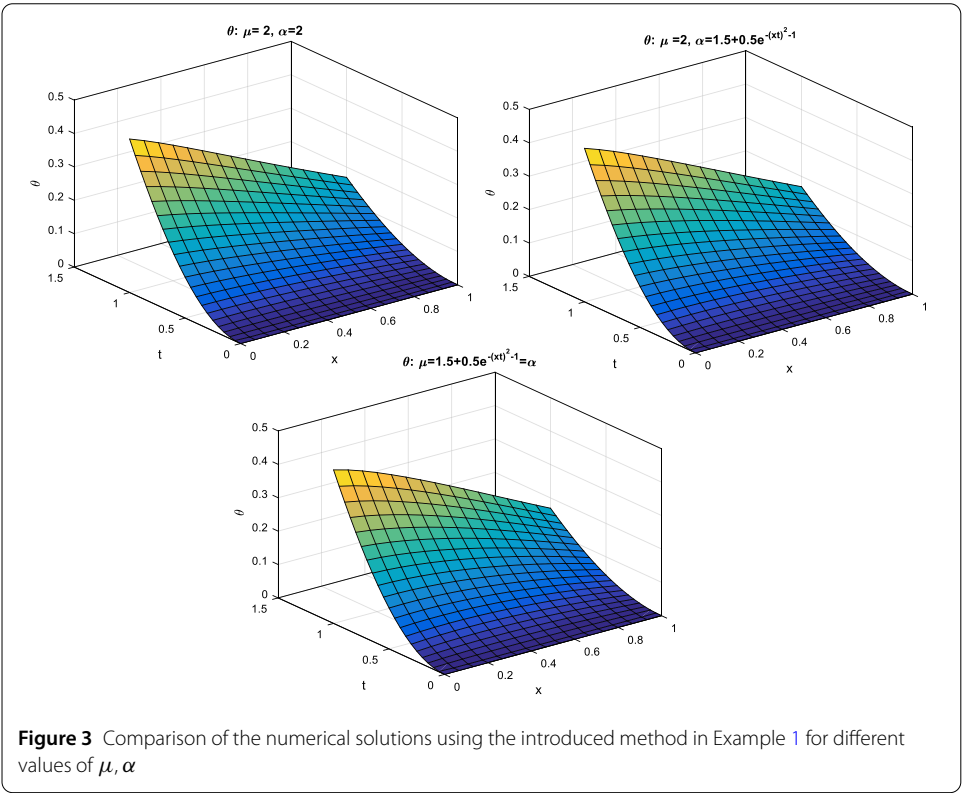


Figure 3 Comparison of the numerical solutions using the introduced method in Example 1 for different values of μ, α

In Fig. 3 we see the approximation of θ which was achieved by the introduced WANSFD scheme for various values of α, μ where $\sigma = 0.5$.

Example 2 ([55]) We treat system (14) for

$$f(x, t) = 2e^{-t} \sin x - e^{-3t} \cos^2 x \sin x, \quad l(x, t) = -e^{-3t} \cos^3 x,$$

and the analytic exact solutions for $\mu = 2, \alpha = 2$ (integer order derivative) are given by

$$u(x, t) = e^{-t} \sin x, \quad \theta(x, t) = e^{-t} \cos x, \quad 0 \leq x \leq \pi.$$

The boundary conditions are

$$u(0, t) = 0, \quad u(\pi, t) = 0, \quad \theta(0, t) = e^{-t}, \quad \theta(\pi, t) = -e^{-t},$$

and the initial conditions are

$$u(x, 0) = \sin x, \quad u_t(x, 0) = -\sin x, \quad \theta(x, 0) = \cos x.$$

From Table 3 we see that E_1 and E_2 are reduced when N and M grow when $\sigma = 0, 0.5, 0 < t \leq 1$ and deduce that the numerical approximation utilizing the explicit NSFDM is not convergent because the condition of stability is not satisfied in this case.

In Fig. 4 we see the approximation of u which was achieved by the introduced WANSFD scheme for various values of μ, α where $\sigma = 0.5$.

In Fig. 5 we see the approximation of θ which was achieved by the introduced WANSFD scheme for various values of μ, α where $\sigma = 0.5$.

Figure 6 illustrates the behavior of the approximations of u obtained by the WANSFD technique when ($\sigma = 0.5$) at $t = 0.5$, for various values of μ and $\alpha = 2$.

Figure 7 illustrates the behavior of the numerical approximations of u obtained by the WANSFD technique when ($\sigma = 0.5$) at $x = 0.7$, for various values of α and $\mu = 2$.

Figure 8 illustrates the behavior of the numerical approximations of θ obtained by the WANSFD method when ($\sigma = 0.5$) at $t = 0.5$, for various values of μ and $\alpha = 2$.

Figure 9 illustrates the behavior of the numerical approximations of θ obtained by the WANSFD method when ($\sigma = 0.5$) at $x = 0.7$, for various values of α and $\mu = 2$.

Table 3 Comparison of E_1 and E_2 , the maximum absolute errors, for different values of N, M in Example 2

N	M	E_1			E_2		
		$\sigma = 1$	$\sigma = 0.5$	$\sigma = 0$	$\sigma = 1$	$\sigma = 0.5$	$\sigma = 0$
10	10	5.2998e-00	4.5278e-02	2.0864e-02	6.2518e-01	2.0721e-02	2.0668e-02
25	25	divergent	6.4189e-03	8.9760e-03	divergent	9.6235e-03	8.9655e-03
40	40	divergent	3.7251e-03	2.5102e-03	divergent	5.1112e-03	3.8879e-03
50	50	divergent	1.0267e-03	9.5183e-04	divergent	8.9926e-04	7.1533e-04

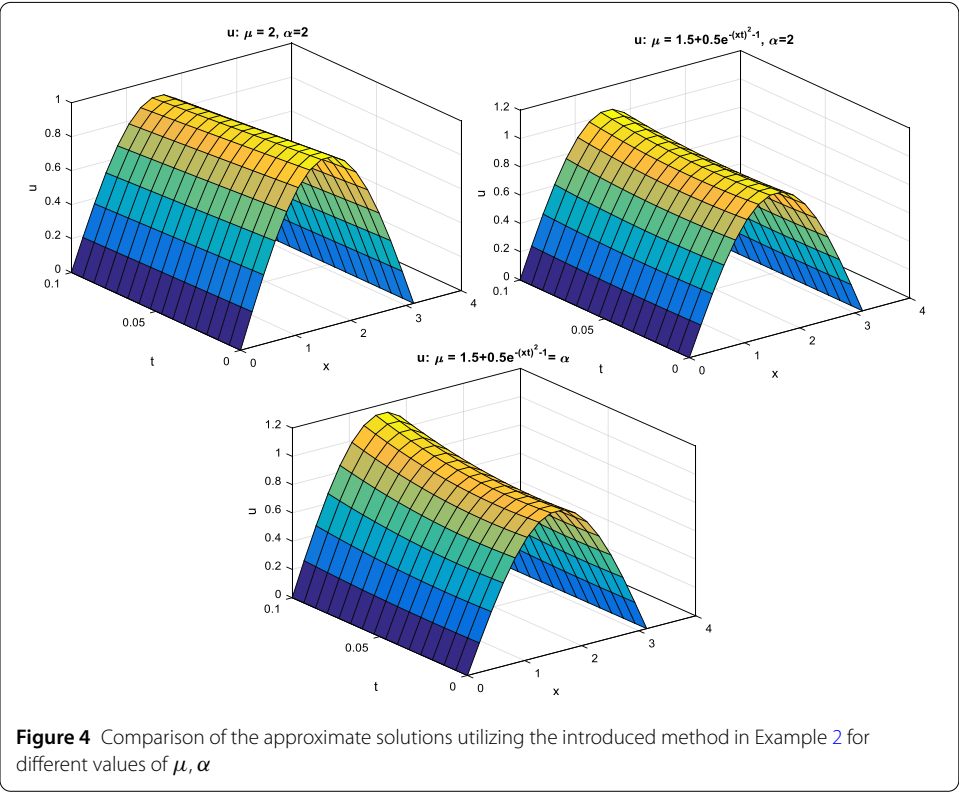


Figure 4 Comparison of the approximate solutions utilizing the introduced method in Example 2 for different values of μ, α

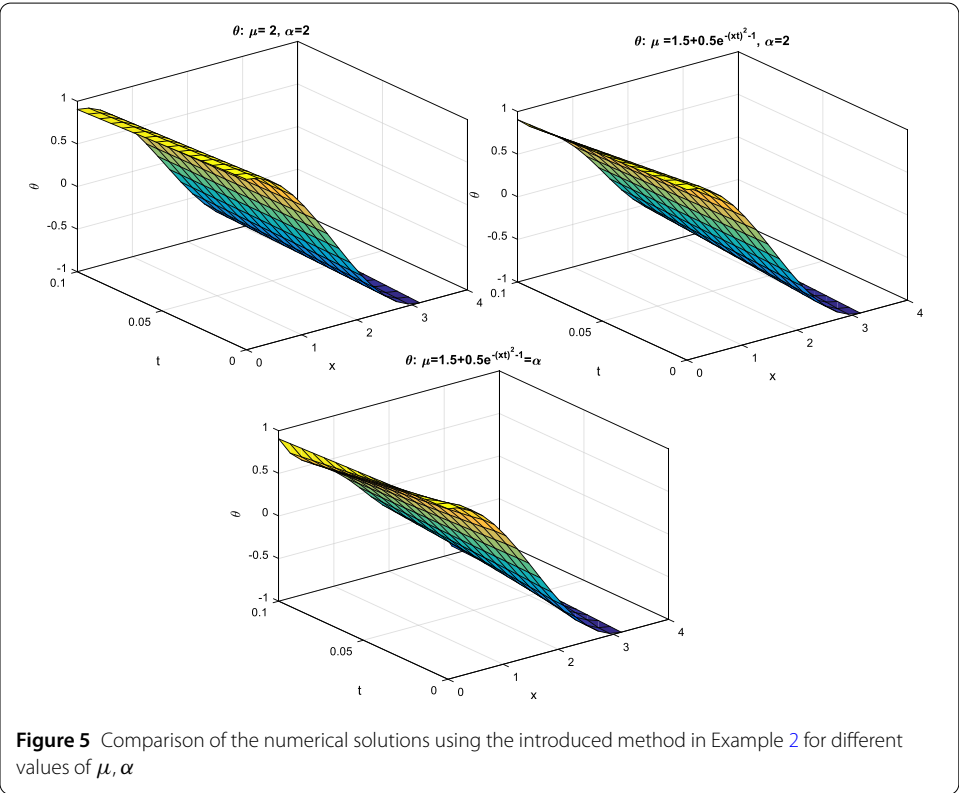


Figure 5 Comparison of the numerical solutions using the introduced method in Example 2 for different values of μ, α

Figure 6 Behavior of the solutions using the introduced schemes in Example 2 for various values of μ when $N = M = 10$ at $t = 0.5$

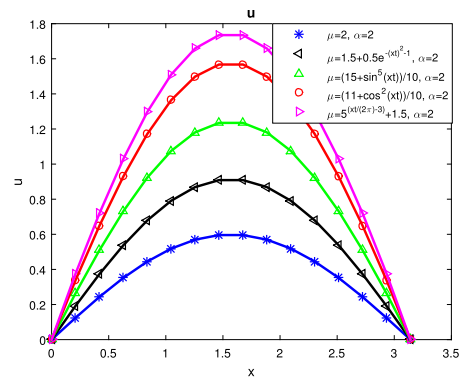


Figure 7 Behavior of the solutions using the introduced schemes in Example 2 for various values of μ when $N = M = 16$ at $x = 0.7$

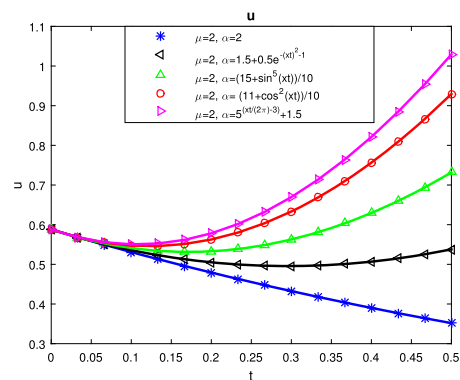


Figure 8 Behavior of the solutions using the introduced schemes in Example 2 for various values of μ when $N = M = 16$ at $t = 0.5$

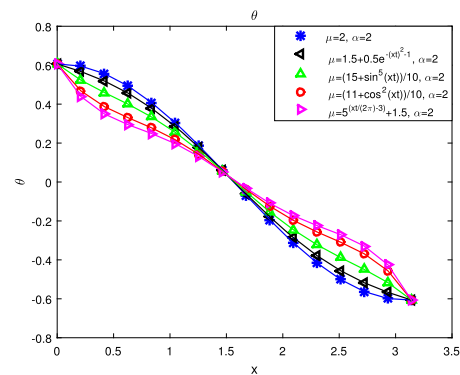
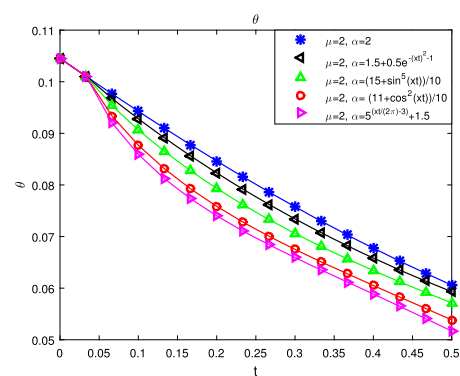


Figure 9 Behavior of the solutions using the introduced schemes in Example 2 for different values of μ when $N = M = 16$ at $x = 0.7$



Example 3 ([12]) Let us consider system (17) where

$$\begin{aligned} f(x, t) &= \frac{2}{1+x^2} - \frac{2(1+t^2)(3x^2-1)}{(1+x^2)^3} \left(2 + \frac{2x(1+t^2)}{(1+x^2)^2} \frac{1+t}{1+x^2} \right) \\ &\quad + \frac{2x(1+t)}{(1+x^2)^2} \left(2 - \frac{2x(1+t^2)}{(1+x^2)^2} \frac{1+t}{1+x^2} \right), \\ l(x, t) &= \frac{1}{1+x^2} - \frac{4xt}{(1+x^2)^2} \left(2 + \frac{2x(1+t^2)}{(1+x^2)^2} \frac{1+t}{1+x^2} \right) \\ &\quad - \frac{2(3x^2-1)(1+t)}{(1+x^2)^2} \frac{1+t}{1+x^2}, \end{aligned} \quad (33)$$

also the analytic exact solutions for $\mu = 2$, $\alpha = 2$ (integer order derivative) are

$$u(x, t) = \frac{1+t^2}{1+x^2}, \quad \theta(x, t) = \frac{1+t}{1+x^2}, \quad 0 \leq x \leq 1.$$

The boundary conditions are

$$u(0, t) = 1+t^2, \quad u(1, t) = \frac{1+t^2}{2}, \quad \theta(0, t) = 1+t, \quad \theta(\pi, t) = \frac{1+t}{2},$$

and the initial conditions are

$$u(x, 0) = \frac{1}{1+x^2}, \quad u_t(x, 0) = 0, \quad \theta(x, 0) = \frac{1}{1+x^2}.$$

From Table 4 we see that E_1 and E_2 are reduced when N and M grow for $\sigma = 0, 0.5$, $0 < t \leq 1$ and deduce that the numerical approximation depending on the explicit NSFDM is not convergent because the condition of stability is not satisfied in this case.

In Fig. 10 we see the approximation of u which was achieved by the introduced WANSFD scheme for various values of α , μ and $\sigma = 0.5$.

In Fig. 11 we see the approximation of θ which was achieved by the introduced WANSFD scheme for various values of α , μ and $\sigma = 0.5$.

Figure 12 illustrates the behavior of the numerical approximations of u obtained by the WANSFD technique when ($\sigma = 0.5$) at $x = 0.3$, for various values of α and $\mu = 2$.

Figure 13 illustrates the behavior of the numerical approximations of θ obtained by the WANSFD method when ($\sigma = 0.5$) at $x = 0.3$, for different values of α and $\mu = 2$.

Table 4 Comparison of E_1 and E_2 , the maximum absolute errors, for different values of N , M in Example 3

N	M	M_1			M_2		
		$\sigma = 1$	$\sigma = 0.5$	$\sigma = 0$	$\sigma = 1$	$\sigma = 0.5$	$\sigma = 0$
6	5	4.5332e+00	8.5664e-02	7.7553e-02	9.8871e-01	3.4288e-02	5.1149e-02
12	10	divergent	3.4421e-02	1.1233e-02	divergent	1.0542e-02	1.01783e-02
24	20	divergent	6.7833e-03	6.4355e-02	divergent	9.3441e-03	8.9178e-03
36	30	divergent	1.0499e-03	7.0998e-03	divergent	5.7334e-03	5.1807e-03

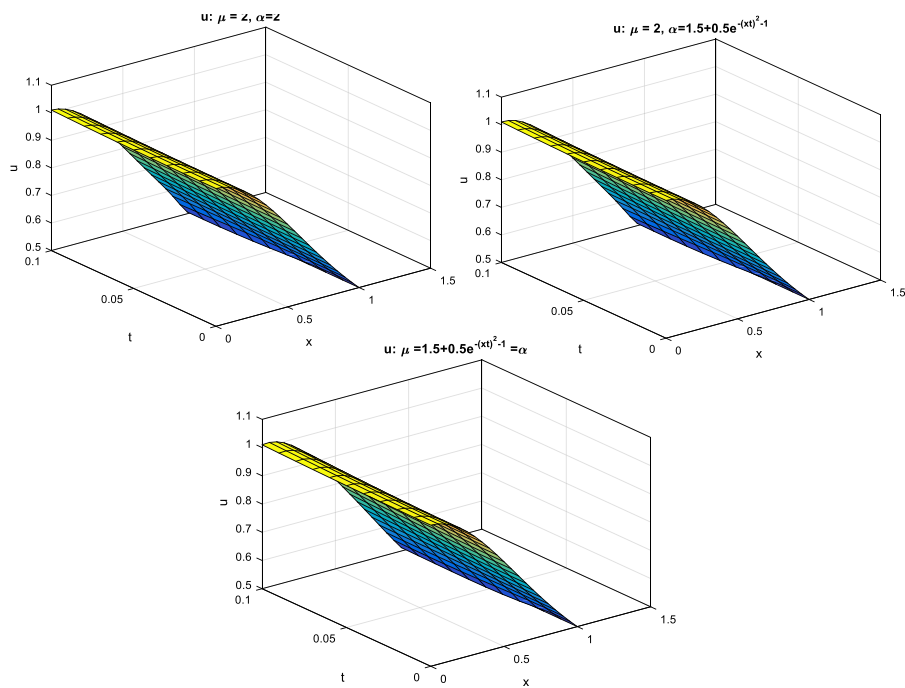


Figure 10 Comparison of the numerical solutions using the introduced method in Example 3 for various values of μ, α

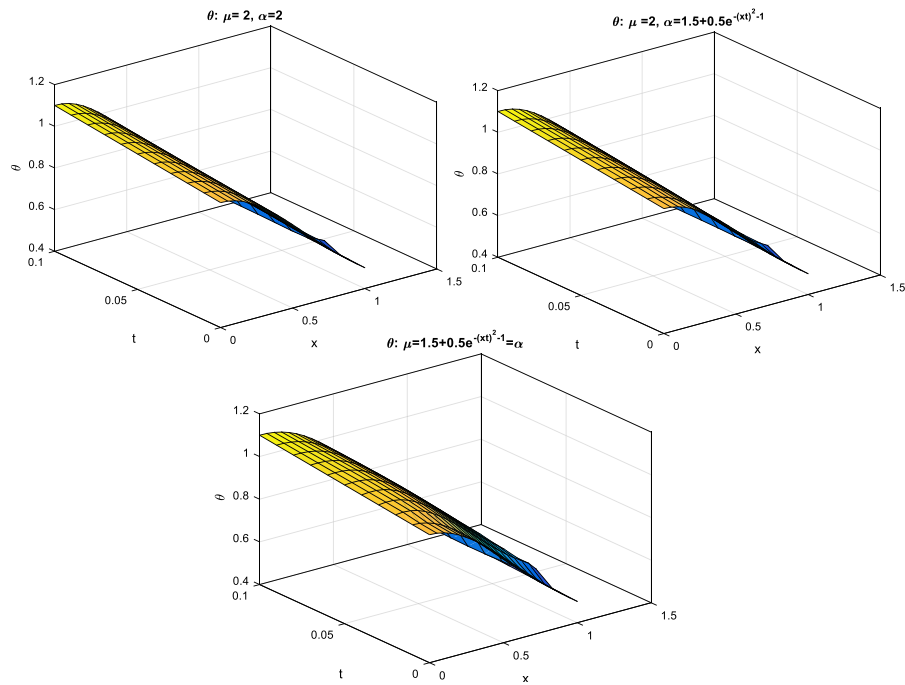


Figure 11 Comparison of the numerical solutions using the introduced method in Example 3 for various values of μ, α

Figure 12 Behavior of the solutions using the introduced schemes in Example 3 for various values of μ when $N = M = 16$ at $x = 0.3$

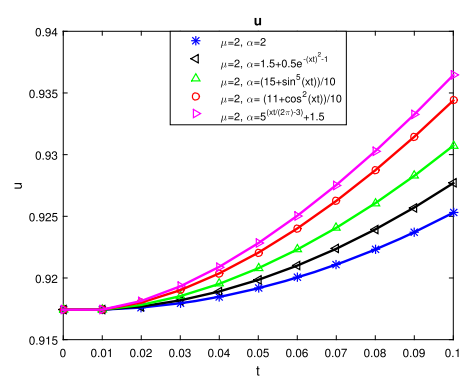
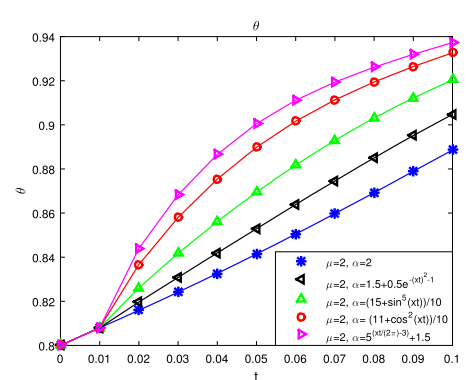


Figure 13 Behavior of the solutions using the introduced schemes in Example 3 for various values of μ when $N = M = 16$ at $x = 0.3$



6 Summary and conclusions

In the paper, we introduced numerical simulations for nonlinear systems of time-space variable-order fractional, coupled hyperbolic and parabolic partial differential equations in one dimension utilizing WANSFDM. Analyzing stability of the proposed scheme is presented by a kind of John von Neumann method, also the relation which determines the truncation error is given. Some test examples are also presented. We compare the numerical results obtained by the used technique with the analytic exact solutions of the proposed model in the case of the standard derivative (integer order). We introduce many figures to show the behaviors of the solutions when μ and α are changed.

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Authors' contributions

The author did all the work himself. The author read and approved the final manuscript.

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