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The controllability of damped fractional differential system with impulses and state delay

Musarrat Nawaz^{1*} , Jiang Wei^{1*}, Jiale Sheng¹ and Azmat Ullaha Khan¹

*Correspondence:
nawazmusarrat@yahoo.com;
jiangwei@ahu.edu.cn

¹School of Mathematical Sciences,
Anhui University, Hefei, China

Abstract

We discuss the controllability for a damped fractional differential system with impulses and state delay, which involves Caputo fractional derivatives. Deriving the condition based on the Gramian matrix defined by the Mittag-Leffler matrix function and Laplace transformation, we establish necessary and sufficient conditions of controllability criteria. Finally, we construct two numerical examples to support the result.

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1 Introduction

The fractional differential equations have proven to be a useful tool for modeling distinct phenomena in various fields of physics, engineering, and economics. Various practical systems can be described more accurately through fractional derivative formulation, as explained in detail based on the theory of fractional calculus [1–6]. Many mathematicians, engineers, and physicists contributed in fractional differential equations theory and its applications. Fractional derivatives have several kinds, such as Caputo, Riemann–Liouville, Grunwald–Letnikov, and Hadamard. Nowadays research on fractional delay differential equations is on the rise, whereas the theory of delay differential equations is well developed [7–15].

The controllability performs an important role in the formulation of current mathematical control theory and engineering, which has a close connection with structural decomposition, quadratic optimal, and so on [16–21]. The controllability is a subjective property in the theory of dynamical systems [22–24]. The controllability of linear systems, nonlinear systems, and stochastic systems with delay has been investigated in [25–31]. Moreover, in recent years, controllability problems for linear and nonlinear fractional differential system are discussed by Matar and Nawaz [32–35], and partial controllability of various semilinear systems is discussed in [36–45]. More recent research work on controllability of various impulsive systems is reported in [46–57].

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Many dynamical systems have been studied by applying discrete- or continuous-time domains. Moreover, many real systems in biology, physics, chemistry, information science, and engineering may face sudden changes at certain instants during continuous dynamical processes. This type of behavior can be formed by impulsive systems. In the past few years the research on impulsive control systems has aroused magnificent interest; the existence of impulses can be observed in biological phenomena containing bursting rhythm models in biology and medicine, thresholds, and frequency-modulated systems. The basic knowledge of a impulsive differential equations can be found in the monographs of Benchohra et al. [58] and Bainov et al. [59]. Feickan et al. [60] discussed the concept and existence of solutions for impulsive fractional differential equations. The controllability and observability for linear fractional impulsive systems with time-invariance are discussed in [61]. Furthermore, the controllability criteria for linear and nonlinear fractional differential systems with state delay and impulses is studied in [62, 63]. The controllability of second-order semilinear impulsive stochastic neutral functional evolution equations is explained in [64]. In addition, the controllability of impulsive neutral functional differential inclusions in Banach spaces is studied by Wan et al. [65]. Moreover, a recent study on neural networks via impulsive control is discussed in [66]. The derivation of the condition in the present study is based on analytical evaluation of the Gramian matrix defined by the Mittage-Leffler matrix function. However, from a numerical point of view, the Mittag-Leffler matrix function is explained by Garrappa [67]. The study of the damped fractional differential system for its controllability results by using the Mittag-Leffler matrix function and Gramian matrix is presented in [68]. Moreover, the Kalman rank criterion is obtained for the controllability of fractional impulse controlled systems in [69], where controllability is dependent on the impulses.

The previous studies mainly focused on different types of fractional impulsive systems. Still no work is reported on the controllability of the damped fractional differential system with impulses and state delay. In this paper, motivated mainly by [62, 68, 70], in this study, we are concerned with the controllability criteria for the damped fractional differential system with impulses and algebraic-based state delay. This paper is organized as follows. Section 2 includes a few fundamental definitions, preliminary results, and lemmas to prove the controllability of a damped fractional differential system with impulses and state delay. In Sect. 3, we obtain a sufficient condition for the controllability of a damped fractional differential system with impulses and state delay by step technique. Two examples in Sect. 4 demonstrate the applicability of the outcomes.

In this study, we consider the controllability of damped fractional differential systems with impulses and state delay

$$\begin{cases} {}^cD^\gamma x(t) - {}^cD^\delta Ax(t) = Bx(t - \tau) + Cu(t), & t \in [0, T] \setminus \{t_1, t_2, \dots, t_k\}, \\ \Delta x(t_i) = x(t_i^+) - x(t_i^-) = I_i(x(t_i)), & i = 1, 2, 3, \dots, k, \\ x(t) = \varphi(t), & t \in [-\tau, 0], \\ x'(t) = \varphi'(t), \end{cases} \quad (1)$$

where ${}^cD^\gamma x(t)$ and ${}^cD^\delta x(t)$ denote the γ th- and δ th-order Caputo fractional derivatives of $x(t)$, $0 < \delta < 1 < \gamma < 2$, $x(t) \in \mathcal{R}^n$ is the state vector, $u(t) \in \mathcal{R}^m$ is the control vector, A , $B \in \mathcal{R}^{n \times n}$ and $C \in \mathcal{R}^{n \times m}$ are any matrices, τ is a positive constant, and $\varphi(t) \in C([- \tau, 0], \mathcal{R}^n)$

is the function that represents the initial state, where $C([-τ, 0], \mathcal{R}^n)$ is the space of all continuous functions from the interval $[-τ, 0]$ into \mathcal{R}^n , $I_i : \mathcal{R}^n \rightarrow \mathcal{R}^n$ is continuous for $i = 1, 2, 3, \dots, k$, where k is an integer,

$$x(t_i^+) = \lim_{\varepsilon \rightarrow 0^+} x(t_i + \varepsilon), \quad x(t_i^-) = \lim_{\varepsilon \rightarrow 0^-} x(t_i - \varepsilon) \quad (2)$$

represent the right and left limits of $x(t)$ at the discontinuity points $t = t_i$,

$$t_{i-1} < i\tau < t_i, \quad i = 1, 2, \dots, k, \quad (3)$$

where $0 = t_0$ and $t_{k+1} = T$, and $x(t_i) = x(t_i^-)$, which means that the solution of system (1) is left continuous at t_i .

2 Preliminaries and essential lemmas

We denote by $C_p([0, T], \mathcal{R}^n)$ the space of all piecewise left-continuous functions from the interval $[0, T]$ into \mathcal{R}^n .

To obtain the main results, we introduce some fundamental definitions and lemmas. The following basic definitions are well known; for more detail, see [1–5].

Definition 2.1 The Riemann–Liouville fractional integral of order $γ > 0$ for a function $f : R^+ \rightarrow R$ is defined as

$$D^{-γ}f(t) = \frac{1}{\Gamma(γ)} \int_0^t (t - θ)^{γ-1} f(θ) dθ,$$

where $Γ$ is the Euler gamma function.

Definition 2.2 ([14]) The Caputo fractional derivative of order $γ$ ($0 ≤ m < γ ≤ m + 1$) for a function $f : R^+ \rightarrow R^n$ is defined as

$${}^cD^γf(t) = \frac{1}{\Gamma(m - γ + 1)} \int_0^t \frac{f^{(m+1)}(θ)}{(t - θ)^{γ-m}} dθ.$$

The Laplace transform of the Caputo fractional derivative is

$$\begin{aligned} L\{f(t); (s)\} &= F(s), \\ L\{{}^cD^γf(t)\}(s) &= s^γ F(s) - \sum_{k=0}^{m-1} f^{(k)}(0^+) s^{γ-1-k}. \end{aligned}$$

In particular, if $0 < γ ≤ 1$, then

$$L\{{}^cD^γf(t)\}(s) = s^γ F(s) - f(0^+) s^{γ-1},$$

and if $1 < γ ≤ 2$, then

$$L\{{}^cD^γf(t)\}(s) = s^γ F(s) - f(0^+) s^{γ-1} - \dot{f}(0^+) s^{γ-2},$$

where s is the complex spectral variable of the Laplace transform.

Definition 2.3 Consider the Mittag-Leffler function

$$E_{\gamma,\delta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\gamma k + \delta)}, \quad z \in C, \gamma > 0, \delta > 0,$$

where C is the complex plane. For $\delta = 1$, the Mittag-Leffler function becomes

$$E_{\gamma}(\lambda z^{\gamma}) = E_{\gamma,1}(\lambda z^{\gamma}) = \sum_{k=0}^{\infty} \frac{\lambda^k z^{k\gamma}}{\Gamma(\gamma k + 1)}, \quad \lambda, z \in C.$$

The Laplace transform of the Mittag-Leffler function is

$$L\{t^{\delta-1} E_{\gamma,\delta}(\pm at^{\gamma})\}(s) = \frac{s^{\gamma-\delta}}{(s^{\gamma} \mp a)}.$$

Particularly, for $\delta = 1$,

$$L\{E_{\gamma}(\pm at^{\gamma})\}(s) = \frac{s^{\gamma-1}}{(s^{\gamma} \mp a)}.$$

Lemma 2.4 ([5]) For any $\gamma, \delta > 0$ and $A \in C^{n \times n}$, we have

$$L\{t^{\delta-1} E_{\gamma,\delta}(\pm At^{\gamma})\}(s) = s^{\gamma-\delta} (s^{\gamma} I \mp A)^{-1}, \quad \Re(s) > \|A\|^{\frac{1}{\alpha}},$$

where $\Re(s)$ is the real part of a complex number s , and I is the identity matrix.

First, we consider the representation of solutions of damped fractional delay differential systems without impulses to obtain the state response of system (1) as follows:

$$\begin{cases} {}^c D^{\gamma} x(t) - {}^c D^{\delta} Ax(t) = Bx(t - \tau) + f(t), & t \in [0, T], \\ x(t) = \varphi(t), & t \in [-\tau, 0]. \end{cases} \quad (4)$$

Lemma 2.5 Let $0 < \delta < 1 < \gamma < 2$. If $f : [0, T] \rightarrow \mathcal{R}^n$ is continuous and exponentially bounded, then the solution of (4) is equivalent to the solution of the system

$$\begin{cases} x(t) = \varphi(0) + tE_{\gamma-\delta,2}(At^{\gamma-\delta})\varphi'(0) + \int_0^t (t-s)^{\gamma-1} E_{\gamma-\delta,\gamma}[A(t-s)^{\gamma-\delta}] \\ \quad \times [Bx(s-\tau) + f(s)] ds, \quad t \in [0, T], \\ \varphi(0) + tE_{\gamma-\delta,2}(At^{\gamma-\delta})\varphi'(0) \\ \quad + \sum_{j=2}^n \int_{(j-2)\tau}^{(j-1)\tau} (t-s)^{\gamma-1} E_{\gamma-\delta,\gamma}[A(t-s)^{\gamma-\delta}][B\varphi_{j-2}(s-\tau)] ds \\ \quad + \int_{(j-1)\tau}^t (t-s)^{\gamma-1} E_{\gamma-\delta,\gamma}[A(t-s)^{\gamma-\delta}][B\varphi_{j-1}(s-\tau)] ds \\ \quad + \int_0^t (t-s)^{\gamma-1} E_{\gamma-\delta,\gamma}[A(t-s)^{\gamma-\delta}]f(s) ds, \quad t \in [(i-1)\tau, i\tau], i = 1, 2, \dots, n, \\ x(t) = \varphi(t), \quad t \in [-\tau, 0]. \end{cases} \quad (5)$$

Proof By implementing the step technique in [71] there is a unique solution to system (4). For $t \in [0, T]$, by taking the Laplace transform with respect to t on both sides of system (4) we get

$$S^{\gamma} [Lx(t)] - s^{\gamma-1} x(0) - s^{\gamma-2} x'(0) - As^{\delta} [Lx(t)] + As^{\delta-1} x(0) = L[Bx(t-\tau) + f(t)], \quad (6)$$

$$\begin{aligned}
[Lx(t)] &= (s^\gamma I - As^\delta)^{-1} s^{\gamma-1} \varphi(0) \\
&\quad - A(s^\gamma I - As^\delta)^{-1} s^{\delta-1} \varphi(0) + (s^\gamma I - As^\delta)^{-1} s^{\gamma-2} (\varphi'(0)) \\
&\quad + (s^\gamma I - As^\delta)^{-1} L[Bx(t-\tau) + f(t)], \\
&= (s^\gamma I - As^\delta)^{-1} s^\gamma L[\varphi(0)] \\
&\quad - A(s^\gamma I - As^\delta)^{-1} s^\delta L[\varphi(0)] + (s^\gamma I - As^\delta)^{-1} s^{\gamma-2} (\varphi'(0)) \\
&\quad + (s^\gamma I - As^\delta)^{-1} L[Bx(t-\tau) + f(t)], \\
&= L[\varphi(0)] + A(s^{\gamma-\delta} I - A)^{-1} L[\varphi(0)] \\
&\quad - A(s^{\gamma-\delta} I - A)^{-1} L[\varphi(0)] + (s^{\gamma-\delta} I - A)^{-1} s^{\gamma-\delta-2} (\varphi'(0)) \\
&\quad + (s^{\gamma-\delta} I - A)^{-1} s^{-\delta} L[Bx(t-\tau) + f(t)], \\
[Lx(t)] &= L[\varphi(0)] + L[tE_{\gamma-\delta,2}(At^{\gamma-\delta})] \varphi'(0) \\
&\quad + L[t^{\gamma-1} E_{\gamma-\delta,\gamma}(At^{\gamma-\delta})] \times [Bx(t-\tau) + f(t)]. \tag{7}
\end{aligned}$$

Now by applying the convolution theorem on (7) we get

$$\begin{aligned}
L[x(t)] &= L[\varphi(0)] + L[tE_{\gamma-\delta,2}(At^{\gamma-\delta})] \varphi'(0) \\
&\quad + L \int_0^t (t-s)^{\gamma-1} E_{\gamma-\delta,\gamma}[A(t-s)^{\gamma-\delta}] [Bx(s-\tau) + f(s)] ds. \tag{8}
\end{aligned}$$

Applying the inverse Laplace transform to equation (8), we get

$$\begin{aligned}
x(t) &= \varphi(0) + tE_{\gamma-\delta,2}(At^{\gamma-\delta}) \varphi'(0) \\
&\quad + \int_0^t (t-s)^{\gamma-1} E_{\gamma-\delta,\gamma}[A(t-s)^{\gamma-\delta}] [Bx(s-\tau) + f(s)] ds, \quad t \in [0, T].
\end{aligned}$$

Denote $x(t) = \varphi(t) = \varphi_0(t)$, $t \in [-\tau, 0]$. Then for $t \in [0, \tau]$,

$$\begin{aligned}
x(t) &= \varphi(0) + tE_{\gamma-\delta,2}(At^{\gamma-\delta}) \varphi'(0) \\
&\quad + \int_0^t (t-s)^{\gamma-1} E_{\gamma-\delta,\gamma}[A(t-s)^{\gamma-\delta}] [B\varphi_0(s-\tau) + f(s)] ds, \quad t \in [0, \tau].
\end{aligned}$$

Denote $\varphi_1(t) = x(t)$, $t \in [0, \tau]$. Then for $t \in [\tau, 2\tau]$,

$$\begin{aligned}
x(t) &= \varphi(0) + tE_{\gamma-\delta,2}(At^{\gamma-\delta}) \varphi'(0) + \int_0^\tau (t-s)^{\gamma-1} E_{\gamma-\delta,\gamma}[A(t-s)^{\gamma-\delta}] \\
&\quad \times [B\varphi_0(s-\tau)] ds + \int_\tau^t (t-s)^{\gamma-1} E_{\gamma-\delta,\gamma}[A(t-s)^{\gamma-\delta}] [B\varphi_1(s-\tau)] ds \\
&\quad + \int_0^t (t-s)^{\gamma-1} E_{\gamma-\delta,\gamma}[A(t-s)^{\gamma-\delta}] f(s) ds.
\end{aligned}$$

Denote $\varphi_2(t) = x(t)$, $t \in [\tau, 2\tau]$. Then for $t \in [2\tau, 3\tau]$,

$$\begin{aligned} x(t) &= \varphi(0) + tE_{\gamma-\delta,2}(At^{\gamma-\delta})\varphi'(0) + \int_0^\tau (t-s)^{\gamma-1}E_{\gamma-\delta,\gamma}[A(t-s)^{\gamma-\delta}] \\ &\quad \times [B\varphi_0(s-\tau)]ds + \int_\tau^{2\tau} (t-s)^{\gamma-1}E_{\gamma-\delta,\gamma}[A(t-s)^{\gamma-\delta}][B\varphi_1(s-\tau)]ds \\ &\quad + \int_{2\tau}^t (t-s)^{\gamma-1}E_{\gamma-\delta,\gamma}[A(t-s)^{\gamma-\delta}][B\varphi_2(s-\tau)]ds \\ &\quad + \int_0^t (t-s)^{\gamma-1}E_{\gamma-\delta,\gamma}[A(t-s)^{\gamma-\delta}]f(s)ds. \end{aligned}$$

Denote $x(t) = \varphi_n(t)$, $t \in [(n-2)\tau, (n-1)\tau]$. Then for $t \in [(n-1)\tau, n\tau] = [(n-1)\tau, T]$,

$$\begin{aligned} x(t) &= \varphi(0) + tE_{\gamma-\delta,2}(At^{\gamma-\delta})\varphi'(0) + \int_0^\tau (t-s)^{\gamma-1}E_{\gamma-\delta,\gamma}[A(t-s)^{\gamma-\delta}] \\ &\quad \times [B\varphi_0(s-\tau)]ds + \int_\tau^{2\tau} (t-s)^{\gamma-1}E_{\gamma-\delta,\gamma}[A(t-s)^{\gamma-\delta}][B\varphi_1(s-\tau)]ds + \dots \\ &\quad + \int_{(n-2)\tau}^{(n-1)\tau} (t-s)^{\gamma-1}E_{\gamma-\delta,\gamma}[A(t-s)^{\gamma-\delta}][B\varphi_{n-2}(s-\tau)]ds \\ &\quad + \int_{(n-1)\tau}^t (t-s)^{\gamma-1}E_{\gamma-\delta,\gamma}[A(t-s)^{\gamma-\delta}][B\varphi_{n-1}(s-\tau)]ds \\ &\quad + \int_0^t (t-s)^{\gamma-1}E_{\gamma-\delta,\gamma}[A(t-s)^{\gamma-\delta}]f(s)ds, \\ x(t) &= \begin{cases} \varphi(0) + tE_{\gamma-\delta,2}(At^{\gamma-\delta})\varphi'(0) \\ \quad + \int_0^t (t-s)^{\gamma-1}E_{\gamma-\delta,\gamma}[A(t-s)^{\gamma-\delta}][B\varphi_0(s-\tau) + f(s)]ds, & t \in [0, \tau], \\ \varphi(0) + tE_{\gamma-\delta,2}(At^{\gamma-\delta})\varphi'(0) \\ \quad + \sum_{j=2}^n \int_{(j-2)\tau}^{(j-1)\tau} (t-s)^{\gamma-1}E_{\gamma-\delta,\gamma}[A(t-s)^{\gamma-\delta}][B\varphi_{j-2}(s-\tau)]ds \\ \quad + \int_{(j-1)\tau}^t (t-s)^{\gamma-1}E_{\gamma-\delta,\gamma}[A(t-s)^{\gamma-\delta}][B\varphi_{j-1}(s-\tau)]ds \\ \quad + \int_0^t (t-s)^{\gamma-1}E_{\gamma-\delta,\gamma}[A(t-s)^{\gamma-\delta}]f(s)ds, & t \in [(i-1)\tau, i\tau], i = 1, 2, \dots, n. \end{cases} \quad \square \end{aligned}$$

Lemma 2.6 Let $0 < \delta < 1 < \gamma < 2$ and $u \in C_p([0, T], \mathcal{R}^m)$. Then system (1) state response representation can be defined as follows. For $t \in [-\tau, 0]$,

$$x(t) = \varphi(t); \tag{9}$$

for $t \in [0, t_1]$,

$$\begin{aligned} x(t) &= \varphi(0) + tE_{\gamma-\delta,2}(At^{\gamma-\delta})\varphi'(0) \\ &\quad + \int_0^t (t-s)^{\gamma-1}E_{\gamma-\delta,\gamma}[A(t-s)^{\gamma-\delta}][B\varphi_0(s-\tau) + Cu(s)]ds; \end{aligned} \tag{10}$$

for $t \in (t_1, \tau]$,

$$\begin{aligned} x(t) &= \varphi(0) + I_1(x(t_1^-)) + tE_{\gamma-\delta,2}(At^{\gamma-\delta})\varphi'(0) \\ &\quad + \int_0^t (t-s)^{\gamma-1}E_{\gamma-\delta,\gamma}[A(t-s)^{\gamma-\delta}][B\varphi_0(s-\tau) + Cu(s)]ds; \end{aligned} \quad (11)$$

for $t \in (\tau, t_2]$,

$$\begin{aligned} x(t) &= \varphi(0) + I_1(x(t_1^-)) + tE_{\gamma-\delta,2}(At^{\gamma-\delta})\varphi'(0) \\ &\quad + \int_0^\tau (t-s)^{\gamma-1}E_{\gamma-\delta,\gamma}[A(t-s)^{\gamma-\delta}]B\varphi_0(s-\tau)ds \\ &\quad + \int_\tau^t (t-s)^{\gamma-1}E_{\gamma-\delta,\gamma}[A(t-s)^{\gamma-\delta}]B\varphi_1(s-\tau)ds \\ &\quad + \int_0^t (t-s)^{\gamma-1}E_{\gamma-\delta,\gamma}[A(t-s)^{\gamma-\delta}]Cu(s)ds; \end{aligned} \quad (12)$$

for $t \in (t_2, 2\tau]$,

$$\begin{aligned} x(t) &= \varphi(0) + I_1(x(t_1^-)) + I_2(x(t_2^-)) + tE_{\gamma-\delta,2}(At^{\gamma-\delta})\varphi'(0) \\ &\quad + \int_0^\tau (t-s)^{\gamma-1}E_{\gamma-\delta,\gamma}[A(t-s)^{\gamma-\delta}]B\varphi_0(s-\tau)ds \\ &\quad + \int_\tau^t (t-s)^{\gamma-1}E_{\gamma-\delta,\gamma}[A(t-s)^{\gamma-\delta}]B\varphi_1(s-\tau)ds \\ &\quad + \int_0^t (t-s)^{\gamma-1}E_{\gamma-\delta,\gamma}[A(t-s)^{\gamma-\delta}]Cu(s)ds, \end{aligned} \quad (13)$$

for $t \in ((i-1)\tau, t_i]$, $i \geq 2$,

$$\begin{aligned} x(t) &= \varphi(0) + \sum_{j=1}^{i-1} I_j(x(t_j^-)) + tE_{\gamma-\delta,2}(At^{\gamma-\delta})\varphi'(0) \\ &\quad + \sum_{j=1}^{i-1} \int_{(j-1)\tau}^{(j)\tau} (t-s)^{\gamma-1}E_{\gamma-\delta,\gamma}[A(t-s)^{\gamma-\delta}][B\varphi_{j-1}(s-\tau)]ds \\ &\quad + \int_{(i-1)\tau}^t (t-s)^{\gamma-1}E_{\gamma-\delta,\gamma}[A(t-s)^{\gamma-\delta}][B\varphi_{i-1}(s-\tau)]ds \\ &\quad + \int_0^t (t-s)^{\gamma-1}E_{\gamma-\delta,\gamma}[A(t-s)^{\gamma-\delta}]Cu(s)ds; \end{aligned} \quad (14)$$

for $t \in (t_i, i\tau]$, $i \geq 2$,

$$\begin{aligned} x(t) &= \varphi(0) + \sum_{j=1}^i I_j(x(t_j^-)) + tE_{\gamma-\delta,2}(At^{\gamma-\delta})\varphi'(0) \\ &\quad + \sum_{j=1}^{i-1} \int_{(j-1)\tau}^{(j)\tau} (t-s)^{\gamma-1}E_{\gamma-\delta,\gamma}[A(t-s)^{\gamma-\delta}][B\varphi_{j-1}(s-\tau)]ds \end{aligned}$$

$$\begin{aligned}
& + \int_{(i-1)\tau}^t (t-s)^{\gamma-1} E_{\gamma-\delta,\gamma} [A(t-s)^{\gamma-\delta}] [B\varphi_{i-1}(s-\tau)] ds \\
& + \int_0^t (t-s)^{\gamma-1} E_{\gamma-\delta,\gamma} [A(t-s)^{\gamma-\delta}] Cu(s) ds. \tag{15}
\end{aligned}$$

Proof The conclusion obviously holds when $t \in [-\tau, 0]$. Denote $x(t) = \varphi(t) = \varphi_0(t)$, $t \in [-\tau, 0]$.

If $t \in [0, t_1]$, then from Lemma 2.5 we have

$$\begin{aligned}
x(t) &= \varphi(0) + tE_{\gamma-\delta,2}(At^{\gamma-\delta})\varphi'(0) \\
& + \int_0^t (t-s)^{\gamma-1} E_{\gamma-\delta,\gamma} [A(t-s)^{\gamma-\delta}] [Bx(s-\tau) + Cu(s)] ds, \\
x(t_1) &= \varphi(0) + t_1 E_{\gamma-\delta,2}(At_1^{\gamma-\delta})\varphi'(0) \\
& + \int_0^{t_1} (t_1-s)^{\gamma-1} E_{\gamma-\delta,\gamma} [A(t_1-s)^{\gamma-\delta}] [B\varphi_0(s-\tau) + Cu(s)] ds.
\end{aligned}$$

If $t \in (t_1, \tau]$, then using [60], we have

$$\begin{aligned}
x(t) &= x(t_1^+) - t_1 E_{\gamma-\delta,2}(At_1^{\gamma-\delta})\varphi'(0) \\
& - \int_0^{t_1} (t_1-s)^{\gamma-1} E_{\gamma-\delta,\gamma} [A(t_1-s)^{\gamma-\delta}] [B\varphi_0(s-\tau) + Cu(s)] ds \\
& + tE_{\gamma-\delta,2}(At^{\gamma-\delta})\varphi'(0) + \int_0^t (t-s)^{\gamma-1} E_{\gamma-\delta,\gamma} [A(t-s)^{\gamma-\delta}] [B\varphi_0(s-\tau) + Cu(s)] ds, \\
x(t) &= x(t_1^-) + I_1(x(t_1^-)) - t_1 E_{\gamma-\delta,2}(At_1^{\gamma-\delta})\varphi'(0) \\
& - \int_0^{t_1} (t_1-s)^{\gamma-1} E_{\gamma-\delta,\gamma} [A(t_1-s)^{\gamma-\delta}] [B\varphi_0(s-\tau) + Cu(s)] ds \\
& + tE_{\gamma-\delta,2}(At^{\gamma-\delta})\varphi'(0) + \int_0^t (t-s)^{\gamma-1} E_{\gamma-\delta,\gamma} [A(t-s)^{\gamma-\delta}] [B\varphi_0(s-\tau) + Cu(s)] ds, \\
x(t) &= \varphi(0) + I_1(x(t_1^-)) + tE_{\gamma-\delta,2}(At^{\gamma-\delta})\varphi'(0) \\
& + \int_0^t (t-s)^{\gamma-1} E_{\gamma-\delta,\gamma} [A(t-s)^{\gamma-\delta}] [B\varphi_0(s-\tau) + Cu(s)] ds.
\end{aligned}$$

Denote $\varphi_1(t) = x(t)$, $t \in [0, \tau]$. If $t \in (\tau, t_2]$, then

$$\begin{aligned}
x(t) &= x(t_1^+) - t_1 E_{\gamma-\delta,2}(At_1^{\gamma-\delta})\varphi'(0) \\
& - \int_0^{t_1} (t_1-s)^{\gamma-1} E_{\gamma-\delta,\gamma} [A(t_1-s)^{\gamma-\delta}] [B\varphi_0(s-\tau) + Cu(s)] ds \\
& + tE_{\gamma-\delta,2}(At^{\gamma-\delta})\varphi'(0) + \int_0^\tau (t-s)^{\gamma-1} E_{\gamma-\delta,\gamma} [A(t-s)^{\gamma-\delta}] [B\varphi_0(s-\tau)] ds \\
& + \int_\tau^t (t-s)^{\gamma-1} E_{\gamma-\delta,\gamma} [A(t-s)^{\gamma-\delta}] [B\varphi_1(s-\tau)] ds \\
& + \int_0^t (t-s)^{\gamma-1} E_{\gamma-\delta,\gamma} [A(t-s)^{\gamma-\delta}] Cu(s) ds,
\end{aligned}$$

$$\begin{aligned}
x(t) &= x(t_1^-) + I_1(x(t_1^-)) - t_1 E_{\gamma-\delta,2}(At_1^{\gamma-\delta})\varphi'(0) \\
&\quad - \int_0^{t_1} (t_1-s)^{\gamma-1} E_{\gamma-\delta,\gamma}[A(t_1-s)^{\gamma-\delta}] [B\varphi_0(s-\tau) + Cu(s)] ds \\
&\quad + t E_{\gamma-\delta,2}(At^{\gamma-\delta})\varphi'(0) + \int_0^\tau (t-s)^{\gamma-1} E_{\gamma-\delta,\gamma}[A(t-s)^{\gamma-\delta}] [B\varphi_0(s-\tau)] ds \\
&\quad + \int_\tau^t (t-s)^{\gamma-1} E_{\gamma-\delta,\gamma}[A(t-s)^{\gamma-\delta}] [B\varphi_1(s-\tau)] ds \\
&\quad + \int_0^t (t-s)^{\gamma-1} E_{\gamma-\delta,\gamma}[A(t-s)^{\gamma-\delta}] Cu(s) ds, \\
x(t) &= \varphi(0) + I_1(x(t_1^-)) + t E_{\gamma-\delta,2}(At^{\delta-\gamma})\varphi'(0) \\
&\quad + \int_0^\tau (t-s)^{\delta-1} E_{\gamma-\delta,\gamma}[A(t-s)^{\gamma-\delta}] B\varphi_0(s-\tau) ds \\
&\quad + \int_\tau^t (t-s)^{\gamma-1} E_{\gamma-\delta,\gamma}[A(t-s)^{\gamma-\delta}] B\varphi_1(s-\tau) ds \\
&\quad + \int_0^t (t-s)^{\gamma-1} E_{\gamma-\delta,\gamma}[A(t-s)^{\gamma-\delta}] Cu(s) ds.
\end{aligned}$$

If $t \in (t_2, 2\tau]$, then

$$\begin{aligned}
x(t) &= x(t_2^+) - t_2 E_{\gamma-\delta,2}(At_2^{\gamma-\delta})\varphi'(0) \\
&\quad + \int_0^\tau (t_2-s)^{\gamma-1} E_{\gamma-\delta,\gamma}[A(t_2-s)^{\gamma-\delta}] B\varphi_0(s-\tau) ds \\
&\quad + \int_\tau^{t_2} (t_2-s)^{\gamma-1} E_{\gamma-\delta,\gamma}[A(t_2-s)^{\gamma-\delta}] B\varphi_1(s-\tau) ds \\
&\quad + \int_0^{t_2} (t_2-s)^{\gamma-1} E_{\gamma-\delta,\gamma}[A(t_2-s)^{\gamma-\delta}] Cu(s) ds + t E_{\gamma-\delta,2}(At^{\gamma-\delta})\varphi'(0) \\
&\quad + \int_0^\tau (t-s)^{\gamma-1} E_{\gamma-\delta,\gamma}[A(t-s)^{\gamma-\delta}] B\varphi_0(s-\tau) ds \\
&\quad + \int_\tau^t (t-s)^{\gamma-1} E_{\gamma-\delta,\gamma}[A(t-s)^{\gamma-\delta}] B\varphi_1(s-\tau) ds \\
&\quad + \int_0^t (t-s)^{\gamma-1} E_{\gamma-\delta,\gamma}[A(t-s)^{\gamma-\delta}] Cu(s) ds, \\
x(t) &= x(t_2^-) + I_2(x(t_2^-)) - t_2 E_{\gamma-\delta,2}(At_2^{\gamma-\delta})\varphi'(0) \\
&\quad + \int_0^\tau (t_2-s)^{\gamma-1} E_{\gamma-\delta,\gamma}[A(t_2-s)^{\gamma-\delta}] B\varphi_0(s-\tau) ds \\
&\quad + \int_\tau^{t_2} (t_2-s)^{\gamma-1} E_{\gamma-\delta,\gamma}[A(t_2-s)^{\gamma-\delta}] B\varphi_1(s-\tau) ds \\
&\quad + \int_0^{t_2} (t_2-s)^{\gamma-1} E_{\gamma-\delta,\gamma}[A(t_2-s)^{\gamma-\delta}] Cu(s) ds + t E_{\gamma-\delta,2}(At^{\gamma-\delta})\varphi'(0) \\
&\quad + \int_0^\tau (t-s)^{\gamma-1} E_{\gamma-\delta,\gamma}[A(t-s)^{\gamma-\delta}] B\varphi_0(s-\tau) ds \\
&\quad + \int_\tau^t (t-s)^{\gamma-1} E_{\gamma-\delta,\gamma}[A(t-s)^{\gamma-\delta}] B\varphi_1(s-\tau) ds \\
&\quad + \int_0^t (t-s)^{\gamma-1} E_{\gamma-\delta,\gamma}[A(t-s)^{\gamma-\delta}] Cu(s) ds,
\end{aligned}$$

$$\begin{aligned}
x(t) = & \varphi(0) + I_1(x(t_1^-)) + I_2(x(t_2^-)) + tE_{\gamma-\delta,2}(At^{\gamma-\delta})\varphi'(0) \\
& + \int_0^\tau (t-s)^{\gamma-1}E_{\gamma-\delta,\gamma}[A(t-s)^{\gamma-\delta}]B\varphi_0(s-\tau)ds \\
& + \int_\tau^t (t-s)^{\gamma-1}E_{\gamma-\delta,\gamma}[A(t-s)^{\gamma-\delta}]B\varphi_1(s-\tau)ds \\
& + \int_0^t (t-s)^{\gamma-1}E_{\gamma-\delta,\gamma}[A(t-s)^{\gamma-\delta}]Cu(s)ds.
\end{aligned}$$

Denote $x(t) = \varphi_i(t)$, $t \in [(i-2)\tau, (i-1)\tau]$. If $t \in ((i-1)\tau, t_i]$, $i \geq 2$, then the same argument implies the following expression:

$$\begin{aligned}
x(t) = & \varphi(0) + \sum_{j=1}^{i-1} I_j(x(t_j^-)) + tE_{\gamma-\delta,2}(At^{\gamma-\delta})\varphi'(0) \\
& + \sum_{j=1}^{i-1} \int_{(j-1)\tau}^{(j)\tau} (t-s)^{\gamma-1}E_{\gamma-\delta,\gamma}[A(t-s)^{\gamma-\delta}][B\varphi_{j-1}(s-\tau)]ds \\
& + \int_{(i-1)\tau}^t (t-s)^{\gamma-1}E_{\gamma-\delta,\gamma}[A(t-s)^{\gamma-\delta}][B\varphi_{i-1}(s-\tau)]ds \\
& + \int_0^t (t-s)^{\gamma-1}E_{\gamma-\delta,\gamma}[A(t-s)^{\gamma-\delta}]Cu(s)ds.
\end{aligned}$$

If $t \in (t_i, i\tau]$, $i \geq 2$, then the same argument implies the following expression:

$$\begin{aligned}
x(t) = & \varphi(0) + \sum_{j=1}^i I_j(x(t_j^-)) + tE_{\gamma-\delta,2}(At^{\gamma-\delta})\varphi'(0) \\
& + \sum_{j=1}^{i-1} \int_{(j-1)\tau}^{(j)\tau} (t-s)^{\gamma-1}E_{\gamma-\delta,\gamma}[A(t-s)^{\gamma-\delta}][B\varphi_{j-1}(s-\tau)]ds \\
& + \int_{(i-1)\tau}^t (t-s)^{\gamma-1}E_{\gamma-\delta,\gamma}[A(t-s)^{\gamma-\delta}][B\varphi_{i-1}(s-\tau)]ds \\
& + \int_0^t (t-s)^{\gamma-1}E_{\gamma-\delta,\gamma}[A(t-s)^{\gamma-\delta}]Cu(s)ds.
\end{aligned}$$

The proof is completed. \square

3 Sufficient and necessary conditions for controllability of system

In this part, we construct sufficient and necessary conditions for controllability criteria of system (1) by using the algebraic method.

Definition 3.1 For any initial function $\varphi \in C([-\tau, 0], \mathcal{R}^n)$ and any state $x_\mu \in \mathcal{R}^n$, system (1) on $[0, \mu]$ ($\mu \in (0, T]$) is called controllable if there exists a control input $u(t) \in C_p([0, \mu], \mathcal{R}^m)$ such that the corresponding solution of (1) satisfies $x(\mu) = x_\mu$.

Theorem 3.2 System (1) is controllable on $[0, \mu]$ if and only if the Gramian matrix

$$W_c[0, \mu] = \int_0^\mu (\mu-s)^{\gamma-1}E_{\gamma-\delta,\gamma}[A(\mu-s)^{\gamma-\delta}]CC^*[E_{\gamma-\delta,\gamma}(A^*(\mu-s)^{\gamma-\delta})]ds \quad (16)$$

is nonsingular for some $\mu \in [0, T]$, where $E_{\gamma-\delta,\gamma}$ is the Mittage-Leffler function, and C^* is the transpose of a matrix C .

Proof First, we demonstrate the sufficiency. If $W_c[0, \mu]$ is nonsingular, then $W_c^{-1}[0, \mu]$ is well defined. For an initial state $\varphi \in C([-t, 0], \mathcal{R}^n)$, when $\mu \in [0, t_1]$, the control function can be taken as

$$\begin{aligned} u(t) = & C^* \left[E_{\gamma-\delta,\gamma}(A^*(\mu-s)^{\gamma-\delta}) \right] W_c^{-1}[0, \mu] \times \left[x_\mu - \varphi(0) - \mu E_{\gamma-\delta,2}(A\mu^{\gamma-\delta})\varphi'(0) \right. \\ & \left. - \int_0^\mu (\mu-\theta)^{\gamma-1} E_{\gamma-\delta,\gamma}[A(\mu-\theta)^{\gamma-\delta}] [B\varphi_0(\theta-\tau) d\theta] \right]. \end{aligned} \quad (17)$$

By changing $t = \mu$ in (10) and inserting (17) we have

$$\begin{aligned} x(\mu) = & \varphi(0) + \mu E_{\gamma-\delta,2}(A\mu^{\gamma-\delta})\varphi'(0) + \int_0^\mu (\mu-s)^{\gamma-1} E_{\gamma-\delta,\gamma}[A(\mu-s)^{\gamma-\delta}] \\ & \times \left\{ B\varphi_0(s-\tau) + CC^* \left[E_{\gamma-\delta,\gamma}(A^*(\mu-s)^{\gamma-\delta}) \right] W_c^{-1}[0, \mu] \right. \\ & \times \left[x_\mu - \varphi(0) - \mu E_{\gamma-\delta,2}(A\mu^{\gamma-\delta})\varphi'(0) \right. \\ & \left. \left. - \int_0^\mu (\mu-\theta)^{\gamma-1} E_{\gamma-\delta,\gamma}[A(\mu-\theta)^{\gamma-\delta}] [B\varphi_0(\theta-\tau) d\theta] \right] \right\} ds, \\ x(\mu) = & \varphi(0) + \mu E_{\gamma-\delta,2}(A\mu^{\gamma-\delta})\varphi'(0) + \int_0^\mu (\mu-s)^{\gamma-1} E_{\gamma-\delta,\gamma}[A(\mu-s)^{\gamma-\delta}] \\ & \times [B\varphi_0(s-\tau)] ds + x_\mu - \varphi(0) - \mu E_{\gamma-\delta,2}(A\mu^{\gamma-\delta})\varphi'(0) \\ & - \int_0^\mu (\mu-\theta)^{\gamma-1} E_{\gamma-\delta,\gamma}[A(\mu-\theta)^{\gamma-\delta}] [B\varphi_0(\theta-\tau) d\theta], \end{aligned}$$

$$x(\mu) = x_\mu.$$

Thus system (1) is controllable on $[0, \mu]$, $\mu \in [0, t_1]$.

For $\mu \in (t_1, \tau]$, the control function can be taken as

$$\begin{aligned} u(t) = & C^* \left[E_{\gamma-\delta,\gamma}(A^*(\mu-s)^{\gamma-\delta}) \right] W_c^{-1}[0, \mu] \\ & \times \left[x_\mu - \varphi(0) - I_1(x(t_1^-)) - \mu E_{\gamma-\delta,2}(A\mu^{\gamma-\delta})\varphi'(0) \right. \\ & \left. - \int_0^\mu (\mu-\theta)^{\gamma-1} E_{\gamma-\delta,\gamma}[A(\mu-\theta)^{\gamma-\delta}] [B\varphi_0(\theta-\tau) d\theta] \right]. \end{aligned} \quad (18)$$

By changing $t = \mu$ in (11) and inserting (18) we have

$$\begin{aligned} x(\mu) = & \varphi(0) + I_1(x(t_1^-)) + \mu E_{\gamma-\delta,2}(A\mu^{\gamma-\delta})\varphi'(0) \\ & + \int_0^\mu (\mu-s)^{\gamma-1} E_{\gamma-\delta,\gamma}[A(\mu-s)^{\gamma-\delta}] \times \left\{ [B\varphi_0(s-\tau) \right. \\ & \left. + CC^* \left[E_{\gamma-\delta,\gamma}(A^*(\mu-s)^{\gamma-\delta}) \right] W_c^{-1}[0, \mu] \right. \\ & \left. \times [x_\mu - \varphi(0) - I_1(x(t_1^-)) - \mu E_{\gamma-\delta,2}(A\mu^{\gamma-\delta})\varphi'(0)] \right\} ds \end{aligned}$$

$$\begin{aligned}
& - \int_0^\mu (\mu - \theta)^{\gamma-1} E_{\gamma-\delta,\gamma} [A(\mu - \theta)^{\gamma-\delta}] [B\varphi_0(\theta - \tau) d\theta] \Big\} ds, \\
x(\mu) &= \varphi(0) + I_1(x(t_1^-)) + \mu E_{\gamma-\delta,2}(A\mu^{\gamma-\delta})\varphi'(0) \\
&+ \int_0^\mu (\mu - s)^{\gamma-1} E_{\gamma-\delta,\gamma} [A(\mu - s)^{\gamma-\delta}] \times [B\varphi_0(s - \tau)] ds \\
&+ x_\mu - \varphi(0) - I_1(x(t_1^-)) - \mu E_{\gamma-\delta,2}(A\mu^{\gamma-\delta})\varphi'(0) \\
&- \int_0^\mu (\mu - \theta)^{\gamma-1} E_{\gamma-\delta,\gamma} [A(\mu - \theta)^{\gamma-\delta}] [B\varphi_0(\theta - \tau) d\theta],
\end{aligned}$$

$$x(\mu) = x_\mu.$$

Hence the system is controllable on $[0, \mu]$, $\mu \in [t_1, \tau]$.

For $\mu \in (\tau, t_2]$, the control function can be taken as

$$\begin{aligned}
u(t) &= C^* [E_{\gamma-\delta,\gamma} (A^*(\mu - s)^{\gamma-\delta})] W_c^{-1}[0, \mu] \\
&\times [x_\mu - \varphi(0) - I_1(x(t_1^-)) - \mu E_{\gamma-\delta,2}(A\mu^{\gamma-\delta})\varphi'(0)] \\
&- \int_0^\tau (\mu - \theta)^{\gamma-1} E_{\gamma-\delta,\gamma} [A(\mu - \theta)^{\gamma-\delta}] B\varphi_0(\theta - \tau) d\theta \\
&- \int_\tau^\mu (\mu - \theta)^{\gamma-1} E_{\gamma-\delta,\gamma} [A(\mu - \theta)^{\gamma-\delta}] B\varphi_1(\theta - \tau) d\theta. \tag{19}
\end{aligned}$$

By changing $t = \mu$ in (12) and inserting (19) we have

$$\begin{aligned}
x(\mu) &= \varphi(0) + I_1(x(t_1^-)) + \mu E_{\gamma-\delta,2}(A\mu^{\gamma-\delta})\varphi'(0) \\
&+ \int_0^\tau (\mu - s)^{\gamma-1} E_{\gamma-\delta,\gamma} [A(\mu - s)^{\gamma-\delta}] B\varphi_0(s - \tau) ds \\
&+ \int_\tau^\mu (\mu - s)^{\gamma-1} E_{\gamma-\delta,\gamma} [A(\mu - s)^{\gamma-\delta}] B\varphi_1(s - \tau) ds \\
&+ \int_0^\mu (\mu - s)^{\gamma-1} E_{\gamma-\delta,\gamma} [A(\mu - s)^{\gamma-\delta}] \Big\{ CC^* [E_{\gamma-\delta,\gamma} (A^*(\mu - s)^{\gamma-\delta})] W_c^{-1}[0, \mu] \\
&\times x_\mu - \varphi(0) - I_1(x(t_1^-)) - \mu E_{\gamma-\delta,2}(A\mu^{\gamma-\delta})\varphi'(0) \\
&- \int_0^\tau (\mu - \theta)^{\gamma-1} E_{\gamma-\delta,\gamma} [A(\mu - \theta)^{\gamma-\delta}] B\varphi_0(\theta - \tau) d\theta \\
&- \int_\tau^\mu (\mu - \theta)^{\gamma-1} E_{\gamma-\delta,\gamma} [A(\mu - \theta)^{\gamma-\delta}] B\varphi_1(\theta - \tau) d\theta \Big\} ds, \\
x(\mu) &= \varphi(0) + I_1(x(t_1^-)) + \mu E_{\gamma-\delta,2}(A\mu^{\gamma-\delta})\varphi'(0) \\
&+ \int_0^\tau (\mu - s)^{\gamma-1} E_{\gamma-\delta,\gamma} [A(\mu - s)^{\gamma-\delta}] B\varphi_0(s - \tau) ds \\
&+ \int_\tau^\mu (\mu - s)^{\gamma-1} E_{\gamma-\delta,\gamma} [A(\mu - s)^{\gamma-\delta}] B\varphi_1(s - \tau) ds \\
&+ x_\mu - \varphi(0) - I_1(x(t_1^-)) - \mu E_{\gamma-\delta,2}(A\mu^{\gamma-\delta})\varphi'(0)
\end{aligned}$$

$$\begin{aligned} & - \int_0^\tau (\mu - \theta)^{\gamma-1} E_{\gamma-\delta,\gamma} [A(\mu - \theta)^{\gamma-\delta}] B\varphi_0(\theta - \tau) d\theta \\ & - \int_\tau^\mu (\mu - \theta)^{\gamma-1} E_{\gamma-\delta,\gamma} [A(\mu - \theta)^{\gamma-\delta}] B\varphi_1(\theta - \tau) d\theta, \end{aligned}$$

$$x(\mu) = x_\mu.$$

Hence system (1) is controllable on $[0, \mu]$, $\mu \in [t_1, \tau]$.

For $\mu \in (t_2, 2\tau]$, the control function can be taken as

$$\begin{aligned} u(t) = & C^* [E_{\gamma-\delta,\gamma} (A^*(\mu - s)^{\gamma-\delta})] W_c^{-1}[0, \mu] \times [x_\mu - \varphi(0) - I_1(x(t_1^-)) - I_2(x(t_2^-))] \\ & - \mu E_{\gamma-\delta,2} (A\mu^{\gamma-\delta}) \varphi'(0) - \int_0^\tau (\mu - \theta)^{\gamma-1} E_{\gamma-\delta,\gamma} [A(\mu - \theta)^{\gamma-\delta}] B\varphi_0(\theta - \tau) d\theta \\ & - \int_\tau^\mu (\mu - \theta)^{\gamma-1} E_{\gamma-\delta,\gamma} [A(\mu - \theta)^{\gamma-\delta}] B\varphi_1(\theta - \tau) d\theta. \end{aligned} \quad (20)$$

By changing $t = \mu$ in (13) and inserting (20) we have

$$\begin{aligned} x(\mu) = & \varphi(0) + I_1(x(t_1^-)) + I_2(x(t_2^-)) + \mu E_{\gamma-\delta,2} (A\mu^{\gamma-\delta}) \varphi'(0) \\ & + \int_0^\tau (\mu - s)^{\gamma-1} E_{\gamma-\delta,\gamma} [A(\mu - s)^{\gamma-\delta}] B\varphi_0(s - \tau) ds \\ & + \int_\tau^\mu (\mu - s)^{\gamma-1} E_{\gamma-\delta,\gamma} [A(\mu - s)^{\gamma-\delta}] B\varphi_1(s - \tau) ds \\ & + \int_0^\mu (\mu - s)^{\gamma-1} E_{\gamma-\delta,\gamma} [A(\mu - s)^{\gamma-\delta}] \{CC^* [E_{\gamma-\delta,\gamma} (A^*(\mu - s)^{\gamma-\delta})] W_c^{-1}[0, \mu] \\ & \times [x_\mu - \varphi(0) - I_1(x(t_1^-)) - I_2(x(t_2^-)) - \mu E_{\gamma-\delta,2} (A\mu^{\gamma-\delta}) \varphi'(0)] \\ & - \int_0^\tau (\mu - \theta)^{\gamma-1} E_{\gamma-\delta,\gamma} [A(\mu - \theta)^{\gamma-\delta}] B\varphi_0(\theta - \tau) d\theta \\ & - \int_\tau^\mu (\mu - \theta)^{\gamma-1} E_{\gamma-\delta,\gamma} [A(\mu - \theta)^{\gamma-\delta}] B\varphi_1(\theta - \tau) d\theta\} ds, \\ x(\mu) = & \varphi(0) + I_1(x(t_1^-)) + I_2(x(t_2^-)) + \mu E_{\gamma-\delta,2} (A\mu^{\gamma-\delta}) \varphi'(0) \\ & + \int_0^\tau (\mu - s)^{\gamma-1} E_{\gamma-\delta,\gamma} [A(\mu - s)^{\gamma-\delta}] B\varphi_0(s - \tau) ds \\ & + \int_\tau^\mu (\mu - s)^{\gamma-1} E_{\gamma-\delta,\gamma} [A(\mu - s)^{\gamma-\delta}] B\varphi_1(s - \tau) ds \\ & + x_\mu - \varphi(0) - I_1(x(t_1^-)) - I_2(x(t_2^-)) - \mu E_{\gamma-\delta,2} (A\mu^{\gamma-\delta}) \varphi'(0) \\ & - \int_0^\tau (\mu - \theta)^{\gamma-1} E_{\gamma-\delta,\gamma} [A(\mu - \theta)^{\gamma-\delta}] B\varphi_0(\theta - \tau) d\theta \\ & - \int_\tau^\mu (\mu - \theta)^{\gamma-1} E_{\gamma-\delta,\gamma} [A(\mu - \theta)^{\gamma-\delta}] B\varphi_1(\theta - \tau) d\theta, \end{aligned}$$

$$x(\mu) = x_\mu.$$

Hence system (1) is controllable on $[0, \mu]$, $\mu \in (t_2, 2\tau]$.

For $\mu \in ((i-1)\tau, t_i]$, $i \geq 2$, the control function can be taken as

$$\begin{aligned} u(t) = & C^* [E_{\gamma-\delta, \gamma}(A^*(\mu-s)^{\gamma-\delta})] W_c^{-1}[0, \mu] \\ & \times \left[x_\mu - \varphi(0) - \sum_{j=1}^{i-1} I_j(x(t_j^-)) - \mu E_{\gamma-\delta, 2}(A\mu^{\gamma-\delta}) \varphi'(0) \right] \\ & - \sum_{j=1}^{i-1} \int_{(j-1)\tau}^{(j)\tau} (\mu-\theta)^{\gamma-1} E_{\gamma-\delta, \gamma}[A(\mu-\theta)^{\gamma-\delta}] [B\varphi_{j-1}(\theta-\tau)] d\theta \\ & - \int_{(i-1)\tau}^{\mu} (\mu-\theta)^{\gamma-1} E_{\gamma-\delta, \gamma}[A(\mu-\theta)^{\gamma-\delta}] [B\varphi_{i-1}(\theta-\tau)] d\theta. \end{aligned} \quad (21)$$

Substituting $t = \mu$ into (14) and inserting (21) we have

$$\begin{aligned} x(\mu) = & \varphi(0) + \sum_{j=1}^{i-1} I_j(x(t_j^-)) + \mu E_{\gamma-\delta, 2}(A\mu^{\gamma-\delta}) \varphi'(0) \\ & + \sum_{j=1}^{i-1} \int_{(j-1)\tau}^{(j)\tau} (\mu-s)^{\gamma-1} E_{\gamma-\delta, \gamma}[A(\mu-s)^{\gamma-\delta}] [B\varphi_{j-1}(s-\tau)] ds \\ & + \int_{(i-1)\tau}^{\mu} (\mu-s)^{\gamma-1} E_{\gamma-\delta, \gamma}[A(\mu-s)^{\gamma-\delta}] [B\varphi_{i-1}(s-\tau)] ds \\ & + \int_0^{\mu} (\mu-s)^{\gamma-1} E_{\gamma-\delta, \gamma}[A(\mu-s)^{\gamma-\delta}] [CC^*[E_{\gamma-\delta, \gamma}(A^*(\mu-s)^{\gamma-\delta})] W_c^{-1}[0, \mu]] \\ & \times \left[x_\mu - \varphi(0) - \sum_{j=1}^{i-1} I_j(x(t_j^-)) - \mu E_{\gamma-\delta, 2}(A\mu^{\gamma-\delta}) \varphi'(0) \right. \\ & \left. - \sum_{j=1}^{i-1} \int_{(j-1)\tau}^{(j)\tau} (\mu-\theta)^{\gamma-1} E_{\gamma-\delta, \gamma}[A(\mu-\theta)^{\gamma-\delta}] [B\varphi_{j-1}(\theta-\tau)] d\theta \right. \\ & \left. - \int_{(i-1)\tau}^{\mu} (\mu-\theta)^{\gamma-1} E_{\gamma-\delta, \gamma}[A(\mu-\theta)^{\gamma-\delta}] [B\varphi_{i-1}(\theta-\tau)] d\theta \right] ds, \\ x(\mu) = & \varphi(0) + \sum_{j=1}^{i-1} I_j(x(t_j^-)) + \mu E_{\gamma-\delta, 2}(A\mu^{\gamma-\delta}) \varphi'(0) \\ & + \sum_{j=1}^{i-1} \int_{(j-1)\tau}^{(j)\tau} (\mu-s)^{\gamma-1} E_{\gamma-\delta, \gamma}[A(\mu-s)^{\gamma-\delta}] [B\varphi_{j-1}(s-\tau)] ds \\ & + \int_{(i-1)\tau}^{\mu} (\mu-s)^{\gamma-1} E_{\gamma-\delta, \gamma}[A(\mu-s)^{\gamma-\delta}] [B\varphi_{i-1}(s-\tau)] ds \\ & + x_\mu - \varphi(0) - \sum_{j=1}^{i-1} I_j(x(t_j^-)) - \mu E_{\gamma-\delta, 2}(A\mu^{\gamma-\delta}) \varphi'(0) \\ & - \sum_{j=1}^{i-1} \int_{(j-1)\tau}^{(j)\tau} (\mu-\theta)^{\gamma-1} E_{\gamma-\delta, \gamma}[A(\mu-\theta)^{\gamma-\delta}] [B\varphi_{j-1}(\theta-\tau)] d\theta \\ & - \int_{(i-1)\tau}^{\mu} (\mu-\theta)^{\gamma-1} E_{\gamma-\delta, \gamma}[A(\mu-\theta)^{\gamma-\delta}] [B\varphi_{i-1}(\theta-\tau)] d\theta, \\ x(\mu) = & x_\mu. \end{aligned}$$

Hence system (1) is controllable on $[0, \mu]$, $\mu \in ((k-1)\tau, t_k]$.

For $\mu \in (t_i, i\tau]$, $i \geq 2$, we have the control function

$$\begin{aligned} u(t) = & C^* [E_{\gamma-\delta, \gamma} (A^*(\mu-s)^{\gamma-\delta})] W_c^{-1}[0, \mu] \\ & \times \left[x_\mu - \varphi(0) - \sum_{j=1}^i I_j(x(t_j^-)) - \mu E_{\gamma-\delta, 2}(A\mu^{\gamma-\delta}) \varphi'(0) \right] \\ & - \sum_{j=1}^{i-1} \int_{(j-1)\tau}^{(j)\tau} (\mu-\theta)^{\gamma-1} E_{\gamma-\delta, \gamma} [A(\mu-\theta)^{\gamma-\delta}] [B\varphi_{j-1}(\theta-\tau)] d\theta \\ & - \int_{(i-1)\tau}^{\mu} (\mu-\theta)^{\gamma-1} E_{\gamma-\delta, \gamma} [A(\mu-\theta)^{\gamma-\delta}] [B\varphi_{i-1}(\theta-\tau)] d\theta. \end{aligned} \quad (22)$$

By changing $t = \mu$ in (15) and inserting (22) we have

$$\begin{aligned} x(\mu) = & \varphi(0) + \sum_{j=1}^i I_j(x(t_j^-)) + \mu E_{\gamma-\delta, 2}(A\mu^{\gamma-\delta}) \varphi'(0) \\ & + \sum_{j=1}^{i-1} \int_{(j-1)\tau}^{(j)\tau} (\mu-s)^{\gamma-1} E_{\gamma-\delta, \gamma} [A(\mu-s)^{\gamma-\delta}] [B\varphi_{j-1}(s-\tau)] ds \\ & + \int_{(i-1)\tau}^{\mu} (\mu-s)^{\gamma-1} E_{\gamma-\delta, \gamma} [A(\mu-s)^{\gamma-\delta}] [B\varphi_{i-1}(s-\tau)] ds \\ & + \int_0^{\mu} (\mu-s)^{\gamma-1} E_{\gamma-\delta, \gamma} [A(\mu-s)^{\gamma-\delta}] CC^* [E_{\gamma-\delta, \gamma} (A^*(\mu-s)^{\gamma-\delta})] W_c^{-1}[0, \mu] \\ & \times \left[x_\mu - \varphi(0) - \sum_{j=1}^i I_j(x(t_j^-)) - \mu E_{\gamma-\delta, 2}(A\mu^{\gamma-\delta}) \varphi'(0) \right. \\ & \left. - \sum_{j=1}^{i-1} \int_{(j-1)\tau}^{(j)\tau} (\mu-\theta)^{\gamma-1} E_{\gamma-\delta, \gamma} [A(\mu-\theta)^{\gamma-\delta}] [B\varphi_{j-1}(\theta-\tau)] d\theta \right. \\ & \left. - \int_{(i-1)\tau}^{\mu} (\mu-\theta)^{\gamma-1} E_{\gamma-\delta, \gamma} [A(\mu-\theta)^{\gamma-\delta}] [B\varphi_{i-1}(\theta-\tau)] d\theta \right] \\ x(\mu) = & \varphi(0) + \sum_{j=1}^i I_j(x(t_j^-)) + \mu E_{\gamma-\delta, 2}(A\mu^{\gamma-\delta}) \varphi'(0) \\ & + \sum_{j=1}^{i-1} \int_{(j-1)\tau}^{(j)\tau} (\mu-s)^{\gamma-1} E_{\gamma-\delta, \gamma} [A(\mu-s)^{\gamma-\delta}] [B\varphi_{j-1}(s-\tau)] ds \\ & + \int_{(i-1)\tau}^{\mu} (\mu-s)^{\gamma-1} E_{\gamma-\delta, \gamma} [A(\mu-s)^{\gamma-\delta}] [B\varphi_{i-1}(s-\tau)] ds \\ & + x_\mu - \varphi(0) - \sum_{j=1}^i I_j(x(t_j^-)) - \mu E_{\gamma-\delta, 2}(A\mu^{\gamma-\delta}) \varphi'(0) \\ & - \sum_{j=1}^{i-1} \int_{(j-1)\tau}^{(j)\tau} (\mu-\theta)^{\gamma-1} E_{\gamma-\delta, \gamma} [A(\mu-\theta)^{\gamma-\delta}] [B\varphi_{j-1}(\theta-\tau)] d\theta \end{aligned}$$

$$-\int_{(i-1)\tau}^{\mu} (\mu - \theta)^{\gamma-1} E_{\gamma-\delta, \gamma} [A(\mu - \theta)^{\gamma-\delta}] [B\varphi_{i-1}(\theta - \tau)] d\theta,$$

$$x(\mu) = x_\mu.$$

Hence system (1) is controllable on $[0, \mu]$. Furthermore, we prove the necessity of Theorem 3.2. Suppose, without loss of generality, that $W_c[0, \mu]$ is singular. For $\mu \in ((i-1)\tau, t_i]$, $i \geq 2$, there exists a nonzero vector z_0 such that

$$z_0^* W_c[0, \mu] z_0 = 0, \quad (23)$$

that is,

$$\int_0^{\mu} z_0^* (\mu - s)^{\gamma-1} E_{\gamma-\delta, \gamma} [A(\mu - s)^{\gamma-\delta}] CC^* [E_{\gamma-\delta, \gamma} (A^*(\mu - s)^{\gamma-\delta})] z_0 ds = 0. \quad (24)$$

Then it follows that

$$z_0^* E_{\gamma-\delta, \gamma} [A(\mu - s)^{\gamma-\delta}] C = 0 \quad (25)$$

on $s \in [0, \mu]$. Since system (1) is controllable, there exist control inputs $u_1(t)$ and $u_2(t)$ such that

$$\begin{aligned} x(\mu) &= \varphi(0) + \sum_{j=1}^{i-1} I_j(x(t_j^-)) + \mu E_{\gamma-\delta, 2} (A\mu^{\gamma-\delta}) \varphi'(0) \\ &\quad + \sum_{j=1}^{i-1} \int_{(j-1)\tau}^{(j)\tau} (\mu - s)^{\gamma-1} E_{\gamma-\delta, \gamma} [A(\mu - s)^{\gamma-\delta}] [B\varphi_{j-1}(s - \tau)] ds \\ &\quad + \int_{(i-1)\tau}^{\mu} (\mu - s)^{\gamma-1} E_{\gamma-\delta, \gamma} [A(\mu - s)^{\gamma-\delta}] [B\varphi_{i-1}(s - \tau)] ds \\ &\quad + \int_0^{\mu} (\mu - s)^{\gamma-1} E_{\gamma-\delta, \gamma} [A(\mu - s)^{\gamma-\delta}] Cu_1(s) ds = 0, \end{aligned} \quad (26)$$

$$\begin{aligned} z_0 &= \varphi(0) + \sum_{j=1}^{i-1} I_j(x(t_j^-)) + \mu E_{\gamma-\delta, 2} (A\mu^{\gamma-\delta}) \varphi'(0) \\ &\quad + \sum_{j=1}^{i-1} \int_{(j-1)\tau}^{(j)\tau} (\mu - s)^{\gamma-1} E_{\gamma-\delta, \gamma} [A(\mu - s)^{\gamma-\delta}] [B\varphi_{j-1}(s - \tau)] ds \\ &\quad + \int_{(i-1)\tau}^{\mu} (\mu - s)^{\gamma-1} E_{\gamma-\delta, \gamma} [A(\mu - s)^{\gamma-\delta}] [B\varphi_{i-1}(s - \tau)] ds \\ &\quad + \int_0^{\mu} (\mu - s)^{\gamma-1} E_{\gamma-\delta, \gamma} [A(\mu - s)^{\gamma-\delta}] Cu_2(s) ds. \end{aligned} \quad (27)$$

Combining (26) and (27) yields

$$z_0 - \int_0^{\mu} (\mu - s)^{\gamma-1} E_{\gamma-\delta, \gamma} [A(\mu - s)^{\gamma-\delta}] C [u_2(s) - u_1(s)] ds = 0. \quad (28)$$

Multiplying both sides of (28) by z_0^* we have

$$z_0^* z_0 - \int_0^\mu (\mu - s)^{\gamma-1} z_0^* E_{\gamma-\delta,\gamma} [A(\mu - s)^{\gamma-\delta}] \times C [u_2(s) - u_1(s)] ds = 0. \quad (29)$$

Since $z_0^* E_{\gamma-\delta,\gamma} [A(\mu - s)^{\gamma-\delta}] C = 0$, we have $z_0^* z_0 = 0$. Thus $z_0 = 0$. This contradiction completes the proof sufficiency.

Next, we establish the necessity. Suppose, without loss of generality, that $W_c[0, \mu]$ is singular. For $t \in (t_i, i\tau]$, $i \geq 2$, the proof is similar, so we omit it. This completes the proof of the theorem. \square

Theorem 3.3 System (1) on $[0, \mu]$ is controllable if and only if $\text{rank}[C|AC|\cdots|A^{n-1}C] = n$.

Proof According to the theorem of Cayley–Hamilton, $tE_{\gamma-\delta,2}(At^{\gamma-\delta})$, $t^{\gamma-1}E_{\gamma-\delta,\gamma}(At^{\gamma-\delta})$ can be represented as

$$tE_{\gamma-\delta,2}(At^{\gamma-\delta}) = \sum_{i=0}^{\infty} \frac{t^i(\gamma-\delta)+1}{\Gamma(i(\gamma-\delta)+2)} A^i = \sum_{i=0}^{n-1} G_{1i}(t) A^i, \quad (30)$$

$$t^{\gamma-1}E_{\gamma-\delta,\gamma}(At^{\gamma-\delta}) = \sum_{i=0}^{\infty} \frac{t^i(\gamma-\delta)+\gamma-1}{\Gamma(i(\gamma-\delta)+\gamma)} A^i = \sum_{i=0}^{n-1} G_{2i}(t) A^i. \quad (31)$$

For $\mu \in [0, t_1]$, we have

$$\begin{aligned} x(\mu) &= \varphi(0) + \mu E_{\gamma-\delta,2}(A\mu^{\gamma-\delta})\varphi'(0) \\ &\quad + \int_0^\mu (\mu - s)^{\gamma-1} E_{\gamma-\delta,\gamma} [A(\mu - s)^{\gamma-\delta}] [B\varphi_0(s - \tau) + Cu(s)] ds, \end{aligned} \quad (32)$$

$$\begin{aligned} x(\mu) &= \varphi(0) + \sum_{i=0}^{n-1} G_{1i}\mu A^i \varphi'(0) \\ &\quad + \sum_{i=0}^{n-1} \int_0^\mu G_{2i}(\mu - s) A^i [B\varphi_0(s - \tau) + Cu(s)] ds. \end{aligned} \quad (33)$$

Let

$$\psi = \varphi(0) + \sum_{i=0}^{n-1} G_{1i}\mu A^i \varphi'(0) + \sum_{i=0}^{n-1} \int_0^\mu G_{2i}(\mu - s) A^i [B\varphi_0(s - \tau)] ds. \quad (34)$$

Combining (33) with (34) we have

$$x(\mu) - \psi = \sum_{i=0}^{n-1} A^i C \int_0^\mu G_{2i}(\mu - s) u(s) ds \quad (35)$$

$$= [C|AC|\cdots|A^{n-1}C] \begin{pmatrix} d_0 \\ d_1 \\ \vdots \\ \vdots \\ d_{n-1} \end{pmatrix}, \quad (36)$$

where $d_i = \int_0^\mu G_{2i}(t)(\mu - s)u(s)ds$, $i = 0, 1, 2, \dots, n - 1$. For arbitrary $\varphi \in C([-\tau, 0], \mathcal{R}^n)$, sufficient and necessary condition to have a control input $u(t)$ satisfying (36) is

$$\text{rank}[C|AC|\cdots|A^{n-1}C] = n.$$

For $\mu \in ((i-1)\tau, t_i]$, we have

$$\begin{aligned} x(\mu) &= \varphi(0) + \sum_{j=1}^{i-1} I_j(x(t_j^-)) + \mu E_{\gamma-\delta, 2}(A\mu^{\gamma-\delta})\varphi'(0) \\ &\quad + \sum_{j=1}^{i-1} \int_{(j-1)\tau}^{(j)\tau} (\mu - s)^{\gamma-1} E_{\gamma-\delta, \gamma}[A(\mu - s)^{\gamma-\delta}][B\varphi_{j-1}(s - \tau)] ds \\ &\quad + \int_{(i-1)\tau}^\mu (\mu - s)^{\gamma-1} E_{\gamma-\delta, \gamma}[A(\mu - s)^{\gamma-\delta}][B\varphi_{i-1}(s - \tau)] ds \\ &\quad + \int_0^\mu (\mu - s)^{\gamma-1} E_{\gamma-\delta, \gamma}[A(t - s)^{\gamma-\delta}]Cu(s) ds, \\ x(\mu) &= \varphi(0) + \sum_{j=1}^{i-1} I_j(x(t_j^-)) + \sum_{i=0}^{n-1} G_{1i}\mu A^i\varphi'(0) \\ &\quad + \sum_{j=1}^{i-1} \sum_{i=0}^{n-1} \int_{(j-1)\tau}^{(j)\tau} G_{2i}(\mu - s)A^i[B\varphi_{j-1}(s - \tau)] ds \\ &\quad + \sum_{i=0}^{n-1} \int_{(i-1)\tau}^\mu G_{2i}(\mu - s)A^i[B\varphi_{i-1}(s - \tau)] ds \\ &\quad + \sum_{i=0}^{n-1} A^i C \int_0^\mu G_{2i}(\mu - s)u(s) ds, \end{aligned} \tag{37}$$

$$\begin{aligned} \psi_1 &= \varphi(0) + \sum_{j=1}^{i-1} I_j(x(t_j^-)) + \sum_{i=0}^{n-1} G_{1i}\mu A^i\varphi'(0) \\ &\quad + \sum_{j=1}^{i-1} \sum_{i=0}^{n-1} \int_{(j-1)\tau}^{(j)\tau} G_{2i}(\mu - s)A^i[B\varphi_{j-1}(s - \tau)] ds \\ &\quad + \sum_{i=0}^{n-1} \int_{(i-1)\tau}^\mu G_{2i}(\mu - s)A^i[B\varphi_{i-1}(s - \tau)] ds. \end{aligned} \tag{38}$$

Combining (37) with (38) we have

$$\begin{aligned} x(\mu) - \psi_1 &= \sum_{i=0}^{n-1} A^i C \int_0^\mu G_{2i}(\mu - s)u(s) ds \\ &= [C|AC|\cdots|A^{n-1}C] \begin{pmatrix} d_0 \\ d_1 \\ \vdots \\ \vdots \\ d_{n-1} \end{pmatrix}. \end{aligned} \tag{39}$$

For arbitrary $\varphi \in C([-\tau, 0], \mathcal{R}^n)$, a sufficient and necessary condition to have a control input $u(t)$ satisfying (39) is that

$$\text{rank}[C|AC|\cdots|A^{n-1}C] = n.$$

For $t \in (t_i, i\tau]$, $i \geq 2$, the proof is similar, and we omit it. Thus the proof is completed. \square

Remark 3.4 System (1) can be controlled only in cases where the resolvent condition $\lambda(\lambda I + Q_w)^{-1} \rightarrow 0$ as $\lambda \rightarrow 0$ holds (here Q_w is the respective Gramian matrix in the non-fractional, nondelay, and nonimpulsive case) since this is equivalent to the rank condition in the finite-dimensional case [24, 61].

4 Example

In this section, we apply the results for the controllability criterion we acquired in the past section.

Example 1 Consider the controllability of the following damped fractional differential system with state delay and impulses:

$$\begin{aligned} {}^cD^{3/2}x(t) - & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} {}^cD^{1/2}x(t) \\ = & \begin{pmatrix} -1 & 2 \\ 1 & 3 \end{pmatrix} x\left(t - \frac{1}{4}\right) + \begin{pmatrix} 1 \\ 2 \end{pmatrix} u(t), \quad t \in [0, 5] \setminus \{1, 2, 3, 4\}, \\ \Delta x(t_k) = & \frac{1}{2}x(t_k^-), \quad t_k = k, \quad k = 1, 2, 3, 4, \\ x(t) = & e^{t+1}, \quad t \in \left[-\frac{1}{4}, 0\right]. \end{aligned}$$

Now by applying Theorem 3.2 we can prove that this system is controllable on $[0, 5]$. Let us take

$$\gamma = 3/2, \quad \delta = 1/2, \quad A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 2 \\ 1 & 3 \end{pmatrix}, \quad C = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

By computation we have

$$\begin{aligned} cc^* &= \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}, \\ E_{1,3/2}A(5-s)^{3/2} &= \sum_{k=0}^1 \frac{A^k(5-s)^{K3/2}}{\Gamma(k+3/2)} \\ &= \frac{1}{\Gamma(3/2)}(I) + \frac{1}{\Gamma(5/2)} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} (5-s)^{3/2}, \\ E_{1,3/2}A^*(5-s)^{3/2} &= \frac{1}{\Gamma(3/2)}(I) + \frac{1}{\Gamma(5/2)} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} (5-s)^{3/2}. \end{aligned}$$

By simple calculation we can see that controllability matrix

$$\begin{aligned} W[0, 5] &= \int_0^5 (5-s)^{\frac{1}{2}} [E_{1,3/2} A(5-s)^{3/2}] c c^* [E_{1,3/2} A^*(5-s)^{3/2}] ds \\ &\approx (5)^3 \begin{pmatrix} \frac{-3\sqrt{5}-32}{\pi} & \frac{128(\sqrt[3]{5})}{81\pi} \\ \frac{-16\sqrt{5}-96}{3\pi} & \frac{-16\sqrt{5}-96}{32\sqrt{5}} \end{pmatrix}. \end{aligned}$$

As $w[0, 5]$ is nonsingular, the conditions stated in Theorem 3.2 are satisfied. Hence the fractional system on $[0, 5]$ is controllable.

Example 2 Consider the controllability of the following damped fractional differential system with state delay and impulses:

$$\begin{aligned} {}^c D^{3/2} x(t) - \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 1 & -3 \end{pmatrix} {}^c D^{1/2} x(t) &= \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 2 & -1 & 2 \end{pmatrix} x\left(t - \frac{\pi}{3}\right) + \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ -1 & 2 \end{pmatrix} u(t), \\ t \in [0, 3\pi] \setminus \left\{ \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi, \frac{5\pi}{2} \right\}, \\ \Delta x(t_k) &= \frac{1}{3} x(t_k^-), \quad t_k = k \frac{\pi}{2}, \quad k = 1, 2, 3, 4, 5, \\ x(t) &= \tan t, \quad t \in \left[-\frac{\pi}{3}, 0\right]. \end{aligned}$$

Now by applying Theorem 3.3 we can prove that this system is controllable on $[0, 3\pi]$. Let us take

$$\begin{aligned} \gamma &= 3/2, \quad \delta = 1/2, \quad A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 1 & -3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 2 & -1 & 2 \end{pmatrix}, \\ C &= \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ -1 & 2 \end{pmatrix}. \end{aligned}$$

Then we can obtain

$$\begin{aligned} \text{rank}[C|AC|\cdots|A^{n-1}C] \\ = \text{rank} \begin{pmatrix} 1 & 0 & 1 & 2 & * & * \\ 1 & 1 & 1 & 1 & * & * \\ -1 & 2 & 6 & -5 & * & * \end{pmatrix} = 3. \end{aligned}$$

Thus by Theorem 3.3 this system is controllable on $[0, 3\pi]$.

5 Conclusion

We have evaluated controllability criteria for a damped fractional differential system with impulses and state delay. By using the step-by-step technique we constructed sufficient and necessary conditions for the controllability of such a system, which demonstrates that

the controllability property is not dependent on impulses, delay, or fractional derivative order.

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