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# Long-time behavior of stochastic reaction–diffusion equation with multiplicative noise

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## Abstract

In this paper, we study the dynamical behavior of the solution for the stochastic reaction–diffusion equation with the nonlinearity satisfying the polynomial growth of arbitrary order $p \geq 2$ and any space dimension $N$. Based on the inductive principle, the higher-order integrability of the difference of the solutions near the initial data is established, and then the (norm-to-norm) continuity of solutions with respect to the initial data in $H_0^1(U)$ is first obtained. As an application, we show the existence of $(L^2(U), L^p(U))$ and $(L^2(U), H_0^1(U))$-pullback random attractors, respectively.

**Keywords:** Stochastic reaction–diffusion equation; Higher-order integrability; Pullback random attractor; Norm-to-norm continuity

## 1 Introduction

We consider the following stochastic reaction–diffusion equation with multiplicative noise:

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u + f(u) = g(x) + bu \circ \frac{dW}{dt}, & x \in U, t \geq 0, \\ u(t)|_{\partial U} = 0, & t \geq 0, \\ u(x, 0) = u_0(x), & x \in U, \end{cases} \tag{1.1}$$

where $U \subset \mathbb{R}^N$ ($N \geq 3$) is a bounded smooth domain, $b$ is a positive constant and $g \in L^2(U)$. "$\circ$" denotes the Stratonovich product and $W(t)$ is a two-sided real-valued Wiener process on a probability space which will be specified later. The nonlinearity $f \in C^1(\mathbb{R}, \mathbb{R})$ satisfies the following conditions:

$$f(0) = 0, \qquad f'(s) \geq -l, \tag{1.2}$$

$$-c_2 + c_1|s|^p \leq f(s)s, \tag{1.3}$$

and

$$\left| f(s_1) - f(s_2) \right| \leq c_3|s_1 - s_2|\left(1 + |s_1|^{p-2} + |s_2|^{p-2}\right), \tag{1.4}$$

where $c_1, c_2, c_3, l$ are some positive constants and $p \in [2, \infty)$.

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As an important mathematical model, stochastic differential equations can describe many different physical phenomena when random spatio-temporal forcing term is taken into account. Some of the key problem for this kind of equation are to establish the existence and regularity of random attractors. The concept of random attractor was introduced in [1, 2], with notable development given in [3–14]. As applications, most other authors extensively investigated the existence of random attractors for some stochastic reaction–diffusion equations; see [15–32] and the references therein.

For instance, provided that $g \equiv 0$ in (1.1), some significant results have been achieved. For instance, Coaraballo and Langa [24] obtained the existence of finite dimensional random attractor in $L^2(U)$ when $f(u) = -\beta u + u^3$. Li et al. [25] used the quasi-continuity and omega-limit compactness introduced in [15] to obtain the $(L^2(U), L^p(U))$-random attractor for the problem (1.1), where $f(u)$ is a polynomial of odd degree with a positive leading coefficient. Assuming that $f(u) = -\beta u + u^3$ and $b = h(t)$ in (1.1), Fan and Chen [27] gave a new method (without transformations) to study the existence of an $L^2(U)$-random attractor. When the nonlinearity $f(u)$ satisfies the polynomial growth of arbitrary order $p \geq 2$, Wang and Tang [29] showed the existence of $(L^2(U), H_0^1(U))$-random attractor for the problem (1.1) exploiting the method of the deterministic systems introduced in [33]. When $g \neq 0$, Zhao [28] proved the existence of $H_0^1(U)$-random attractors for (1.1) by using the quasi-continuity ([15]) along with the compactness of an omega-limit set.

Inspired by the above papers, we will continue studying the asymptotic behavior for the stochastic reaction–diffusion equation with multiplicative noise. Especially, we are interested in understanding the integrability and continuity of the solutions of Eq. (1.1) with the forcing term $g \neq 0$.

On the one hand, we know that obtaining certain higher-order integrability and regularity are significant for better understanding the dynamical systems. When $b \equiv 0$ and the forcing term $g$ belongs to $L^2(U)$ or $H^{-1}(U)$, the solutions of the equation in the deterministic system are at most in $H^2(U) \cap L^{2p-2}(U)$ or $H_0^1(U) \cap L^p(U)$ and have no regularities. As regards the stochastic system, if the initial data $u_0$ and forcing term $g$ belong to $L^2(U)$, then the solution $u$ with the initial data $u(0) = u_0$ belongs to $L^2(U) \cap H_0^1(U) \cap L^p(U)$ only and has no higher regularity because of the random noise term. Compared with the case $g \equiv 0$ mentioned above (from [25]), the case $g \neq 0$ is even more complicated. The reason is that the regularity and integrability of the solutions depend not only on the growth exponent $p$, but also on the regularity and integrability of $g$. Therefore, a natural question is: can we get some higher integrability when $g \neq 0$?

On the other hand, comparing with verifying the (norm-to-norm) continuity and asymptotic compactness, it is easy to check the quasi-continuity and the flattening conditions for most of the dynamical systems, especially in the space $H_0^1(U)$ and $L^p(U)$ ($p > 2$); see [34, 35] for details. For the deterministic autonomous reaction diffusion equations, the authors [36] first proved the continuity of solutions in $H_0^1(U)$ for any space dimension $N$ and any growth exponent $p \geq 2$ by the method of differentiating the equation. However, for the stochastic case, since the Wiener processes $W(t)$ are continuous but are not differentiable functions in $\mathbb{R}$, we cannot use such a method to obtain the continuity in $H_0^1(U)$. Thus, for any space dimension $N$ and any growth exponent $p \geq 2$, we address the question whether or not we can obtain the continuity of solutions in $H_0^1(U)$ by some new kinds of estimates.

In order to answer the above two problems, we follow the ideas from [21] to obtain our main results, in which the authors investigated the high-order integrability of difference of solutions and existence of random attractors for the reaction–diffusion equations with additive noise.

The remainder article is arranged as follows. In Sect. 2, we first recall some definitions and known results about the pullback random attractors, then we give the well-posedness of a solution and the existence of random attractors in $L^2(U)$. In Sect. 3, we establish the higher-order integrability of the difference of the solutions near the initial time and get the continuity of solutions in $H_0^1(U)$. Furthermore, as an application of above continuity and higher-order integrability results of solutions, we show $(L^2(U), L^p(U))$ and $(L^2(U), H_0^1(U))$ $\mathcal{D}$-pullback random attractors for the problem (1.1).

## 2 Preliminaries

Throughout the paper, we denote the norm of a Banach space $X$ by $\|\cdot\|_X$. For the sake of convenience, we denote the norm of $L^r(U)$ ($r \geq 1$, $r \neq 2$) by $\|\cdot\|_{L^r(U)}$. The inner product and norm of $L^2(U)$ will be written as $(\cdot, \cdot)$ and $\|\cdot\|$, respectively.

### 2.1 Random dynamical system

In this subsection, we collect some definitions and known results regarding pullback attractors for random dynamical systems from [1–5, 7, 8, 17, 18, 21, 37].

Next, let $(X, \|\cdot\|_X)$ be a separable Banach space with Borel $\sigma$-algebra $\mathcal{B}(X)$. We use $(\Omega, \mathcal{F}, \mathbb{P})$ and $(X, d)$ to denote a probability space and a completely separable metric space, respectively. If $Y$ and $Z$ are two nonempty subsets of $X$, then we use $\text{dist}_X(Y, Z)$ to denote their Hausdorff semi-distance, i.e., $\text{dist}_X(Y, Z) = \sup_{y \in Y} \inf_{z \in Z} \|y - z\|_X$ for any $Y \subset X$, $Z \subset X$.

**Definition 2.1** Let $\theta : \mathbb{R} \times \Omega \to \Omega$ be a $(\mathcal{B}(\mathbb{R}) \times \mathcal{F}, \mathcal{F})$-measurable mapping. We say $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})$ is a metric dynamical system if $\theta_0$ is the identity on $\Omega$, $\theta_{s+t} = \theta_t \circ \theta_s$ for all $t, s \in \mathbb{R}$, and $\theta_t \mathbb{P} = \mathbb{P}$ for all $t \in \mathbb{R}$.

**Definition 2.2** Let $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})$ be a metric dynamical system. If the cocycle mapping $\Phi : \mathbb{R}^+ \times \Omega \times X \to X$ satisfies the following properties:

(i) $\Phi : \mathbb{R}^+ \times \Omega \times X \to X$ satisfies $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(X), \mathcal{B}(X))$-measurable;

(ii) $\Phi(0, \omega, x) = x$, $\forall \omega \in \Omega$, $x \in X$;

(iii) $\Phi(t, \theta_s \omega, \Phi(s, \omega, x)) = \Phi(t + s, \omega, x)$, $\forall s, t \in \mathbb{R}^+$, $x \in X$, $\omega \in \Omega$,

then $\Phi$ is called a random dynamical system. Furthermore, $\Phi$ is called a continuous random dynamical system if $\Phi$ is continuous with respect to $x$ for $t \geq 0$ and $\omega \in \Omega$.

**Definition 2.3** A set-valued map $K : \Omega \to 2^X$ is called a random set in $X$ if the mapping $\omega \mapsto \text{dist}(x, K(\omega))$ is $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$ measurable for all $x \in X$. A random set $K : \Omega \to 2^X$ is called a random closed set if $K(\omega)$ is closed, nonempty for each $\omega \in \Omega$.

**Definition 2.4** A random set $K : \Omega \to 2^X$ is called a bounded random set if there is a random variable $r(\omega) \geq 0$, $\omega \in \Omega$ such that

$$\text{diam}(K(\omega)) = \sup\{\|x\|_X : x \in K(\omega)\} \leq r(\omega), \quad \text{for all } \omega \in \Omega.$$

A bounded random set $\mathcal{K} := \{K(\omega)\}_{\omega \in \Omega}$ is said to be tempered $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})$ if for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$\lim_{t \to +\infty} e^{-\beta t} \operatorname{diam}\bigl(K(\theta_{-t}\omega)\bigr) = 0, \quad \text{for all } \beta > 0.$$

**Definition 2.5** Let $\mathcal{D}$ be a collection of random sets in $X$. Then a random set $K \in \mathcal{D}$ is called a $\mathcal{D}$-pullback absorbing set for a random dynamical system $(\theta, \Phi)$ if for any random set $D \in \mathcal{D}$ and $\mathbb{P}$-a.e. $\omega \in \Omega$, there exists $T = T_D(\omega) > 0$ such that

$$\Phi\bigl(t, \theta_{-t}\omega, D(\theta_{-t}\omega)\bigr) \subseteq K(\omega), \quad \text{for all } t \geq T.$$

**Definition 2.6** Let $\mathcal{D}$ be a collection of random sets in $X$. Then $\Phi$ is said to be $\mathcal{D}$-pullback asymptotically compact in $X$ if for all $\mathbb{P}$-a.e. $\omega \in \Omega$, the sequence

$$\bigl\{\Phi(t_n, \theta_{-t_n}\omega, x_n)\bigr\}_{n=1}^{\infty} \text{ has a convergent subsequence in } X$$

whenever $t_n \to \infty$ and $x_n \in K(\theta_{-t_n}\omega)$ with $K(\omega) \in \mathcal{D}$.

**Definition 2.7** Let $\mathcal{D}$ be a collection of some families of nonempty subsets of $X$. Then $A = \{A(\omega)\}_{\omega \in \Omega} \in \mathcal{D}$ is called a $\mathcal{D}$-pullback attractor for a random dynamical system $\Phi$ if the following conditions (i)–(iii) are fulfilled:

(i) $A$ is a compact random set, that is, $\omega \mapsto \operatorname{dist}(x, A(\omega))$ is measurable for every $x \in X$ and $A(\omega)$ is nonempty and compact in $X$ for $\mathbb{P}$-a.e. $\omega \in \Omega$;

(ii) $A$ is invariant, that is, $\Phi(t, \omega, A(\omega)) = A(\theta_t\omega)$, for $\mathbb{P}$-a.e. $\omega \in \Omega$ and every $t > 0$;

(iii) for every $D = \{D(\omega)\}_{\omega \in \Omega} \in \mathcal{D}$,

$$\lim_{t \to +\infty} \operatorname{dist}_X\bigl(\Phi\bigl(t, \theta_{-t}\omega, D(\theta_{-t}\omega)\bigr), A(\omega)\bigr) = 0, \quad \mathbb{P}\text{-almost surely,}$$

where $\operatorname{dist}_X$ is Hausdorff semi-metric given by $\operatorname{dist}_X(Y, Z) = \sup_{y \in Y} \inf_{z \in Z} \|y - z\|_X$ for any $Y \subseteq X$ and $Z \subseteq X$.

**Theorem 2.8** ([3]) *Let $\mathcal{D}$ be an inclusion-closed collection of some families of nonempty subsets of $X$. Suppose that $\Phi$ be a continuous random dynamical system on $X$ over $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t \in \mathbb{R}})$. If there exists a closed random absorbing set $K \in \mathcal{D}$ and $\Phi$ is $\mathcal{D}$-pullback asymptotically compact in $X$, then $\Phi$ has a unique $\mathcal{D}$-random attractor $A \in \mathcal{D}$,*

$$A(\omega) = \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} \Phi\bigl(t, \theta_{-t}\omega, K(\theta_{-t}\omega)\bigr)}, \quad \omega \in \Omega.$$

## 2.2 Well-posedness of random dynamical system generated by (1.1)

We consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where

$$\Omega = \bigl\{\omega \in C(\mathbb{R}, \mathbb{R}), \omega(0) = 0\bigr\}.$$

$\mathcal{F}$ a is Borel $\sigma$-algebra induced by the compact-open topology of $\Omega$ and $\mathbb{P}$ is the corresponding Wiener measure. Then we will identify $\omega(t)$ with $W(t)$, that is,

$$W(t) = W(t, \omega) = \omega(t), \quad t \in \mathbb{R}.$$

The time shift is simply defined by

$$\theta_t \omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad \text{for all } \omega \in \Omega, t \in \mathbb{R},$$

then $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t \in \mathbb{R}})$ is a metric dynamical system.

Now we convert the problem (1.1) into a deterministic system with a random parameter. For this purpose, we consider the Ornstein–Uhlenbeck process

$$z(\theta_t \omega) = -\int_{-\infty}^{0} e^{\tau}(\theta_t \omega)(\tau) \, d\tau, \quad t \in \mathbb{R},$$

and it solves the Itô equation

$$dz + z \, dt = dW(t). \tag{2.1}$$

From [16, 38], it is known that the random variable $z(\omega)$ is tempered, and there exists a $\theta_t$-invariant set $\tilde{\Omega} \subset \Omega$ of full $\mathbb{P}$ measure such that for every $\omega \in \tilde{\Omega}$, $t \mapsto z(\theta_t \omega)$ is continuous in $t$ and

$$\lim_{t \to \pm\infty} \frac{|z(\theta_t \omega)|}{|t|} = 0, \qquad \lim_{t \to \pm\infty} \frac{1}{t} \int_0^t z(\theta_s \omega) \, ds = 0. \tag{2.2}$$

Furthermore, there is a tempered random variable $r_1(\omega) > 0$ such that

$$\left| z(\theta_t \omega) \right| \leq e^{\frac{|t|}{2}} r_1(\omega). \tag{2.3}$$

Setting $\alpha(\omega) = e^{-bz(\omega)}$, it is clear that both $\alpha(\omega)$ and $\alpha^{-1}(\omega)$ are tempered, $\alpha(\theta_t \omega)$ is continuous with respect to $t$ for $\mathbb{P}$-a.e. $\omega \in \Omega$. Thus, applying Proposition 4.3.3 in [5], we find that there is a $\frac{\lambda_1}{2}$-slowly varying random variable $r_2(\omega) > 0$ such that

$$\frac{1}{r_2(\omega)} \leq \alpha(\omega) \leq r_2(\omega), \tag{2.4}$$

where $r_2(\omega)$ satisfies, for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$e^{-\frac{\lambda_1}{2}|t|} r_2(\omega) \leq r_2(\theta_t \omega) \leq e^{\frac{\lambda_1}{2}|t|} r_2(\omega), \quad t \in \mathbb{R}. \tag{2.5}$$

From (2.3)–(2.5), we have

$$e^{-\frac{\lambda_1}{2}|t|} r_2^{-1}(\omega) \leq \alpha(\theta_t \omega) \leq e^{\frac{\lambda_1}{2}|t|} r_2(\omega), \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega, t \in \mathbb{R}, \tag{2.6}$$

where $\lambda_1$ is the first eigenvalue of $-\Delta$ with Dirichlet boundary condition.

Choosing $r(\omega) = \max\{r_1(\omega), r_2(\omega)\}$, we will, respectively, convert (2.3) and (2.6) into the forms

$$\left| z(\theta_t \omega) \right| \leq e^{\frac{|t|}{2}} r(\omega) \tag{2.7}$$

and

$$e^{-\frac{\lambda_1}{2}|t|}r^{-1}(\omega) \le \alpha(\theta_t\omega) \le e^{\frac{\lambda_1}{2}|t|}r(\omega), \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega, t \in \mathbb{R}, \tag{2.8}$$

where $r(\omega)$ is also tempered.

Next, in order to show that the problem (1.1) generates a random dynamical system, we let $v(t) = \alpha(\theta_t\omega)u(t)$ and $\alpha(\theta_t\omega) = e^{-bz(\theta_t\omega)}$. Then, applying (2.1), we will convert (1.1) into the following deterministic equation with random variable:

$$\begin{cases} \frac{\partial v}{\partial t} - \Delta v + \alpha(\theta_t\omega)f(\alpha^{-1}(\theta_t\omega)v) = \alpha(\theta_t\omega)g(x) + bz(\theta_t\omega)v, & x \in U, t \ge 0, \\ v(t)|_{\partial U} = 0, & t \ge 0, \\ v(0,\omega) = v_0(\omega) = \alpha(\omega)u_0. \end{cases} \tag{2.9}$$

From [25], it is well known that for $\mathbb{P}$-a.e. $\omega \in \Omega$, for all $v_0(\omega) \in L^2(U)$ and $g \in L^2(U)$, the problem (2.9) satisfying the condition (1.2)–(1.4) has a unique solution,

$$v(\cdot, \omega, v_0) \in C([0,\infty), L^2(U)) \cap L^p([0,\infty), L^p(U)) \cap L^2([0,\infty), H_0^1(U)).$$

Furthermore, $v(t, \omega, v_0)$ is continuous with respect to $v_0$ in $L^2(U)$ for all $t > 0$ and $\mathbb{P}$-a.e. $\omega \in \Omega$. Thus, we know that $u(t) = \alpha^{-1}(\theta_t\omega)v(t)$ is a solution of (1.1) with $u_0 = \alpha^{-1}(\omega)v_0$. Denote the mapping $\Phi : \mathbb{R}^+ \times \Omega \times L^2(U) \to L^2(U)$ by

$$\Phi(t, \omega, u_0) = u(t, \omega, u_0) = \alpha^{-1}(\theta_t\omega)v(t, \omega, \alpha(\omega)u_0),$$

then $\Phi(t, \omega, u_0)$ satisfies conditions (i)–(iii) in Definition 2.2 and is continuous. Therefore, $\Phi$ is a continuous random dynamical system.

### 2.3 Random attractor in $L^2(U)$

In this subsection, we give some estimates of solutions to obtain our main results.

**Lemma 2.9** *Assume that $g \in L^2(U)$ and (1.2)–(1.4) hold. Let $D \in \mathcal{D}$ and $u_0 \in D(\omega)$. Then for $\mathbb{P}$-a.e. $\omega \in \Omega$, there exists $T_D(\omega) > 0$ and the tempered functions $\rho_i(\omega) > 0$ ($i = 1, 2, 3$) such that the solution $v(t, \omega, v_0(\omega))$ of (2.9) with $v_0(\omega) = \alpha(\omega)u_0(\omega)$ satisfies, for all $t > T_D(\omega)$,*

$$\left\| v(t, \theta_{-t}\omega, v_0(\theta_{-t}\omega)) \right\|^2 \le \rho_1(\omega); \tag{2.10}$$

$$\int_t^{t+1} \left\| v(s, \theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega)) \right\|_{L^p(U)}^p \, ds \le \rho_2(\omega); \tag{2.11}$$

*and*

$$\int_t^{t+1} \left\| \nabla v(s, \theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega)) \right\|^2 \, ds \le \rho_3(\omega). \tag{2.12}$$

*Proof* Taking the inner product of (2.9) with $v$ in $L^2(U)$, we find that

$$\frac{1}{2}\frac{d}{dt}\|v\|^2 + \|\nabla v\|^2 = -(\alpha(\theta_t\omega)f(\alpha^{-1}(\theta_t\omega)v), v) + (\alpha(\theta_t\omega)g(x), v)$$

$$+ (bvz(\theta_t\omega), v). \tag{2.13}$$

By using (1.3), we have

$$\left(\alpha(\theta_t\omega)f\left(\alpha^{-1}(\theta_t\omega)v\right),v\right) \geq c_1\alpha^{2-p}(\theta_t\omega)\|v\|^p_{L^p(U)} - c_2|U|\alpha^2(\theta_t\omega). \tag{2.14}$$

At the same time, applying Hölder's inequality and Young's inequality, we conclude that

$$\left(\alpha(\theta_t\omega)g(x),v\right) \leq \frac{\alpha^2(\theta_t\omega)}{2\lambda_1}\|g\|^2 + \frac{\lambda_1}{2}\|v\|^2 \tag{2.15}$$

and

$$\left|\left(bvz(\theta_t\omega),v\right)\right| \leq b\left|z(\theta_t\omega)\right|\|v\|^2, \tag{2.16}$$

where $\lambda_1$ is the first eigenvalue of $-\Delta$ with Dirichlet boundary value in (2.15).

Thus, (2.13)–(2.16) imply that

$$\frac{d}{dt}\|v\|^2 + 2\|\nabla v\|^2 + 2c_1\alpha^{2-p}(\theta_t\omega)\|v\|^p_{L^p(U)}$$
$$\leq 2c_2|U|\alpha^2(\theta_t\omega) + \frac{\|g\|^2}{\lambda_1}\alpha^2(\theta_t\omega) + \lambda_1\|v\|^2 + 2b\left|z(\theta_t\omega)\right|\|v\|^2. \tag{2.17}$$

Using the Poincaré inequality $\|\nabla v\|^2 \geq \lambda_1\|v\|^2$ in the above result, we have

$$\frac{d}{dt}\|v\|^2 + \left(\lambda_1 - 2b\left|z(\theta_t\omega)\right|\right)\|v\|^2 + 2c_1\alpha^{2-p}(\theta_t\omega)\|v\|^p_{L^p(U)}$$
$$\leq 2c_2|U|\alpha^2(\theta_t\omega) + \frac{\|g\|^2}{\lambda_1}\alpha^2(\theta_t\omega). \tag{2.18}$$

Then, applying Gronwall's lemma, we get

$$\left\|v\left(t,\omega,v_0(\omega)\right)\right\|^2 \leq e^{2b\int_0^t |z(\theta_\tau\omega)|\,d\tau - \lambda_1 t}\|v_0\|^2$$
$$+ 2c_2|U|\int_0^t e^{2b\int_s^t |z(\theta_\tau\omega)|\,d\tau + \lambda_1(s-t)}\alpha^2(\theta_s\omega)\,ds$$
$$+ \frac{\|g\|^2}{\lambda_1}\int_0^t e^{2b\int_s^t |z(\theta_\tau\omega)|\,d\tau + \lambda_1(s-t)}\alpha^2(\theta_s\omega)\,ds. \tag{2.19}$$

Substituting $\omega$ by $\theta_{-t}\omega$ for above inequality and using (2.8), we find that

$$\left\|v\left(t,\theta_{-t}\omega,v_0(\theta_{-t}\omega)\right)\right\|^2 \leq e^{2b\int_0^t |z(\theta_{\tau-t}\omega)|\,d\tau - \lambda_1 t}\left\|v_0(\theta_{-t}\omega)\right\|^2$$
$$+ 2c_2|U|\int_0^t e^{2b\int_s^t |z(\theta_{\tau-t}\omega)|\,d\tau + \lambda_1(s-t)}\alpha^2(\theta_{s-t}\omega)\,ds$$
$$+ \frac{\|g\|^2}{\lambda_1}\int_0^t e^{2b\int_s^t |z(\theta_{\tau-t}\omega)|\,d\tau + \lambda_1(s-t)}\alpha^2(\theta_{s-t}\omega)\,ds$$
$$\leq e^{2b\int_{-t}^0 |z(\theta_\tau\omega)|\,d\tau - \lambda_1 t}\left\|v_0(\theta_{-t}\omega)\right\|^2$$
$$+ \left(2c_2|U| + \frac{\|g\|^2}{\lambda_1}\right)r^2(\omega)\int_{-\infty}^0 e^{2b\int_s^0 |z(\theta_\tau\omega)|\,d\tau}\,ds.$$

It is obvious that $e^{2b\int_{-t}^{0}|z(\theta_\tau\omega)|\,d\tau}$ is tempered, that is, there exists a random variable $r_3(\omega)$ such that $e^{2b\int_{-t}^{0}|z(\theta_\tau\omega)|\,d\tau} \le r_3(\omega)$. In fact,

$$e^{-\beta t}e^{2b\int_{-t}^{0}|z(\theta_\tau\omega)|\,d\tau} = e^{-\beta t}e^{2bt\cdot\frac{1}{-t}\int_{0}^{-t}|z(\theta_\tau\omega)|\,d\tau} \le e^{-\beta t}e^{2bt\cdot\frac{\beta}{4b}} = e^{-\frac{\beta t}{2}}$$

from (2.2) and $z(\theta_t\omega)$ is tempered, where $\beta$ is a positive constant. Notice that $D(\omega) \in \mathcal{D}$ is tempered, then $v_0(\theta_{-t}\omega) \in D(\theta_{-t}\omega)$ is also tempered. Moreover, it follows from the properties of the Ornstein–Uhlenbeck process that

$$\int_{-\infty}^{0} e^{2b\int_{s}^{0}|z(\theta_\tau\omega)|\,d\tau}\,ds < \infty.$$

Hence, combining with the above results, we set

$$\rho_1(\omega) = e^{-\lambda_1 t}r_3(\omega)\big\|v_0(\theta_{-t}\omega)\big\|^2 + \left(2c_2|U| + \frac{\|g\|^2}{\lambda_1}\right)r^2(\omega)\int_{-\infty}^{0} e^{2b\int_{s}^{0}|z(\theta_\tau\omega)|\,d\tau}\,ds,$$

then (2.10) holds.

Next, we will prove that (2.11) holds. Integrating (2.17) over $[t, t+1]$ with respect to $t$ and replacing $\omega$ by $\theta_{-t-1}\omega$, we obtain

$$\int_{t}^{t+1} \alpha^{2-p}(\theta_{s-t-1}\omega)\big\|v(s)\big\|_{L^p(U)}^p\,ds$$
$$\le \frac{1}{2c_1}\left(\big\|v(t)\big\|^2 + \left(2c_2|U| + \frac{\|g\|^2}{\lambda_1}\right)\int_{t}^{t+1}\alpha^2(\theta_{s-t-1}\omega)\,ds\right)$$
$$\le \frac{1}{2c_1}\left(\int_{t}^{t+1}\left(\lambda_1 + 2b\big|z(\theta_{s-t-1}\omega)\big|\right)\big\|v(s)\big\|^2\,ds\right). \tag{2.20}$$

Since

$$\int_{t}^{t+1}\alpha^{2-p}(\theta_{s-t-1}\omega)\big\|v(s)\big\|_{L^p(U)}^p\,ds = \int_{-1}^{0}\alpha^{2-p}(\theta_s\omega)\big\|v(s+t+1)\big\|_{L^p(U)}^p\,ds$$
$$\ge r^{2-p}(\omega)\int_{-1}^{0}e^{\frac{(p-2)\lambda_1}{2}s}\big\|v(s+t+1)\big\|_{L^p(U)}^p\,ds$$
$$\ge \left(r(\omega)e^{\frac{\lambda_1}{2}}\right)^{2-p}\int_{t}^{t+1}\big\|v(s)\big\|_{L^p(U)}^p\,ds, \tag{2.21}$$

we can get

$$\int_{t}^{t+1}\big\|v(s)\big\|_{L^p(U)}^p\,ds$$
$$\le \frac{(r(\omega)e^{\frac{\lambda_1}{2}})^{p-2}}{2c_1}\left(\big\|v(t)\big\|^2 + \left(2c_2|U| + \frac{\|g\|^2}{\lambda_1}\right)\int_{-1}^{0}\alpha^2(\theta_s\omega)\,ds\right)$$
$$+ \frac{(r(\omega)e^{\frac{\lambda_1}{2}})^{p-2}}{2c_1}\left(\lambda_1\int_{t}^{t+1}\big\|v(s)\big\|^2\,ds + 2b\int_{-1}^{0}\big|z(\theta_s\omega)\big|\big\|v(s+t+1)\big\|^2\,ds\right)$$
$$\le \frac{(r(\omega)e^{\frac{\lambda_1}{2}})^{p-2}}{2c_1}\left(\left(1 + \lambda_1 + 2be^{\frac{1}{2}}r(\omega)\right)\rho_1(\omega) + \left(2c_2|U| + \frac{\|g\|^2}{\lambda_1}\right)e^{\lambda_1}r^2(\omega)\right),$$

by using (2.7), (2.8), (2.10) and combining (2.20) with (2.21). So (2.11) holds if we choose

$$\rho_2(\omega) = \frac{(r(\omega)e^{\frac{\lambda_1}{2}})^{p-2}}{2c_1}\left(\left(1 + \lambda_1 + 2be^{\frac{1}{2}}r(\omega)\right)\rho_1(\omega) + \left(2c_2|U| + \frac{\|g\|^2}{\lambda_1}\right)e^{\lambda_1}r^2(\omega)\right).$$

Finally, taking $t \geq T_D(\omega)$ and $s \in (t, t+1)$, integrating (2.17) from $s$ to $t+1$, it follows that

$$\begin{aligned}
\|v(t+1)\|^2 &+ 2\int_s^{t+1}\|\nabla v(\tau)\|^2\,d\tau \\
&\leq \|v(s)\|^2 + \left(2c_2|U| + \frac{\|g\|^2}{\lambda_1}\right)\int_s^{t+1}\alpha^2(\theta_\tau\omega)\,d\tau \\
&\quad + \lambda_1\int_s^{t+1}\|v(\tau)\|^2\,d\tau + 2b\int_s^{t+1}|z(\theta_\tau\omega)|\|v(\tau)\|^2\,d\tau.
\end{aligned} \tag{2.22}$$

Again integrating (2.22) over $[t, t+1]$ with respect to $s$ and replacing $\omega$ by $\theta_{-t-1}\omega$, we infer that

$$\begin{aligned}
\int_t^{t+1}\|\nabla v(\tau)\|^2\,d\tau &\leq \frac{1}{2}\int_t^{t+1}\|v(s)\|^2\,ds + \left(c_2|U| + \frac{\|g\|^2}{2\lambda_1}\right)\int_t^{t+1}\alpha^2(\theta_{\tau-t-1}\omega)\,d\tau \\
&\quad + \frac{\lambda_1}{2}\int_t^{t+1}\|v(\tau)\|^2\,d\tau + b\int_t^{t+1}|z(\theta_{\tau-t-1}\omega)|\|v(\tau)\|^2\,d\tau \\
&\leq \frac{1}{2}\int_t^{t+1}\|v(s)\|^2\,ds + \left(c_2|U| + \frac{\|g\|^2}{2\lambda_1}\right)\int_{-1}^0\alpha^2(\theta_\tau\omega)\,d\tau \\
&\quad + \frac{\lambda_1}{2}\int_t^{t+1}\|v(\tau)\|^2\,d\tau + b\int_{-1}^0|z(\theta_\tau\omega)|\|v(\tau+t+1)\|^2\,d\tau \\
&\leq \left(\frac{1+\lambda_1}{2} + 2be^{\frac{1}{2}}r(\omega)\right)\rho_1(\omega) + \left(c_2|U| + \frac{\|g\|^2}{2\lambda_1}\right)e^{\lambda_1}r^2(\omega)
\end{aligned}$$

by using (2.7), (2.8) and (2.10). Thus, let

$$\rho_3(\omega) = \left(\frac{1+\lambda_1}{2} + 2be^{\frac{1}{2}}r(\omega)\right)\rho_1(\omega) + \left(c_2|U| + \frac{\|g\|^2}{2\lambda_1}\right)e^{\lambda_1}r^2(\omega),$$

then (2.12) holds. □

**Lemma 2.10** *Assume that $g \in L^2(U)$ and (1.2)–(1.4) hold. Let $D \in \mathcal{D}$ and $u_0(\omega) \in D$. Then for $\mathbb{P}$-a.e. $\omega \in \Omega$, we have $T_D(\omega) > 0$ such that the solution $u(t, \omega, u_0(\omega))$ of (1.1) satisfies, for all $t > T_D(\omega)$,*

$$\|u(t, \theta_{-t}\omega, u_0(\theta_{-t}\omega))\|^2 \leq r^2(\omega)\rho_1(\omega) \tag{2.23}$$

*and*

$$\|\nabla u(t, \theta_{-t}\omega, u_0(\theta_{-t}\omega))\|^2 \leq \rho_4(\omega). \tag{2.24}$$

*Proof* Combining (2.10) with $v(t, \omega, v_0(\omega)) = \alpha(\theta_t\omega)u(t, \omega, u_0(\omega))$, we have

$$\|u(t, \theta_{-t}\omega, u_0(\theta_{-t}\omega))\|^2 = \|\alpha^{-1}(\omega)v(t, \theta_{-t}\omega, v_0(\theta_{-t}\omega))\|^2 \leq r^2(\omega)\rho_1(\omega).$$

Now, multiplying (2.9) by $-\Delta v$ and integrating over $U$, we find that

$$
\frac{1}{2}\frac{d}{dt}\|\nabla v\|^2 + \|\Delta v\|^2 + \big(\alpha(\theta_t\omega)f\big(\alpha^{-1}(\theta_t\omega)v(t)\big), -\Delta v\big)
$$
$$
= \big(\alpha(\theta_t\omega)g(x), -\Delta v\big) + (bzv, -\Delta v). \tag{2.25}
$$

By using (1.2), Hölder's inequality and Young's inequality, we have

$$
\big(\alpha(\theta_t\omega)f\big(\alpha^{-1}(\theta_t\omega)v(t)\big), -\Delta v\big) \geq -l\|\nabla v\|^2, \tag{2.26}
$$

$$
\big|\big(\alpha(\theta_t\omega)g(x), -\Delta v\big)\big| \leq \frac{\alpha^2(\theta_t\omega)}{4}\|g\|^2 + \|\Delta v\|^2, \tag{2.27}
$$

and

$$
\big|(bzv, -\Delta v)\big| \leq b|z|\|\nabla v\|^2. \tag{2.28}
$$

Thus, it follows from (2.25)–(2.28) that

$$
\frac{d}{dt}\|\nabla v\|^2 \leq 2l\|\nabla v\|^2 + 2b|z|\|\nabla v\|^2 + \frac{\alpha^2(\theta_t\omega)}{2}\|g\|^2. \tag{2.29}
$$

Now, taking $t \geq T_D(\omega)$ and $s \in (t, t+1)$, integrating (2.29) from $s$ to $t+1$, we get

$$
\big\|\nabla v(t+1)\big\|^2 \leq 2l\int_s^{t+1}\big\|\nabla v(\tau)\big\|^2\,d\tau + 2b\int_s^{t+1}\big|z(\theta_\tau\omega)\big|\big\|\nabla v(\tau)\big\|^2\,d\tau
$$
$$
+ \frac{\|g\|^2}{2}\int_s^{t+1}\alpha^2(\theta_\tau\omega)\,d\tau + \big\|\nabla v(s)\big\|^2. \tag{2.30}
$$

Integrating (2.30) over $[t, t+1]$ with respect to $s$ and replacing $\omega$ by $\theta_{-t-1}\omega$, we deduce that

$$
\big\|\nabla v\big(t+1, \theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega)\big)\big\|^2
$$
$$
\leq (1+2l)\int_t^{t+1}\big\|\nabla v\big(s, \theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega)\big)\big\|^2\,ds
$$
$$
+ \frac{\|g\|^2}{2}\int_t^{t+1}\alpha^2(\theta_{\tau-t-1}\omega)\,d\tau
$$
$$
+ 2b\int_t^{t+1}\big|z(\theta_{\tau-t-1}\omega)\big|\big\|\nabla v\big(\tau, \theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega)\big)\big\|^2\,d\tau
$$
$$
\leq (1+2l)\rho_3(\omega) + \frac{\|g\|^2}{2}\int_{-1}^0\alpha^2(\theta_\tau\omega)\,d\tau
$$
$$
+ 2b\int_{-1}^0\big|z(\theta_\tau\omega)\big|\big\|\nabla v\big(\tau+t+1, \theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega)\big)\big\|^2\,d\tau
$$
$$
\leq \big(1+2l+2b\sqrt{e}r(\omega)\big)\rho_3(\omega) + \frac{e^{\lambda_1}}{2}r^2(\omega)\|g\|^2
$$

by using (2.7), (2.8) and (2.12). Choosing

$$
\rho_4(\omega) = \big(1+2l+2b\sqrt{e}r(\omega)\big)\rho_3(\omega) + \frac{e^{\lambda_1}}{2}r^2(\omega)\|g\|^2,
$$

and combining with

$$v(t) = \alpha(t)u(t),$$

we complete the proof.                                                                                    □

Combining the boundedness of solutions in $H_0^1(U)$ given in Lemma 2.10 with the Sobolev compact embedding $H_0^1(U) \hookrightarrow L^2(U)$, it is easy to obtain the compactness of solutions in $L^2(U)$. Thus, by Theorem 2.8 we obtain the following result.

**Lemma 2.11** *Assume that $g \in L^2(U)$ and* (1.2)–(1.4) *hold. Then the continuous random dynamical system $\Phi$ generated by* (1.1) *has a unique $\mathcal{D}$-random attractor $A$, that is, for $\mathbb{P}$-a.e. $\omega \in \Omega$, $A$ is nonempty, compact, invariant and $\mathcal{D}$-pullback attracting in the topology of $L^2(U)$.*

## 3  Uniform estimates of solutions

In this section, the estimates on the higher order integrability for the difference of solutions near initial time will be given. At the same time, we also prove other corresponding results. For the sake of convenience, we choose $C$ as the positive constant which may be different from line to line or in the same line in our paper.

### 3.1  Higher order integrability near initial time

**Theorem 3.1** *Assume that* (1.2)–(1.4) *hold, and $b > 0$ and $u_{0i} \in D(\omega)$ ($i = 1, 2$) is the initial data. Then, for any $T > 0$, any $k = 1, 2, \dots$ and $\mathbb{P}$-a.e. $\omega \in \Omega$, there exist positive constants $M_k(\omega) = M(l, k, b, N, T, r(\omega), r_3(\omega), \lambda, \|u_{0i}\|)$, such that*

$$t^{\frac{N}{N-2}} \left\| t^{b_k} \bar{u}(t) \right\|_{L^{2(\frac{N}{N-2})^k}(U)}^{2(\frac{N}{N-2})^k} \leq M_k(\omega), \quad \text{for all } t \in [0, T],$$

*and*

$$\int_0^T \left( \int_U \left| t^{b_{k+1}} \bar{u}(t) \right|^{2(\frac{N}{N-2})^{k+1}} dx \right)^{\frac{N-2}{N}} dt \leq M_k(\omega),$$

*where $\bar{u}(t) = \Phi(t, \omega)u_{01} - \Phi(t, \omega)u_{02}$ and*

$$b_1 = 1 + \frac{1}{2}, \qquad b_2 = 1 + \frac{1}{2} + 1 \quad and \quad b_{k+1} = b_k + \frac{1 + \frac{N}{N-2}}{2(\frac{N}{N-2})^{k+1}} \quad for \ k = 2, 3, \dots. \quad (3.1)$$

*Proof* We see that $\bar{u}(t)$ satisfies the equation

$$\begin{cases} \frac{\partial \bar{u}}{\partial t} - \Delta \bar{u} + f(u_1(t)) - f(u_2(t)) = b\bar{u} \circ \frac{dW}{dt}, & (x,t) \in U \times (0, T), \\ \bar{u}(t)|_{\partial U} = 0, & t \geq 0, \\ \bar{u}(0, \omega) = u_{01}(\omega) - u_{02}(\omega), \end{cases} \quad (3.2)$$

where $u_i(t) = \Phi(t, \omega, u_{0i}(\omega))$ ($i = 1, 2$) is the solution of Eq. (1.1) with initial data $u_{0i}$.

Due to $v(t) = \alpha(\theta_t\omega)u(t)$ with $\alpha(\theta_t\omega) = e^{-bz(\theta_t\omega)}$, we convert Eq. (3.2) into the equation

$$
\begin{cases}
\frac{\partial \bar{v}}{\partial t} - \Delta \bar{v} + \alpha(\theta_t\omega)(f(u_1(t)) - f(u_2(t))) = bz\bar{v}, & (x,t) \in U \times (0,T), \\
\bar{v}(t)|_{\partial U} = 0, & t \geq 0, \\
\bar{v}(0,\omega) = v_{01}(\omega) - v_{02}(\omega).
\end{cases}
\tag{3.3}
$$

Our proof will be completed in two steps.

We can justify the following estimates by means of the Faedo–Galerkin approximation procedure.

• For the case $k = 1$. Taking the inner product of (3.3) with $\bar{v}$ in $L^2(U)$, we find that

$$
\frac{1}{2}\frac{d}{dt}\|\bar{v}\|^2 + \|\nabla \bar{v}\|^2 = -\big(\alpha(\theta_t\omega)\big(f\big(u_1(t)\big) - f\big(u_2(t)\big)\big), \bar{v}\big) + b|z|\|\bar{v}\|^2, \quad t \in (0,T).
\tag{3.4}
$$

By using (1.2), we have the following estimate:

$$
-\big(\alpha(\theta_t\omega)\big(f\big(u_1(t)\big) - f\big(u_2(t)\big)\big), \bar{v}\big) \leq l\|\bar{v}\|^2.
\tag{3.5}
$$

It follows from (3.4) and (3.5) that

$$
\frac{d}{dt}\|\bar{v}\|^2 + 2\|\nabla\bar{v}\|^2 \leq 2l\|\bar{v}\|^2 + 2b|z|\|\bar{v}\|^2.
$$

By Gronwall's lemma, we conclude

$$
\big\|\bar{v}(t)\big\|^2 \leq e^{2lt} e^{2b\int_0^t |z(\theta_\tau\omega)|\,d\tau} \big\|\bar{v}(0)\big\|^2.
\tag{3.6}
$$

It is obvious that $e^{2b\int_0^t |z(\theta_\tau\omega)|\,d\tau}$ is tempered, that is, there exists a random variable $r_4(\omega)$ such that $e^{2b\int_0^t |z(\theta_\tau\omega)|\,d\tau} \leq r_4(\omega)$. In fact,

$$
e^{-\beta t} e^{2b\int_0^t |z(\theta_\tau\omega)|\,d\tau} = e^{-\beta t} e^{2bt \cdot \frac{1}{t}\int_0^t |z(\theta_\tau\omega)|\,d\tau} \leq e^{-\beta t} e^{2bt \cdot \frac{\beta}{4b}} = e^{-\frac{\beta t}{2}},
$$

from (2.2), where $\beta$ is a proper positive constant.

Then, (3.6) is equivalent to the following form:

$$
\big\|\bar{v}(t)\big\|^2 \leq r_4(\omega) e^{2lt} \big\|\bar{v}(0)\big\|^2, \quad \forall t \in [0,T].
\tag{3.7}
$$

Furthermore,

$$
\begin{aligned}
\int_0^T & \big\|\nabla\bar{v}(s)\big\|^2\,ds \\
&\leq \frac{1}{2}\big\|\bar{v}(0)\big\|^2 + l\int_0^T \big\|\bar{v}(s)\big\|^2\,ds + b\int_0^T |z|\big\|\bar{v}(s)\big\|^2\,ds \\
&\leq \frac{1}{2}\big\|\bar{v}(0)\big\|^2 + lr_4(\omega)\int_0^T e^{2ls}\big\|\bar{v}(0)\big\|^2\,ds + br(\omega)r_4(\omega)\int_0^T e^{\frac{s}{2}} e^{2ls}\big\|\bar{v}(0)\big\|^2\,ds \\
&\leq \frac{1}{2}\big\|\bar{v}(0)\big\|^2 + lr_4(\omega)\frac{e^{2lT} - 1}{2l}\big\|\bar{v}(0)\big\|^2 + br(\omega)r_4(\omega)\frac{e^{(2l+1)T} - 1}{2l + 1}\big\|\bar{v}(0)\big\|^2 \\
&= C\big(l, b, T, r(\omega), r_4(\omega)\big)\big\|\bar{v}(0)\big\|^2.
\end{aligned}
\tag{3.8}
$$

And then it follows from $\|\bar{v}\|_{L^{\frac{2N}{N-2}}(U)} \leq c\|\nabla\bar{v}\|$ ($c$ is the embedding constant) that

$$\int_0^T \|\bar{v}(s)\|_{L^{\frac{2N}{N-2}}(U)}^2 \, ds \leq cC\big(l,b,T,r(\omega),r_4(\omega)\big)\|\bar{v}(0)\|^2. \tag{3.9}$$

So,

$$\int_0^T \|s^{b_1}\bar{v}(s)\|_{L^{\frac{2N}{N-2}}(U)}^2 \, ds = \int_0^T s^{2b_1}\|\bar{v}(s)\|_{L^{\frac{2N}{N-2}}(U)}^2 \, ds$$

$$\leq T^{2b_1}\int_0^T \|\bar{v}(s)\|_{L^{\frac{2N}{N-2}}(U)}^2 \, ds$$

$$\leq C\big(l,b,c,b_1,T,r(\omega),r_4(\omega)\big)\|\bar{v}(0)\|^2. \tag{3.10}$$

Taking the inner product of (3.3) with $|\bar{v}|^{\frac{2N}{N-2}-2}\cdot\bar{v}$ in $L^2(U)$ again, we obtain

$$\frac{N-2}{2N}\frac{d}{dt}\|\bar{v}(t)\|_{L^{\frac{2N}{N-2}}(U)}^{\frac{2N}{N-2}} + \frac{\frac{2N}{N-2}-1}{(\frac{N}{N-2})^2}\int_U \big|\nabla\big|\bar{v}(t)\big|^{\frac{N}{N-2}}\big|^2 \, dx$$

$$\leq \big(l+b|z|\big)\|\bar{v}(t)\|_{L^{\frac{2N}{N-2}}(U)}^{\frac{2N}{N-2}},$$

it follows that

$$\frac{d}{dt}\|\bar{v}(t)\|_{L^{\frac{2N}{N-2}}(U)}^{\frac{2N}{N-2}} + \frac{2(N+2)}{N}\int_U \big|\nabla\big|\bar{v}(t)\big|^{\frac{N}{N-2}}\big|^2 \, dx$$

$$\leq \frac{2N(l+b|z|)}{N-2}\|\bar{v}(t)\|_{L^{\frac{2N}{N-2}}(U)}^{\frac{2N}{N-2}}. \tag{3.11}$$

Multiplying both sides of (3.11) with $t^{\frac{3N}{N-2}}$, for a.e. $t\in(0,T)$, yields

$$t^{\frac{3N}{N-2}}\frac{d}{dt}\|\bar{v}(t)\|_{L^{\frac{2N}{N-2}}(U)}^{\frac{2N}{N-2}} + t^{\frac{3N}{N-2}}\frac{2(N+2)}{N}\int_U \big|\nabla\big|\bar{v}(t)\big|^{\frac{N}{N-2}}\big|^2 \, dx$$

$$\leq t^{\frac{3N}{N-2}}\frac{2N(l+b|z|)}{N-2}\|\bar{v}(t)\|_{L^{\frac{2N}{N-2}}(U)}^{\frac{2N}{N-2}}.$$

At the same time, we see that

$$\frac{d}{dt}\|t^{b_1}\bar{v}(t)\|_{L^{\frac{2N}{N-2}}(U)}^{\frac{2N}{N-2}} = \frac{d}{dt}\int_U t^{\frac{3N}{N-2}}\big|\bar{v}(t)\big|^{\frac{2N}{N-2}} \, dx$$

$$= t^{\frac{3N}{N-2}}\frac{d}{dt}\|\bar{v}(t)\|_{L^{\frac{2N}{N-2}}(U)}^{\frac{2N}{N-2}} + \frac{3N}{N-2}t^{\frac{3N}{N-2}-1}\|\bar{v}(t)\|_{L^{\frac{2N}{N-2}}(U)}^{\frac{2N}{N-2}}.$$

Therefore, for a.e. $t\in(0,T)$, we have

$$\frac{d}{dt}\|t^{b_1}\bar{v}(t)\|_{L^{\frac{2N}{N-2}}(U)}^{\frac{2N}{N-2}} + \frac{2(N+2)}{N}\int_U \big|\nabla\big|t^{b_1}\bar{v}(t)\big|^{\frac{N}{N-2}}\big|^2 \, dx$$

$$\leq \frac{2N(l+b|z|)}{N-2}\|t^{b_1}\bar{v}(t)\|_{L^{\frac{2N}{N-2}}(U)}^{\frac{2N}{N-2}} + \frac{3N}{N-2}t^{-1}\|t^{b_1}\bar{v}(t)\|_{L^{\frac{2N}{N-2}}(U)}^{\frac{2N}{N-2}}$$

$$\leq C(l,N,b)\big(1+|z(\theta_t\omega)|+t^{-1}\big)\big\|t^{b_1}\bar{v}(t)\big\|^{\frac{2N}{N-2}}_{L^{\frac{2N}{N-2}}(U)}$$

$$\leq C(l,N,b)\big(1+e^{\frac{t}{2}}r(\omega)+t^{-1}\big)\big\|t^{b_1}\bar{v}(t)\big\|^{\frac{2N}{N-2}}_{L^{\frac{2N}{N-2}}(U)}, \tag{3.12}$$

where $C(l,N,b)=\max\{\frac{2Nl}{N-2},\frac{2Nb}{N-2},\frac{3N}{N-2}\}$. Thus, for a.e. $t\in(0,T)$,

$$t\frac{d}{dt}\big\|t^{b_1}\bar{v}(t)\big\|^{\frac{2N}{N-2}}_{L^{\frac{2N}{N-2}}(U)}\leq C(l,N,b)\big(t+te^{\frac{t}{2}}r(\omega)+1\big)\big\|t^{b_1}\bar{v}(t)\big\|^{\frac{2N}{N-2}}_{L^{\frac{2N}{N-2}}(U)}$$

$$\leq C(l,N,b)\big(T+Te^{\frac{T}{2}}r(\omega)+1\big)\big\|t^{b_1}\bar{v}(t)\big\|^{\frac{2N}{N-2}}_{L^{\frac{2N}{N-2}}(U)} \tag{3.13}$$

and

$$t\frac{d}{dt}\big\|t^{b_1}\bar{v}(t)\big\|^{2}_{L^{\frac{2N}{N-2}}(U)}=t\frac{d}{dt}\left(\int_U\big|t^{b_1}\bar{v}(t)\big|^{\frac{2N}{N-2}}dx\right)^{\frac{N-2}{N}}$$

$$=t\frac{N-2}{N}\left(\int_U\big|t^{b_1}\bar{v}(t)\big|^{\frac{2N}{N-2}}dx\right)^{\frac{N-2}{N}-1}\frac{d}{dt}\big\|t^{b_1}\bar{v}(t)\big\|^{\frac{2N}{N-2}}_{L^{\frac{2N}{N-2}}(U)}$$

$$\leq\frac{N-2}{N}\left(\int_U\big|t^{b_1}\bar{v}(t)\big|^{\frac{2N}{N-2}}dx\right)^{\frac{N-2}{N}-1}$$

$$\cdot C(l,N,b)\big(T+Te^{\frac{T}{2}}r(\omega)+1\big)\big\|t^{b_1}\bar{v}(t)\big\|^{\frac{2N}{N-2}}_{L^{\frac{2N}{N-2}}(U)}$$

$$\leq C(l,N,b)\big(T+Te^{\frac{T}{2}}r(\omega)+1\big)\frac{N-2}{N}\big\|t^{b_1}\bar{v}(t)\big\|^{2}_{L^{\frac{2N}{N-2}}(U)}. \tag{3.14}$$

For any fixed $t\in(0,T)$, integrating (3.14) from 0 to $t$, we have

$$\int_0^t s\frac{d}{ds}\big\|s^{b_1}\bar{v}(t)\big\|^{2}_{L^{\frac{2N}{N-2}}(U)}\,ds=t\big\|t^{b_1}\bar{v}(t)\big\|^{2}_{L^{\frac{2N}{N-2}}(U)}-\int_0^t\big\|s^{b_1}\bar{v}(t)\big\|^{2}_{L^{\frac{2N}{N-2}}(U)}\,ds$$

$$\leq C(l,N,b)\big(T+Te^{\frac{T}{2}}r(\omega)+1\big)\frac{N-2}{N}$$

$$\cdot\int_0^t\big\|s^{b_1}\bar{v}(s)\big\|^{2}_{L^{\frac{2N}{N-2}}(U)}\,ds.$$

Then, using (3.10), we have

$$t\big\|t^{b_1}\bar{v}(t)\big\|^{2}_{L^{\frac{2N}{N-2}}(U)}\leq\big(C(l,N,b)\big(T+Te^{\frac{T}{2}}r(\omega)+1\big)+1\big)$$

$$\cdot\int_0^T\big\|s^{b_1}\bar{v}(s)\big\|^{2}_{L^{\frac{2N}{N-2}}(U)}\,ds$$

$$\leq\big(C(l,N,b)\big(T+Te^{\frac{T}{2}}r(\omega)+1\big)+1\big)$$

$$\cdot C\big(l,b,c,b_1,T,r(\omega),r_4(\omega)\big)\big\|\bar{v}(0)\big\|^{2}$$

and

$$t^{\frac{N}{N-2}}\big\|t^{b_1}\bar{v}(t)\big\|^{\frac{2N}{N-2}}_{L^{\frac{2N}{N-2}}(U)}\leq M_1'(\omega), \tag{3.15}$$

where

$$M_1'(\omega) = \left(C(l,N,b)\left(T + Te^{\frac{T}{2}}r(\omega) + 1\right) + 1\right)^{\frac{N}{N-2}}$$
$$\cdot \left(C\left(l,b,T,c,b_1,r(\omega),r_4(\omega)\right)\right)^{\frac{N}{N-2}} \left\|\bar{v}(0)\right\|^{\frac{2N}{N-2}}.$$

Hence, for a.e. $t \in (0,T)$, we get

$$t^{\frac{N}{N-2}} \left\|t^{b_1}\bar{u}(t)\right\|_{L^{\frac{2N}{N-2}}(U)}^{\frac{2N}{N-2}} = t^{\frac{N}{N-2}} \left\|t^{b_1}\alpha^{-1}(\theta_t\omega)\bar{v}(t)\right\|_{L^{\frac{2N}{N-2}}(U)}^{\frac{2N}{N-2}}$$

$$\leq \left(\alpha(\theta_t\omega)\right)^{-\frac{2N}{N-2}} M_1'(\omega)$$

$$\leq \left(e^{\frac{\lambda}{2}T}r(\omega)\right)^{\frac{2N}{N-2}} M_1'(\omega) = M_1''(\omega) \tag{3.16}$$

from (2.8), (3.15) and $\bar{v}(t) = \alpha(\theta_t\omega)\bar{u}(t)$.

Multiplying (3.12) by $t^{\frac{2N}{N-2}}$ and combining with (3.15), we have

$$t^{\frac{2N}{N-2}} \frac{d}{dt} \left\|t^{b_1}\bar{v}(t)\right\|_{L^{\frac{2N}{N-2}}(U)}^{\frac{2N}{N-2}} + t^{\frac{2N}{N-2}} \frac{2(N+2)}{N} \int_U \left|\nabla\left|t^{b_1}\bar{v}(t)\right|^{\frac{N}{N-2}}\right|^2 dx$$

$$= t^{\frac{2N}{N-2}} \frac{d}{dt} \left\|t^{b_1}\bar{v}(t)\right\|_{L^{\frac{2N}{N-2}}(U)}^{\frac{2N}{N-2}} + \frac{2(N+2)}{N} \int_U \left|\nabla\left|t^{b_1+1}\bar{v}(t)\right|^{\frac{N}{N-2}}\right|^2 dx$$

$$\leq C(l,N,b)\left(t^{\frac{N}{N-2}} + t^{\frac{N}{N-2}}e^{\frac{t}{2}}r(\omega) + t^{\frac{N}{N-2}-1}\right)t^{\frac{N}{N-2}} \left\|t^{b_1}\bar{v}(t)\right\|_{L^{\frac{2N}{N-2}}(U)}^{\frac{2N}{N-2}}$$

$$\leq C(l,N,b)\left(t^{\frac{N}{N-2}} + t^{\frac{N}{N-2}}e^{\frac{t}{2}}r(\omega) + t^{\frac{N}{N-2}-1}\right)M_1'(\omega). \tag{3.17}$$

Integrating (3.17) over $[0,T]$ with respect to $t$, we see that

$$\frac{2(N+2)}{N} \int_0^T \int_U \left|\nabla\left|t^{b_1+1}\bar{v}(t)\right|^{\frac{N}{N-2}}\right|^2 dx\,dt$$

$$\leq \frac{2N}{N-2} \int_0^T t^{\frac{2N}{N-2}-1} \left\|t^{b_1}\bar{v}(t)\right\|_{L^{\frac{2N}{N-2}}(U)}^{\frac{2N}{N-2}} dt$$

$$+ C(l,N,b)M_1'(\omega) \int_0^T \left(t^{\frac{N}{N-2}} + t^{\frac{N}{N-2}}e^{\frac{t}{2}}r(\omega) + t^{\frac{N}{N-2}-1}\right) dt$$

$$\leq \frac{2N}{N-2} M_1'(\omega) \int_0^T t^{\frac{N}{N-2}-1} dt$$

$$+ C(l,N,b)M_1'(\omega) \int_0^T \left(t^{\frac{N}{N-2}} + t^{\frac{N}{N-2}}e^{\frac{t}{2}}r(\omega) + t^{\frac{N}{N-2}-1}\right) dt$$

$$\leq C\left(l,N,b,T,r(\omega)\right)M_1'(\omega). \tag{3.18}$$

At the same time, combining with $\|\bar{v}\|_{L^{\frac{2N}{N-2}}(U)} \leq c\|\nabla\bar{v}\|$ with (3.18), we conclude

$$\int_0^T \left(\int_U \left(\left|t^{b_2}\bar{v}(t)\right|^{\frac{N}{N-2}}\right)^{\frac{2N}{N-2}} dx\right)^{\frac{N-2}{N}} dt \leq c \int_0^T \int_U \left|\nabla\left|t^{b_1+1}\bar{v}(t)\right|^{\frac{N}{N-2}}\right|^2 dx\,dt$$

$$\leq C\left(l,N,b,T,c,r(\omega)\right)M_1'(\omega). \tag{3.19}$$

Hence, for a.e. $t \in (0, T)$, we get

$$
\int_0^T \left( \int_U \left( \left| t^{b_2} \bar{u}(t) \right|^{\frac{N}{N-2}} \right)^{\frac{2N}{N-2}} dx \right)^{\frac{N-2}{N}} dt
$$

$$
= \int_0^T \left( \alpha(\theta_t \omega) \right)^{-\frac{2N}{N-2}} \left( \int_U \left( \left| t^{b_2} \bar{v}(t) \right|^{\frac{N}{N-2}} \right)^{\frac{2N}{N-2}} dx \right)^{\frac{N-2}{N}} dt
$$

$$
\leq \left( e^{\lambda_1 T} r^2(\omega) \right)^{\frac{N}{N-2}} C\left( l, N, b, T, c, r(\omega) \right) M_1'(\omega)
$$

$$
\leq C\left( l, N, b, T, c, \lambda_1, r(\omega) \right) M_1'(\omega) = M_1'''(\omega) \tag{3.20}
$$

from (2.8), (3.19) and $\bar{v}(t) = \alpha(\theta_t \omega)\bar{u}(t)$.

Set $M_1(\omega) = \max\{M_1''(\omega), M_1'''(\omega)\}$, we show that $(A_1)$ and $(B_1)$ hold from (3.16) and (3.20).

• Assume that $(A_k)$ and $(B_k)$ hold for $k \geq 2$. Next, we will prove that $(A_{k+1})$ and $(B_{k+1})$ hold.

Taking the inner product of (3.3) with $|\bar{v}|^{2(\frac{N}{N-2})^{k+1}-2} \cdot \bar{v}$, we find that

$$
\frac{1}{2} \left( \frac{N-2}{N} \right)^{k+1} \frac{d}{dt} \left\| \bar{v}(t) \right\|_{L^{2(\frac{N}{N-2})^{k+1}}(U)}^{2(\frac{N}{N-2})^{k+1}} + \frac{2(\frac{N}{N-2})^{k+1}-1}{(\frac{N}{N-2})^{2(k+1)}} \int_U \left| \nabla \left| \bar{v}(t) \right|^{(\frac{N}{N-2})^{k+1}} \right|^2 dx
$$

$$
\leq \left( l + b|z| \right) \left\| \bar{v}(t) \right\|_{L^{2(\frac{N}{N-2})^{k+1}}(U)}^{2(\frac{N}{N-2})^{k+1}},
$$

that is, for a.e. $t \in (0, T)$

$$
\frac{d}{dt} \left\| \bar{v} \right\|_{L^{2(\frac{N}{N-2})^{k+1}}(U)}^{2(\frac{N}{N-2})^{k+1}} + 2 \frac{2(\frac{N}{N-2})^{k+1}-1}{(\frac{N}{N-2})^{k+1}} \int_U \left| \nabla \left| \bar{v}(t) \right|^{(\frac{N}{N-2})^{k+1}} \right|^2 dx
$$

$$
\leq \frac{2(l + b|z|)}{(\frac{N-2}{N})^{k+1}} \left\| \bar{v}(t) \right\|_{L^{2(\frac{N}{N-2})^{k+1}}(U)}^{2(\frac{N}{N-2})^{k+1}}. \tag{3.21}
$$

Multiplying both sides of (3.21) with $t^{2(\frac{N}{N-2})^{k+1} \cdot b_{k+1}}$, it follows that

$$
\frac{d}{dt} \left( t^{2(\frac{N}{N-2})^{k+1} \cdot b_{k+1}} \left\| \bar{v}(t) \right\|_{L^{2(\frac{N}{N-2})^{k+1}}(U)}^{2(\frac{N}{N-2})^{k+1}} \right) + 2 \frac{2(\frac{N}{N-2})^{k+1}-1}{(\frac{N}{N-2})^{k+1}} \int_U \left| \nabla \left| t^{b_{k+1}} \bar{v}(t) \right|^{(\frac{N}{N-2})^{k+1}} \right|^2 dx
$$

$$
\leq \frac{2(l + b|z|)}{(\frac{N-2}{N})^{k+1}} \left\| t^{b_{k+1}} \bar{v}(t) \right\|_{L^{2(\frac{N}{N-2})^{k+1}}(U)}^{2(\frac{N}{N-2})^{k+1}}
$$

$$
+ 2 \left( \frac{N}{N-2} \right)^{k+1} b_{k+1} t^{2(\frac{N}{N-2})^{k+1} \cdot b_{k+1} - 1} \left\| \bar{v}(t) \right\|_{L^{2(\frac{N}{N-2})^{k+1}}(U)}^{2(\frac{N}{N-2})^{k+1}},
$$

that is,

$$
\frac{d}{dt} \left\| t^{b_{k+1}} \bar{v}(t) \right\|_{L^{2(\frac{N}{N-2})^{k+1}}(U)}^{2(\frac{N}{N-2})^{k+1}} + 2 \frac{2(\frac{N}{N-2})^{k+1}-1}{(\frac{N}{N-2})^{k+1}} \int_U \left| \nabla \left| t^{b_{k+1}} \bar{v}(t) \right|^{(\frac{N}{N-2})^{k+1}} \right|^2 dx
$$

$$
\leq C(l, N, k, b) \left( 1 + \left| z(\theta_t \omega) \right| + t^{-1} \right) \left\| t^{b_{k+1}} \bar{v}(t) \right\|_{L^{2(\frac{N}{N-2})^{k+1}}(U)}^{2(\frac{N}{N-2})^{k+1}}, \tag{3.22}
$$

where $C(l, N, k, b) = \max\{ \frac{2l}{(\frac{N-2}{N})^{k+1}}, \frac{2b}{(\frac{N-2}{N})^{k+1}}, 2(\frac{N}{N-2})^{k+1} b_{k+1} \}$.

Firstly, from (3.22) we deduce that, for all $t \in [0, T]$,

$$t\frac{d}{dt}\left\|t^{b_{k+1}}\bar{v}(t)\right\|_{L^{2(\frac{N}{N-2})^{k+1}}(U)}^{2(\frac{N}{N-2})^k}$$

$$= t\frac{d}{dt}\left(\int_U \left|t^{b_{k+1}}\bar{v}(t)\right|^{2(\frac{N}{N-2})^{k+1}} dx\right)^{\frac{N-2}{N}}$$

$$= t\frac{N-2}{N}\left(\int_U \left|t^{b_{k+1}}\bar{v}(t)\right|^{2(\frac{N}{N-2})^{k+1}}\right)^{\frac{N-2}{N}-1}\frac{d}{dt}\left\|t^{b_{k+1}}\bar{v}(t)\right\|_{L^{2(\frac{N}{N-2})^{k+1}}(U)}^{2(\frac{N}{N-2})^{k+1}}$$

$$\leq C(l,N,k,b)\left(T + Te^{\frac{T}{2}} + 1\right)\left\|t^{b_{k+1}}\bar{v}(t)\right\|_{L^{2(\frac{N}{N-2})^{k+1}}(U)}^{2(\frac{N}{N-2})^k}. \tag{3.23}$$

Integrating (3.23) over $[0, t]$, for all $t \in [0, T]$, we have

$$t\left\|t^{b_{k+1}}\bar{v}(t)\right\|_{L^{2(\frac{N}{N-2})^{k+1}}(U)}^{2(\frac{N}{N-2})^k} \leq \left(C(l,N,k,b)\left(T + Te^{\frac{T}{2}} + 1\right) + 1\right)$$

$$\cdot \int_0^T \left\|t^{b_{k+1}}\bar{v}(t)\right\|_{L^{2(\frac{N}{N-2})^{k+1}}(U)}^{2(\frac{N}{N-2})^k} dt. \tag{3.24}$$

Using (2.8) and $(B_k)$, we also find that

$$\int_0^T \left\|t^{b_{k+1}}\bar{v}(t)\right\|_{L^{2(\frac{N}{N-2})^{k+1}}(U)}^{2(\frac{N}{N-2})^k} dt$$

$$= \int_0^T \left(\alpha(\theta_s\omega)\right)^{-2(\frac{N}{N-2})^k}\left\|t^{b_{k+1}}\bar{u}(t)\right\|_{L^{2(\frac{N}{N-2})^{k+1}}(U)}^{2(\frac{N}{N-2})^k} dt$$

$$\leq \left(e^{\lambda_1 T}r^2(\omega)\right)^{(\frac{N}{N-2})^k}\int_0^T \left\|t^{b_{k+1}}\bar{u}(t)\right\|_{L^{2(\frac{N}{N-2})^{k+1}}(U)}^{2(\frac{N}{N-2})^k} dt$$

$$\leq \left(e^{\lambda_1 T}r^2(\omega)\right)^{(\frac{N}{N-2})^k}M_k(\omega). \tag{3.25}$$

So, from (3.24)–(3.25) we obtain

$$t\left\|t^{b_{k+1}}\bar{v}(t)\right\|_{L^{2(\frac{N}{N-2})^{k+1}}(U)}^{2(\frac{N}{N-2})^k} \leq \left(C(l,N,k,b)\left(T + Te^{\frac{T}{2}} + 1\right) + 1\right)$$

$$\cdot \left(e^{\lambda_1 T}r^2(\omega)\right)^{(\frac{N}{N-2})^k}M_k(\omega). \tag{3.26}$$

In addition, we can get

$$\left(e^{-\lambda_1 T}r^{-2}(\omega)\right)^{(\frac{N}{N-2})^k}t\left\|t^{b_{k+1}}\bar{u}(t)\right\|_{L^{2(\frac{N}{N-2})^{k+1}}(U)}^{2(\frac{N}{N-2})^k}$$

$$\leq t\left\|t^{b_{k+1}}\bar{v}(t)\right\|_{L^{2(\frac{N}{N-2})^{k+1}}(U)}^{2(\frac{N}{N-2})^k} \tag{3.27}$$

by using (2.8) and $\bar{v}(t) = \alpha(\theta_t\omega)\bar{u}(t)$. Thus, from (3.26) and (3.27), we arrive at

$$t\left\|t^{b_{k+1}}\bar{u}(t)\right\|_{L^{2(\frac{N}{N-2})^{k+1}}(U)}^{2(\frac{N}{N-2})^k} \leq \left(C(l,N,k,b)\left(T + Te^{\frac{T}{2}} + 1\right) + 1\right)$$

$$\cdot \left(e^{\lambda_1 T}r^2(\omega)\right)^{2(\frac{N}{N-2})^k}M_k(\omega),$$

which implies that

$$t^{\frac{N}{N-2}} \left\| t^{b_{k+1}} \bar{u} \right\|_{L^{2(\frac{N}{N-2})^{k+1}}(U)}^{2(\frac{N}{N-2})^{k+1}}$$

$$\leq \left[ \left( C(l,N,k,b) \left( T + Te^{\frac{T}{2}} + 1 \right) + 1 \right) \left( e^{\lambda_1 T} r^2(\omega) \right)^{2(\frac{N}{N-2})^k} M_k \right]^{\frac{N}{N-2}}$$

$$= M_{k+1}''(\omega). \tag{3.28}$$

Secondly, after obtaining (3.28), we will prove $(B_{k+1})$ by (3.22). Multiplying both sides of (3.22) by $t^{1+\frac{N}{N-2}}$, we find that

$$t^{1+\frac{N}{N-2}} \frac{d}{dt} \left\| t^{b_{k+1}} \bar{v}(t) \right\|_{L^{2(\frac{N}{N-2})^{k+1}}(U)}^{2(\frac{N}{N-2})^{k+1}}$$

$$+ 2 \frac{2(\frac{N}{N-2})^{k+1} - 1}{(\frac{N}{N-2})^{k+1}} \int_U \left| \nabla \left| t^{b_{k+1} + \frac{1+\frac{N}{N-2}}{2(\frac{N}{N-2})^{k+1}}} \bar{v}(t) \right|^{(\frac{N}{N-2})^{k+1}} \right|^2 dx$$

$$\leq C(l,N,k,b) \left( t + t \left| z(\theta_t \omega) \right| + 1 \right) t^{\frac{N}{N-2}} \left\| t^{b_{k+1}} \bar{v}(t) \right\|_{L^{2(\frac{N}{N-2})^{k+1}}(U)}^{2(\frac{N}{N-2})^{k+1}}. \tag{3.29}$$

Then, applying (2.8), (3.28) and the definition of $b_{k+2}$, we obtain

$$t^{1+\frac{N}{N-2}} \frac{d}{dt} \left\| t^{b_{k+1}} \bar{v}(t) \right\|_{L^{2(\frac{N}{N-2})^{k+1}}(U)}^{2(\frac{N}{N-2})^{k+1}}$$

$$+ 2 \frac{2(\frac{N}{N-2})^{k+1} - 1}{(\frac{N}{N-2})^{k+1}} \int_U \left| \nabla \left| t^{b_{k+2}} \bar{v}(t) \right|^{(\frac{N}{N-2})^{k+1}} \right|^2 dx$$

$$\leq C(l,N,k,b) \left( t + te^{\frac{t}{2}} r(\omega) + 1 \right) \left( e^{\lambda_1 T} r^2(\omega) \right)^{(\frac{N}{N-2})^{k+1}} M_{k+1}''(\omega). \tag{3.30}$$

Integrating (3.30) over $[0, T]$ and applying (3.28), we obtain that

$$2 \frac{2(\frac{N}{N-2})^{k+1} - 1}{(\frac{N}{N-2})^{k+1}} \int_0^T \int_U \left| \nabla \left| t^{b_{k+2}} \bar{v}(t) \right|^{(\frac{N}{N-2})^{k+1}} \right|^2 dx\, dt$$

$$\leq C(l,N,k,b) \left( e^{\lambda_1 T} r^2(\omega) \right)^{(\frac{N}{N-2})^{k+1}} M_{k+1}''(\omega) \int_0^T \left( t + te^{\frac{t}{2}} r(\omega) + 1 \right) dt$$

$$+ \frac{2N-2}{N-2} \int_0^T t^{\frac{N}{N-2}} \left\| t^{b_{k+1}} \bar{v}(t) \right\|_{L^{2(\frac{N}{N-2})^{k+1}}(U)}^{2(\frac{N}{N-2})^{k+1}} dt$$

$$\leq C(l,N,k,b) \left( e^{\lambda_1 T} r^2(\omega) \right)^{(\frac{N}{N-2})^{k+1}} \left( T^2 + T + \left( 2Te^T + 4 \right) r(\omega) \right) M_{k+1}''(\omega)$$

$$+ \frac{2N-2}{N-2} T \left( e^{\lambda_1 T} r^2(\omega) \right)^{(\frac{N}{N-2})^{k+1}} M_{k+1}''(\omega)$$

$$\leq C \left( l,N,k,b,\lambda_1,T,r(\omega) \right) M_{k+1}''(\omega),$$

which, combining with $\left( \int_U |\bar{v}|^{\frac{2N}{N-2}} dx \right)^{\frac{N-2}{N}} \leq c \int_U |\nabla \bar{v}|^2 dx$, leads to

$$\int_0^T \left( \int_U \left| t^{b_{k+2}} \bar{v}(t) \right|^{2(\frac{N}{N-2})^{k+2}} dx \right)^{\frac{N}{N-2}} dt \leq c \int_0^T \int_U \left| \nabla \left| t^{b_{k+2}} \bar{v}(t) \right|^{(\frac{N}{N-2})^{k+1}} \right|^2 dx\, dt$$

$$\leq C \left( l,N,k,b,\lambda_1,T,c,r(\omega) \right) M_{k+1}''(\omega). \tag{3.31}$$

Similar to (3.20), from (3.31) we also obtain

$$
\begin{aligned}
\int_0^T &\left( \int_U \left| t^{b_{k+2}} \bar{u}(t) \right|^{2(\frac{N}{N-2})^{k+2}} dx \right)^{\frac{N}{N-2}} dt \\
&= \int_0^T \alpha^{-2(\frac{N}{N-2})^{k+1}} (\theta_t \omega) \left( \int_U \left| t^{b_{k+2}} \bar{v}(t) \right|^{2(\frac{N}{N-2})^{k+2}} dx \right)^{\frac{N}{N-2}} dt \\
&\leq \left( e^{\lambda_1 T} r^2(\omega) \right)^{(\frac{N}{N-2})^{k+1}} C\left(l, N, k, b, \lambda_1, T, c, r(\omega)\right) M''_{k+1}(\omega).
\end{aligned}
\tag{3.32}
$$

Set

$$
M_{k+1}(\omega) = \max\left\{ M''_{k+1}, \left( e^{\lambda_1 T} r^2(\omega) \right)^{(\frac{N}{N-2})^{k+1}} C\left(l, N, k, b, \lambda_1, T, c, r(\omega)\right) M''_{k+1} \right\}.
$$

Therefore, (3.28) and (3.32) show that $(A_{k+1})$ and $(B_{k+1})$ hold, respectively. We finished the proof. $\qquad\square$

**Theorem 3.2** (($L^2, L^{2+\delta}$) attraction) *Assume that* (1.2)–(1.4) *hold and $b > 0$. $A \in \mathcal{D}$ is the* ($L^2, L^2$) *$\mathcal{D}$-pullback random attractor obtained in Lemma* 2.11. *Then the random set $A \in \mathcal{D}$ is also $\mathcal{D}$-pullback attracting in the topology of $L^{2+\delta}$ for any $\delta \in [0, \infty)$, that is, for every random set $D \in \mathcal{D}$,*

$$
\lim_{t \to +\infty} \mathrm{dist}_{L^{2+\delta}} \left( \phi(t, \theta_{-t}\omega, D(\theta_{-t}\omega)), A(\omega) \right) = 0, \quad \mathbb{P}\text{-almost surely,}
\tag{3.33}
$$

*where $\mathrm{dist}_{L^{\delta+2}}$ means that*

$$
\mathrm{dist}_{L^{2+\delta}}(A, B) = \sup_{a \in A} \inf_{b \in B} \|a - b\|_{L^{2+\delta}}
$$

*for any two subset $A, B$ in $L^2(U)$.*

*Proof* The proof is similar to the proof of Theorem 4.5 (from [21]), so we omit it. $\qquad\square$

### 3.2 $L^p$-Pullback attracting set

In this subsection, we will make uniform estimates for the solutions of Eq. (1.1) so that we prove the existence of a bounded random absorbing set in $L^p(U)$ ($p \geq 2$).

**Lemma 3.3** (Random absorbing set in $L^p$) *Assume that* (1.2)–(1.4) *hold and $b > 0$. Then there exists a random absorbing set $B \in \mathcal{D}$ such that for any random set $D \in \mathcal{D}$ and $\mathbb{P}$-a.e. $\omega \in \Omega$, we have $T_D^1(\omega) > T_D(\omega)$ such that*

$$
\Phi\left( t, \theta_{-t}\omega, D(\theta_{-t}\omega) \right) \subset B(\omega), \quad \text{for all } t \geq T_D^1(\omega),
\tag{3.34}
$$

*and*

$$
B(\omega) \text{ is bounded in } L^p(U).
$$

*Proof* Taking the inner product of (2.9) with $|v|^{p-2}v$ in $L^2(U)$, we find that

$$\frac{1}{p}\frac{d}{dt}\|v\|_{L^p(U)}^p + (p-1)\int_U |v|^{p-2}|\nabla v|^2\,dx$$
$$= -\big(\alpha(\theta_t\omega)f\big(\alpha^{-1}(\theta_t\omega)v\big), |v|^{p-2}v\big) + \big(bvz, |v|^{p-2}v\big)$$
$$+ \big(\alpha(\theta_t\omega)g(x), |v|^{p-2}v\big). \tag{3.35}$$

Using (1.3), Hölder's inequality and Young's inequality, we have

$$\big(\alpha(\theta_t\omega)f\big(\alpha^{-1}(\theta_t\omega)v\big), |v|^{p-2}v\big) \geq c_4\alpha^{2-p}(\theta_t\omega)\|v\|_{L^{2p-2}(U)}^{2p-2} - c_5|U|\alpha^p(\theta_t\omega), \tag{3.36}$$

where $c_4$, $c_5$ are positive constants, and

$$\big(\alpha(\theta_t\omega)g(x), |v|^{p-2}v\big) \leq \frac{1}{2c_4}\alpha^p(\theta_t\omega)\|g\|^2 + \frac{c_4}{2}\alpha^{2-p}(\theta_t\omega)\|v\|_{L^{2p-2}(U)}^{2p-2} \tag{3.37}$$

and

$$\big(bvz, |v|^{p-2}v\big) \leq b|z|\|v\|_{L^p(U)}^p. \tag{3.38}$$

From (3.35)–(3.38), we get

$$\frac{d}{dt}\|v\|_{L^p(U)}^p + \frac{c_4}{2}\alpha^{2-p}(\theta_t\omega)\|v\|_{L^{2p-2}(U)}^{2p-2} \leq c_5 p|U|\alpha^p(\theta_t\omega) + \frac{p}{2c_4}\alpha^p(\theta_t\omega)\|g\|^2$$
$$+ bp|z|\|v\|_{L^p(U)}^p. \tag{3.39}$$

Now, choosing $t \geq T_D(\omega)$ ($T_D(\omega)$ to be the positive number in Lemma 2.9 and integrating (3.39) over $(s, t+1)$ with respect to $t$, we obtain

$$\big\|v\big(t+1, \omega, v_0(\omega)\big)\big\|_{L^p(U)}^p \leq \big\|v\big(s, \omega, v_0(\omega)\big)\big\|_{L^p(U)}^p$$
$$+ \bigg(c_5 p|U| + \frac{p\|g\|^2}{2c_4}\bigg)\int_s^{t+1}\alpha^p(\theta_\tau\omega)\,d\tau$$
$$+ bp\int_s^{t+1}\big|z(\theta_\tau\omega)\big|\big\|v\big(\tau, \omega, v_0(\omega)\big)\big\|_{L^p(U)}^p\,d\tau. \tag{3.40}$$

Next, integrating (3.40) over $(t, t+1)$ with respect to $s$, we have

$$\big\|v\big(t+1, \omega, v_0(\omega)\big)\big\|_{L^p(U)}^p \leq \int_t^{t+1}\big\|v\big(s, \omega, v_0(\omega)\big)\big\|_{L^p(U)}^p\,ds$$
$$+ \bigg(c_5 p|U| + \frac{p\|g\|^2}{2c_4}\bigg)\int_t^{t+1}\alpha^p(\theta_\tau\omega)\,d\tau$$
$$+ bp\int_t^{t+1}\big|z(\theta_\tau\omega)\big|\big\|v\big(\tau, \omega, v_0(\omega)\big)\big\|_{L^p(U)}^p\,d\tau.$$

Replacing $\omega$ by $\theta_{-t-1}\omega$ and using (2.8) and (2.11), we conclude that

$$\|v(t+1,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega)\|^p_{L^p(U)}$$

$$\leq \int_t^{t+1} \left\| v\big(s,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega)\big) \right\|^p_{L^p(U)} ds$$

$$+ \left( c_5 p|U| + \frac{p\|g\|^2}{2c_4} \right) \int_{-1}^0 \alpha^p(\theta_\tau\omega)\,d\tau$$

$$+ bp \int_{-1}^0 \left| z(\theta_\tau\omega) \right| \left\| v\big(\tau+t+1,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega)\big) \right\|^p_{L^p(U)} d\tau$$

$$\leq \rho_2(\omega) + \left( c_5 p|U| + \frac{p\|g\|^2}{2c_4} \right) r^p(\omega) \int_{-1}^0 e^{-\frac{p\lambda_1}{2}\tau}\,d\tau$$

$$+ bpe^{\frac{1}{2}}r(\omega) \int_t^{t+1} \left\| v\big(\tau,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega)\big) \right\|^p_{L^p(U)} d\tau$$

$$\leq \rho_2(\omega) + \left( c_5 p|U| + \frac{p\|g\|^2}{2c_4} \right) \frac{2}{p\lambda_1} e^{-\frac{p\lambda_1}{2}} r^p(\omega) + bpe^{\frac{1}{2}}r(\omega)\rho_2(\omega),$$

that is,

$$\|v(t+1,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega)\|^p_{L^p(U)} \leq \rho_4(\omega). \tag{3.41}$$

Then, from (3.41), for $t \geq T_D^1(\omega) \geq T_D(\omega)$,

$$\|u(t+1,\theta_{-t-1}\omega,u_0(\theta_{-t-1}\omega)\|^p_{L^p(U)}$$

$$= \|\alpha^{-1}(\theta_{-1}\omega)v(t+1,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega)\|^p_{L^p(U)}$$

$$\leq e^{\frac{p\lambda_1}{2}} r^p(\omega)\rho_4(\omega),$$

where $\rho_5(\omega) = e^{\frac{p\lambda_1}{2}} r^p(\omega)\rho_4(\omega)$, that is, for $\omega \in \Omega$,

$$B(\omega) = \left\{ u \in L^p(U) : \|u\|^p_{L^p(U)} \leq \rho_5(\omega) \right\}.$$

Therefore, $B(\omega)$ is a random absorbing set for $\Phi$ in $L^p(U)$. $\qquad\square$

**Theorem 3.4** *Assume that* (1.2)–(1.4) *hold. The* $(L^2, L^2)$ $\mathcal{D}$-*pullback random attractor* $A \in \mathcal{D}$ *is also a* $(L^2, L^p)$ $\mathcal{D}$-*pullback random attractor, that is,* $A(\omega)$ *is compact in* $L^p(U)$ *for* $\mathbb{P}$-*a.e.* $\omega \in \Omega$, $A$ *is* $\Phi$-*invariant and* $\mathcal{D}$-*pullback attracting every random set* $D \in \mathcal{D}$ *in the topology of* $L^p(U)$.

*Proof* Using the interpolation inequality, Theorem 3.1, (2.4), (2.8) and (3.7), we have the following inequality:

$$\|\Phi(t,\omega,u_n(\omega)) - \Phi(t,\omega,u_0(\omega))\|^2_{L^p(U)}$$

$$\leq \|\Phi(t,\omega,u_n(\omega)) - \Phi(t,\omega,u_0(\omega))\|^{2-2\theta}_{L^{2(\frac{N}{N-2})^{k_0}}(U)}$$

$$\cdot \|\Phi(t,\omega,u_n(\omega)) - \Phi(t,\omega,u_0(\omega))\|^{2\theta}$$

$$\leq \frac{M_k^{\frac{2-2\theta}{2(\frac{N}{N-2})k_0}}(\omega)}{t^{(2-2\theta)+\frac{2-2\theta}{2(\frac{N}{N-2})k_0-1}}} \cdot e^{\lambda_1\theta t}r^{2\theta}(\omega)r_4^\theta(\omega)e^{2l\theta t}\left\|\bar{v}(0)\right\|^{2\theta}$$

$$\leq \frac{\tilde{M}_k(\omega)}{t^{r_0}} \cdot e^{(2l+\lambda_1)t}r^{4\theta}(\omega)r_4^\theta(\omega)\left\|u_n(\omega) - u_0(\omega)\right\|^{2\theta},$$

where $r_0$ is given by Theorem 3.5.

From the above result and Lemma 3.3, it is obvious that the $(L^2(U), L^2(U))$ $\mathcal{D}$-pullback random attractor $A \in \mathcal{D}$ is compact in $L^p(U)$ for $\mathbb{P}$-a.e. $\omega \in \Omega$. □

### 3.3 Continuity of solutions in $H_0^1(U)$

**Theorem 3.5** *Assume that* (1.2)–(1.4) *hold. If* $\{u_n(\omega)\}_n^\infty$ *are bounded in* $L^p(U)$ *and* $u_n(\omega) \to u_0(\omega)$ *in* $L^2(U)$ *as* $n \to \infty$, *then,* $\mathbb{P}$-*a.e.* $\omega \in \Omega$, *for any* $t > 0$,

$$\Phi(t, \omega, u_n(\omega)) \to \Phi(t, \omega, u_0(\omega)), \quad \text{in } H_0^1(U) \text{ as } n \to \infty. \tag{3.42}$$

*That is, the following estimate holds*:

$$\left\|\Phi(t, \omega, u_n(\omega)) - \Phi(t, \omega, u_0(\omega))\right\|_{H_0^1(U)}^2$$

$$\leq e^{\lambda_1 t}r^2(\omega)C\left(c_3, p, \theta, t, r_0, \lambda_1, \rho_1(\omega), \tilde{M}(\omega), M_{k_0}, r_3(\omega), \left\|\bar{v}_n(0)\right\|^2\right), \tag{3.43}$$

*where* $\theta \in (0, 1)$ *is the exponent of the interpolation inequality* $\|\cdot\|_{L^{4p-6}} \leq \|\cdot\|_{L_{1-\theta}^{2(\frac{N}{N-2})k_0}}^{\infty} \|\cdot\|$ *with* $k_0 \in \mathbb{N}$ *satisfying* $2(\frac{N}{N-2})^{k_0} > 4p - 6$, *and* $r_0 = (\frac{N}{N-2})\frac{2-2\theta}{2(\frac{N}{N-2})k_0} + (2 - 2\theta)b_{k_0}$.

*Proof* If we set $\bar{u}_n(t) = \Phi(t, \omega, u_n(\omega)) - \Phi(t, \omega, u_0(\omega))$ $(n = 1, 2, \ldots)$, then $\bar{u}_n(t)$ satisfies the following equation:

$$\begin{cases} \frac{\partial \bar{u}_n}{\partial t} - \Delta\bar{u}_n + f(u_n(t)) - f(u(t)) = b\bar{u}_n(t) \circ \frac{dW}{dt}, & (x, t) \in U \times (0, t), \\ \bar{u}_n(t)|_{\partial U} = 0, \quad t \geq 0, \\ \bar{u}_n(0) = u_n(\omega) - u_0(\omega), \end{cases} \tag{3.44}$$

where $u_n(t) = \Phi(t, \omega, u_n(\omega))$ $(n = 1, 2)$ and $u(t) = \Phi(t, \omega, u_0(\omega))$.

Thanks to $v_n(t) = \alpha(\theta_t\omega)u_n(t)$ and $\alpha(\theta_t\omega) = e^{-bz(\theta_t\omega)}$, we may convert Eq. (3.44) into the following equation:

$$\begin{cases} \frac{\partial \bar{v}_n}{\partial t} - \Delta\bar{v}_n + \alpha(\theta_t\omega)(f(\alpha^{-1}(\theta_t\omega)v_n(t)) - f(\alpha^{-1}(\theta_t\omega)v(t))) = bz\bar{v}_n, \\ \quad (x, t) \in U \times (0, t), \\ \bar{v}_n(t)|_{\partial U} = 0, \quad t \geq 0, \\ \bar{v}_n(0) = v_n(\omega) - v_0(\omega), \end{cases} \tag{3.45}$$

where $v_n(t) = \Phi(t, \omega, v_n(\omega))$ $(n = 1, 2)$ and $v(t) = \Phi(t, \omega, v_0(\omega))$.

Firstly, it follows from (3.39) and (3.41) that

$$\frac{c_4}{2}\int_0^t \alpha^{2-p}(\theta_s\omega)\left\|v_n(s)\right\|_{L^{2p-2}(U)}^{2p-2} ds$$

$$\leq \left\|v_n(0)\right\|_{L^p(U)}^p + \left(\frac{p}{2c_4}\|g\|^2 + c_5p|U|\right)\int_0^t \alpha^p(\theta_s\omega) ds$$

$$+ bp \int_0^t |z(\theta_s\omega)| \|v_n(s)\|_{L^p(U)}^p \, ds$$

$$\leq \|v_n(0)\|_{L^p(U)}^p + \left(\frac{p}{2c_4}\|g\|^2 + c_5 p|U|\right) \frac{2e^{\lambda_1 t}}{\lambda_1} r^p(\omega)$$

$$+ bpe^{\frac{t}{2}} r(\omega)\rho_4(\omega). \tag{3.46}$$

Combining (3.46) with the inequality

$$\frac{c_4}{2}\left(e^{-\frac{\lambda_1}{2}t}r^{-1}(\omega)\right)^{p-2} \int_0^t \|v_n(s)\|_{L^{2p-2}(U)}^{2p-2} \, ds$$

$$\leq \frac{c_4}{2} \int_0^t \alpha^{2-p}(\theta_s\omega)\|v_n(s)\|_{L^{2p-2}(U)}^{2p-2} \, ds,$$

we obtain

$$\int_0^t \|v_n(s)\|_{L^{2p-2}(U)}^{2p-2} \, ds$$

$$\leq C\left(p, c_4, c_5, t, \lambda_1, |U|, \|g\|^2\right)\left(\|v_n(0)\|_{L^p(U)}^p + r^p(\omega)\right). \tag{3.47}$$

Therefore, we also obtain a similar estimate about $v(t)$, that is, we have the following result:

$$\int_0^t \|v_n(s)\|_{L^{2p-2}(U)}^{2p-2} \, ds + \int_0^t \|v(s)\|_{L^{2p-2}(U)}^{2p-2} \, ds \leq \tilde{M}(\omega), \tag{3.48}$$

where $\tilde{M}(\omega)$ depends on $p$, $c_4$, $c_5$, $t$, $\lambda_1$, $|U|$, $r(\omega)$, $\|g\|^2$, $\|v_n(0)\|_{L^p(U)}^p$.

In addition, using (3.7), we get

$$\int_0^t \|\bar{v}_n(s)\|^2 \, ds \leq \frac{e^{2lt} - 1}{2l} r_4(\omega)\|\bar{v}_n(0)\|^2, \quad \forall t \geq 0. \tag{3.49}$$

Using (3.8), for all $t \geq 0$,

$$\int_0^t \|\nabla \bar{v}(s)\|^2 \, ds \leq \frac{1}{2}\|\bar{v}(0)\|^2 + l \int_0^t \|\bar{v}(s)\|^2 \, ds + b \int_0^t |z(\theta_t\omega)| \|\bar{v}(s)\|^2 \, ds. \tag{3.50}$$

Next, taking the inner product of (3.45) with $-\Delta \bar{v}_n$, we find that

$$\frac{1}{2}\frac{d}{dt}\|\nabla \bar{v}_n\|^2 + \|\Delta \bar{v}_n\|^2 = -\left(\alpha(\theta_t\omega)\left(f\left(\alpha^{-1}(\theta_t\omega)v_n(t)\right) + (bz\bar{v}_n, -\Delta\bar{v}_n)\right.\right.$$

$$\left.\left. - f\left(\alpha^{-1}(\theta_t\omega)v(t)\right)\right), \Delta\bar{v}_n\right). \tag{3.51}$$

Using the conditions (1.2)–(1.4), the Hölder inequality and the Young inequality, we have

$$\left|\left(\alpha(\theta_t\omega)\left(f\left(\alpha^{-1}(\theta_t\omega)v_n(t)\right) - f\left(\alpha^{-1}(\theta_t\omega)v(t)\right)\right), \Delta\bar{v}_n\right)\right|$$

$$\leq c_3 \int_U |\bar{v}_n||\Delta\bar{v}_n|\left(1 + \left|\alpha^{-1}v_n\right|^{p-2} + \left|\alpha^{-1}v\right|^{p-2}\right) dx$$

$$\leq c_3\left(e^{\frac{\lambda_1}{2}t}r(\omega)\right)^{p-2} \int_U |\bar{v}_n||\Delta\bar{v}_n|\left(1 + |v_n|^{p-2} + |v|^{p-2}\right) dx$$

$$\leq c_3 \big(e^{\frac{\lambda_1}{2}t}r(\omega)\big)^{p-2}\|\bar{v}_n\|_{L^{4p-6}}\|\Delta\bar{v}_n\|C(p)\big(1+\|v_n\|_{L^{2p-3}}^{p-2}+\|v\|_{L^{2p-3}}^{p-2}\big)$$

$$\leq c_3^2\big(e^{\frac{\lambda_1}{2}t}r(\omega)\big)^{2p-4}\frac{C^2(p)}{2}\big(\|v_n\|_{L^{2p-3}}^{2p-4}+\|v\|_{L^{2p-3}}^{2p-4}\big)\|\bar{v}_n\|_{L^{4p-6}}^2$$

$$+\frac{\|\Delta\bar{v}_n\|^2}{2} \tag{3.52}$$

and

$$\big|(bz\bar{v}_n,-\Delta\bar{v}_n)\big|\leq\frac{b^2|z|^2}{2}\|\bar{v}_n\|^2+\frac{\|\Delta\bar{v}_n\|^2}{2}. \tag{3.53}$$

From (3.51)–(3.53), we obtain

$$\frac{d}{dt}\|\nabla\bar{v}_n\|^2\leq c_3^2\big(e^{\frac{\lambda_1}{2}t}r(\omega)\big)^{2p-4}C^2(p)\|v_n\|_{L^{2p-3}(U)}^{2p-4}\|\bar{v}_n\|_{L^{4p-6}(U)}^2$$

$$+c_3^2\big(e^{\frac{\lambda_1}{2}t}r(\omega)\big)^{2p-4}C^2(p)\|v\|_{L^{2p-3}(U)}^{2p-4}\|\bar{v}_n\|_{L^{4p-6}(U)}^2$$

$$+b^2|z|^2\|\bar{v}_n\|^2. \tag{3.54}$$

Due to $2(\frac{N}{N-2})^k\to\infty$ as $k\to\infty$, we set $k_0=[\log_{\frac{N}{N-2}}(2p-3)]+1\in\mathbb{N}$ such that

$$2\left(\frac{N}{N-2}\right)^{k_0}>4p-6.$$

Exploiting the interpolation inequality, we have

$$\|\bar{v}_n\|_{L^{4p-6}}\leq\|\bar{v}_n\|_{L^{2(\frac{N}{N-2})^{k_0}}(U)}^{1-\theta}\|\bar{v}_n\|^{\theta},$$

where $\theta\in(0,1)$ depends on $p$, $k_0$.

Thus, we conclude that

$$\frac{d}{dt}\|\nabla\bar{v}_n\|^2\leq c_3^2\big(e^{\frac{\lambda_1}{2}t}r(\omega)\big)^{2p-4}C^2(p)\big(\|v_n\|_{L^{2p-3}(U)}^{2p-4}+\|v\|_{L^{2p-3}(U)}^{2p-4}\big)$$

$$\cdot\|\bar{v}_n\|_{L^{2(\frac{N}{N-2})^{k_0}}(U)}^{2-2\theta}\|\bar{v}_n\|^{2\theta}+b^2|z|^2\|\bar{v}_n\|^2,\quad\text{for a.e. }(0,t). \tag{3.55}$$

Set

$$r_0=\left(\frac{N}{N-2}\right)\frac{2-2\theta}{2(\frac{N}{N-2})^{k_0}}+(2-2\theta)b_{k_0}.$$

Multiplying both sides of (3.55) with $t^{r_0}$, we get

$$t^{r_0}\frac{d}{dt}\|\nabla\bar{v}_n\|^2\leq c_3^2\big(e^{\frac{\lambda_1}{2}t}r(\omega)\big)^{2p-4}C^2(p)\big(\|v_n\|_{L^{2p-3}(U)}^{2p-4}+\|v\|_{L^{2p-3}(U)}^{2p-4}\big)$$

$$\cdot\Big(t^{\frac{N}{N-2}}\big\|t^{b_{k_0}}\bar{v}_n\big\|_{L^{2(\frac{N}{N-2})^{k_0}}(U)}^{2(\frac{N}{N-2})^{k_0}}\Big)^{\frac{2-2\theta}{2(\frac{N}{N-2})^{k_0}}}\|\bar{v}_n\|^{2\theta}$$

$$+t^{r_0}b^2|z|^2\|\bar{v}_n\|^2, \tag{3.56}$$

where $b_{k_0}$ is given by (3.1).

Moreover, due to Theorem 3.1, we know that there exists a constant $M_{k_0}(\omega)$ such that

$$
\left( s^{\frac{N}{N-2}} \left\| s^{b_{k_0}} \bar{v}_n(s) \right\|_{L^{2(\frac{N}{N-2})^{k_0}}(U)}^{2(\frac{N}{N-2})^{k_0}} \right)^{\frac{2-2\theta}{2(\frac{N}{N-2})^{k_0}}}
$$

$$
\leq \left( e^{\frac{\lambda_1 s}{2}} r(\omega) M_{k_0}(\omega) \right)^{2-2\theta}, \quad n = 1, 2, \ldots, s \in [0, t]. \tag{3.57}
$$

So, for $n = 1, 2, \ldots$, from (3.56)–(3.57) we obtain the following estimate:

$$
s^{r_0} \frac{d}{ds} \left\| \nabla \bar{v}_n(s) \right\|^2 \leq c_3^2 \left( e^{\lambda_1 s} r^2(\omega) \right)^{p-\theta-1} C^2(p) M_{k_0}^{2-2\theta} \left\| v_n(s) \right\|_{L^{2p-3}(U)}^{2p-4} \left\| \bar{v}_n(s) \right\|^{2\theta}
$$

$$
+ c_3^2 \left( e^{\lambda_1 s} r^2(\omega) \right)^{p-\theta-1} C^2(p) M_{k_0}^{2-2\theta} \left\| v(s) \right\|_{L^{2p-3}(U)}^{2p-4} \left\| \bar{v}_n(s) \right\|^{2\theta}
$$

$$
+ s^{r_0} b^2 \left| z(\theta_s \omega) \right|^2 \left\| \bar{v}_n(s) \right\|^2. \tag{3.58}
$$

Next, multiplying both sides of (3.58) with $s$, for a.e. $s \in [0, t]$, we find that

$$
s^{1+r_0} \frac{d}{ds} \left\| \nabla \bar{v}_n(s) \right\|^2 \leq c_3^2 \left( e^{\lambda_1 t} r^2(\omega) \right)^{p-\theta-1} C^2(p) M_{k_0}^{2-2\theta} t \left\| v_n(s) \right\|_{L^{2p-3}(U)}^{2p-4} \left\| \bar{v}_n(s) \right\|^{2\theta}
$$

$$
+ c_3^2 \left( e^{\lambda_1 t} r^2(\omega) \right)^{p-\theta-1} C^2(p) M_{k_0}^{2-2\theta} t \left\| v(s) \right\|_{L^{2p-3}(U)}^{2p-4} \left\| \bar{v}_n(s) \right\|^{2\theta}
$$

$$
+ t^{1+r_0} b^2 \left| z(\theta_s \omega) \right|^2 \left\| \bar{v}_n(s) \right\|^2. \tag{3.59}
$$

Integrating (3.59) over $[0, t]$ with respect to $s$, for $n = 1, 2, \ldots$, we have

$$
t^{1+r_0} \left\| \nabla \bar{v}_n(t) \right\|^2
$$

$$
\leq c_3^2 \left( e^{\lambda_1 t} r^2(\omega) \right)^{p-\theta-1} C^2(p) M_{k_0}^{2-2\theta} t \int_0^t \left\| v_n(s) \right\|_{L^{2p-3}(U)}^{2p-4} \left\| \bar{v}_n(s) \right\|^{2\theta} ds
$$

$$
+ c_3^2 \left( e^{\lambda_1 t} r^2(\omega) \right)^{p-\theta-1} C^2(p) M_{k_0}^{2-2\theta} t \int_0^t \left\| v(s) \right\|_{L^{2p-3}(U)}^{2p-4} \left\| \bar{v}_n(s) \right\|^{2\theta} ds
$$

$$
+ (1 + r_0) t^{r_0} \int_0^t \left\| \nabla \bar{v}_n(s) \right\|^2 ds + t^{1+r_0} b^2 e^t r^2(\omega) \int_0^t \left\| \bar{v}_n(s) \right\|^2 ds
$$

$$
= I_1 + I_2 + I_3. \tag{3.60}
$$

Note that we can obtain that

$$
\int_0^t \left\| v_n(s) \right\|_{L^{2p-3}(U)}^{2p-4} \left\| \bar{v}_n(s) \right\|^{2\theta} ds
$$

$$
\leq \left( \int_0^t \left\| v_n(s) \right\|_{L^{2p-3}(U)}^{2p-3} ds \right)^{\frac{2p-4}{2p-3}} \left( \int_0^t \left\| \bar{v}_n(s) \right\|^{2\theta(2p-3)} ds \right)^{\frac{1}{2p-3}} \tag{3.61}
$$

by the Hölder inequality for $I_1$. Combining with (3.48), (3.49), (3.61) and the interpolation inequality, we have

$$
\int_0^t \left( \left\| v_n(s) \right\|_{L^{2p-3}(U)}^{2p-4} + \left\| v(s) \right\|_{L^{2p-3}(U)}^{2p-4} \right) \left\| \bar{v}_n(s) \right\|^{2\theta} ds
$$

$$
\leq \left( \int_0^t \left\| v_n(s) \right\|_{L^{2p-3}(U)}^{2p-3} ds + \int_0^t \left\| v(s) \right\|_{L^{2p-3}(U)}^{2p-3} ds \right)^{\frac{2p-4}{2p-3}}
$$

$$\cdot \left( \int_0^t \left\| \bar{v}_n(s) \right\|^{2\theta(2p-3)} ds \right)^{\frac{1}{2p-3}}$$

$$\leq \left( \int_0^t \left\| v_n(s) \right\|_{L^{2p-2}(U)}^{(1-\theta)(2p-3)} \left\| v_n(s) \right\|^{\theta(2p-3)} ds \right.$$

$$+ \left. \int_0^t \left\| v(s) \right\|_{L^{2p-2}(U)}^{(1-\theta)(2p-3)} \left\| v(s) \right\|^{\theta(2p-3)} ds \right)^{\frac{2p-4}{2p-3}}$$

$$\cdot \frac{e^{2l\theta(2p-3)t} - 1}{2l\theta(2p-3)} r_4(\omega) \left\| \bar{v}_n(0) \right\|^{2\theta}$$

$$\leq C\left(p, \theta, \rho_1(\omega), \tilde{M}(\omega), r_4(\omega)\right) \frac{e^{2l\theta(2p-3)t} - 1}{2l\theta(2p-3)} \left\| \bar{v}_n(0) \right\|^{2\theta}.$$

So

$$I_1 \leq c_3^2 t \left( e^{\lambda_1 t} r^2(\omega) \right)^{p-\theta-1} C^2(p) M_{k_0}^{2-2\theta} C\left(p, \theta, \rho_1(\omega), \tilde{M}(\omega), r_4(\omega)\right)$$

$$\cdot \frac{e^{2l\theta(2p-3)t} - 1}{2l\theta(2p-3)} \left\| \bar{v}_n(0) \right\|^{2\theta}. \tag{3.62}$$

For the estimates of $I_2$, $I_3$, applying (3.49)–(3.50), we deduce that

$$I_2 \leq (1 + r_0) t^{r_0} \left( \frac{1}{2} \left\| \bar{v}_n(0) \right\|^2 + \left( l + b e^{\frac{T}{2}} r(\omega) \right) \int_0^t \left\| \bar{v}_n(s) \right\|^2 ds \right)$$

$$\leq (1 + r_0) t^{r_0} \left( \frac{1}{2} \left\| \bar{v}_n(0) \right\|^2 + \left( l + b e^{\frac{T}{2}} r(\omega) \right) \frac{e^{2lt} - 1}{2l} r_4(\omega) \left\| \bar{v}_n(0) \right\|^2 \right) \tag{3.63}$$

and

$$I_3 \leq t^{1+r_0} b^2 e^t r^2(\omega) \frac{e^{2lt}}{2l} r_4(\omega) \left\| \bar{v}_n(0) \right\|^2. \tag{3.64}$$

Thus it follows from (3.62)–(3.64) that (3.60) holds. That is,

$$\left\| \nabla \bar{v}_n(t) \right\|^2 \leq C\left(c_3, p, \theta, t, r_0, \lambda_1, \rho_1(\omega), \tilde{M}(\omega), M_{k_0}, r_4(\omega), \left\| \bar{v}_n(0) \right\|^2 \right). \tag{3.65}$$

Applying $\bar{v}_n(t) = \alpha(\theta_t \omega) \bar{u}_n(t)$ and (3.65), we can get

$$\left\| \nabla \bar{u}_n(t) \right\|^2 = \left\| \nabla \left( \alpha^{-1}(\theta_t \omega) \bar{v}_n(t) \right) \right\|^2 \leq e^{\lambda_1 t} r^2(\omega) \left\| \nabla \bar{v}_n(t) \right\|^2$$

$$\leq e^{\lambda_1 t} r^2(\omega) C\left(c_3, p, \theta, t, r_0, \lambda_1, \rho_1(\omega), \tilde{M}(\omega), M_{k_0}, r_4(\omega), \left\| \bar{v}_n(0) \right\|^2 \right).$$

Thus, we finish the proof of (3.43), which implies that (3.42) holds. □

**Theorem 3.6** (($L^2, H_0^1$) attraction) *Assume that* (1.2)–(1.4) *hold.* $A \in \mathcal{D}$ *is the* ($L^2, L^2$) $\mathcal{D}$-*pullback random attractor obtained in Lemma* 2.11. *Then, the random set* $A \in \mathcal{D}$ *is also* $\mathcal{D}$-*pullback attracting in the topology of* $H_0^1(U)$, *that is, for every random set* $D \in \mathcal{D}$,

$$\lim_{t \to +\infty} \text{dist}_{H_0^1} \left( \phi\left(t, \theta_{-t}\omega, D(\theta_{-t}\omega)\right), A(\omega) \right) = 0, \quad \mathbb{P}\text{-almost surely.} \tag{3.66}$$

*Proof* Based on Theorem 3.1 and Theorem 3.5, we can utilize the same approach with Theorem 5.5 of [21] and obtain this result. So we omit it. □

Combining Theorem 3.5 with Theorem 3.6 and the existence of the absorbing set (Lemma 2.10) in $H_0^1(U)$, we easily find the existence of a $(L^2, H_0^1)$ $\mathcal{D}$-pullback random attractor.

**Theorem 3.7** *The $(L^2, L^2)$ $\mathcal{D}$-pullback random attractor A is also a $(L^2, H_0^1)$ $\mathcal{D}$-pullback random attractor.*

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