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Stability analysis of imitative dynamics in two-community with multiple delays

Wenjun Hu^{1,2} and Gang Zhang^{2*}

*Correspondence:

gangzhang@hebtu.edu.cn

²School of Mathematical Sciences,
Hebei Normal University,
Shijiazhuang, P.R. China
Full list of author information is
available at the end of the article

Abstract

Evolutionary game dynamics has become a hot topic in recent years. In this paper, we combine game theory and dynamical systems to investigate the stability of imitative dynamics in two-community with different kinds of delays. First, we establish a new model of imitative dynamics in two-community. Then, three sufficient conditions of the asymptotical stability for the neutrally stable strategy of the imitative dynamics are obtained. Finally, numerical simulations are consistent with the theoretical analysis.

Keywords: Imitative dynamics; Neutrally stable strategy; Asymptotical stability; Delay

1 Introduction

Evolutionary game dynamics propose a basic framework to study and predict the evolutionary behavior in physics [1, 2], biology [3–5], economics [6, 7], and social networks [8, 9]. The most important approach focusing on the evolutionary game dynamics is replicator dynamic, which was first defined for single species by Taylor and Jonker [10], and then has been widely researched by many scholars [11–14]. In fact, other evolutionary dynamics have also been researched. For example, imitative dynamics [15, 16], Brown–von Neumann–Nash dynamics [17], logit dynamics [18, 19], pairwise comparison dynamics [20], etc.

Imitative dynamic is a generalization of replicator dynamic, which can spread the successful strategy more likely than replicator dynamic. In recent years, imitative dynamic has been discussed in different fields by many researchers [21–25]. Cheung [21] studied the imitative dynamics for games with continuous strategy space, they obtained global convergence and local stability results for imitative dynamics. Wang *et al.* [22] investigated the imitation dynamics with delay, they obtained a sufficient condition for two-phenotype imitative model and gave some numerical simulations for three-phenotype imitative model. Hu *et al.* [25] researched the imitative dynamics with discrete delay, they obtained some sufficient conditions of stability in imitative dynamics.

Time delay can represent those processes requiring time to occur such as incubation and transportation, which is ubiquitous in physics, biology, or social systems. Since the important effect of the time delay, many researchers have studied the replicator dynamics with

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time delay [26–30]. Tao *et al.* [26] first proposed the effect of time delay and evolutionary stable strategy, they found that time delay would affect the stability of the replicator dynamics. Nesrine *et al.* [27] researched the discrete and continuous delays in replicator dynamics, they derived some necessary and sufficient delay-independent conditions for replicator dynamics. Wesson *et al.* [28] discussed the Hopf bifurcation in two-strategy and three-strategy delayed replicator dynamics, they showed that the bifurcation and limit cycles would exhibit in delayed replicator dynamics.

As we all know, the research in this field is mainly focused on the same community where the individual has no choice to interact with the opponent in the other communities. Nevertheless, in social networks and biology [31, 32], the population is formed by different communities which can be called clusters. In this case, the individual may interact with an opponent inside or outside the community. Nesrine *et al.* [32] considered evolutionary dynamics in interacting communities, they researched the two-community two-strategy model and obtained some sufficient conditions for stability. However, the delay between two communities is the same, it may be different in two communities in an actual system.

Based on previous literature, we aim to generalize the model in [32] and discuss (i) the delays are diverse in two communities; (ii) a new signification parameter α ($0 < \alpha < 1$). In our model, the dynamics can be considered as a stochastic combination with diverse delays. We research the effect of diversity delay on the stability in a new model and obtain three sufficient conditions for stability in imitative dynamics with different delays. Our results show that the interior equilibrium in imitative dynamics is asymptotically stable when the delay is smaller than the critical value. The present paper is structured as follows. First, we recall the main concepts of evolutionary dynamics, imitative dynamics, and asymmetric delay in Sect. 2. Then the model and stability analysis of the neutrally stable strategy is researched in Sect. 3. At last, the numerical analysis is given by snowdrift games in Sect. 4.

2 Preliminary

In this section, we recall neutrally stable strategy, two-community imitative dynamics, and asymmetric delay.

2.1 Neutrally stable strategy

In the previous literature, we have seen that the evolutionarily stable strategy (ESS) [33] is important in the evolutionary dynamics. In this paper, a special kind of ESS called neutrally stable strategy (NSS) is discussed.

Definition 2.1 ([34]) A strategy $x \in \Delta$ is an NSS if, for every strategy $y \in \Delta$, there exists some $\bar{\varepsilon}_y \in (0, 1)$ such that the inequality

$$u[x, \varepsilon y + (1 - \varepsilon)x] \geq u[y, \varepsilon y + (1 - \varepsilon)x]$$

holds for all $\varepsilon \in (0, \bar{\varepsilon}_y)$, where Δ denotes the mixed-strategy set, $u[x, y]$ is the payoff when strategy x interacts with strategy y .

The definition means that NSS is different from the ESS, while the former requires that no mutant thrive in the sense of earning a higher payoff than the incumbent strategy, the latter requires that no mutant strategy persist in the sense of earning an equal or higher payoff.

2.2 Two-community imitative dynamics

The imitative dynamics can describe the evolution of various strategies in the communities. In two-community, the individual from each community may interact with an opponent from the same or different communities. Assume that p_i denotes the probability that the opponent is from the same community i ($i = 1, 2$), and $\{M_i, N_i\}$ is the strategy set in each community. The population profile can be defined by $x = (x_i, y_i)$, where x_i is the frequency of the strategy M_i in community i and y_i is the frequency of the strategy N_i in community i . The payoff matrix of the two-strategy in community is

$$J = \begin{matrix} & \begin{matrix} M_1 & N_1 & M_2 & N_2 \end{matrix} \\ \begin{matrix} M_1 \\ N_1 \\ M_2 \\ N_2 \end{matrix} & \begin{pmatrix} a_{11} & b_{11} & a_{12} & b_{12} \\ c_{11} & d_{11} & c_{12} & d_{12} \\ a_{21} & b_{21} & a_{22} & b_{22} \\ c_{21} & d_{21} & c_{22} & d_{22} \end{pmatrix} \end{matrix}.$$

From the payoff matrix, one can obtain the payoff of each strategy as follows:

$$\begin{aligned} f_{M_1} &= a_{11}x_1 + b_{11}y_1 + a_{12}x_2 + b_{12}y_2, & f_{N_1} &= c_{11}x_1 + d_{11}y_1 + c_{12}x_2 + d_{12}y_2, \\ f_{M_2} &= a_{21}x_1 + b_{21}y_1 + a_{22}x_2 + b_{22}y_2, & f_{N_2} &= c_{21}x_1 + d_{21}y_1 + c_{22}x_2 + d_{22}y_2. \end{aligned}$$

The imitative dynamics in two-community can be written as follows:

$$\begin{cases} \frac{\dot{x}_1(t)}{x_1(t)} = [p_1 y_1(t)(f_{N_1} - f_{M_1})_+ + (1 - p_1)(x_2(t)(f_{M_2} - f_{M_1})_+ + y_2(t)(f_{N_2} - f_{M_1})_+)], \\ \frac{\dot{y}_1(t)}{y_1(t)} = [p_1 x_1(t)(f_{M_1} - f_{N_1})_+ + (1 - p_1)(x_2(t)(f_{M_2} - f_{N_1})_+ + y_2(t)(f_{N_2} - f_{N_1})_+)], \\ \frac{\dot{x}_2(t)}{x_2(t)} = [p_2 y_2(t)(f_{N_2} - f_{M_2})_+ + (1 - p_2)(x_1(t)(f_{M_1} - f_{M_2})_+ + y_1(t)(f_{N_1} - f_{M_2})_+)], \\ \frac{\dot{y}_2(t)}{y_2(t)} = [p_2 x_2(t)(f_{N_1} - f_{N_2})_+ + (1 - p_2)(x_1(t)(f_{M_1} - f_{N_2})_+ + y_1(t)(f_{N_1} - f_{N_2})_+)], \end{cases} \quad (2.1)$$

where $f(x)_+ = f(x)$ if $f(x) \geq 0$ and $f(x) = 0$ if $f(x) < 0$.

Since $x_1(t) + y_1(t) = 1$, $x_2(t) + y_2(t) = 1$, Eq. (2.1) can be calculated and simplified as follows:

$$\begin{cases} \frac{\dot{x}_1(t)}{x_1(t)} = (1 - x_1(t))[p_1 A_{11}x_1(t) + (1 - p_1)A_{12}x_2(t) + C_1], \\ \frac{\dot{x}_2(t)}{x_2(t)} = (1 - x_2(t))[p_2 A_{22}x_2(t) + (1 - p_2)A_{21}x_1(t) + C_2], \end{cases} \quad (2.2)$$

where the parameters are presented in Table 1.

Table 1 Parameters in the model

Par.	Value
A_{11}	$a_{11} - b_{11} - c_{11} + d_{11}$
A_{22}	$a_{22} - b_{22} - c_{22} + d_{22}$
A_{12}	$a_{12} - b_{12} - c_{12} + d_{12}$
A_{21}	$a_{21} - b_{21} - c_{21} + d_{21}$
C_1	$p_1(b_{11} - d_{11}) + (1 - p_1)(b_{12} - d_{12})$
C_2	$p_2(b_{22} - d_{22}) + (1 - p_2)(b_{21} - d_{21})$

Eq. (2.1) shows the existence of nine equilibrium points

$$\left\{ (0,0), (1,1), (0,1), (1,0), \left(0, -\frac{C_2}{p_2 A_{22}}\right), \left(-\frac{C_2}{p_2 A_{22}}, 0\right), \right. \\ \left. \left(1, -\frac{(1-p_2)A_{12} + C_2}{p_2 A_{22}}\right), \left(-\frac{(1-p_1)A_{12} + C_1}{p_1 A_{11}}, 1\right), \left(\frac{Q_1}{Q}, \frac{Q_2}{Q}\right) \right\},$$

where

$$Q = p_1 p_2 A_{11} A_{22} - (1-p_1)(1-p_2)A_{12} A_{21}, \\ Q_1 = (1-p_1)A_{12} C_2 - p_2 A_{22} C_1, \quad Q_2 = (1-p_2)A_{21} C_1 - p_1 A_{11} C_2.$$

We know that only one point is an interior equilibrium point, and denote the equilibrium as $x^* = (x_1^*, x_2^*)$. Refer to [32], one can obtain the stability of Eq. (2.2).

Lemma 1 ([32]) (i) *The interior equilibrium point x^* in dynamics (2.2) is asymptotically stable if $A_{11} < 0$, $A_{22} < 0$, and $Q > 0$.* (ii) *The fully mixed NSS is asymptotically stable.*

2.3 Asymmetric delay

Assume that $x = (x_1, x_2)$ is the population profile, and $\bar{x} \equiv (\bar{x}_1, \bar{x}_2)$ is the symmetric delay, where $\bar{x}_i \equiv x_i(t - \tau)$. The asymmetric delay is the case where only interactions between individuals with different strategies are delayed, while interactions between individuals with the same strategy may be instantaneous.

Therefore, we define

$$\bar{x}^1 \equiv (x_1, \bar{x}_2), \quad \bar{x}^2 \equiv (\bar{x}_1, x_2).$$

For ease of notation, we would introduce an incremental formula between two strategies:

$$\bar{x}_\alpha \equiv \bar{x} + \alpha(\bar{x}^i - \bar{x}), \quad \alpha \in [0, 1].$$

Especially, (i) $\alpha = 0$ means the increment is zero, i.e., the fixed delay, which has been researched in [30]; (ii) $\alpha = 1$ means the case of asymmetric delay; (iii) when values of α are between 0 and 1, the dynamics can be regarded as a stochastic combination. While interactions with individuals adopting the other strategy are always delayed by τ , interactions with individuals adopting one's own strategy are instantaneous with probability α and delayed with probability $1 - \alpha$.

3 The main result

In this section, we consider two-community imitative dynamics with diverse delays.

3.1 Two-community with asymmetric delay

In this subsection, we assume that the strategies take an asymmetric delay τ . The corresponding imitative dynamics for the two-community is given as follows:

$$\begin{cases} \frac{\dot{x}_1(t)}{x_1(t)} = (1 - x_1(t))[p_1 A_{11}(\alpha x_1(t) + (1 - \alpha)x_1(t - \tau)) + (1 - p_1)A_{12}x_2(t - \tau) + C_1], \\ \frac{\dot{x}_2(t)}{x_2(t)} = (1 - x_2(t))[p_2 A_{22}(\alpha x_1(t) + (1 - \alpha)x_2(t - \tau)) + (1 - p_2)A_{21}x_1(t - \tau) + C_2], \end{cases} \quad (3.1)$$

where A_{11} , A_{22} , A_{12} , A_{21} , C_1 , C_2 are given in Table 1.

For convenience of research, we introduce a small perturbation around $\mathbf{x}^*$: $z_1(t) = x_1(t) - x_1^*$, $z_2(t) = x_2(t) - x_2^*$. Then we can obtain the linearized form of Eq. (3.1):

$$\begin{cases} \dot{z}_1(t) = \delta_1[p_1A_{11}(\alpha z_1(t) + (1-\alpha)z_1(t-\tau)) + (1-p_1)A_{12}z_2(t-\tau)], \\ \dot{z}_2(t) = \delta_2[p_2A_{22}(\alpha z_2(t) + (1-\alpha)z_2(t-\tau)) + (1-p_2)A_{21}z_1(t-\tau)], \end{cases} \quad (3.2)$$

where $\delta_1 = x_1^*(1-x_1^*)$ and $\delta_2 = x_2^*(1-x_2^*)$. Linear Eq. (3.2) is of the form

$$\dot{x}(t) = Ax(t-\tau),$$

which has been discussed in [35]. Similar to the results in the literature, we obtain the theorem for the stability of the mixed NSS for Eq. (3.1).

Theorem 1 *The mixed NSS $\mathbf{x}^*$ in Eq. (3.1) is asymptotically stable if*

$$\tau < \bar{\tau} = \min \left\{ \frac{1}{k_1(1-\alpha)}, \frac{k_1}{2(k_2(1-\alpha) + k_3)} \right\},$$

where

$$k_1 = -(\delta_1 p_1 A_{11} + \delta_2 p_2 A_{22}), \quad k_2 = \delta_1 \delta_2 p_1 p_2 A_{11} A_{22},$$

$$k_3 = -\delta_1 \delta_2 (1-p_1)(1-p_2) A_{12} A_{21}.$$

Proof Taking the Laplace transform of system (3.2), we can obtain the corresponding characteristic equation as follows:

$$\begin{aligned} & \lambda^2 - (\delta_1 p_1 A_{11} + \delta_2 p_2 A_{22})(e^{-\lambda\tau} + \alpha(1 - e^{-\lambda\tau}))\lambda \\ & + \delta_1 \delta_2 [p_1 p_2 A_{11} A_{22}(e^{-\lambda\tau} + \alpha(1 - e^{-\lambda\tau}))^2 - (1-p_1)(1-p_2) A_{12} A_{21} e^{-2\lambda\tau}] = 0. \end{aligned}$$

By substituting the exponential term with a Taylor series expansion and keeping only linear terms in τ in the above equation, one can obtain the simplified characteristic equation

$$(1 - k_1(1-\alpha)\tau)\lambda^2 + (k_1 + 2k_3\tau - 2k_2(1-\alpha)\tau)\lambda + k_2 + k_3 = 0. \quad (3.3)$$

The zero solution is asymptotically stable if all solutions of (3.2) have negative real parts, i.e., $\mathbf{x}^*$ is asymptotically stable if all the roots of the characteristic equation have negative real parts. In this case, the roots of Eq. (3.2) have negative real parts if their product is positive and their sum is negative:

$$\begin{cases} \frac{k_2 + k_3}{1 - k_1(1-\alpha)\tau} > 0, \\ -\frac{k_1 - 2k_2(1-\alpha)\tau - 2k_3\tau}{1 - k_1(1-\alpha)\tau} < 0. \end{cases}$$

By Lemma 1, one can obtain $k_1 > 0$, $k_2 + k_3 = \delta_1 \delta_2 Q > 0$. Therefore, the condition (i) yields $\tau < \frac{1}{k_1(1-\alpha)}$. Condition (ii) yields:

if $(k_2(1-\alpha) - k_3) > 0$, then $\tau < \frac{k_1}{2(k_2(1-\alpha) + k_3)}$;

if $(k_2(1-\alpha) + k_3) < 0$, the conclusion would be established for all τ .

In a sum, we can obtain $\tau < \bar{\tau} = \min\{\frac{1}{k_1(1-\alpha)}, \frac{k_1}{2(k_2(1-\alpha)+k_3)}\}$. $\square$

Remark 3.1 The result in *Theorem 1* gives an upper bound on strategic delays for which NSS remains asymptotically stable in two-community with asymmetric delay.

3.2 Two-community with asymmetric and diverse delays

In this subsection, we research two kinds of different delays in two-community.

3.2.1 Two-community with asymmetric delay

Assume that τ_1 denotes the delay associated with the inter-community interactions and τ_2 denotes that two individuals from different communities get involved in an interaction.

The corresponding imitative dynamics for the two-community is given by

$$\begin{cases} \frac{\dot{x}_1(t)}{x_1(t)} = (1 - x_1(t))[p_1 A_{11}(\alpha x_1(t) + (1 - \alpha)x_1(t - \tau_1)) + (1 - p_1)A_{12}x_2(t - \tau_2) + C_1], \\ \frac{\dot{x}_2(t)}{x_2(t)} = (1 - x_2(t))[p_2 A_{22}(\alpha x_2(t) + (1 - \alpha)x_2(t - \tau_1)) + (1 - p_2)A_{21}x_1(t - \tau_2) + C_2], \end{cases} \quad (3.4)$$

where $A_{11}, A_{22}, A_{12}, A_{21}, C_1, C_2$ are given in *Sect. 2.2*. Using the similar linearization in the above subsection, we can obtain the linearized form of *Eq. (3.4)* as follows:

$$\begin{cases} \dot{z}_1(t) = \delta_1[p_1 A_{11}(\alpha z_1(t) + (1 - \alpha)z_1(t - \tau_1)) + (1 - p_1)A_{12}z_2(t - \tau_2)], \\ \dot{z}_2(t) = \delta_2[p_2 A_{22}(\alpha z_2(t) + (1 - \alpha)z_2(t - \tau_1)) + (1 - p_2)A_{21}z_1(t - \tau_2)]. \end{cases} \quad (3.5)$$

Linear *Eq. (3.5)* is of the form

$$\dot{x}(t) = Ax(t - \tau_1) + Bx(t - \tau_2).$$

Similar to the results in [35], we obtain the theorem on the stability of the NSS for *Eq. (3.4)*.

Theorem 2 If (i) $\tau_1 < \bar{\tau} = \frac{1}{k_1(1-\alpha)}$ and (ii) $k_2(1 - \alpha)\tau_1 + k_3\tau_2 < \frac{k_1}{2}$, the mixed NSS $\mathbf{x}^*$ in *Eq. (3.4)* is asymptotically stable, where k_1, k_2 , and k_3 are given in *Theorem 1*.

Proof Similar to the proof in *Theorem 1*, the corresponding characteristic equation can be written as follows:

$$(1 - k_1(1 - \alpha)\tau_1)\lambda^2 + (k_1 + 2k_3\tau_2 - 2k_2(1 - \alpha)\tau_1)\lambda + k_2 + k_3 = 0. \quad (3.6)$$

The roots of *Eq. (3.6)* have negative real parts if their product is positive and their sum is negative, i.e.,

$$\begin{cases} \frac{k_2 + k_3}{1 - k_1(1 - \alpha)\tau_1} > 0, \\ -\frac{k_1 - 2k_2(1 - \alpha)\tau_1 - 2k_3\tau_2}{1 - k_1(1 - \alpha)\tau_1} < 0. \end{cases}$$

Using similar methods in the proof of *Theorem 1*, one can obtain $\tau_1 < \bar{\tau} = \frac{1}{k_1(1-\alpha)}$ and (ii) $k_2(1 - \alpha)\tau_1 + k_3\tau_2 < \frac{k_1}{2}$. $\square$

Remark 3.2 The result in *Theorem 2* gives an upper bound of τ_1 and the linear combination of τ_1 and τ_2 for which the mixed NSS remains asymptotically stable in two-community with different delays.

3.2.2 Two-community with diverse delays

Let τ_i be the delay associated with the inter community i , and τ_{12} be the delay when two individuals from different communities get involved in an interaction.

The corresponding imitative dynamics for the two-community are given by

$$\begin{cases} \frac{\dot{x}_1(t)}{x_1(t)} = (1 - x_1(t))[p_1 A_{11}(\alpha x_1(t) + (1 - \alpha)x_1(t - \tau_1)) \\ \quad + C_1 + (1 - p_1)A_{12}x_2(t - \tau_{12})], \\ \frac{\dot{x}_2(t)}{x_2(t)} = (1 - x_2(t))[p_2 A_{22}(\alpha x_2(t) + (1 - \alpha)x_2(t - \tau_2)) \\ \quad + C_2 + (1 - p_2)A_{21}x_1(t - \tau_{12})]. \end{cases} \quad (3.7)$$

Similarly, one can obtain the linearized form of Eq. (3.7) as follows:

$$\begin{cases} \dot{z}_1(t) = \delta_1[p_1 A_{11}(\alpha z_1(t) + (1 - \alpha)z_1(t - \tau_1)) + (1 - p_1)A_{12}z_2(t - \tau_{12})], \\ \dot{z}_2(t) = \delta_2[p_2 A_{22}(\alpha z_2(t) + (1 - \alpha)z_2(t - \tau_2)) + (1 - p_2)A_{21}z_1(t - \tau_{12})]. \end{cases} \quad (3.8)$$

Linear Eq. (3.8) is of the form

$$\dot{x}(t) = Ax(t - \tau_1) + Bx(t - \tau_2) + Cx(t - \tau_3).$$

Similar to the results, we obtain the theorem on the stability of the NSS for Eq. (3.7).

Theorem 3 If (i) $k_{11}\tau_1 + k_{12}\tau_2 < \bar{\tau} = \frac{1}{(1-\alpha)}$ and (ii) $k_2(1 - \alpha)(\tau_1 + \tau_2) + 2k_3\tau_{12} < k_1$, the mixed NSS $\mathbf{x}^*$ in Eq. (3.7) is asymptotically stable, where $k_{11} = -\delta_1 p_1 A_{11}$ and $k_{12} = -\delta_2 p_2 A_{22}$, and k_1, k_2 , and k_3 are given in Theorem 1.

Using similar methods in the proof of Theorem 2, one can obtain (i) $k_{11}\tau_1 + k_{12}\tau_2 < \bar{\tau} = \frac{1}{(1-\alpha)}$ and (ii) $k_2(1 - \alpha)(\tau_1 + \tau_2) + 2k_3\tau_{12} < k_1$.

Remark 3.3 The result in Theorem 3 gives an upper bound of the linear combination of τ_1 , τ_2 , and τ_3 for which the mixed NSS remains asymptotically stable in two-community with different delays.

4 Numerical simulation

In this section, we give numerical simulation to verify that our results are effective. We consider the classical snowdrift game [36, 37], where the player may use a cooperate (C) strategy or a defect (D) strategy. According to the relevant literature, we can get the payoff matrix of the snowdrift game as follows:

$$\begin{array}{cc} & \begin{matrix} C & D \end{matrix} \\ \begin{matrix} C \\ D \end{matrix} & \begin{pmatrix} b - \frac{c}{2} & b - c \\ b & 0 \end{pmatrix}, \quad b > c > 0, \end{array}$$

where b represents the payoff of going home and c represents the cost of shoveling snow. We apply our model to the snowdrift game played in a population composed of two communities of players having different behaviors. Players in community 1 are more selfless, and players in community 2 are more selfish.

In this case, the payoff between communities 1 and 2 is described by the matrix

$$J = \begin{matrix} & \begin{matrix} C_1 & D_1 & C_2 & D_2 \end{matrix} \\ \begin{matrix} C_1 \\ D_1 \\ C_2 \\ D_2 \end{matrix} & \begin{pmatrix} b - \frac{c}{2} & b - c & b - \omega c & b - c \\ b & 0 & b & 0 \\ b - (1 - \omega)c & b - c & b - \frac{c}{2} & b - c \\ b & 0 & b & 0 \end{pmatrix} \end{matrix},$$

where ω ($0.5 < \omega < 1$) is the proposition of the cost when a selfless player competes with a selfish player.

Example 1 For the parameter values

$$b = 5, \quad c = 3, \quad \omega = 0.7, \quad p_1 = 0.6, \quad p_2 = 0.7,$$

we can conclude that the mixed intermediate ESS is $s^* = (s_1^*, s_2^*) = (0.4357, 0.6616)$. When the initial (arbitrary) is given by $(0.65, 0.35)$, the maximum value of the delay for the intermediate NSS is asymptotically stable is $\bar{\tau} = 2.3478$. In Fig. 1, we observe a convergence to the intermediate NSS s^* when $\tau < \bar{\tau}$. However, the oscillation phenomenon appears when $\tau > \bar{\tau}$ (see Fig. 1). At the same time, the result implies that the amplitude and the period of the solution will increase with the increase of τ (see Fig. 2).

Example 1 explains the effect of the same delay on the stability of the NSS, which is consistent with *Theorem 1*.

Figure 2 The effect of τ in the replicator dynamics **Figure 3** The stability and instability of the mixed NSS with two delays

Example 2 The parameter values are given as follows:

$$b = 5, \quad c = 3, \quad \omega = 0.7, \quad p_1 = 0.6, \quad p_2 = 0.7.$$

According to the conditions in *Theorem 2*, we obtain that the stability should satisfy two conditions: (i) $\tau_1 + \tau_2 < 10.0931$ and (ii) $0.5163\tau_1 + 0.5485\tau_2 < 2.5$. In Fig. 3, it is clear that the intermediate NSS is stable when τ_1 and τ_2 satisfy the conditions, but when the linear combination of τ_1 and τ_2 does not satisfy condition (i) or (ii), then s^* is unstable.

Example 2 verifies the effect of different delays (case 1) on the stability of the mixed NSS, which is consistent with *Theorem 2*.

Example 3 We discuss the effect of different delays (case 2) on the stability of the intermediate ESS. The parameter values are given as follows:

$$b = 5, \quad c = 3, \quad \omega = 0.7, \quad p_1 = 0.8, \quad p_2 = 0.75, \quad \alpha = 0.4.$$

Assume that (i) $0.6967\tau_1 + 0.6219\tau_2 < 1.6667$ and (ii) $0.26(\tau_1 + \tau_2) - 0.07\tau_{12} < 1.3186$. In Fig. 4, it is clear that the NSS is stable when τ_1 and τ_2 satisfy the conditions, and when the linear combination of τ_1 and τ_2 does not satisfy condition (i) or (ii), then s^* is unstable.

Example 3 describes the effect of off-diagonal and different delay (case 2) on the stability of the mixed NSS, which is consistent with Theorem 3.

5 Conclusion

In this paper, the stability of the NSS in two-community has been investigated for the imitative dynamics with diverse delays. Compared with the model in [32], we introduce a new signification parameter α and discuss the different delays in different communities, which is more meaningful in actual social networks. Three sufficient conditions for the imitative dynamics with diverse delays are derived. The results indicate that the different delays have different effects in the community, which is different from the results in [32].

Furthermore, we will research the two-community imitative dynamics with continuous distributed delays. At the same time, the current theoretical model and numerical simulation are only considered in the imitative dynamics; we plan to discuss the stability for other dynamics such as logit dynamics or best-response dynamics.

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Availability of data and materials

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the manuscript and typed, read, and approved the final manuscript.

Author details

¹Department of Mathematics, Luliang University, Lüliang, P.R. China. ²School of Mathematical Sciences, Hebei Normal University, Shijiazhuang, P.R. China.

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