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Approximation of functions in generalized Zygmund class by double Hausdorff matrix

H.K. Nigam[1], M. Mursaleen[2,3,4*] and Supriya Rani[1]

*Correspondence:
mursaleenm@gmail.com
[2]Department of Medical Research, China Medical University Hospital, China Medical University (Taiwan), Taichung, Taiwan
[3]Department of Mathematics, Aligarh Muslim University, Aligarh, India
Full list of author information is available at the end of the article

Abstract
In the present work, we emphasize, for the first time, the error estimation of a two-variable function $g(y,z)$ in the generalized Zygmund class $Y_r^{(\xi)}$ $(r \geq 1)$ using the double Hausdorff matrix means of its double Fourier series. In fact, in this work, we establish two theorems on error estimation of a two-variable function of g in the generalized Zygmund class.

Keywords: Error estimation; Generalized Zygmund class; Double Hausdorff ($\Delta_{H_{pq}}$) summability means; Double Fourier series (DFS)

1 Introduction

The study of the error estimation of a function of single variable g in Lipschitz spaces, Hölder spaces, generalized Hölder spaces, Besov spaces, Zygmund spaces, and generalized Zygmund spaces with different single means, and various product summability means of Fourier series and conjugate Fourier series have been considered as a center of creative study for the researchers [1–13] in the past few decades. The error estimation of a two-variable function $g(y,z)$ in a Hölder space with the double matrix means of the double Fourier series and its generalization for n-variable functions in Hölder spaces using multiple Fourier series were obtained in [14], and the degree of approximation of Nòrlund means of double Fourier series continuous two-variable functions was obtained in [15]. The above review of research shows that the studies of error estimation of a two-variable function $g(y,z)$ in the generalized Zygmund class $Y_r^{(\xi)}$ $(r \geq 1)$ using double Hausdorff means of double Fourier series have not been initiated so far.

The basic theory of Hausdorff transformations for double sequences came into being by Adams [16] in 1933. Later, a few authors investigated double Hausdorff matrices; see, for example, Ramanujan [17] and Ustina [18]. Consequently, we consider the error estimation of two-variable functions $g(y,z)$ in the generalized Zygmund class $Y_r^{(\xi)}$ $(r \geq 1)$ by the double Hausdorff summability means of its double Fourier series.

We establish two theorems on the degree of approximation of a two-variable function in the generalized Zygmund class $Y_r^{(\xi)}$ $(r \geq 1)$ by the double Hausdorff summability means of its double Fourier series.

Let $z : \mathbb{N} \times \mathbb{N} \mapsto \mathbb{C}$ be a double sequence of complex numbers, and let $(s_{p,q})$ be the double sequence defined by

$$s_{p,q} := \sum_{i=1}^{p} \left(\sum_{j=1}^{q} z_{i,j} \right).$$

The pair (z, s) is called a double series and is denoted by the symbol $\sum_{p,q=1}^{\infty} z_{p,q}$.

Let $\sum_{p=1}^{\infty} \sum_{q=1}^{\infty} z_{p,q}$ be an infinite double series having the (p,q)th partial sum $s_{p,q} = \sum_{i=1}^{p} (\sum_{j=1}^{q} z_{i,j})$.

Let $g(y)$ be a 2π-periodic Lebesgue-integrable function of y over the interval $(-\pi, \pi)$. The Fourier series of $g(y)$ is given by

$$g(y) \sim \frac{a_0}{2} + \sum_{q=1}^{\infty} (a_q \cos qy + b_q \sin qy), \tag{1}$$

and the conjugate series to (1) is given by

$$\sum_{q=1}^{\infty} (a_q \cos qy - b_q \sin qy). \tag{2}$$

It is well known that corresponding conjugate function of (2) is defined as

$$\tilde{g}(y) = \frac{1}{\pi} \int_0^{\pi} \frac{g(y+l) - g(y-l)}{2 \tan \frac{l}{2}} \, dl.$$

Let $g(y,z)$ be a function of (y,z), periodic with respect to y and with respect to z, in each case with period 2π, integrable in the Lebesgue sense and summable in the square $Q(-\pi, -\pi; \pi, \pi)$.

The double Fourier series of a function $g(y,z)$ is given by

$$\begin{aligned} g(y,z) &\sim \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \beta_{p,q} [\tau_{p,q} \cos py \cos qz + \psi_{p,q} \sin py \cos qz \\ &\quad + \rho_{p,q} \cos py \sin qz + \zeta_{p,q} \sin py \sin qz] \\ &= \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \tau_{p,q} A_{p,q}(y,z), \end{aligned} \tag{3}$$

where

$$\beta_{p,q} = \begin{cases} \frac{1}{4} & \text{for } p = 0, q = 0 \text{ or } p = q = 0, \\ \frac{1}{2} & \text{for } p > 0, q = 0 \text{ and } p = 0, q > 0, \\ 1 & \text{for } p > 0, q > 0, \end{cases} \tag{4}$$

and the coefficients $\tau_{p,q}$, $\psi_{p,q}$, $\rho_{p,q}$, and $\zeta_{p,q}$ are calculated by the formulae

$$\begin{aligned}
\tau_{p,q} &= \frac{1}{\pi^2}\int\int_Q g(y,z)\cos py\cos qz\,dy\,dz,\\
\psi_{p,q} &= \frac{1}{\pi^2}\int\int_Q g(y,z)\sin py\cos qz\,dy\,dz,\\
\rho_{p,q} &= \frac{1}{\pi^2}\int\int_Q g(y,z)\cos py\sin qz\,dy\,dz,\\
\zeta_{p,q} &= \frac{1}{\pi^2}\int\int_Q g(y,z)\sin py\sin qz\,dy\,dz
\end{aligned} \tag{5}$$

for $p = 0,1,2,\dots$ and $q = 0,1,2,\dots$. The quantities

$$\begin{aligned}
s_{p,q}(y,z) = \sum_{j=0}^{p}\sum_{k=0}^{q}[&\tau_{j,k}\cos jy\cos kz + \psi_{j,k}\sin jy\cos kz\\
&+ \rho_{j,k}\cos jy\sin kz + \zeta_{j,k}\sin jy\sin kz]
\end{aligned}$$

($p = 0,1,2,\dots$; $q = 0,1,2,\dots$) are called the partial sums of the double Fourier series.

According to (5), we know that

$$s_{p,q}(y,z) - g(y,z) = \frac{1}{\pi^2}\int\int_Q g(y+s,z+l)\frac{[\sin(p+\frac{1}{2})s][\sin(q+\frac{1}{2})l]}{4\sin\frac{s}{2}\sin\frac{l}{2}}\,ds\,dl.$$

The double Hausdorff matrix has the entries

$$h_{pqij} = \binom{p}{i}\binom{q}{j}\Delta_1^{p-i}\Delta_2^{q-j}\mu_{ij},$$

where $\{\mu_{ij}\}$ is any real or complex sequence, and for any sequence μ_{ij}, the operator Δ is defined by

$$\Delta_{ij}\mu_{ij} := \mu_{ij} - \mu_{i+1,j} - \mu_{i,j+1} + \mu_{i+1,j+1}$$

and

$$\Delta_1^{p-i}\Delta_2^{q-j}\mu_{ij} = \sum_{s=0}^{p-1}\sum_{l=0}^{q-j}(-1)^{i+j}\binom{p-i}{s}\binom{q-j}{l}\mu_{i+s,j+l}.$$

Necessary and sufficient condition for double Hausdorff matrices to be conservative is the existence of a function $\chi(s,l) \in BV[0,1]\times[0,1]$ such that

$$\int_0^1\int_0^1 \left|d\chi(s,l)\right| < \infty$$

and

$$\mu_{pq} = \int_0^1\int_0^1 s^p l^q\,d\chi(s,l).$$

Without loss of generality, we may assume that $\chi(0,0) = 0$. If, in addition, we have $\chi(1,1) = 1$, and the continuity conditions

$$\chi(s,+0) = \chi(s,0), \qquad \chi(s,+0) = \lim_{l\to 0} \chi(s,l),$$
$$\chi(+0,l) = \chi(0,l), \qquad \chi(+0,l) = \lim_{s\to 0} \chi(s,l)$$

are also satisfied, so that $\mu_{00} = 1$.

We say that μ_{pq} is a regular moment constant [16, 17].

Let $\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} b_{p,q}$ be a double series with $s_{p,q} = \sum_{j=0}^{p}\sum_{k=0}^{q} b_{j,k}$ as its (p,q)th partial sums.

The double Hausdorff mean $t_{p,q}$ is given by

$$t_{p,q} = \sum_{j=0}^{p}\sum_{k=0}^{q} h_{p,q,j,k} s_{j,k}. \tag{6}$$

The double series $\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} b_{p,q}$ with the sequence of (p,q)th partial sums $(s_{p,q})$ is said to be summable by the double Hausdorff summability method or summable $(H_{p,q})$ if $t_{p,q}$ tends to a limit s as $p \to \infty$ and $q \to \infty$.

The norm $\|\cdot\|_r$ is defined as

$$\|f\|_r := \begin{cases} \{\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} |g(y,z)|^r\,dy\,dz\}^{1/r} & \text{for } 1 \le r < \infty, \\ \operatorname{ess\,sup}_{\substack{0<y<2\pi \\ 0<z<2\pi}} |g(y,z)| & \text{for } r = \infty. \end{cases}$$

Let $\xi : [-\pi,-\pi;\pi,\pi] \to \mathbb{R}\times\mathbb{R}$ be an arbitrary function with $\xi(s,l) > 0$ for $0 < s < 2\pi$, $0 < l < 2\pi$ and such that $\lim_{\substack{s\to 0^+ \\ l\to 0^+}} \xi(s,l) = \xi(0,0) = 0$.

We define

$$Y_r^{(\xi)} := \left\{ g \in L^r : \sup_{\substack{s\neq 0 \\ l\neq 0}} \frac{\|g(\cdot+s,\cdot+l) + g(\cdot+s,\cdot-l) + g(\cdot-s,\cdot+l) + g(\cdot-s,\cdot-l) + 4g(\cdot,\cdot)\|_r}{\xi(s,l)} \right.$$
$$\left. < \infty \right\}$$

and

$$\|g\|_r^{(\xi)} := \|g\|_r + \sup_{\substack{s\neq 0 \\ l\neq 0}} \frac{\|g(\cdot+s,\cdot+l) + g(\cdot+s,\cdot-l) + g(\cdot-s,\cdot+l) + g(\cdot-s,\cdot-l) + 4g(\cdot,\cdot)\|_r}{\xi(s,l)}.$$

Clearly, $\|\cdot\|_r^{(\xi)}$ is a norm on $Y_r^{(\xi)}$.

Hence the Zygmund space $(Y_r^{(\xi)})$ is a Banach space under the norm $\|\cdot\|_r^{(\xi)}$. The completeness of the space $Y_r^{(\xi)}$ can be discussed considering the completeness of L^r $(r \ge 1)$. We refer to [19] for more detail on the Zygmund space.

We write

$$\begin{aligned}\phi(s,l) &= \phi(y,z:s,l) \\ &= \frac{1}{4}\big[g(y+s,z+l) + g(y+s,z-l) + g(y-s,z+l) + g(y-s,z-l) - 4g(y,z)\big];\end{aligned}$$

$$\Phi(y,z) = \int_0^y \int_0^z \left|\phi(v,w)\right| dv\,dw;$$

$$K_{p,q}(s,l) = \frac{1}{4\pi^2} \sum_{j=0}^{p} h_{p,j} \frac{\sin(j+\frac{1}{2})s}{\sin\frac{s}{2}} \sum_{k=0}^{q} h_{q,k} \frac{\sin(k+\frac{1}{2})l}{\sin\frac{l}{2}}.$$

Remark 1.1 A double Hausdorff matrix method reduces to

(i) (C,1,1) summability mean if $\sigma_p = \frac{1}{p+1}$ and $\sigma_q = \frac{1}{q+1}$ and

(ii) (E,r,r) summability mean if $E_p^r = \frac{1}{(1+r)^p}\binom{p}{j} r^{p-j}$ and $E_q^r = \frac{1}{(1+r)^q}\binom{q}{k} r^{q-k}$.

2 Theorems

Theorem 2.1 *Let g be a function of* (y,z) *periodic* (*in each case, with period* 2π) *with respect to y and z, Lebesgue integrable on Q, and belonging to the class* $Y_r^{(\xi)}$, $r \geq 1$. *Then the error estimate of g by the* $\Delta_{H_{p,q}}$ *method of its DFS is given by*

$$\left\| t_{p,q}^{\Delta H_{p,q}}(y,z) - g(y,z) \right\|_r^{(\eta)} = O\left(\frac{p+q+6}{(p+1)(q+1)} \int_{\frac{1}{p+1}}^{\pi} \int_{\frac{1}{q+1}}^{\pi} \frac{\xi(s,l)}{\eta(s,l)s^2l^2} \right),$$

where ξ *and* η *denote the moduli of continuity of second order such that* $\xi(s,l)/\eta(s,l)$ *is positive and nondecreasing.*

Theorem 2.2 *In addition to the conditions of Theorem* 2.1, *if* $\xi(s,l)/sl\eta(s,l)$ *is nonincreasing, then the error estimate of g in* $Y_r^{(\xi)}$ ($r \geq 1$) *by the* $\Delta_{H_{p,q}}$ *method of its DFS is given by*

$$\left\| t_{p,q}^{\Delta H_{p,q}}(y,z) - g(y,z) \right\|_r^{(\eta)} = O\left(\frac{\xi(\frac{1}{p+1},\frac{1}{q+1})}{\eta(\frac{1}{p+1},\frac{1}{q+1})} (p+q+6) \log \pi (p+q+2) \right).$$

3 Lemmas

Lemma 3.1 $K_{pq}(s,l) = O((p+1)(q+1))$ *for* $0 < s \leq \frac{1}{p+1}$ *and* $0 < l \leq \frac{1}{q+1}$.

Proof For $0 < s \leq \frac{1}{p+1}$, $0 < l \leq \frac{1}{l+1}$, $\sin\frac{l}{2} \geq \frac{l}{\pi}$, and $\sin ql \leq ql$, we get

$$\begin{aligned}
\left|K_{pq}(s,l)\right| &= \frac{1}{4\pi^2} \left| \sum_{j=0}^{p} h_{p,j} \frac{\sin(j+\frac{1}{2})s}{\sin\frac{s}{2}} \sum_{k=0}^{q} h_{q,k} \frac{\sin(k+\frac{1}{2})l}{\sin\frac{l}{2}} \right| \\
&= \frac{1}{4\pi^2} \left| \sum_{j=0}^{p} \int_0^1 \binom{p}{j} v^j (1-v)^{p-j}\, d\chi(v) \frac{\sin(j+\frac{1}{2})s}{\sin\frac{s}{2}} \right. \\
&\quad \left. \times \sum_{k=0}^{q} \int_0^1 \binom{q}{k} w^k (1-w)^{q-k}\, d\chi(w) \frac{\sin(k+\frac{1}{2})l}{\sin\frac{l}{2}} \right| \\
&= \frac{1}{4\pi^2} \left| \sum_{j=0}^{p} \int_0^1 \binom{p}{j} v^j (1-v)^{p-j}\, d\chi(v) \frac{(2s+1)\frac{s}{2}}{\frac{s}{\pi}} \right. \\
&\quad \left. \times \sum_{k=0}^{q} \int_0^1 \binom{q}{k} w^k (1-w)^{q-k}\, d\chi(w) \frac{(2k+1)\frac{l}{2}}{\frac{l}{\pi}} \right|
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{16}\left|\sum_{j=0}^{p}\int_0^1\binom{p}{j}v^j(1-v)^{p-j}\,d\chi(v)(2s+1)\right.\\
&\quad\left.\times\sum_{k=0}^{q}\int_0^1\binom{q}{k}w^k(1-w)^{q-k}\,d\chi(w)(2k+1)\right|. \qquad (7)
\end{aligned}
$$

Since

$$
\begin{aligned}
&\sum_{j=0}^{p}\binom{p}{j}v^j(1-v)^{p-j}(2s+1)=(2pv+1)\quad\text{and}\\
&\sum_{k=0}^{q}\binom{q}{k}w^k(1-w)^{q-k}(2k+1)=(2qw+1),
\end{aligned}
\qquad (8)
$$

from (7) and (8) we get

$$
\begin{aligned}
\left|K_{pq}(s,l)\right| &= \frac{1}{16}\left|\int_0^1(2pv+1)\,dv\int_0^1(2qw+1)\,dw\right|\\
&= O\bigl((p+1)(q+1)\bigr). \qquad \square
\end{aligned}
$$

Lemma 3.2 *$K_{pq}(s,l)=O((q+1)\frac{1}{(p+1)s^2})$ for $\frac{1}{p+1}<s\le\pi$ and $0<l\le\frac{1}{q+1}$.*

Proof Since $\frac{1}{p+1}<s\le\pi$, $\sin\frac{s}{2}\ge\frac{s}{\pi}$, $\sin^2 qs\le 1$, $\sup_{0\le v\le 1}|j'(v)|=M$, $0<l\le\frac{1}{q+1}$, $\sin\frac{l}{2}\ge\frac{l}{\pi}$, and $\sin ql\le ql$, we get

$$
\begin{aligned}
\left|K_{pq}(s,l)\right| &= \frac{1}{4\pi^2}\left|\sum_{j=0}^{p}\int_0^1\binom{p}{j}v^j(1-v)^{p-j}\,d\chi(v)\frac{\sin(j+\frac{1}{2})s}{\sin\frac{s}{2}}\right.\\
&\quad\left.\times\sum_{k=0}^{q}\int_0^1\binom{q}{k}w^k(1-w)^{q-k}\,d\chi(w)\frac{\sin(k+\frac{1}{2})l}{\sin\frac{l}{2}}\right|\\
&= \frac{1}{4\pi^2}\left|\sum_{j=0}^{p}\int_0^1\binom{p}{j}v^j(1-v)^{p-j}\,d\chi(v)\frac{\sin(j+\frac{1}{2})s}{\frac{s}{\pi}}\right.\\
&\quad\left.\times\sum_{k=0}^{q}\int_0^1\binom{q}{k}w^k(1-w)^{q-k}\,d\chi(w)\frac{(k+\frac{1}{2})l}{\frac{l}{\pi}}\right|\\
&= \frac{1}{8s}\left|\sum_{j=0}^{p}\int_0^1\binom{p}{j}v^j(1-v)^{p-j}\,d\chi(v)\sin\left(j+\frac{1}{2}\right)s\right.\\
&\quad\left.\times\sum_{k=0}^{q}\int_0^1\binom{q}{k}w^k(1-w)^{q-k}\,d\chi(w)(2k+1)\right|\\
&\le \frac{M}{8s}\left|\sum_{j=0}^{p}\int_0^1\binom{p}{j}v^j(1-v)^{p-j}e^{i(j+\frac{1}{2})s}\,dv\int_0^1(2qw+1)\,dw\right|\\
&\quad\text{since }\sum_{k=0}^{q}\binom{q}{k}w^k(1-w)^{q-k}(2k+1)=(2qw+1)
\end{aligned}
$$

$$\leq \frac{M}{8s}\left|\sum_{j=0}^{p}\int_0^1\binom{p}{j}v^j(1-v)^{p-j}e^{i(j+\frac{1}{2})s}\,dv(q+1)\right|$$
$$\leq \frac{MO(q+1)}{8s}\left|\sum_{j=0}^{p}\int_0^1\binom{p}{j}v^j(1-v)^{p-j}e^{i(j+\frac{1}{2})s}\,dv\right|. \tag{9}$$

Now

$$\begin{aligned}
&\sum_{j=0}^{p}\int_0^1\binom{p}{j}v^j(1-v)^{p-j}e^{(j+\frac{1}{2})s}\,dv\\
&= (1-v)^p\int_0^1\sum_{j=0}^{p}\binom{p}{j}\left(\frac{v}{1-v}\right)^j \mathrm{Im}\left\{e^{i(j+\frac{1}{2})s}\right\}dv\\
&= (1-v)^p\int_0^1\sum_{j=0}^{p}\binom{p}{j}\left(\frac{v}{1-v}\right)^j \mathrm{Im}\left\{e^{ijs}\cdot e^{\frac{is}{2}}\right\}dv\\
&= (1-v)^p e^{\frac{is}{2}}\int_0^1\sum_{j=0}^{p}\binom{p}{j}\left(\frac{ve^{is}}{1-v}\right)^j dv\\
&= (1-v)^p\,\mathrm{Im}\int_0^1\left[e^{\frac{is}{2}}\left\{\binom{p}{0}+\binom{p}{1}\frac{ve^{is}}{1-v}+\binom{p}{2}\left(\frac{ve^{is}}{1-v}\right)^2+\cdots+\binom{p}{p}\left(\frac{ve^{is}}{1-v}\right)^p\right\}dv\right]\\
&= \mathrm{Im}\left[e^{\frac{is}{2}}\int_0^1\left\{\binom{p}{0}(1-v)^{p-0}+\binom{p}{1}ve^{is}(1-v)^{p-1}+\cdots+\binom{p}{p}\left(ve^{is}\right)^p(1-v)^{p-p}\right\}dv\right]\\
&= \mathrm{Im}\left[e^{\frac{is}{2}}\int_0^1\left(1-v+ve^{is}\right)^p dv\right]\\
&= \mathrm{Im}\left[e^{\frac{is}{2}}\int_0^1\left\{1+v\left(e^{is}-1\right)\right\}^p dv\right]\\
&= \mathrm{Im}\left[\frac{e^{i(p+1)s}-1}{(1+p)(e^{\frac{is}{2}}-e^{\frac{-is}{2}})}\right]\\
&= \mathrm{Im}\left[\frac{e^{i(p+1)s}-1}{(p+1)2i\sin\frac{s}{2}}\right]\\
&= \mathrm{Im}\left[\frac{\cos(p+1)s+i\sin(p+1)s-1}{2i(p+1)\sin\frac{s}{2}}\right]\\
&= \frac{\sin^2(p+1)\frac{s}{2}}{(p+1)\sin\frac{s}{2}}.
\end{aligned} \tag{10}$$

Now, from equations (9) and (10), we get

$$\begin{aligned}
\left|K_{pq}(s,l)\right| &\leq \frac{MO(q+1)}{8s}\left|\frac{\sin^2(p+1)\frac{s}{2}}{(p+1)\frac{s}{2}}\right|\\
&= \frac{MO(q+1)}{8s}\left|\frac{1}{(p+1)\frac{s}{2}}\right|\\
&= O\left((q+1)\frac{1}{(p+1)s^2}\right).
\end{aligned}$$

□

Lemma 3.3 $K_{pq}(s,l)=O((p+1)\frac{1}{(q+1)l^2})$ *for* $0<s\leq\frac{1}{p+1}$ *and* $\frac{1}{q+1}<l\leq\pi$.

Proof Since $0 < s \leq \frac{1}{p+1}$, $\sin\frac{s}{2} \geq \frac{s}{\pi}$, $\sin ps \leq ps$, $\frac{1}{q+1} < l \leq \pi$, $\sin\frac{l}{2} \geq \frac{l}{\pi}$, $\sin^2 ql \leq 1$, and $\sup_{0\leq w\leq 1}|k'(w)| = N$, we get

$$\begin{aligned}
\left|K_{pq}(s,l)\right| &= \frac{1}{4\pi^2}\left|\sum_{j=0}^{p}\int_0^1\binom{p}{j}v^j(1-v)^{p-j}\,d\chi(v)\frac{\sin(j+\frac{1}{2})s}{\sin\frac{s}{2}}\right. \\
&\quad\left.\times\sum_{k=0}^{q}\int_0^1\binom{q}{k}w^k(1-w)^{l-k}\,d\chi(w)\frac{\sin(k+\frac{1}{2})l}{\sin\frac{l}{2}}\right| \\
&= \frac{1}{4\pi^2}\left|\sum_{j=0}^{p}\int_0^1\binom{p}{j}v^j(1-v)^{p-j}\,d\chi(v)\frac{(j+\frac{1}{2})s}{\frac{s}{\pi}}\right. \\
&\quad\left.\times\sum_{k=0}^{q}\int_0^1\binom{q}{k}w^k(1-w)^{q-k}\,d\chi(w)\frac{\sin(k+\frac{1}{2})l}{\frac{l}{\pi}}\right| \\
&= \frac{1}{8l}\left|\sum_{j=0}^{p}\int_0^1\binom{p}{j}v^j(1-v)^{p-j}\,d\chi(v)(2j+1)\right. \\
&\quad\left.\times\sum_{k=0}^{q}\int_0^1\binom{q}{k}w^k(1-w)^{q-k}\,d\chi(w)\sin\left(k+\frac{1}{2}\right)l\right| \\
&\leq \frac{N}{8l}\left|\int_0^1(2pv+1)\,dv\sum_{k=0}^{q}\binom{q}{k}w^k(1-w)^{q-k}e^{i(k+\frac{1}{2})l}\,dw\right| \\
&\quad\text{since }\sum_{j=0}^{p}\binom{p}{j}v^j(1-v)^{p-j}(2s+1) = (2pv+1) \\
&\leq \frac{N}{8l}\left|(p+1)\sum_{k=0}^{q}\binom{q}{k}w^k(1-w)^{q-k}e^{i(k+\frac{1}{2})l}\,dw\right| \\
&\leq \frac{NO(p+1)}{8l}\left|\sum_{k=0}^{q}\int_0^1\binom{q}{k}w^k(1-w)^{q-k}e^{i(k+\frac{1}{2})l}\,dw\right|.
\end{aligned} \tag{11}$$

Now

$$\begin{aligned}
&\sum_{k=0}^{q}\int_0^1\binom{q}{k}w^k(1-w)^{q-k}e^{(k+\frac{1}{2})l}\,dw \\
&= (1-w)^q\int_0^1\sum_{k=0}^{q}\binom{q}{k}\left(\frac{w}{1-w}\right)^k\operatorname{Im}\left\{e^{i(k+\frac{1}{2})l}\right\}dw \\
&= (1-w)^q\int_0^1\sum_{k=0}^{q}\binom{q}{k}\left(\frac{w}{1-w}\right)^k\operatorname{Im}\left\{e^{ikl}\cdot e^{\frac{il}{2}}\right\}dw \\
&= (1-w)^q e^{\frac{il}{2}}\int_0^1\sum_{k=0}^{q}\binom{q}{k}\left(\frac{we^{il}}{1-w}\right)^k dw \\
&= (1-w)^q\operatorname{Im}\int_0^1\left[e^{\frac{il}{2}}\left\{\binom{q}{0}+\binom{q}{1}\frac{we^{il}}{1-w}+\binom{q}{2}\left(\frac{we^{il}}{1-w}\right)^2+\cdots\right.\right. \\
&\quad\left.\left.+\binom{q}{q}\left(\frac{we^{il}}{1-w}\right)^q\right\}dw\right]
\end{aligned}$$

$$\begin{aligned}
&= \operatorname{Im}\left[e^{\frac{il}{2}} \int_0^1 \left\{\binom{q}{0}(1-w)^{q-0} + \binom{q}{1} we^{il}(1-w)^{q-1} + \cdots \right.\right.\\
&\quad \left.\left. + \binom{q}{q}\left(we^{il}\right)^q (1-w)^{q-q}\right\} dw\right] \\
&= \operatorname{Im}\left[e^{\frac{il}{2}} \int_0^1 \left(1 - w + we^{il}\right)^q dw\right] \\
&= \operatorname{Im}\left[e^{\frac{il}{2}} \int_0^1 \left\{1 + w\left(e^{il} - 1\right)\right\}^q dw\right] \\
&= \operatorname{Im}\left[\frac{e^{i(q+1)l} - 1}{(1+q)(e^{\frac{il}{2}} - e^{\frac{-il}{2}})}\right] \\
&= \operatorname{Im}\left[\frac{e^{i(q+1)l} - 1}{(q+1)2i\sin\frac{l}{2}}\right] \\
&= \operatorname{Im}\left[\frac{\cos(q+1)l + i\sin(q+1)l - 1}{2i(q+1)\sin\frac{l}{2}}\right] \\
&= \frac{\sin^2(q+1)\frac{l}{2}}{(q+1)\sin\frac{l}{2}}.
\end{aligned} \tag{12}$$

Now from equations (11) and (12) we get

$$\begin{aligned}
\left|K_{pq}(s,l)\right| &\leq \frac{NO(p+1)}{8l}\left|\frac{\sin^2(q+1)\frac{l}{2}}{(q+1)\frac{l}{2}}\right| \\
&= \frac{NO(p+1)}{8l}\left|\frac{1}{(q+1)\frac{l}{2}}\right| \\
&= O\left((p+1)\frac{1}{(q+1)l^2}\right).
\end{aligned}$$

□

Lemma 3.4 $K_{pq}(s,l) = O(\frac{1}{(p+1)s^2}\frac{1}{(q+1)l^2})$ *for* $\frac{1}{p+1} < s \leq \pi$ *and* $\frac{1}{q+1} < l \leq \pi$.

Proof Since $\frac{1}{p+1} < s \leq \pi$, $\sin\frac{s}{2} \geq \frac{s}{\pi}$, $\sin^2 ps \leq 1$, $\sup_{0\leq v\leq 1}|j'(v)| = M$, $\frac{1}{q+1} < l \leq \pi$, $\sin\frac{l}{2} \geq \frac{l}{\pi}$, $\sin^2 ql \leq 1$, and $\sup_{0\leq w\leq 1}|k'(w)| = N$, using (10) and (12), we get

$$\begin{aligned}
\left|K_{pq}(s,l)\right| &= \frac{1}{4\pi^2}\left|\sum_{j=0}^{p}\int_0^1 \binom{p}{j} v^j(1-v)^{p-j}\,d\chi(v)\frac{\sin(j+\frac{1}{2})s}{\sin\frac{s}{2}}\right. \\
&\quad \left.\times \sum_{k=0}^{q}\int_0^1 \binom{q}{k} w^k(1-w)^{q-k}\,d\chi(w)\frac{\sin(k+\frac{1}{2})l}{\sin\frac{l}{2}}\right| \\
&= \frac{1}{4\pi^2}\left|\sum_{j=0}^{p}\int_0^1 \binom{p}{j} v^j(1-v)^{p-j}\,d\chi(v)\frac{(j+\frac{1}{2})s}{\frac{s}{\pi}}\right. \\
&\quad \left.\times \sum_{k=0}^{q}\int_0^1 \binom{q}{k} w^k(1-w)^{q-k}\,d\chi(w)\frac{(k+\frac{1}{2})l}{\frac{l}{\pi}}\right| \\
&= \frac{1}{16sl}\left|\sum_{j=0}^{p}\int_0^1 \binom{p}{j} v^j(1-v)^{p-j}\,d\chi(v)\left(j+\frac{1}{2}\right)s\right.
\end{aligned}$$

$$\begin{aligned}
&\times \sum_{k=0}^{q}\int_0^1 \binom{q}{k} w^k(1-w)^{q-k}\,d\chi(w)(2k+1)\Bigg| \\
&\leq \frac{MN}{16sl}\left|\sum_{j=0}^{p}\int_0^1 \binom{p}{j} v^j(1-v)^{p-j}e^{i(j+\frac{1}{2})s}\,dv \sum_{k=0}^{q}\int_0^1 \binom{q}{k} w^k(1-w)^{q-k}e^{i(k+\frac{1}{2})l}\,dw\right| \\
&\leq \frac{MN}{16sl}\left|\frac{\sin^2(p+1)\frac{s}{2}}{(p+1)\frac{s}{2}}\frac{\sin^2(q+1)\frac{l}{2}}{(q+1)\frac{l}{2}}\right| \\
&= \frac{MN}{16sl}\left|\frac{1}{(p+1)\frac{s}{2}}\frac{1}{(q+1)\frac{l}{2}}\right| \\
&= O\left(\frac{1}{(p+1)s^2}\frac{1}{(q+1)l^2}\right). \qquad (13)
\end{aligned}$$

□

Lemma 3.5 *Let $g(y,z) \in Y_r^{(\xi)}$. Then for $0 < s \leq \pi$ and $0 < l \leq \pi$,*

(i) $\|\phi(\cdot,s,\cdot,l)\|_r = O(\xi(s,l))$;

(ii) $\|\phi(\cdot+v,\cdot+w,s,l)+\phi(\cdot+v,\cdot-w,s,l)+\phi(\cdot-v,\cdot+w,s,l)+\phi(\cdot-v,\cdot+w,s,l)+\phi(\cdot-v,\cdot-w,s,l)-4\phi(\cdot s,\cdot l)\|_r = \begin{cases} 8(\xi(s,l)), \\ 8(\xi(v,w)); \end{cases}$

(iii) *If $\xi(s,l)$ and $\eta(s,l)$ are as defined in Theorem 2.1, then* $\|\phi(\cdot+v,\cdot+w,s,l)+\phi(\cdot+v,\cdot-w,s,l)+\phi(\cdot-v,\cdot+w,s,l)+\phi(\cdot-v,\cdot+w,s,l)+\phi(\cdot-v,\cdot-w,s,l)-4\phi(\cdot s,\cdot l)\|_r = 8(\eta(v,w)\frac{\xi(s,l)}{\eta(s,l)})$, *where*

$$\begin{aligned}
\phi(y,z;s,l) = \frac{1}{4}\big[&g(y+s,z+l)+g(y+s,z-l)+g(y-s,z+l) \\
&+g(y-s,z-l)-4g(y,z)\big].
\end{aligned}$$

Proof (i) Since

$$\phi(y,z;s,l) = \frac{1}{4}\big[g(y+s,z+l)+g(y+s,z-l)+g(y-s,z+l)+g(y-s,z-l)-4g(y,z)\big],$$

applying Minkowski's inequality, we have

$$\begin{aligned}
\big\|\phi(\cdot,s,\cdot,l)\big\|_r &\leq \frac{1}{4}\big\|g(y+s,z+l)+g(y+s,z-l) \\
&\quad +g(y-s,z+l)+g(y-s,z-l)-4g(y,z)\big\|_r \\
&= O\big(\xi(s,l)\big).
\end{aligned}$$

□

Proof (ii) Clearly,

$$\begin{aligned}
&\big|\phi(y+v,z+w,s,l)+\phi(y+v,z-w,s,l)+\phi(y-v,z+w,s,l) \\
&\quad +\phi(y-v,z-w,s,l)-4\phi(y,z,s,l)\big| \\
&\leq \big|g(y+v+s,z+w+l)+g(y+v+s,z+w-l)+g(y+v-s,z+w+l) \\
&\quad +g(y+v-s,z+w-l)-4g(y+v,z+w)\big| \\
&\quad +\big|g(y+v+s,z-w+l)+g(y+v+s,z-w-l)+g(y+v-s,z-w+l) \\
&\quad +g(y+v-s,z-w-l)-4g(y+v,z-w)\big|
\end{aligned}$$

$$
\begin{aligned}
&+ \left|g(y-v+s,z+w+l)+g(y-v+s,z+w-l)+g(y-v-s,z+w+l)\right.\\
&\left.+ g(y-v-s,z+w-l)-4g(y-v,z+w)\right|\\
&+ \left|g(y-v+s,z-w+l)+g(y-v+s,z-w-l)+g(y-v-s,z-w+l)\right.\\
&\left.+ g(y-v-s,z-w-l)-4g(y-v,z-w)\right|\\
&- 4\left|g(y+s,z+l)+g(y+s,z-l)+g(y-s,z+l)+g(y-s,z-l)-4g(y,z)\right|.
\end{aligned}
$$

Applying Minkowski's inequality, we have

$$
\begin{aligned}
&\left\|\phi(\cdot+v,\cdot+w,s,l)+\phi(\cdot+v,\cdot-w,s,l)+\phi(\cdot-v,\cdot+w,s,l)\right.\\
&\qquad\left.+\phi(\cdot-v,\cdot-w,s,l)-4\phi(\cdot,s,\cdot,l)\right\|_r\\
&\quad\le \left\|g(\cdot+v+s,\cdot+w+l)+g(\cdot+v+s,\cdot+w-l)+g(\cdot+v-s,\cdot+w+l)\right.\\
&\qquad\left.+g(\cdot+v-s,\cdot+w-l)-4g(\cdot+v,\cdot+w)\right\|_r\\
&\qquad+\left\|g(\cdot+v+s,\cdot-w+l)+g(\cdot+v+s,\cdot-w-l)+g(\cdot+v-s,\cdot-w+l)\right.\\
&\qquad\left.+f(\cdot+v-s,\cdot-w-l)-4g(\cdot+v,\cdot-w)\right\|_r\\
&\qquad+\left\|g(\cdot-v+s,\cdot+w+l)+g(\cdot-v+s,\cdot+w-l)+g(\cdot-v-s,\cdot+w+l)\right.\\
&\qquad\left.+g(\cdot-v-s,\cdot+w-l)-4g(\cdot-v,\cdot+w)\right\|_r\\
&\qquad+\left\|g(\cdot-v+s,\cdot-w+l)+g(\cdot-v+s,\cdot-w-l)+g(\cdot-v-s,\cdot-w+l)\right.\\
&\qquad\left.+g(\cdot-v-s,\cdot-w-l)-4g(\cdot-v,\cdot-w)\right\|_r\\
&\qquad-4\left\|g(\cdot+s,\cdot+l)+g(\cdot+s,\cdot-l)+g(\cdot-s,\cdot+l)+g(\cdot-s,\cdot-l)-4g(\cdot,\cdot)\right\|_r\\
&\quad=8\bigl(\xi(s,l)\bigr).
\end{aligned}
$$

Also,

$$
\begin{aligned}
&\left\|\phi(\cdot+v,\cdot+w,s,l)+\phi(\cdot+v,\cdot-w,s,l)+\phi(\cdot-v,\cdot+w,s,l)\right.\\
&\qquad\left.+\phi(\cdot-v,\cdot-w,s,l)-4\phi(\cdot+s,\cdot+l)\right\|_r\\
&\quad\le \left\|g(\cdot+s+v,\cdot+l+w)+g(\cdot+s+v,\cdot+l-w)+g(\cdot+s-v,\cdot+l+w)\right.\\
&\qquad\left.+g(\cdot+s-v,\cdot+l-w)-4g(\cdot+s,\cdot+l)\right\|_r\\
&\qquad+\left\|g(\cdot+s+v,\cdot-l+w)+g(\cdot+s+v,\cdot-l-w)+g(\cdot+s-v,\cdot-l+w)\right.\\
&\qquad\left.+g(\cdot+s-v,\cdot-l-w)-4g(\cdot+s,\cdot-l)\right\|_r\\
&\qquad+\left\|g(\cdot-s+v,\cdot+l+w)+g(\cdot-s+v,\cdot+l-w)+g(\cdot-s-v,\cdot+l+w)\right.\\
&\qquad\left.+g(\cdot-s-v,\cdot+l-w)-4g(\cdot-s,\cdot+l)\right\|_r\\
&\qquad+\left\|g(\cdot-s+v,\cdot-l+w)+g(\cdot-s+v,\cdot-l-w)+g(\cdot-s-v,\cdot-l+w)\right.\\
&\qquad\left.+g(\cdot-s-v,\cdot-l-w)-4g(\cdot-s,\cdot-l)\right\|_r\\
&\qquad-4\left\|g(\cdot+v,\cdot+w)+g(\cdot+v,\cdot-w)+g(\cdot-v,\cdot+w)+g(\cdot-v,\cdot-w)-4g(\cdot,\cdot)\right\|_r\\
&\quad=8\bigl(\xi(v,w)\bigr).
\end{aligned}
$$

□

Proof (iii) Since η is positive and nondecreasing, $s \leq v$, $l \leq w$, using Lemma 3.5(ii), we obtain

$$\begin{aligned}
&\big\| \phi(\cdot + v, \cdot + w, s, l) + \phi(\cdot + v, \cdot - w, s, l) + \phi(\cdot - v, \cdot + w, s, l) \\
&\quad + \phi(\cdot - v, \cdot - w, s, l) - 4\phi(\cdot s, \cdot l) \big\|_r \\
&= O\big(\xi(s,l)\big) \\
&= 8\left(\eta(s,l)\left(\frac{\xi(s,l)}{\eta(s,l)}\right)\right) \\
&= 8\left(\eta(v,w)\left(\frac{\xi(s,l)}{\eta(s,l)}\right)\right).
\end{aligned}$$

Since $\frac{\xi(s,l)}{\eta(s,l)}$ is positive and nondecreasing, if $v \geq s$, $w \geq l$, then $\frac{\xi(s,l)}{\eta(s,l)} \geq \frac{\xi(v,w)}{\eta(v,w)}$. Then by Lemma 3.5(ii) we get that

$$\begin{aligned}
&\big\| \phi(\cdot + v, \cdot + w, s, l) + \phi(\cdot + v, \cdot - w, s, l) + \phi(\cdot - v, \cdot + w, s, l) \\
&\quad + \phi(\cdot - v, \cdot - w, s, l) - 4\phi(\cdot s, \cdot l) \big\|_r \\
&= 8\big(\xi(s,l)\big) \\
&= 8\left(\eta(v,w)\frac{\xi(s,l)}{\eta(s,l)}\right).
\end{aligned}$$

□

4 Proof of main theorem

4.1 Proof of Theorem 2.1

$$s_{p,q}(y,z) - g(y,z) = \frac{1}{\pi^2}\int_0^\pi \int_0^\pi \phi(y,z,v,w)\frac{[\sin(p+\frac{1}{2})s][\sin(q+\frac{1}{2})l]}{4\sin\frac{s}{2}\sin\frac{l}{2}}\,ds\,dl. \tag{14}$$

Taking into account (14) and that $\tau_{p,q}(y,z)$ is double Hausdorff matrix means of $s_{p,q}(y,z)$, we write

$$\begin{aligned}
\tau_{p,q}(y,z) - g(y,z) &= \sum_{j=0}^{p}\sum_{k=0}^{q} h_{p,q,j,k}\big\{s_{j,k}(y,z) - g(y,z)\big\} \\
&= \frac{1}{\pi^2}\int_0^\pi \int_0^\pi \phi(y,z,v,w)\sum_{j=0}^{p}\sum_{k=0}^{q} h_{p,q,j,k}\frac{[\sin(j+\frac{1}{2})s][\sin(k+\frac{1}{2})l]}{4\sin\frac{s}{2}\sin\frac{l}{2}}\,ds\,dl \\
&= \int_0^\pi \int_0^\pi \phi(y,z,s,l)K_{pq}(s,l)\,ds\,dl.
\end{aligned}$$

Let

$$\begin{aligned}
l_{p,q}(y,z) &= t_{p,q}^{\Delta H_{p,q}}(y,z) - g(y,z) \\
&= \int_0^\pi \int_0^\pi \phi(y,z,s,l)K_{pq}(s,l)\,ds\,dl.
\end{aligned}$$

Then

$$\begin{aligned}
&l_{p,q}(y+v,z+w)+l_{p,q}(y+v,z-w)+l_{p,q}(y-v,z+w)+l_{p,q}(y-v,z-w)-4l_{p,q}(y,z)\\
&\quad=\int_0^{\pi}\int_0^{\pi}\big(\phi(y+v,z+w,s,l)+\phi(y+v,z-w,s,l)+\phi(y-v,z+w,s,l)\\
&\quad\quad+\phi(y-v,z-w,s,l)-4\phi(y,z,s,l)\big)K_{pq}(s,l)\,ds\,dl.
\end{aligned}$$

By generalized Minkowski's inequality

$$\begin{aligned}
&\big\|l_{p,q}(\cdot+v,\cdot+w)+l_{p,q}(\cdot+v,\cdot-w)+l_{p,q}(\cdot-v,\cdot+w)+l_{p,q}(\cdot-v,\cdot-w)-4l_{p,q}(\cdot,\cdot)\big\|_r\\
&\quad\leq\int_0^{\pi}\int_0^{\pi}\big\|\phi(\cdot+v,\cdot+w,s,l)+\phi(\cdot+v,\cdot-w,s,l)+\phi(\cdot-v,\cdot+w,s,l)\\
&\quad\quad+\phi(\cdot-v,\cdot-w,s,l)-4\phi(\cdot,s,\cdot,l)\big\|_r K_{pq}(s,l)\,ds\,dl\\
&\quad=\bigg(\int_0^{\frac{1}{p+1}}\int_0^{\frac{1}{q+1}}+\int_0^{\frac{1}{p+1}}\int_{\frac{1}{q+1}}^{\pi}+\int_{\frac{1}{p+1}}^{\pi}\int_0^{\frac{1}{q+1}}+\int_{\frac{1}{p+1}}^{\pi}\int_{\frac{1}{q+1}}^{\pi}\bigg)\\
&\quad\quad\times\big\|\phi(\cdot+v,\cdot+w,s,l)+\phi(\cdot+v,\cdot-w,s,l)+\phi(\cdot-v,\cdot+w,s,l)\\
&\quad\quad+\phi(\cdot-v,\cdot-w,s,l)-4\phi(\cdot,s,\cdot,l)\big\|_r K_{pq}(s,l)\,ds\,dl\\
&\quad=I_1+I_2+I_3+I_4.
\end{aligned}\tag{15}$$

Using Lemma 3.1 and Lemma 3.5(iii), we obtain

$$\begin{aligned}
I_1&=\int_0^{\frac{1}{p+1}}\int_0^{\frac{1}{q+1}}\big\|\phi(\cdot+v,\cdot+w,s,l)+\phi(\cdot+v,\cdot-w,s,l)+\phi(\cdot-v,\cdot+w,s,l)\\
&\quad+\phi(\cdot-v,\cdot-w,s,l)+\phi(\cdot-v,\cdot-w,s,l)-4\phi(\cdot,s,\cdot,l)\big\|_r K_{pq}(s,l)\,ds\,dl\\
&=8\bigg(\int_0^{\frac{1}{p+1}}\int_0^{\frac{1}{q+1}}\eta(v,w)\frac{\xi(s,l)}{\eta(s,l)}(p+1)(q+1)\,ds\,dl\bigg)\\
&=8\bigg((p+1)(q+1)\eta(v,w)\int_0^{\frac{1}{p+1}}\int_0^{\frac{1}{q+1}}\frac{\xi(s,l)}{\eta(s,l)}\,ds\,dl\bigg)\\
&=8\bigg((p+1)(q+1)\eta(v,w)\bigg\{\int_0^{\frac{1}{p+1}}\frac{\xi(\frac{1}{p+1},l)}{\eta(\frac{1}{q+1},l)}\bigg\}\,dl\int_0^{\frac{1}{q+1}}\,ds\bigg)\\
&=8\bigg((p+1)\eta(v,w)\bigg\{\int_0^{\frac{1}{p+1}}\frac{\xi(\frac{1}{p+1},l)}{\eta(\frac{1}{p+1},l)}\bigg\}\,dl\bigg)\\
&=8\bigg((p+1)\eta(v,w)\frac{\xi(\frac{1}{p+1},\frac{1}{q+1})}{\eta(\frac{1}{p+1},\frac{1}{q+1})}\bigg\{\int_0^{\frac{1}{p+1}}\,dl\bigg\}\bigg),\\
I_1&=8\bigg(\eta(v,w)\frac{\xi(\frac{1}{p+1},\frac{1}{q+1})}{\xi(\frac{1}{p+1},\frac{1}{q+1})}\bigg).
\end{aligned}\tag{16}$$

Using Lemma 3.3 and Lemma 3.5(iii), we obtain

$$\begin{aligned}
I_2 &= \int_0^{\frac{1}{p+1}} \int_{\frac{1}{q+1}}^{\pi} \big\| \phi(\cdot + v, \cdot + w, s, l) + \phi(\cdot + v, \cdot - w, s, l) + \phi(\cdot - v, \cdot + w, s, l) \\
&\quad + \phi(\cdot - v, \cdot - w, s, l) + \phi(\cdot - v, \cdot - w, s, l) - 4\phi(\cdot, s, \cdot, l) \big\|_r K_{pq}(s,l)\, ds\, dl \\
&= 8\biggl(\int_0^{\frac{1}{p+1}} \int_{\frac{1}{q+1}}^{\pi} \eta(v,w) \frac{\xi(s,l)}{\eta(s,l)} (p+1) \frac{1}{(q+1)l^2}\, ds\, dl \biggr) \\
&= 8\biggl(\biggl(\frac{p+1}{q+1} \biggr) \eta(v,w) \int_{\frac{1}{q+1}}^{\pi} \frac{\xi(\frac{1}{p+1}, l)}{\eta(\frac{1}{p+1}, l)} \frac{1}{l^2}\, dl \biggr).
\end{aligned} \tag{17}$$

Using Lemma 3.2 and Lemma 3.5(iii), we obtain

$$\begin{aligned}
I_3 &= \int_{\frac{1}{p+1}}^{\pi} \int_0^{\frac{1}{q+1}} \big\| \phi(\cdot + v, \cdot + w, s, l) + \phi(\cdot + v, \cdot - w, s, l) + \phi(\cdot - v, \cdot + w, s, l) \\
&\quad + \phi(\cdot - v, \cdot - w, s, l) + \phi(\cdot - v, \cdot - w, s, l) - 4\phi(\cdot, s, \cdot, l) \big\|_r K_{pq}(s,l)\, ds\, dl \\
&= 8\biggl(\int_{\frac{1}{p+1}}^{\pi} \int_0^{\frac{1}{q+1}} \eta(v,w) \frac{\xi(s,l)}{\eta(s,l)} (q+1) \frac{1}{(p+1)s^2}\, ds\, dl \biggr) \\
&= 8\biggl(\biggl(\frac{q+1}{p+1} \biggr) \eta(v,w) \int_{\frac{1}{p+1}}^{\pi} \frac{\xi(s, \frac{1}{q+1})}{\eta(s, \frac{1}{q+1})} \frac{1}{s^2}\, ds \biggr).
\end{aligned} \tag{18}$$

Using Lemma 3.4 and Lemma 3.5(iii), we obtain

$$\begin{aligned}
I_4 &= \int_{\frac{1}{p+1}}^{\pi} \int_{\frac{1}{q+1}}^{\pi} \big\| \phi(\cdot + v, \cdot + w, s, l) + \phi(\cdot + v, \cdot - w, s, l) + \phi(\cdot - v, \cdot + w, s, l) \\
&\quad + \phi(\cdot - v, \cdot - w, s, l) + \phi(\cdot - v, \cdot - w, s, l) - 4\phi(\cdot, s, \cdot, l) \big\|_r K_{pq}(s,l)\, ds\, dl \\
&= 8\biggl(\int_{\frac{1}{p+1}}^{\pi} \int_{\frac{1}{q+1}}^{\pi} \eta(v,w) \frac{\xi(s,l)}{\eta(s,l)} \frac{1}{(p+1)s^2} \frac{1}{(q+1)l^2}\, ds\, dl \biggr) \\
&= 8\biggl(\frac{1}{(p+1)} \frac{1}{(q+1)} \eta(v,w) \int_{\frac{1}{p+1}}^{\pi} \int_{\frac{1}{q+1}}^{\pi} \frac{\xi(s,l)}{\eta(s,l)} \frac{1}{s^2} \frac{1}{l^2}\, ds\, dl \biggr).
\end{aligned} \tag{19}$$

By (15), (16), (17), (18), and (19), we have

$$\begin{aligned}
&\big\| l_{p,q}(\cdot + v, \cdot + w) + l_{p,q}(\cdot + v, \cdot - w) + l_{p,q}(\cdot - v, \cdot + w) + l_{p,q}(\cdot - v, \cdot - w) - 4 l_{p,q}(\cdot, \cdot) \big\|_r \\
&\quad = 8\biggl(\eta(v,w) \frac{\eta(\frac{1}{p+1}, \frac{1}{q+1})}{\eta(\frac{1}{p+1}, \frac{1}{q+1})} \biggr) + 8\biggl(\biggl(\frac{p+1}{q+1} \biggr) \eta(v,w) \int_{\frac{1}{q+1}}^{\pi} \frac{\xi(\frac{1}{p+1}, l)}{\eta(\frac{1}{p+1}, l)} \frac{1}{l^2}\, dl \biggr) \\
&\qquad + 8\biggl(\biggl(\frac{q+1}{p+1} \biggr) \eta(v,w) \int_{\frac{1}{p+1}}^{\pi} \frac{\xi(s, \frac{1}{q+1})}{\eta(s, \frac{1}{q+1})} \frac{1}{s^2}\, ds \biggr) \\
&\qquad + 8\biggl(\frac{1}{(p+1)(q+1)} \eta(v,w) \int_{\frac{1}{p+1}}^{\pi} \int_{\frac{1}{q+1}}^{\pi} \frac{\xi(s,l)}{\eta(s,l)} \frac{1}{s^2} \frac{1}{l^2}\, ds\, dl \biggr).
\end{aligned} \tag{20}$$

Thus

$$\begin{aligned}
&\sup_{\substack{v\neq 0\\ w\neq 0}} \frac{\|l_{p,q}(\cdot+v,\cdot+w)+l_{p,q}(\cdot+v,\cdot-w)+l_{p,q}(\cdot-v,\cdot+w)+l_{p,q}(\cdot-v,\cdot-w)-4l_{p,q}(\cdot,\cdot)\|_r}{\eta(v,w)}\\
&\quad= 8\left(\frac{\xi(\frac{1}{p+1},\frac{1}{q+1})}{\eta(\frac{1}{p+1},\frac{1}{q+1})}\right)+O\left(\left(\frac{p+1}{q+1}\right)\int_{\frac{1}{q+1}}^{\pi}\frac{\xi(\frac{1}{p+1},l)}{\eta(\frac{1}{p+1},l)}\frac{1}{l^2}\,dl\right)\\
&\qquad+8\left(\left(\frac{q+1}{p+1}\right)\int_{\frac{1}{p+1}}^{\pi}\frac{\xi(s,\frac{1}{q+1})}{\eta(s,\frac{1}{q+1})}\frac{1}{s^2}\,ds\right)\\
&\qquad+O\left(\frac{1}{(p+1)(q+1)}\int_{\frac{1}{p+1}}^{\pi}\int_{\frac{1}{q+1}}^{\pi}\frac{\xi(s,l)}{\eta(s,l)}\frac{1}{s^2}\frac{1}{l^2}\,ds\,dl\right).
\end{aligned}\tag{21}$$

Using Lemmas 3.1–3.4 and Lemma 3.5(i), we obtain

$$\begin{aligned}
&\|l_{p,q}(\cdot,\cdot)\|_r\\
&\quad=\|t_{p,q}^{\Delta H_{p,q}}-g\|_r\\
&\quad\leq\left(\int_0^{\frac{1}{p+1}}\int_0^{\frac{1}{q+1}}+\int_0^{\frac{1}{p+1}}\int_{\frac{1}{q+1}}^{\pi}+\int_{\frac{1}{p+1}}^{\pi}\int_0^{\frac{1}{q+1}}+\int_{\frac{1}{p+1}}^{\pi}\int_{\frac{1}{q+1}}^{\pi}\right)\|\phi(\cdot,s,\cdot,l)\|_r|K_{pq}(s,l)|\,ds\,dl\\
&\quad=\int_0^{\frac{1}{p+1}}\int_0^{\frac{1}{q+1}}\|\phi(\cdot,s\cdot,l)\|_r|K_{pq}(s,l)|\,ds\,dl+\int_0^{\frac{1}{p+1}}\int_{\frac{1}{q+1}}^{\pi}\|\phi(\cdot,s,\cdot,l)\|_r|K_{pq}(s,l)|\,ds\,dl\\
&\qquad+\int_{\frac{1}{p+1}}^{\pi}\int_0^{\frac{1}{q+1}}\|\phi(\cdot,s\cdot,l)\|_r|K_{pq}(s,l)|\,ds\,dl+\int_{\frac{1}{p+1}}^{\pi}\int_{\frac{1}{q+1}}^{\pi}\|\phi(\cdot,s,\cdot,l)\|_r|K_{pq}(s,l)|\,ds\,dl\\
&\quad=O\left((p+1)(q+1)\int_0^{\frac{1}{p+1}}\int_0^{\frac{1}{q+1}}\xi(s,l)\,ds\,dl\right)+O\left(\left(\frac{p+1}{q+1}\right)\int_0^{\frac{1}{p+1}}\int_{\frac{1}{q+1}}^{\pi}\frac{\xi(s,l)}{l^2}\,ds\,dl\right)\\
&\qquad+O\left(\left(\frac{q+1}{p+1}\right)\int_{\frac{1}{p+1}}^{\pi}\int_0^{\frac{1}{q+1}}\frac{\xi(s,l)}{s^2}\,ds\,dl\right)+O\left(\frac{1}{(p+1)(q+1)}\int_{\frac{1}{p+1}}^{\pi}\int_{\frac{1}{q+1}}^{\pi}\frac{\xi(s,l)}{s^2l^2}\,ds\,dl\right)\\
&\quad=O\left(\xi\left(\frac{1}{p+1},\frac{1}{q+1}\right)\right)+O\left(\left(\frac{p+1}{q+1}\right)\int_0^{\frac{1}{p+1}}\int_{\frac{1}{q+1}}^{\pi}\frac{\xi(s,l)}{l^2}\,ds\,dl\right)\\
&\qquad+O\left(\left(\frac{q+1}{p+1}\right)\int_{\frac{1}{p+1}}^{\pi}\int_0^{\frac{1}{q+1}}\frac{\xi(s,l)}{s^2}\,ds\,dl\right)+O\left(\frac{1}{(p+1)(q+1)}\int_{\frac{1}{p+1}}^{\pi}\int_{\frac{1}{q+1}}^{\pi}\frac{\xi(s,l)}{s^2l^2}\,ds\,dl\right)\\
&\quad=O\left(\xi\left(\frac{1}{p+1},\frac{1}{q+1}\right)\right)+O\left(\left(\frac{p+1}{q+1}\right)\int_{\frac{1}{q+1}}^{\pi}\frac{\xi(\frac{1}{p+1},t)}{l^2}\,dl\right)\\
&\qquad+O\left(\left(\frac{q+1}{p+1}\right)\int_{\frac{1}{p+1}}^{\pi}\frac{\xi(s,\frac{1}{q+1})}{s^2}\,ds\right)\\
&\qquad+O\left(\frac{1}{(p+1)(q+1)}\int_{\frac{1}{p+1}}^{\pi}\int_{\frac{1}{q+1}}^{\pi}\frac{\xi(s,l)}{s^2l^2}\,ds\,dl\right),
\end{aligned}\tag{22}$$

$$\begin{aligned}
&\|l_{p,q}(\cdot,\cdot)\|_r^{(\eta)}\\
&\quad=\|l_{p,q}(\cdot,\cdot)\|_r
\end{aligned}$$

$$
\begin{aligned}
&+ \sup_{\substack{v\neq 0\\ w\neq 0}} \frac{\|l_{p,q}(\cdot+v,\cdot+w)+l_{p,q}(\cdot+v,\cdot-w)+l_{p,q}(\cdot-v,\cdot+w)+l_{p,q}(\cdot-v,\cdot-w)-4l_{p,q}(\cdot,\cdot)\|_r}{\eta(v,w)}\\
&= O\biggl(\xi\biggl(\frac{1}{p+1},\frac{1}{q+1}\biggr)\biggr) + O\biggl(\biggl(\frac{p+1}{q+1}\biggr)\int_{\frac{1}{q+1}}^{\pi}\frac{\xi(\frac{1}{p+1},l)}{l^2}\,dl\biggr)\\
&\quad + O\biggl(\biggl(\frac{q+1}{p+1}\biggr)\int_{\frac{1}{p+1}}^{\pi}\frac{\xi(s,\frac{1}{q+1})}{s^2}\,ds\biggr) + O\biggl(\frac{1}{(p+1)(q+1)}\int_{\frac{1}{p+1}}^{\pi}\int_{\frac{1}{q+1}}^{\pi}\frac{\xi(s,l)}{s^2l^2}\,ds\,dl\biggr)\\
&\quad + 8\biggl(\frac{\xi(\frac{1}{p+1},\frac{1}{q+1})}{\eta(\frac{1}{p+1},\frac{1}{q+1})}\biggr) + 8\biggl(\biggl(\frac{p+1}{q+1}\biggr)\int_{\frac{1}{q+1}}^{\pi}\frac{\xi(\frac{1}{p+1},l)}{\eta(\frac{1}{p+1},l)}\frac{1}{l^2}\,dl\biggr)\\
&\quad + 8\biggl(\biggl(\frac{q+1}{p+1}\biggr)\int_{\frac{1}{p+1}}^{\pi}\frac{\xi(s,\frac{1}{q+1})}{\eta(s,\frac{1}{q+1})}\frac{1}{s^2}\,ds\biggr)\\
&\quad + 8\biggl(\frac{1}{(p+1)(q+1)}\int_{\frac{1}{p+1}}^{\pi}\int_{\frac{1}{q+1}}^{\pi}\frac{\xi(s,l)}{\eta(s,l)}\frac{1}{s^2l^2}\biggr).
\end{aligned}
\tag{23}
$$

Since $\xi(s,l)=\frac{\xi(s,l)}{\eta(s,l)}$ and $\eta(s,l)\le\eta(\pi,\pi)\frac{\xi(s,l)}{\eta(s,l)}$, $0<s\le\pi$, $0<l\le\pi$, we get

$$
\begin{aligned}
\|l_{p,q}(\cdot,\cdot)\|_r^{(\eta)} &= O\biggl(\frac{\xi(\frac{1}{p+1},\frac{1}{q+1})}{\eta(\frac{1}{p+1},\frac{1}{q+1})}\biggr) + O\biggl(\biggl(\frac{p+1}{q+1}\biggr)\int_{\frac{1}{q+1}}^{\pi}\frac{\xi(\frac{1}{p+1},l)}{\eta(\frac{1}{p+1},l)}\frac{1}{l^2}\,dl\biggr)\\
&\quad + O\biggl(\biggl(\frac{q+1}{p+1}\biggr)\int_{\frac{1}{p+1}}^{\pi}\frac{\xi(s,\frac{1}{q+1})}{\eta(s,\frac{1}{q+1})}\frac{1}{s^2}\,ds\biggr)\\
&\quad + O\biggl(\frac{1}{(p+1)(q+1)}\int_{\frac{1}{p+1}}^{\pi}\int_{\frac{1}{q+1}}^{\pi}\frac{\xi(s,l)}{\eta(s,l)}\frac{1}{s^2l^2}\,ds\,dl\biggr).
\end{aligned}
\tag{24}
$$

Since ξ and η are the Zygmund moduli of continuity, $\frac{\xi(s,l)}{\eta(s,l)}$ is positive and nondecreasing, and therefore

$$
\begin{aligned}
\biggl(\frac{p+1}{q+1}\biggr)\int_{\frac{1}{q+1}}^{\pi}\frac{\xi(\frac{1}{p+1},l)}{\eta(\frac{1}{p+1},l)}\frac{1}{l^2}\,dl &\ge \frac{\xi(\frac{1}{p+1},\frac{1}{q+1})}{\eta(\frac{1}{p+1},\frac{1}{q+1})}\biggl(\frac{p+1}{q+1}\biggr)\int_{\frac{1}{q+1}}^{\pi}\frac{dl}{l^2}\\
&\ge \frac{\xi(\frac{1}{p+1},\frac{1}{q+1})}{2\eta(\frac{1}{p+1},\frac{1}{q+1})}(p+1).
\end{aligned}
\tag{25}
$$

Then

$$
O\biggl(\biggl(\frac{p+1}{q+1}\biggr)\int_{\frac{1}{q+1}}^{\pi}\frac{\xi(\frac{1}{p+1},l)}{\eta(\frac{1}{p+1},l)}\frac{1}{l^2}\,dl\biggr) = O\biggl(\frac{(\frac{p+1}{2})\xi(\frac{1}{p+1},\frac{1}{q+1})}{\eta(\frac{1}{p+1},\frac{1}{q+1})}\biggr).
\tag{26}
$$

Since ξ and η are the Zygmund moduli of continuity, $\frac{\xi(s,l)}{\eta(s,l)}$ is positive and nondecreasing, and therefore

$$
\begin{aligned}
\biggl(\frac{q+1}{p+1}\biggr)\int_{\frac{1}{p+1}}^{\pi}\frac{\xi(s,\frac{1}{q+1})}{\eta(s,\frac{1}{q+1})}\frac{1}{s^2}\,ds &\ge \frac{\xi(\frac{1}{p+1},\frac{1}{q+1})}{\eta(\frac{1}{p+1},\frac{1}{q+1})}\biggl(\frac{q+1}{p+1}\biggr)\int_{\frac{1}{p+1}}^{\pi}\frac{ds}{s^2}\\
&\ge \frac{\xi(\frac{1}{p+1},\frac{1}{q+1})}{2\eta(\frac{1}{p+1},\frac{1}{q+1})}(q+1).
\end{aligned}
\tag{27}
$$

Then

$$O\left(\left(\frac{q+1}{p+1}\right)\int_{\frac{1}{p+1}}^{\pi}\frac{\xi(s,\frac{1}{q+1})}{\eta(s,\frac{1}{q+1})}\frac{1}{s^2}\,ds\right)=O\left(\frac{(\frac{q+1}{2})\xi(\frac{1}{p+1},\frac{1}{q+1})}{\eta(\frac{1}{p+1},\frac{1}{q+1})}\right). \tag{28}$$

Since ξ and η are the Zygmund moduli of continuity, $\frac{\xi(s,l)}{\eta(s,l)}$ is positive and nondecreasing, and therefore

$$\begin{aligned}\frac{1}{(p+1)(q+1)}\int_{\frac{1}{p+1}}^{\pi}\int_{\frac{1}{q+1}}^{\pi}\frac{\xi(s,l)}{\eta(s,l)}\frac{1}{s^2l^2}\,ds\,dl &\geq \frac{\xi(\frac{1}{p+1},\frac{1}{q+1})}{\eta(\frac{1}{p+1},\frac{1}{q+1})}\frac{1}{(p+1)(q+1)}\int_{\frac{1}{p+1}}^{\pi}\int_{\frac{1}{q+1}}^{\pi}\frac{ds}{s^2}\frac{dl}{l^2}\\ &\geq \frac{\xi(\frac{1}{p+1},\frac{1}{q+1})}{4\eta(\frac{1}{p+1},\frac{1}{q+1})}.\end{aligned} \tag{29}$$

Then

$$O\left(\frac{\xi(\frac{1}{p+1},\frac{1}{q+1})}{\eta(\frac{1}{p+1},\frac{1}{q+1})}\right)=O\left(\frac{1}{(p+1)(q+1)}\int_{\frac{1}{p+1}}^{\pi}\int_{\frac{1}{q+1}}^{\pi}\frac{\xi(s,l)}{\eta(s,l)}\frac{1}{s^2}\frac{1}{l^2}\,ds\,dl\right). \tag{30}$$

By equations (24), (26), (28), and (30) we get

$$\begin{aligned}\left\|l_{p,q}(\cdot,\cdot)\right\|_r^{(\eta)} &= O\left(\frac{\xi(\frac{1}{p+1},\frac{1}{q+1})}{\eta(\frac{1}{p+1},\frac{1}{q+1})}\right)+O\left(\frac{(\frac{p+1}{2})\xi(\frac{1}{p+1},\frac{1}{q+1})}{\eta(\frac{1}{p+1},\frac{1}{q+1})}\right)\\ &\quad+O\left(\frac{(\frac{q+1}{2})\xi(\frac{1}{p+1},\frac{1}{q+1})}{\eta(\frac{1}{p+1},\frac{1}{q+1})}\right)+O\left(\frac{1}{(p+1)(q+1)}\int_{\frac{1}{p+1}}^{\pi}\int_{\frac{1}{q+1}}^{\pi}\frac{\xi(s,l)}{\eta(s,l)}\frac{1}{s^2}\frac{1}{l^2}\,ds\,dl\right)\\ &= O\left(1+\frac{p+1}{2}+\frac{q+1}{2}\right)\frac{\xi(\frac{1}{p+1},\frac{1}{q+1})}{\eta(\frac{1}{p+1},\frac{1}{q+1})}\\ &\quad+O\left(\frac{1}{(p+1)(q+1)}\int_{\frac{1}{p+1}}^{\pi}\int_{\frac{1}{q+1}}^{\pi}\frac{\xi(s,l)}{\eta(s,l)}\frac{1}{s^2}\frac{1}{l^2}\,ds\,dl\right)\\ &= O\left(1+\frac{p+1}{2}+\frac{q+1}{2}\right)O\left(\frac{1}{(p+1)(q+1)}\int_{\frac{1}{p+1}}^{\pi}\int_{\frac{1}{q+1}}^{\pi}\frac{\xi(s,l)}{\eta(s,l)}\frac{1}{s^2}\frac{1}{l^2}\,ds\,dl\right)\\ &\quad+O\left(\frac{1}{(p+1)(q+1)}\int_{\frac{1}{p+1}}^{\pi}\int_{\frac{1}{q+1}}^{\pi}\frac{\xi(s,l)}{\eta(s,l)}\frac{1}{s^2}\frac{1}{l^2}\,ds\,dl\right)\\ &= O\left(1+\frac{p+1}{2}+\frac{q+1}{2}+1\right)O\left(\frac{1}{(p+1)(q+1)}\int_{\frac{1}{p+1}}^{\pi}\int_{\frac{1}{q+1}}^{\pi}\frac{\xi(s,l)}{\eta(s,l)}\frac{1}{s^2}\frac{1}{l^2}\,ds\,dl\right)\\ &= O\left(\frac{p+q+6}{(p+1)(q+1)}\right)O\left(\int_{\frac{1}{p+1}}^{\pi}\int_{\frac{1}{q+1}}^{\pi}\frac{\xi(s,l)}{\eta(s,l)}\frac{1}{s^2}\frac{1}{l^2}\,ds\,dl\right).\end{aligned} \tag{31}$$

4.2 Proof of Theorem 2.2

$$E_{p,q}(g)=\left\|l_{p,q}(\cdot,\cdot)\right\|_r^{(\eta)}=O\left(\frac{p+q+6}{(p+1)(q+1)}\right)O\left(\int_{\frac{1}{p+1}}^{\pi}\int_{\frac{1}{q+1}}^{\pi}\frac{\xi(s,l)}{\eta(s,l)}\frac{1}{s^2}\frac{1}{l^2}\,ds\,dl\right).$$

Since $\frac{\xi(s,l)}{l\eta(s,l)}$ is positive and nonincreasing, by the second mean value theorem of integral calculus we have

$$\begin{aligned}E_{p,q}(g) &= O\left(\frac{p+q+6}{(p+1)(q+1)}\right)O\left(\int_{\frac{1}{p+1}}^{\pi}\frac{1}{s^2}\left(\int_{\frac{1}{q+1}}^{\pi}\frac{\xi(s,l)}{\eta(s,l)}\frac{1}{l^2}\,dl\right)ds\right)\\ &= O\left(\frac{p+q+6}{(p+1)(q+1)}\right)O\left(\int_{\frac{1}{p+1}}^{\pi}\frac{1}{s^2}\left((q+1)\frac{\xi(s,\frac{1}{q+1})}{\eta(s,\frac{1}{q+1})}\int_{\frac{1}{q+1}}^{\pi}\frac{dl}{l}\right)ds\right)\\ &= O\left(\frac{p+q+6}{(p+1)}\right)O\left(\int_{\frac{1}{p+1}}^{\pi}\frac{1}{s^2}\left(\frac{\xi(s,\frac{1}{q+1})}{\eta(s,\frac{1}{q+1})}\log\pi(q+1)\right)ds\right).\end{aligned} \tag{32}$$

Again, since $\frac{\xi(s,l)}{s\eta(s,l)}$ is positive and nonincreasing, by the second mean value theorem of integral calculus we have

$$\begin{aligned}E_{p,q}(f) &= O\left(\frac{p+q+6}{(p+1)}\right)O\left(\log\pi(q+1)\int_{\frac{1}{p+1}}^{\pi}\frac{\xi(s,\frac{1}{p+1})}{\eta(s,\frac{1}{q+1})}\frac{1}{s^2}\,ds\right)\\ &= O\left(\frac{p+q+6}{(p+1)}\right)O\left(\log\pi(q+1)(p+1)\frac{\xi(\frac{1}{p+1},\frac{1}{q+1})}{\eta(\frac{1}{p+1},\frac{1}{q+1})}\int_{\frac{1}{p+1}}^{\pi}\frac{ds}{s}\right)\\ &= O\left((p+q+6)\log\pi(q+1)\frac{\xi(\frac{1}{p+1},\frac{1}{q+1})}{\eta(\frac{1}{p+1},\frac{1}{q+1})}\log\pi(p+1)\right)\\ &= O\left(\frac{\xi(\frac{1}{p+1},\frac{1}{nq+1})}{\eta(\frac{1}{p+1},\frac{1}{q+1})}(p+q+6)\log\pi(p+q+2)\right).\end{aligned} \tag{33}$$

5 Corollaries

Corollary 5.1 *Following Remark* 1.1(i), *we obtain*

$$\left\|t_{p,q}^{(C,1,1)}(y,z)-g(y,z)\right\|_r^{(\eta)} = O\left(\frac{p+q+6}{(p+1)(q+1)}\int_{\frac{1}{p+1}}^{\pi}\int_{\frac{1}{q+1}}^{\pi}\frac{\xi(s,l)}{\eta(s,l)s^2l^2}\right),$$

where ξ and η denote the moduli of continuity of second order such that $\xi(s,l)/\eta(s,l)$ is positive and nondecreasing.

Corollary 5.2 *Following the Remark* 1.1(i), *we obtain*

$$\left\|t_{p,q}^{(E,r,r)}(y,z)-g(y,z)\right\|_r^{(\eta)} = O\left(\frac{p+q+6}{(p+1)(q+1)}\int_{\frac{1}{p+1}}^{\pi}\int_{\frac{1}{q+1}}^{\pi}\frac{\xi(s,l)}{\eta(s,l)s^2l^2}\right),$$

where ξ and η denote the moduli of continuity of second order such that $\xi(s,l)/\eta(s,l)$ is positive and nondecreasing.

6 Conclusion

In this paper, we established the error estimate of a two-variable function $g(y,z)$ in the generalized Zygmund class $Y_r^{(\xi)}$ ($r \geq 1$) using the double Hausdorff matrix means of its double Fourier series. We have proved two results on error estimates of a two-variable function of g in the generalized Zygmund class.

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Authors' information
H.K. Nigam, hknigam@cusb.ac.in; Supriya Rani, supriya@cusb.ac.in.

Author details
[1]Department of Mathematics, Central University of South Bihar, Gaya, India. [2]Department of Medical Research, China Medical University Hospital, China Medical University (Taiwan), Taichung, Taiwan. [3]Department of Mathematics, Aligarh Muslim University, Aligarh, India. [4]Department of Computer Science and Information Engineering, Asia University, Taichung, Taiwan.

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