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A dynamic model of viruses with the effect of removable media on a computer network with heterogeneous immunity

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Abstract

In this paper, we build a mathematical model to study the impact of external removable devices on a network with weakly- and strongly-protected computers. The model describes the dynamics between weak, strong, infected computers and susceptible, infected removable media. Analytical investigations of the model produce two equilibrium points: virus-free and endemic. Moreover, we investigate the local and global stability of both equilibria. The existence and stability conditions of the equilibrium points depend primarily on the basic reproduction number (R_0) of the model. Furthermore, we perform numerical simulations to substantiate the analytical results. Also, a sensitivity analysis is carried out to examine the critical parameters that lead to strategies to control the dissipation of viruses.

Keywords: Computer virus; Propagation model; Equilibrium; Local stability; Global stability

1 Introduction

The development of communication networks has made computers increasingly present in our daily lives. The human dependence on many electronic devices, such as mobile phones and laptops, is increasing every day since almost everything can be accessed online. For these reasons, a computer virus has become a major problem for individuals, institutions and even governments. A virus is a malicious program developed to harm the operating systems of computers and mobile phones. It plays the same role as a biological virus in the human body. Computer viruses slow down or crash the operating system, erase data, steal information, disrupt normal operations, spy on users with webcams, microphones, and other damages. According to the Ponemon Institute, 7 out of 10 organizations say that security risks increased dramatically in 2017 [1]. In 2017, the Wannacry virus infected more than 400,000 devices from 100,000 groups in at least 150 countries, at a total damage cost of about 4 billion [2].

The propagation of computer viruses is similar to the spread of infectious diseases in which viruses are transmitted through communication. Computer viruses are usually transmitted in networks via e-mail messages and when downloading an infected file. In

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addition, connecting a computer to removable devices such as USB, mobile phones and hard drives is another way to spread viruses. Anti-virus software always strives to protect computers from viruses. However, due to the emergence of new viruses, this keeps the anti-virus programs lagging behind the new ones. As a result, this requires the user to update their anti-virus software regularly.

Mathematical experiments are necessary to understand the dynamics of computer virus spread in order to improve the safety and reliability of a computer network. Cohen [3] and Murray [4] were among the first to pave the way for the use of mathematical models of infectious diseases to explore the dynamics of the spread of computer viruses due to their similarity. Consequently, the computer's population is divided into compartments according to their status such as susceptible (S), latent (L), infectious (I), recovered (R), antidotal (A), and breaking out (B). By considering the dynamics between these compartments, different types of models have been studied [5–15]. In particular, Kephart and White [5] presented an SIS model of the biological epidemic to explore the way computer viruses spread on the Internet. L. Yang et al. [9] proposed an SLB model assuming that both latent and broken computers posses infectivity; however, the cure rate of latent computers is lower than the broken out computers. Zhu et al. [16] investigated theoretically, a computer virus model SIRS with an anti-virus strategy known as the countermeasure competing that was proposed in [17]. Their work provided a theoretical approach to assess the efficiency of different deployment strategies for anti-virus software. In [7], Gan et al. incorporated the probability of immunization into the classic SIRS model to examine the impact of vaccination (i.e. the measure that an uninfected computer has the latest version of anti-virus software installed) on the spread of computer viruses. Further, in [8], they studied the same model; however, expressing the infection rate as a generic nonlinear incidence rate. Khanh and Huy [11] investigated an SLIR model of computer network where some nodes have antidote rates to represent a vulnerability to viruses in the system. Meanwhile, in [12, 13], antidotal computers are expressed as a separate compartment generating an SAIR model. Recently, Upadhyay and Singh [18] proposed a model with two different classes of computer nodes, the attacking and targeting nodes. They studied the virus propagation from an attack node into the targeted system.

Furthermore, Liu et al. [6] proposed a different mathematical model that examines the impact of user security awareness on the spread of infectious malware. To understand how user security awareness affects the spread of malware, they have divided computers that are vulnerable to viruses into two compartments in terms of protection. They suggested two levels of protection: weakly-protected and strongly-protected, where they assumed that the latter has a lower infection rate than the former.

Other mathematical models studied the dynamics between computers and external removable devices because these devices could be affected by viruses [19], which result in computer infection. For example, Zhu et al. [20] expressed explicitly in their SIR model compartments that represents removable devices. A similar model is given in [21] but with an added compartment expressing latent computers. However, in [22, 23], the effect of removable media is analyzed without expressing it as a separate compartment. Also, in [24], Gan and Yang incorporated the effects of removable storage devices as well as anti-virus software in their SLIR model but not as separate compartments. Recently, in [25], Gan et al. studied the impact of network topology and removable devices on the trans-

mission of viruses. They compared the theoretical results of the model with numerical experiments based on the Oregon routing network in order to verify their model.

In this paper, we build a mathematical model to study the effect of external removable devices on a network with weakly- and strongly-protected computers (i.e. heterogeneous immunity). Our model is an extended model to the compartment model given in [6]. We added to the model in [6] two compartments representing susceptible and infected removable media. We believe that a lack of user awareness regarding the correct use of removable devices plays a fundamental role in the spread of viruses. Connecting infected removable media to the computer causes the virus to spread. In particular, computers with weak protection will be more vulnerable to infection with the virus than those with strong protection. Therefore, to explore the effect of removable devices on the spread of viruses, we need to include removable devices in the dynamics of the model as explicit compartments. Moreover, we assume that strongly-protected computers are unlikely to possess an infection without being first weakly-protected. Therefore, we modify the model in [6] by removing the terms that demonstrate infection of strongly-protected computers due to the connection with infected computers. The study aims to investigate the impact of user awareness on the computer network during a virus outbreak.

The rest of this work is organized as follows. Section 2 deals with formulating the mathematical model with its notations and assumptions. In Sect. 3, the equilibrium points and their stabilities are investigated. Moreover, Sect. 4 illustrates numerical experiments of the model and demonstrates the analysis of the sensitivity of its parameters. Finally, a brief conclusion is given in Sect. 5.

2 Mathematical model

We consider the impact of removable devices in a network with weakly- and strongly-protected computers. We introduce two compartments: susceptible and infected removable devices to the model in [6]. Accordingly, the proposed model is divided into five compartments denoted as follows: $S(t)$, strongly-protected computers (S-node); $W(t)$, weakly-protected computers (W-node); $I(t)$, infected computers (I-node); $R_S(t)$, susceptible removable devices (R_S-node), and $R_I(t)$, infected removable devices (R_I-node). We assume that the total number of computers and removable devices in the network are $N(t)$ and $R_N(t)$, respectively.

Strongly-protected computers are installed with anti-virus software that is updated continuously. Weakly-protected computers are either installed with outdated anti-virus software or without security products. An S-node is infected only when its anti-virus software is outdated; thus, it becomes a W-node. On the contrary, when the anti-virus software of a W-node is updated or installed, it then becomes an S-node. Virus infection is caused due to the connection between the I and W nodes. Also, viruses are transmitted to W-node when an infected removable device is connected to it. When W-node is infected, it becomes I-node. Meanwhile, if I-node is cleaned from all viruses and its anti-virus software is updated, it then becomes an S-node. An infected removable device, R_I-node, is cleaned when it is scanned by anti-virus software that is installed on a strongly-protected computer, S-node; therefore, it becomes an R_S-node. On the other hand, once the susceptible removable device is connected to an infected computer, it is infected again (see Fig. 1).

User security awareness plays a vital role in the dynamics of the model. High security awareness is regarded when users continuously update their anti-virus software and scan

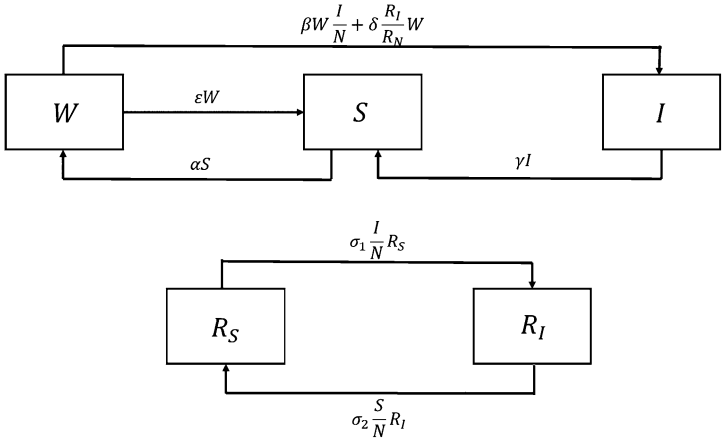

Figure 1 The dynamics of the model

Table 1 Characterization of notations

Notation	Meaning	Unit
$S(t)$	Strongly-protected susceptible computers	In number
$W(t)$	Weakly-protected susceptible computers	In number
$I(t)$	Infected computers	In number
$R_S(t)$	Susceptible removable devices	In number
$R_I(t)$	Infected removable devices	In number
$N(t)$	The total number of computers	In number
$R_N(t)$	The total number of removable devices	In number
β	The infection rate of W-node caused by an infected computer	Day^{-1}
δ	The infection rate of W-node caused by an infected removable device	Day^{-1}
ϵ	The rate that W-node enters S-node	Day^{-1}
α	The rate that S-node converts to W-node	Day^{-1}
γ	The recovery rate of infected computers	Day^{-1}
σ_1	The infection rate of R_S-node caused by an infected computer	Day^{-1}
σ_2	The recovery rate of infected removable devices	Day^{-1}

all removable devices when connected to their computers, and vice versa is considered as acts of low security awareness.

The state variables S, W, I, R_S and R_I are non-negative and the parameters α, β, ϵ, δ, γ, σ_1, σ_2 are positive and lie in the interval $(0, 1]$. A summary of the model's notations is given in Table 1.

The model is based on the following reasonable assumptions:

(H1) The network in this model is static which means that the total number of nodes over the network is invariant.

(H2) An up-to-data anti-virus software is powerful enough to keep S-node computers immune from viruses.

(H3) Every W-node gets infected with probability β per day due to possible connection with I-node.

(H4) Due to the contact with an infected removable device, W-node computers become infected with constant probability δ per day.

(H5) When anti-virus software is expired or not updated, the computers in S-node go to W-node with rate α.

(H6) W-node computers go to S-node with rate ϵ, when installed by an updated anti-virus software.

(H7) Each infected computer is successfully cured by the effect of anti-virus software with rate γ.

(H8) An infected removable device can become susceptible if it is connected to a strongly-protected computer with rate σ_2 and a susceptible one can become infected if it is connected to an infected computer with rate σ_1.

According to the above assumptions, one can immediately describe the dynamics of the model by the following system of nonlinear ordinary differential equations:

$$\begin{aligned}
&\dot{W} = -\beta W\frac{I}{N} - \delta\frac{R_I}{R_N}W - \epsilon W + \alpha S,\\
&\dot{S} = \epsilon W + \gamma I - \alpha S,\\
&\dot{I} = \beta W\frac{I}{N} + \delta\frac{R_I}{R_N}W - \gamma I,\\
&\dot{R}_S = \sigma_2\frac{S}{N}R_I - \sigma_1\frac{I}{N}R_S,\\
&\dot{R}_I = \sigma_1\frac{I}{N}R_S - \sigma_2\frac{S}{N}R_I,
\end{aligned} \tag{1}$$

where S, W, I, N, R_S, R_I and R_N are abbreviations of $S(t)$, $W(t)$, $I(t)$, $N(t)$, $R_S(t)$, $R_I(t)$ and $R_N(t)$, respectively.

From the assumption (H1), the network is static, thus, the total number of computers connected to the network is constant, i.e., $N(t) = W(t) + S(t) + I(t) = N_0 = constant$ for all $t \geq 0$. This is also the case for the total number of removable devices in the network, i.e., $R_N(t) = R_S(t) + R_I(t) = R_{N0} = constant$. Therefore, model (1) can be normalized by setting the state variables as follows: $w = W/N$, $s = S/N$, $i = I/N$, $R_s = R_S/R_N$ and $R_i = R_I/R_N$. Consequently, model (1) has the equivalent form

$$\begin{aligned}
&\dot{w} = -\beta wi - \delta R_i w - \epsilon w + \alpha s,\\
&\dot{s} = \epsilon w + \gamma i - \alpha s,\\
&\dot{i} = \beta wi + \delta R_i w - \gamma i,\\
&\dot{R}_s = \sigma_2 s R_i - \sigma_1 i R_s,\\
&\dot{R}_i = \sigma_1 i R_s - \sigma_2 s R_i.
\end{aligned} \tag{2}$$

By using the identities $w + s + i = 1$ and $R_s + R_i = 1$ in (2), we can facilitate the study of the model by examining a reduced subsystem which is mainly the compartments where the virus appears. Thus, the model can be expressed by the following limiting system:

$$\begin{aligned}
&\dot{w} = -\beta wi - \delta R_i w - \epsilon w + \alpha(1 - w - i),\\
&\dot{i} = \beta wi + \delta R_i w - \gamma i,\\
&\dot{R}_i = \sigma_1 i(1 - R_i) - \sigma_2 R_i(1 - w - i).
\end{aligned} \tag{3}$$

Let the feasible region for system (3) be

$$\Omega = \{(w, i, R_i) : w \geq 0, i \geq 0, R_i \geq 0, w + i \leq 1, R_i \leq 1\}.$$

It follows, from system (3), that

$$\dot{w}|_{(w=0)} = \alpha(1 - i) \geq 0, \qquad \dot{i}|_{(i=0)} = \delta R_i w \geq 0, \qquad \dot{R}_i|_{(R_i=0)} = \sigma_1 i \geq 0.$$

This implies that, for $t \geq 0$, all solutions that are non-negative remain non-negative.

Now, from system (3), we have

$$\begin{aligned}(w + i + R_i)' &= -\epsilon w + \alpha(1 - w - i) - \gamma i + \sigma_1(1 - R_i) - \sigma_2 R_i(1 - w - i),\\ &\leq -\epsilon w - \gamma i. \end{aligned} \tag{4}$$

On the boundary of Ω, i.e., when $w + i = 1$ and $R_i = 1$, we find that the vector field in (4) points into the interior of Ω. Hence, Ω is positively invariant, that is, every solution of model (3), with initial conditions in Ω, remains there for all $t \geq 0$.

3 Mathematical analysis of the model

In this section, we find the equilibria of model (3). Also, we use the next generation method [26] to calculate the basic reproductive number. Moreover, we examine the local stability of the equilibrium points using the linearization method [27] and Routh–Hurwitz' criterion [28]. The global stability is investigated using Castillo-Chavez et al.'s theorem [29] and Lyapunov's theorem [28].

3.1 Equilibrium points and basic reproductive number

In general, the equilibrium points are obtained by equating the rates in system (3) to zero, that is,

$$\begin{aligned} 0 &= -\beta w i - \delta R_i w - \epsilon w + \alpha(1 - w - i),\\ 0 &= \beta w i + \delta R_i w - \gamma i,\\ 0 &= \sigma_1 i(1 - R_i) - \sigma_2 R_i(1 - w - i). \end{aligned} \tag{5}$$

By letting $i = 0$ in system (5), we obtain the virus-free equilibrium point $E_0 = (\frac{\alpha}{\epsilon+\alpha}, 0, 0)$, which exists always. Next, we apply the next generation method on system (3) in order to compute the basic reproductive number R_0. Let $x = (i, R_i)^T$, then system (3) can be written as

$$x' = F(x) - V(x),$$

where

$$F(x) = \begin{pmatrix} \beta w i + \delta R_i w \\ 0 \end{pmatrix},$$

$$V(x) = \begin{pmatrix} \gamma i \\ -\sigma_1(1 - R_i) + \sigma_2 R_i(1 - w - i) \end{pmatrix}.$$

The Jacobian matrices of $F(x)$ and $V(x)$ at the virus-free equilibrium point E_0 are, respectively,

$$f = \begin{pmatrix} \beta\frac{\alpha}{\epsilon+\alpha} & \delta\frac{\alpha}{\epsilon+\alpha} \\ 0 & 0 \end{pmatrix}, \qquad v = \begin{pmatrix} \gamma & 0 \\ -\sigma_1 & \sigma_2(1-\frac{\alpha}{\epsilon+\alpha}) \end{pmatrix}.$$

Consequently, the next generation matrix is $G = f.v^{-1}$, i.e.,

$$G = \begin{pmatrix} \frac{\beta\alpha\sigma_2\epsilon+\sigma_1\delta\alpha(\epsilon+\alpha)}{\gamma\sigma_2\epsilon(\epsilon+\alpha)} & \frac{\delta\alpha}{\sigma_2\epsilon} \\ 0 & 0 \end{pmatrix}.$$

It follows that the spectral radius of G is the basic reproductive number, thus,

$$R_0 = \frac{\beta\alpha\sigma_2\epsilon + \sigma_1\delta\alpha(\epsilon+\alpha)}{\gamma\sigma_2\epsilon(\epsilon+\alpha)}.$$

Now, when $i \neq 0$, the solution to system (5) gives the unique endemic equilibrium point of model (3), $E^* = (w^*, i^*, R_i^*)$, where

$$w^* = \frac{\alpha(1-i^*)d}{d(\beta i^* + (\epsilon+\alpha)) + \delta\sigma_1 i^*},$$
$$i^* = \frac{\sqrt{b^2-4ac}-b}{2a},$$
$$R_i^* = \frac{\sigma_1 i^*}{d}.$$

Here,

$$a = \beta(\alpha+\gamma)\big[\epsilon(\sigma_1-\sigma_2) + \alpha\sigma_1 + \sigma_2\gamma\big],$$
$$b = \beta\sigma_2\epsilon(\alpha+\gamma) - \beta\alpha\big[\sigma_1(\epsilon+\alpha) + \sigma_2(\gamma-\epsilon)\big] + \delta\sigma_1(\epsilon+\alpha)(\alpha+\gamma) + \gamma(\epsilon+\alpha)\big[\sigma_1(\epsilon+\alpha) + \sigma_2(\gamma-\epsilon)\big],$$
$$c = -\delta\sigma_1(\epsilon+\alpha)\alpha + \gamma(\epsilon+\alpha)\sigma_2\epsilon - \beta\alpha\sigma_2\epsilon,$$
$$d = \sigma_2\epsilon\big(1-i^*\big) + \big[\sigma_1(\epsilon+\alpha) + \sigma_2\gamma\big]i^*.$$

If $R_0 > 1$, then $c < 0$ and if $\sigma_1 > \sigma_2$, then $a > 0$. This means that $\sqrt{b^2-4ac} > b$, which leads to $i^* > 0$. As a result, w^*, R_i^* and d are positive since $0 < i^* < 1$. Consequently, E^* exists when $R_0 > 1$ and $\sigma_1 > \sigma_2$.

3.2 Local stability of the equilibrium points

Theorem 1 *If $R_0 < 1$, the virus-free equilibrium point E_0 is locally asymptotically stable. If $R_0 = 1$, E_0 is locally stable. If $R_0 > 1$, E_0 is unstable.*

Proof By linearizing system (3), we obtain the following Jacobian matrix evaluated at the equilibrium $E_0 = (\frac{\alpha}{\epsilon+\alpha}, 0, 0)$:

$$J = \begin{pmatrix} -\epsilon-\alpha & -\beta w_0-\alpha & -\delta w_0 \\ 0 & \beta w_0-\gamma & \delta w_0 \\ 0 & \sigma_1 & -\sigma_2(1-w_0) \end{pmatrix}.$$

One eigenvalue of the Jacobian matrix can be found easily, namely, $\lambda_1 = -(\epsilon+\alpha)$. The rest of the eigenvalues, $\lambda_{2,3}$, satisfy the equation:

$$\lambda^2 + A_1\lambda + A_2 = 0, \tag{6}$$

where

$$A_1 = -\beta w_0 + \gamma + \sigma_2(1-w_0),$$
$$A_2 = \frac{1}{\epsilon+\alpha}\left(-\frac{\beta\alpha\sigma_2\epsilon}{\epsilon+\alpha} + \gamma\sigma_2\epsilon - \sigma_1\delta\alpha\right).$$

Solving (6) yields $\lambda_{2,3} = (-A_1 \pm \sqrt{A_1^2 - 4A_2})/2$. If $\beta\alpha < \gamma(\epsilon+\alpha)$, then $A_1 > 0$ and if $\beta\alpha\sigma_2\epsilon + \sigma_1\delta\alpha(\epsilon+\alpha) < \gamma\sigma_2\epsilon(\epsilon+\alpha)$, i.e., $R_0 < 1$, then $A_2 > 0$. Consequently, $A_1 > \sqrt{A_1^2 - 4A_2}$ which means that $\lambda_{2,3} < 0$. Notice that the condition $R_0 < 1$ leads to $\beta\alpha < \gamma(\epsilon+\alpha)$ since

$$\beta\alpha\sigma_2\epsilon + \sigma_1\delta\alpha(\epsilon+\alpha) < \gamma\sigma_2\epsilon(\epsilon+\alpha),$$
$$\beta\alpha + \frac{\sigma_1\delta\alpha(\epsilon+\alpha)}{\sigma_2\epsilon} < \gamma(\epsilon+\alpha),$$
$$\beta\alpha < \gamma(\epsilon+\alpha).$$

Therefore, under the condition $R_0 < 1$, all the eigenvalues have negative real parts, this proves that E_0 is locally asymptotically stable when $R_0 < 1$. On the other hand, when $R_0 > 1$, then $A_2 < 0$, thus, Eq. (6) has at least one root with positive real part and. Hence, E_0 is unstable provided $R_0 > 1$. If $R_0 = 1$, then the eigenvalues are $\lambda_{1,2} < 0$ and $\lambda_3 = 0$, thus, E_0 is locally stable. □

Theorem 2 *E^* is locally asymptotically stable with respect to Ω if $\gamma < (\alpha+\epsilon)$.*

Proof The characteristic equation of the Jacobian matrix of the linearized system of (3) at E^* is given by

$$\det\begin{pmatrix} \lambda+\beta i^*+\delta R_i^*+(\epsilon+\alpha) & \beta w^*+\alpha & \delta w^* \\ -\beta i^*-\delta R_i^* & \lambda-\beta w^*+\gamma & -\delta w^* \\ -\sigma_2 R_i^* & -\sigma_1(1-R_i^*)-\sigma_2 R_i^* & \lambda+\sigma_1 i^*+\sigma_2(1-w^*-i^*) \end{pmatrix} = 0, \tag{7}$$

which is equivalent to

$$\det\begin{pmatrix} \lambda + a_{11} & a_{12} & a_{13} \\ -a_{21} & \lambda + a_{22} & -a_{23} \\ -a_{31} & -a_{32} & \lambda + a_{33} \end{pmatrix} = 0,$$

where $a_{11} = \beta i^* + \delta R_i^* + (\epsilon + \alpha)$, $a_{12} = \beta w^* + \alpha$, $a_{13} = \delta w^*$, $a_{21} = \beta i^* + \delta R_i^*$, $a_{22} = -\beta w^* + \gamma$, $a_{23} = \delta w^*$, $a_{31} = \sigma_2 R_i^*$, $a_{32} = (\sigma_2 - \sigma_1)R_i^* + \sigma_1$, $a_{33} = (\sigma_1 - \sigma_2)i^* + \sigma_2(1 - w^*)$.

Substituting $w^* = \frac{\gamma i^*}{\beta i^* + \delta R_i^*}$ in a_{22}, we get

$$a_{22} = \frac{\gamma \delta R_i^*}{\beta i^* + \delta R_i^*} > 0.$$

Moreover, since $\sigma_1 > \sigma_2$ is the existence condition of E^*, then a_{32} and a_{33} are positive. Clearly, the rest of the a are positive also.

By expanding the determinant we get the following cubic equation in λ:

$$\lambda^3 + C_1\lambda^2 + C_2\lambda + C_3 = 0, \tag{8}$$

where

$$C_1 = a_{11} + a_{22} + a_{33},$$

$$C_2 = a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33} + a_{13}a_{31} - a_{23}a_{32} + a_{12}a_{21},$$

$$C_3 = a_{13}a_{21}a_{32} + a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} + a_{11}a_{22}a_{33} + a_{12}a_{21}a_{33}.$$

According to the Hurwitz criteria,

$$H_1 = C_1 = (a_{11} + a_{22} + a_{33}) > 0,$$

$$\begin{aligned} H_2 &= C_1C_2 - C_3 \\ &= (a_{11} + a_{33})a_{13}a_{31} - (a_{22} + a_{33})a_{23}a_{32} + (a_{11} + a_{22})a_{11}a_{22} + (a_{11} + a_{33})a_{11}a_{33} \\ &\quad + (a_{22} + a_{33})a_{23}a_{32} + 2a_{11}a_{22}a_{33} - a_{13}a_{21}a_{32} - a_{12}a_{23}a_{31} + (a_{11} + a_{22})a_{12}a_{21}. \end{aligned}$$

Taking the term $(a_{22} + a_{33})$ as a common factor and evaluating the term $a_{22}a_{33} - a_{23}a_{32}$, we have

$$\begin{aligned} a_{22}a_{33} - a_{23}a_{32} &= \left(-\beta w^* + \gamma\right)\left(\frac{\sigma_1 i^*}{R_i^*}\right) - \left(\delta w^*\right)\left(-(\sigma_1 - \sigma_2)R_i^* + \sigma_1\right) \\ &> \left(\delta\sigma_1 w^* - \delta\sigma_1 w^*\right) \\ &= 0. \end{aligned}$$

Here, we used $(\sigma_1 - \sigma_2)i^* + \sigma_2(1 - w^*) = \frac{\sigma_1 i^*}{R_i^*}$ and $i^* = \frac{\delta w^* R_i^*}{\gamma - \beta w^*}$. Therefore, $(a_{22} + a_{33})(a_{22}a_{33} - a_{23}a_{32}) \geq 0$.

Next, we evaluate the terms

$$\begin{aligned} &a_{11}a_{22}a_{33} + a_{11}a_{13}a_{31} - a_{12}a_{23}a_{31} \\ &\quad = \left(\beta i^* + \delta R_i^* + (\epsilon + \alpha)\right)\left(-\beta w^* + \gamma\right)\left(\sigma_1 i^* + \sigma_2\left(1 - i^* - w^*\right)\right) \end{aligned}$$

$$
\begin{aligned}
&+ \big(\delta w^*\big)\big(\beta i^* + \delta R_i^* + (\epsilon+\alpha)\big)\big(\sigma_2 R_i^*\big) - \big(\delta w^*\big)\big(\beta w^* + \alpha\big)\big(\sigma_2 R_i^*\big)\\
&= \big(\beta i^* + (\epsilon+\alpha)\big)\big(-\beta w^* + \gamma\big)\big(\sigma_1 i^* + \sigma_2\big(1 - i^* - w^*\big)\big)\\
&\quad + \big(\delta R_i^*\big)\big(-\beta w^* + \gamma\big)\big(\sigma_1 i^* + \sigma_2\big(1 - i^*\big)\big) + \delta\beta\sigma_2 R_i^* w^{*2} - \gamma\delta\sigma_2 R_i^* w^*\\
&\quad + \big(\delta w^*\big)\big(\beta i^* + \delta R_i^* + \epsilon\big)\big(\sigma_2 R_i^*\big) - \delta\beta\sigma_2 R_i^* w^{*2}.
\end{aligned}
$$

By substituting $i^* = \frac{\delta w^* R_i^*}{\gamma - \beta w^*}$ in the above equation and since $\sigma_1 > \sigma_2$ leads to $-\sigma_2 > -\sigma_1$, we get

$$
\begin{aligned}
&a_{11}a_{22}a_{33} + a_{11}a_{13}a_{31} - a_{12}a_{23}a_{31}\\
&\geq \beta i^*\big(-\beta w^* + \gamma\big)\big(\sigma_1 i^* + \sigma_2\big(1 - i^* - w^*\big)\big) + (\epsilon+\alpha)\big(-\beta w^* + \gamma\big)\\
&\quad \times \left(\sigma_1\left(\frac{\delta w^* R_i^*}{\gamma - \beta w^*}\right)\right) + (\epsilon+\alpha)\big(-\beta w^* + \gamma\big)\big(\sigma_2\big(1 - i^* - w^*\big)\big) + \big(\delta R_i^*\big)\\
&\quad \times \big(-\beta w^* + \gamma\big)\big(\sigma_1 i^* + \sigma_2\big(1 - i^*\big)\big) + \big(\delta w^*\big)\big(\beta i^* + \delta R_i^* + \epsilon\big)\big(\sigma_2 R_i^*\big)\\
&\quad - \gamma\delta\sigma_1 R_i^* w^*\\
&= \beta i^*\big(-\beta w^* + \gamma\big)\big(\sigma_1 i^* + \sigma_2\big(1 - i^* - w^*\big)\big) + (\epsilon+\alpha-\gamma)\big(\sigma_1\delta w^* R_i^*\big)\\
&\quad + (\epsilon+\alpha)\big(-\beta w^* + \gamma\big)\big(\sigma_2\big(1 - i^* - w^*\big)\big) + \big(\delta R_i^*\big)\big(-\beta w^* + \gamma\big)\\
&\quad \times \big(\sigma_1 i^* + \sigma_2\big(1 - i^*\big)\big) + \big(\delta w^*\big)\big(\beta i^* + \delta R_i^* + \epsilon\big)\big(\sigma_2 R_i^*\big).
\end{aligned}
$$

Thus, $a_{11}a_{22}a_{33} + a_{11}a_{13}a_{31} - a_{12}a_{23}a_{31} \geq 0$ if $\gamma < \epsilon + \alpha$. Lastly, we evaluate the terms

$$
\begin{aligned}
&a_{11}a_{22}a_{33} - a_{13}a_{21}a_{32}\\
&= \left(\frac{\gamma i^*}{w^*} + (\epsilon+\alpha)\right)\big(\gamma - \beta w^*\big)\left(\frac{\sigma_1 i^*}{R_i^*}\right) - \delta w^*\left(\frac{\gamma i^*}{w^*}\right)\big((\sigma_2 - \sigma_1)R_i^* + \sigma_1\big)\\
&\geq \left(\frac{\gamma i^*}{w^*} + (\epsilon+\alpha)\right)\big(\gamma - \beta w^*\big)\left(\frac{\sigma_1\big(\frac{\delta w^* R_i^*}{\gamma - \beta w^*}\big)}{R_i^*}\right) - \delta w^*\left(\frac{\gamma i^*}{w^*}\right)(\sigma_1)\\
&= \gamma\sigma_1\delta i^* + (\epsilon+\alpha)\sigma_1\delta w^* - \gamma\sigma_1\delta i^*\\
&= (\epsilon+\alpha)\sigma_1\delta w^*\\
&> 0.
\end{aligned}
$$

Hence, $H_2 > 0$ if $\gamma < \epsilon + \alpha$. Calculating H_3, we get

$$
H_3 = C_3 H_2 = (a_{13}a_{21}a_{32} + a_{13}a_{22}a_{31} + a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} + a_{12}a_{21}a_{33})H_2.
$$

We take the terms

$$
\begin{aligned}
&a_{13}a_{21}a_{32} + a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32}\\
&= \delta w^*\left(\frac{\gamma i^*}{w^*}\right)\big((\sigma_2 - \sigma_1)R_i^* + \sigma_1\big) + \big(\beta i^* + \delta R_i^* + (\epsilon+\alpha)\big)\big(-\beta w^* + \gamma\big)\\
&\quad \times \left(\frac{\sigma_1 i^*}{R_i^*}\right) - \delta w^*\left(\frac{\gamma i^*}{w^*} + (\epsilon+\alpha)\right)\big((\sigma_2 - \sigma_1)R_i^* + \sigma_1\big)
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\beta i^* + \delta R_i^* + (\epsilon+\alpha)\right)\left(-\beta w^* + \gamma\right)\left(\frac{\sigma_1 i^*}{R_i^*}\right) - \delta w^*(\epsilon+\alpha) \\
&\quad \times \left((\sigma_2-\sigma_1)R_i^* + \sigma_1\right) \\
&\geq \left(\beta i^* + \delta R_i^* + (\epsilon+\alpha)\right)\left(-\beta w^* + \gamma\right)\left(\frac{\sigma_1\left(\frac{\delta w^* R_i^*}{\gamma-\beta w^*}\right)}{R_i^*}\right) - \delta w^*(\epsilon+\alpha)(\sigma_1) \\
&= \left(\beta i^* + \delta R_i^* + (\epsilon+\alpha)\right)\left(\sigma_1 \delta w^*\right) - \delta w^*(\epsilon+\alpha)(\sigma_1) \\
&= \left(\beta i^* + \delta R_i^*\right)\left(\sigma_1 \delta w^*\right) \\
&> 0.
\end{aligned}
$$

Hence, $C_3 > 0$ and therefore H_3 is positive. Since $H_1 > 0$, $H_2 > 0$ and $H_3 > 0$, all the eigenvalues of Eq. (8) have negative real parts. Thus, if $\gamma < \epsilon + \alpha$, then the endemic equilibrium point E^* is locally asymptotically stable. □

3.3 Global stability of the equilibrium points

First, we present some theories that we will use to investigate the stability of the virus-free equilibrium point of system (3).

Lemma 1 ([29]) *Consider a disease model system written in the form*:

$$
\begin{aligned}
&\frac{dX}{dt} = F(X,Y), \\
&\frac{dY}{dt} = G(X,Y), \quad G(X,0) = 0,
\end{aligned}
\tag{9}
$$

where $X \in R^m$ *denotes* (*its components*) *the number of uninfected individuals and* $Y \in R^n$ *denotes* (*its components*) *the number of infected individuals including latent, infectious, etc.* $U_0 = (x_0, 0)$ *denotes the disease-free equilibrium of system* (9). *Assume the conditions* (C1) *and* (C2) *below*:

(C1) *For* $\frac{dX}{dt} = F(X,0)$, x_0 *is globally asymptotically stable*,

(C2) $G(X,Y) = AY - \hat{G}(X,Y)$, *with* $\hat{G}(X,Y) \geq 0$ *for* $(X,Y) \in \Omega$,

where $A = \frac{\partial G}{\partial Y}(x_0, 0)$ *has all non-negative off-diagonal elements and* Ω *is the region where the model makes biological sense*.

If system (9) satisfies the above two conditions then the following theorems hold.

Theorem 3 ([29]) *The fixed point* $U_0 = (x_0, 0)$ *is a globally asymptotic stable equilibrium of* (9) *provided that* $R_0 < 1$ *and that assumptions* (Cl) *and* (C2) *are satisfied*.

Theorem 4 *The virus-free equilibrium* E_0 *of system* (3) *is globally asymptotically stable with respect to* Ω *if* $\sigma_2 < \sigma_1$, $R_0 < 1$ *and the assumptions in Lemma* 1 *are satisfied*.

Proof Apply Lemma 1 to system (3). Consider $X = w$ and $Y = \left[\begin{smallmatrix} i \\ R_i \end{smallmatrix}\right]$.

When $i = R_i = 0$, the uninfected subsystem becomes

$$
\frac{dw}{dt} = \alpha - (\epsilon+\alpha)w,
\tag{10}
$$

which has the solution

$$w(t) = \frac{\alpha}{\epsilon+\alpha} + e^{-(\epsilon+\alpha)t}\left(w(0) - \frac{\alpha}{\epsilon+\alpha}\right).$$

Clearly, $w(t) \to \frac{\alpha}{\epsilon+\alpha} = w_0$ as $t \to \infty$ regardless of the initial condition value $w(0)$. Thus, condition (C1) from Lemma 1 is satisfied.

Next, we can write the right hand side of the infectious subsystem as

$$\begin{aligned}\frac{dY}{dt} &= G(X,Y) = \begin{bmatrix} \beta w i + \delta w R_i - \gamma i \\ \sigma_1 i - \sigma_1 i R_i - \sigma_2 R_i(1-w-i) \end{bmatrix} \\ &= \begin{bmatrix} \beta w_0 - \gamma & \delta w_0 \\ \sigma_1 & -\sigma_2 w_0 \end{bmatrix} \begin{bmatrix} i \\ R_i \end{bmatrix} - \begin{bmatrix} 0 \\ \sigma_1 i R_i - \sigma_2 i R_i \end{bmatrix} \\ &= AY - \hat{G}(X,Y),\end{aligned}$$

where

$$A = \begin{bmatrix} \beta w_0 - \gamma & \delta w_0 \\ \sigma_1 & -\sigma_2 w_0 \end{bmatrix}, \qquad \hat{G} = \begin{bmatrix} 0 \\ \sigma_1 i R_i - \sigma_2 i R_i \end{bmatrix}.$$

Clearly, A has all non-negative off-diagonal elements, and $\hat{G}(X,Y) \geq 0$ for $(X,Y) \in \Omega$ if $\sigma_1 > \sigma_2$. Thus, the condition (C2) in Lemma 1 hold for system (3) when $\sigma_1 > \sigma_2$. Hence E_0 is globally asymptotically stable if $\sigma_2 < \sigma_1$ and $R_0 < 1$. □

Next, we use Lyapunov's theorem to examine the global stability of the endemic equilibrium point of system (3).

Theorem 5 *The endemic equilibrium E^* of system (3) is globally stable with respect to Ω if $\beta > \alpha$.*

Proof Define the Lyapunov function as

$$\begin{aligned}L(w,i,R_i) = {} & \frac{\sigma_1(1+R_i^*)}{\alpha}\left(w - w^* - w^* \ln\frac{w}{w^*}\right) + \frac{\sigma_1(1+R_i^*)}{\alpha}\left(i - i^* - i^* \ln\frac{i}{i^*}\right) \\ & + \left(R_i - R_i^* - R_i^* \ln\frac{R_i}{R_i^*}\right).\end{aligned}$$

Clearly, L is a positive definite function. Computing the derivative of L along the solutions of system (3), we obtain

$$\begin{aligned}L' &= \frac{\sigma_1(1+R_i^*)}{\alpha}\left(1 - \frac{w^*}{w}\right)w' + \frac{\sigma_1(1+R_i^*)}{\alpha}\left(1 - \frac{i^*}{i}\right)i' + \left(1 - \frac{R_i^*}{R_i}\right)R_i' \\ &= \frac{\sigma_1(1+R_i^*)}{\alpha}\left(1 - \frac{w^*}{w}\right)\left(-\beta w i - \delta R_i w - \epsilon w + \alpha(1-w-i)\right) \\ &\quad + \frac{\sigma_1(1+R_i^*)}{\alpha}\left(1 - \frac{i^*}{i}\right)(\beta w i + \delta R_i w - \gamma i) \\ &\quad + \left(1 - \frac{R_i^*}{R_i}\right)\left(\sigma_1 i(1-R_i) - \sigma_2 R_i(1-w-i)\right). \qquad (11)\end{aligned}$$

Since E^* is an equilibrium point to system (3),

$$\alpha = \beta w^* i^* + \delta R_i^* w^* + (\epsilon + \alpha) w^* + \alpha i^*, \tag{12}$$

$$\gamma = \frac{1}{i^*}\left(\beta w^* i^* + \delta R_i^* w^*\right), \tag{13}$$

$$\sigma_2 = \frac{\sigma_1 i^*}{R_i^*} - \sigma_1 i^* + \sigma_2 w^* + \sigma_2 i^*. \tag{14}$$

Thus, using (12)–(14) in (11), we have

$$\begin{aligned}
L' = {} & \frac{\sigma_1(1+R_i^*)}{\alpha}\left(1-\frac{w^*}{w}\right)\left(-\beta w i - \delta R_i w - (\epsilon+\alpha)w - \alpha i + \beta w^* i^* + \delta R_i^* w^* + (\epsilon+\alpha)w^*\right. \\
& \left. + \alpha i^*\right) + \frac{\sigma_1(1+R_i^*)}{\alpha}\left(1-\frac{i^*}{i}\right)\left(\beta w i + \delta R_i w - \frac{i}{i^*}\left(\beta w^* i^* + \delta R_i^* w^*\right)\right) \\
& + \left(1-\frac{R_i^*}{R_i}\right)(\sigma_1 i(1-R_i) + \sigma_2 R_i(w+i) - R_i\left(\frac{\sigma_1 i^*}{R_i^*} - \sigma_1 i^* + \sigma_2\left(w^*+i^*\right)\right) \\
= {} & \frac{\sigma_1(1+R_i^*)}{\alpha}\left(1-\frac{w^*}{w}\right)\left[\beta w^* i^*\left(1-\frac{wi}{w^* i^*}\right) + \delta R_i^* w^*\left(1-\frac{R_i w}{R_1^* w^*}\right)\right. \\
& \left. + (\epsilon+\alpha)w^*\left(1-\frac{w}{w^*}\right) + \alpha i^*\left(1-\frac{i}{i^*}\right)\right] \\
& + \frac{\sigma_1(1+R_i^*)}{\alpha}\left(1-\frac{i^*}{i}\right)\left[\beta w^* i^*\left(\frac{wi}{w^* i^*} - \frac{i}{i^*}\right) + \delta R_i^* w^*\left(\frac{R_i w}{R_i^* w^*} - \frac{i}{i^*}\right)\right] \\
& + \left(1-\frac{R_i^*}{R_i}\right)\left[\sigma_1 i^*\left(\frac{i}{i^*} - \frac{R_i}{R_i^*}\right) + \sigma_1 R_i^* i^*\left(\frac{R_i}{R_i^*} - \frac{iR_i}{i^* R_i^*}\right) + \sigma_2 R_i^* w^*\left(\frac{R_i w}{R_i^* w^*} - \frac{R_i}{R_i^*}\right)\right. \\
& \left. + \sigma_2 R_i^* i^*\left(\frac{iR_i}{i^* R_i^*} - \frac{R_i}{R_i^*}\right)\right].
\end{aligned} \tag{15}$$

Collecting and simplifying terms yield

$$\begin{aligned}
L' = {} & \frac{\sigma_1(1+R_i^*)}{\alpha}\left[\beta w^* i^*\left(2 - \frac{w^*}{w} - \frac{w}{w^*}\right)\right. \\
& + \delta R_i^* w^*\left(4 - \frac{w^*}{w} - \frac{i}{i^*} - \frac{R_i^*}{R_i} - \frac{R_i w i^*}{R_i^* w^* i} + \frac{R_i^*}{R_i} + \frac{R_i}{R_i^*} - 2\right) \\
& \left. + (\epsilon+\alpha)w^*\left(2 - \frac{w^*}{w} - \frac{w}{w^*}\right) + \alpha i^*\left(1 - \frac{i}{i^*} - \frac{w^*}{w} + \frac{w^* i}{w i^*}\right)\right] \\
& + \sigma_1 i^*\left(1 - \frac{R_i}{R_i^*} + \frac{i}{i^*} - \frac{R_i^* i}{R_i i^*}\right) + \sigma_1 R_i^* i^*\left(-1 + \frac{R_i}{R_i^*} + \frac{i}{i^*} - \frac{R_i i}{R_i^* i^*}\right) \\
& + \sigma_2 R_i^* w^*\left(1 + \frac{R_i w}{R_i^* w^*} - \frac{w}{w^*} - \frac{R_i}{R_i^*}\right) + \sigma_2 R_i^* i^*\left(1 - \frac{R_i}{R_i^*} - \frac{i}{i^*} + \frac{R_i i}{R_i^* i^*}\right).
\end{aligned} \tag{16}$$

From the inequality of arithmetic and geometric means, we have

$$2 - \frac{w^*}{w} - \frac{w}{w^*} \le 0,$$

$$4 - \frac{w^*}{w} - \frac{i}{i^*} - \frac{R_i^*}{R_i} - \frac{R_i w i^*}{R_i^* w^* i} \le 0.$$

Employing the above inequalities and recollecting terms give

$$\begin{aligned} L' \leq{}& C + \frac{R_i^*}{R_i}\left(\frac{\sigma_1\delta R_i^* w^*(1+R_i^*)}{\alpha}\right) \\ &+ \frac{R_i}{R_i^*}\left(\frac{\sigma_1\delta R_i^* w^*(1+R_i^*)}{\alpha} + \sigma_1 i^*\left(R_i^*-1\right) - \sigma_2 R_i^*\left(w^*+i^*\right)\right) \\ &+ \frac{i}{i^*}\left(-\sigma_1 i^*\left(1+R_i^*\right) + \sigma_1 i^*\left(1+R_i^*\right) - \sigma_2 R_i^* i^*\right) - \frac{w^*}{w}\sigma_1 i^*\left(1+R_i^*\right) - \frac{w}{w^*}\sigma_2 R_i^* w^* \\ &+ \frac{w^* i}{w i^*}\left(\sigma_1 i^*\left(1+R_i^*\right)\right) - \sigma_1 i^*\frac{R_i^* i}{R_i i^*} + \frac{R_i i}{R_i^* i^*}(\sigma_2-\sigma_1)R_i^* i^* + \sigma_2 R_i^* w^*\frac{R_i w}{R_i^* w^*}, \end{aligned} \tag{17}$$

where

$$C = -\frac{2\sigma_1\delta R_i^* w^*(1+R_i^*)}{\alpha} + \sigma_1 i^*\left(1+R_i^*\right) + \sigma_1 i^*\left(1-R_i^*\right) + \sigma_2 R_i^*\left(w^*+i^*\right).$$

Again using the fact that E^* is an equilibrium point, we have

$$\begin{aligned} C &\leq \sigma_1 i^*\left(1+R_i^*\right) + \sigma_2 R_i^* \\ &= \frac{\sigma_1}{\alpha}\left(1+R_i^*\right)\left(-\beta w^* i^* - \delta R_i^* i^* - \epsilon w^* + \alpha - \alpha w^*\right) \\ &\quad + \frac{\sigma_2}{\delta w^*}\left(-\beta w^* i^* - \epsilon w^* + \alpha - \alpha\left(w^*+i^*\right)\right). \end{aligned}$$

Since $\beta > \beta w^* i^*$, we have $C \leq 0$ if $\beta > \alpha$. Accordingly,

$$\begin{aligned} L' \leq{}& \frac{R_i^*}{R_i}\frac{\sigma_1(1+R_i^*)}{\alpha}\left(-\beta w^* i^* - \epsilon w^* + \alpha - \alpha\left(w^*+i^*\right)\right) + \frac{R_i}{R_i^*}\left(\frac{\sigma_1(1+R_i^*)}{\alpha}\left[-\beta w^* i^*\right.\right. \\ &\left.\left. - \epsilon w^* + \alpha - \alpha\left(w^*+i^*\right)\right] - \sigma_2 R_i^*\right) - \frac{i}{i^*}\sigma_2 R_i^* i^* + \frac{w}{w^*}\left(-\sigma_1 i^*\left(1+R_i^*\right) - \sigma_2 R_i^* w^*\right) \\ &+ \frac{w^* i}{w i^*}\frac{\sigma_1(1+R_i^*)}{\alpha}\left(-\beta w^* i^* - \delta R_i^* i^* - \epsilon w^* + \alpha - \alpha w^*\right) - \sigma_1 i^*\frac{R_i^* i}{R_i i^*} \\ &+ \frac{R_i i}{R_i^* i^*}(\sigma_2-\sigma_1)R_i^* i^* + \frac{R_i w}{R_i^* w^*}\frac{\sigma_2}{\delta}\left(-\beta w^* i^* - \epsilon w^* + \alpha - \alpha\left(w^*+i^*\right)\right). \end{aligned} \tag{18}$$

Since E^* exists when $\sigma_1 > \sigma_2$, it follows that $L' \leq 0$ if $\beta > \alpha$. Hence, E^* is globally stable in Ω. □

4 Numerical experiments

In this section, some numerical examples are conducted with the aid of MATLAB. In particular, we consider two different specifications for the parameters to substantiate the analytical results in the previous section. All the simulations are based on a network size of $N = 100$ computers and $R_N = 100$ removable devices. Consequently, the results are expressed in terms of percentage of the total network size.

Example 1 (Virus-free equilibrium point) Let the parameters in model (3) be as follows: $\epsilon = 0.2$, $\alpha = 0.04$, $\gamma = 0.1$, $\sigma_1 = 0.05$, $\sigma_2 = 0.03$, $\delta = 0.03$, $\beta = 0.02$. Here, the threshold is $R_0 = 0.1333$, which is less than unity. Therefore, the system in (3) will approach the virus-free equilibrium point E_0 according to Theorem 1.

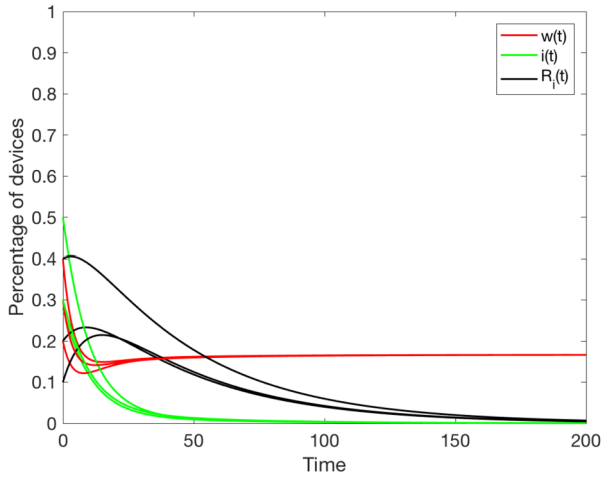

Figure 2 Time variation of compartments in model (3) with different initial conditions for $R_0 < 1$

Figure 2 shows the time evolution of the compartments in model (3) for different initial conditions with parameters set as in Example 1. It can be seen that both the infected computers and the infected removable devices eventually tend to extinction. However, weak computers tend to an equilibrium level with time. Thus, the solution curves converge to the virus-free equilibrium $E_0 = (0.1667, 0, 0)$. Hence, the numerical solution agrees with the analytical result in Theorem 1.

Example 2 (Endemic equilibrium point) In this example, we set the parameters to different values as follows: $\epsilon = 0.25$, $\alpha = 0.3$, $\gamma = 0.1$, $\sigma_1 = 0.4$, $\sigma_2 = 0.3$, $\delta = 0.5$, $\beta = 0.7$. Calculating the threshold, we obtain $R_0 = 11.818 > 1$. Also, the condition $\sigma_1 > \sigma_2$ is satisfied, thus, the endemic equilibrium E^* exists. Moreover, the parameters fulfill the conditions: $\gamma < \epsilon + \alpha$ (Theorem 2) and $\beta > \alpha$ (Theorem 5), as a result, the system in (3) approximates the endemic equilibrium E^*.

The time variation of the compartments in model (3) with parameters specified as in Example 2 is displayed in Fig. 3. We see that, for different initial conditions, the infected computers and removable devices eventually reach an equilibrium level that is higher than the equilibrium level reached by weak computers. Consequently, the solution curves converge to the endemic equilibrium $E^* = (0.0777, 0.6431, 0.7544)$. Hence, the simulations are consistent with the qualitative analysis in Theorem 2.

4.1 Sensitivity analysis

The basic reproductive number R_0 plays a very important role in the design of efficient control strategies. Specifically, if R_0 is less than unity, the virus outbreak dies out. Consequently, a reduction in the numeric value of R_0 will be the main goal of all control strategies. Therefore, it is crucial to take various actions to control the system parameters so that R_0 is remarkably below one. To examine the sensitivity of R_0 to the model's parameters, we vary R_0 with respect to one parameter each time. Accordingly, we have the following

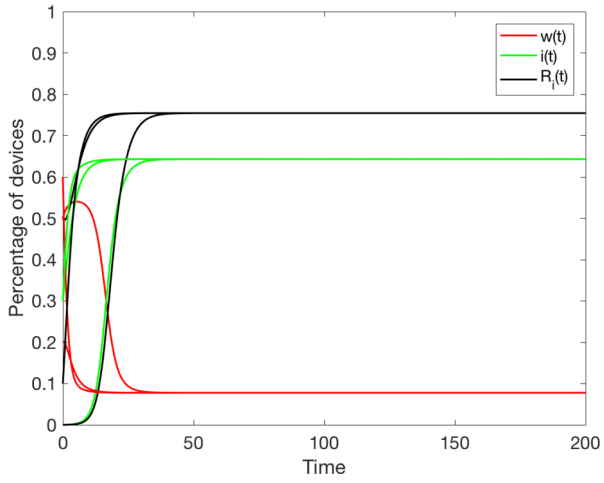

Figure 3 Time variation of compartments in model (3) with different initial conditions for $R_0 > 1$

variations of R_0, taking into account that $0 < \alpha, \epsilon, \gamma, \delta, \beta, \sigma_1, \sigma_2 \leq 1$:

$$\frac{\partial R_0}{\partial \beta} = \frac{\alpha}{\gamma(\epsilon+\alpha)} > 0,$$

$$\frac{\partial R_0}{\partial \delta} = \frac{\sigma_1 \alpha}{\gamma \sigma_2 \epsilon} > 0,$$

$$\frac{\partial R_0}{\partial \sigma_1} = \frac{\delta \alpha}{\gamma \sigma_2 \epsilon} > 0,$$

$$\frac{\partial R_0}{\partial \alpha} = \frac{\epsilon^2 \beta \sigma_2 + \sigma_1 \delta(\epsilon+\alpha)^2}{\gamma \sigma_2 \epsilon(\epsilon+\alpha)^2} > 0,$$

$$\frac{\partial R_0}{\partial \gamma} = \frac{-[\beta \alpha \sigma_2 \epsilon + \sigma_1 \delta \alpha(\epsilon+\alpha)]}{\gamma^2 \sigma_2 \epsilon(\epsilon+\alpha)} < 0,$$

$$\frac{\partial R_0}{\partial \sigma_2} = \frac{-\sigma_1 \delta \alpha}{\epsilon \gamma \sigma_2^2} < 0,$$

$$\frac{\partial R_0}{\partial \epsilon} = \frac{-[\sigma_1 \delta \alpha(\epsilon+\alpha)(\alpha+1) + \sigma_2 \alpha \beta \epsilon^2]}{\gamma \sigma_2 \epsilon^2(\epsilon+\alpha)^2} < 0.$$

In the above calculations, we considered that all the variables of R_0 are constant except for one. We can see that R_0 decreases with increasing γ, ϵ and σ_2. On the other hand, R_0 has a proportional increase relationship with the parameters: β, α, σ_1 and δ. Figure 4 demonstrates these results. Hence, from the sensitivity analysis, we conclude that the basic parameters to control the outbreak of viruses are: ϵ, the rate of weak computers becoming strong; γ, the recovery rate of infected computers; and σ_2, the recovery rate of infected removable devices. These parameters are primarily concerned with user awareness toward immunizing computers and removable devices against viruses. Therefore, it is essential to protect all computers with regularly updated anti-virus software that is scheduled to be scanned every day. Also, we are to check each removable device against viruses before use.

Furthermore, Fig. 5 illustrates the impact of high user awareness on network security through the key parameters: γ, ϵ and σ_2. We see that the increase in these parameters leads to a decrease in both i-node and R_i-node. Moreover, a further increase in the pa-

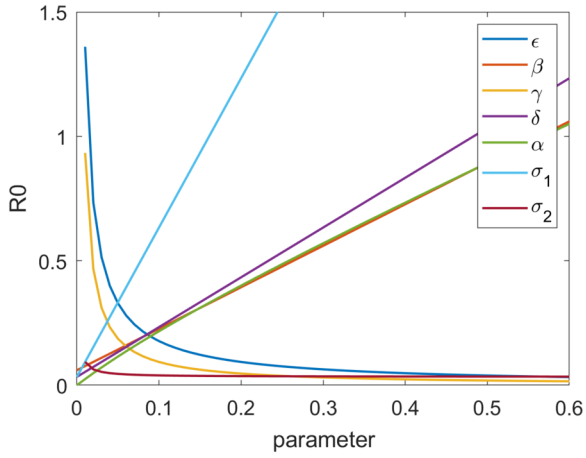

Figure 4 The basic reproduction number R_0 of model (3) versus the model's parameters

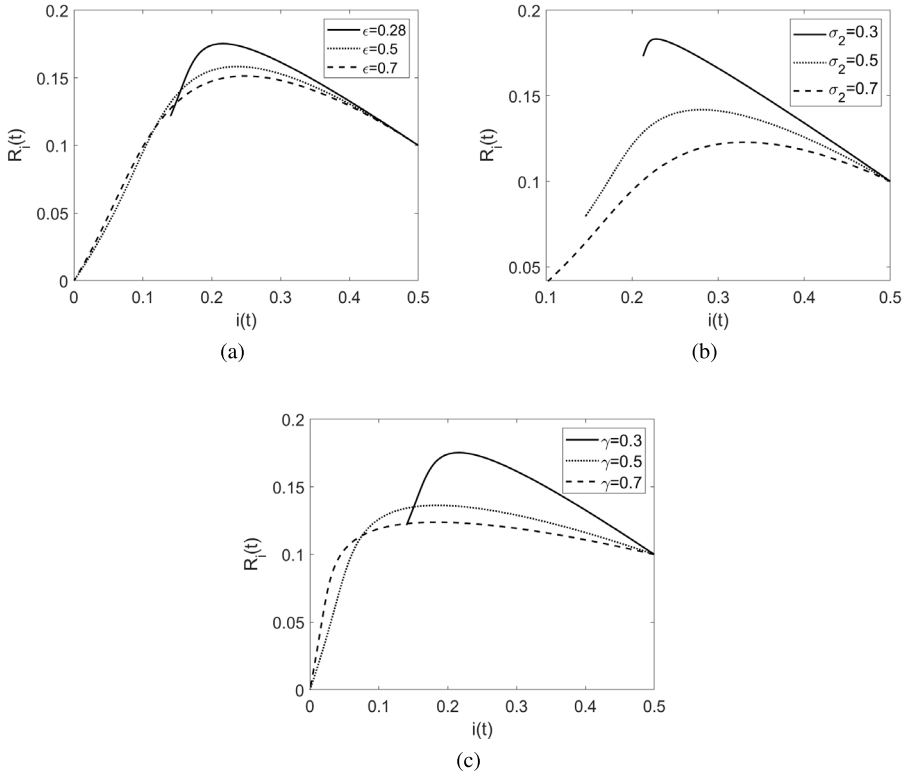

Figure 5 Evolution of i and R_i for: (**a**) different values of ϵ with $\alpha = 0.56$, $\gamma = 0.3$, $\sigma_1 = 0.1$, $\sigma_2 = 0.3$, $\delta = 0.2$, $\beta = 0.4$, (**b**) different values of σ_2 with $\epsilon = 0.28$, $\alpha = 0.56$, $\gamma = 0.3$, $\sigma_1 = 0.1$, $\delta = 0.33$, $\beta = 0.4$, (**c**) different values of γ with $\epsilon = 0.28$, $\alpha = 0.56$, $\sigma_1 = 0.1$, $\sigma_2 = 0.3$, $\delta = 0.2$, $\beta = 0.4$

rameters yields the virus-free equilibrium. On the contrary, Fig. 6 demonstrates the effect of low user awareness on network security. Actions such as plugging a removable device into an infected computer and neglecting anti-virus updates are represented in the parameters: σ_1 and α, that is, the infection rate of a removable device and the rate of strong computers becoming weak. The figure shows that the increase in these parameters rises

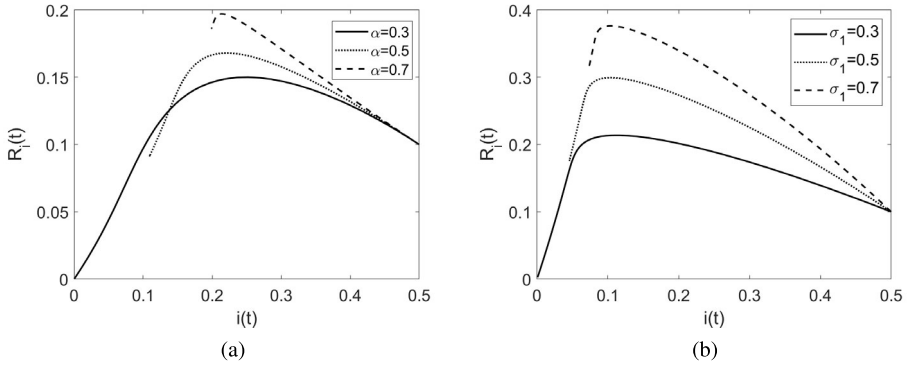

Figure 6 Evolution of i and R_i for: (**a**) different values of α with $\epsilon = 0.28$, $\gamma = 0.3$, $\sigma_1 = 0.1$, $\sigma_2 = 0.3$, $\delta = 0.2$, $\beta = 0.4$, (**b**) different values of σ_1 with $\epsilon = 0.28$, $\alpha = 0.56$, $\gamma = 0.7$, $\sigma_2 = 0.3$, $\delta = 0.2$, $\beta = 0.4$

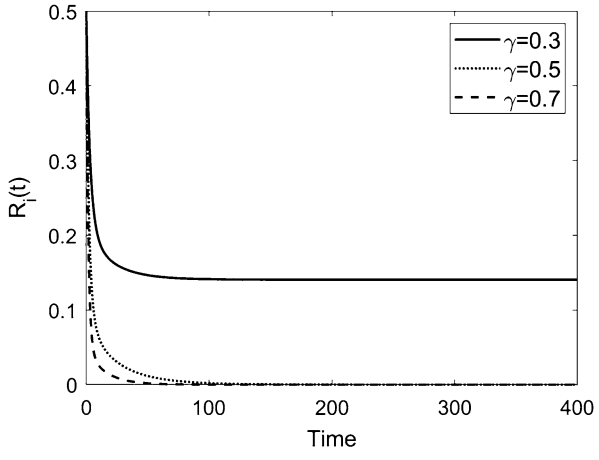

Figure 7 Time variation of the removable device compartment, $R_i(t)$, with different values of γ (recovery rate)

the proportion of infected computers and removable devices. Higher values of σ_1 and α lead to the endemic equilibrium. Finally, Fig. 7 depicts the effect of γ on the size of infected removable devices. We see that when the recovery rate of infected computers increases, a decrease in the percentage of infected removable devices results.

Based on the sensitivity analysis and simulation results, the following suggestions are proposed to control the virus outbreak:

- Installing effective anti-virus software and updating it regularly. This reduces the infectious rate β and the W-node conversion rate α. At the same time, it increases the rate ϵ and the recovery rate γ.
- Filtering removable devices with anti-virus software and disconnecting them from the computer whenever unused. This minimizes the infectious rates σ_1 and δ, while maximizing the recovery rate σ_2.

5 Conclusion

In this paper, we extended the model of computer virus propagation in [6] by introducing two new compartments representing removable devices, because these devices play a crucial role in transmitting viruses. The model consists of two populations: computers

and removable media. We aimed to investigate the influence of diverse levels of protection on the spread of viruses. We assumed that infected computers and infected removable devices posses infectivity. Up-to-date anti-virus software is powerful enough to keep computers immune from viruses. The model was analyzed qualitatively using the stability theory of nonlinear ordinary differential equations. As a result, two equilibrium points were produced: virus-free and endemic equilibrium points. The stability of both equilibrium points was examined. We found that when $R_0 < 1$, the virus-free equilibrium was locally asymptotically stable, and when $\sigma_2 < \sigma_1$, it was globally asymptotically stable. On the other hand, when $R_0 > 1$, $\sigma_2 < \sigma_1$, and $\gamma < (\alpha + \epsilon)$, then the endemic equilibrium was locally asymptotically stable, and it is globally stable when $\beta > \alpha$. Furthermore, we showed that numerical simulations were consistent with the analytical results. In addition, a sensitivity analysis was performed to understand the role of each parameter in dissipating viruses. We found that an increase in the parameters γ, ϵ and σ_2 leads to a virus-free equilibrium, a state of high user awareness. However, the rise in the parameters σ_1 and α leads to an endemic equilibrium that represents a decrease in user awareness.

In conclusion, user awareness plays an essential role in limiting the spread of viruses. Ongoing educational campaigns are recommended regarding the correct use of removable devices and the protection of computers with updated antivirus software.

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Authors' contributions
SA and WB contributed to the design and implementation of the research, to the analysis of the results and to the writing of the manuscript. All authors read and approved the final manuscript.

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