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# Asymptotically almost periodic solutions for certain differential equations with piecewise constant arguments

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## Abstract

It is well known that differential equations with piecewise constant arguments is a class of functional differential equations, which has fascinated many scholars in recent years. These delay differential equations have been successfully applied to diverse models in real life, especially in biology, physics, economics, etc. In this work, we are interested in the existence and uniqueness of asymptotically almost periodic solution for certain differential equation with piecewise constant arguments. Due to the particularity of the equations, we cannot use the traditional method to convert it into the difference equation with exponential dichotomy. Through constructing Cauchy matrix of the investigated system to find the corresponding Green matrix of the difference equation, we need the concept of exponential dichotomy and the Banach contraction fixed point theorem of the corresponding system. Then we give some sufficient conditions to obtain the existence and uniqueness of asymptotically almost periodic solutions for these systems.

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**Keywords:** Asymptotically almost periodic; Differential equations with piecewise constant arguments; Banach contraction fixed point theorem; Exponential dichotomy

## 1 Introduction

In recent years, the delay differential equations have been successfully applied to various models in many fields, especially in biology, physics, and economy.

In 1977, Myshkis [29] proposed a differential equation with noncontinuous variables

$$x'(t) = g(t, x(t), x(h(t))),$$

where  $h$  is a deviated function with piecewise constant arguments such as  $h(t) = [t]$  or  $h(t) = 2[\frac{t+1}{2}]$ ,  $[ \cdot ]$  denotes the largest integer function. These equations are called differential equations with piecewise constant arguments, abbreviated as DEPCA. The research work on DEPCA was first initiated by Shah and Wiener in 1983 [39]. A year later, Cooke and Wiener studied DEPCA with time delay in their work [13]. Because

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the differential equation with piecewise constant arguments describes the hybrid dynamical system (continuous and discrete combination) and combines the properties of differential equations and difference equations, so the differential equation with piecewise constant arguments is more abundant than general ordinary differential equation, and it is more difficult to study. DEPCA has shown important applications in medicine, physics, and other scientific fields, which is why DEPCA has attracted so much attention (see [5, 7, 12, 14, 15, 24, 28, 32, 37, 38, 43, 45, 48, 50] and the references therein). Most of these works focused on some qualitative properties of the solutions, such as the existence, uniqueness, boundedness, periodicity, almost periodicity, pseudo-almost periodicity, stability, oscillation, and so on. (see [1, 5, 6, 8–11, 18, 22, 23, 25–27, 30–33, 44, 50, 53, 56, 60, 62, 63, 66] and the references therein).

Compared with the almost periodic solution of the differential equation, the corresponding results of the asymptotically almost periodic solutions are very few. In recent years, the existence of asymptotically almost periodic solution is one of the topics with great interest to many mathematicians in the theory of differential equations (see [16, 17, 19, 32, 35, 40–42, 52, 54, 55, 57–59, 61, 67] and the references therein). Moreover, asymptotically almost periodic function is a generalization of almost periodic function, so it is more general to discuss the asymptotically almost periodic solutions of differential equations in practical problems.

In 2015, Samuel Castillo [8] studied the following systems:

$$y'(t) = A(t)y(t) + B(t)y(\gamma^0(t)) + f(t), \quad t \in R \tag{1.1}$$

and

$$y'(t) = A(t)y(t) + B(t)y(\gamma^0(t)) + F(t, y_\gamma(t)), \quad t \in R, \tag{1.2}$$

where

$$y_\gamma(t) = (y(\gamma^{p_1}(t)), y(\gamma^{p_2}(t)), \dots, y(\gamma^{p_l}(t))),$$

$$p_1, p_2, \dots, p_l \in N \cup \{0\},$$

$\gamma^{p_i}(t), i = 1, 2, \dots, l$ , denotes step functions. Equations (1.1) and (1.2) can be regarded a perturbation of the following linear homogeneous equation:

$$z'(t) = A(t)z(t) + B(t)z(\gamma^0(t)), \tag{1.3}$$

where the matrices  $A, B : R \rightarrow R^{q \times q}$  and  $f : R \rightarrow R^q$  is a continuous function,  $F : R \times R^q \rightarrow R^q$  is a continuous function and satisfies the Lipschitz condition (see  $(H_5)$ ). For  $p \in Z$ , let  $\gamma^p : R \rightarrow R$  be a step function such that  $\frac{\gamma^p}{J_n} = t_{n-p}$ , where  $J_n = [t_n, t_{n+1}]$  for all  $n \in Z$ . The author Samuel Castillo [8] gave some sufficient conditions to obtain the existence and uniqueness of the almost periodic solutions for systems (1.1) and (1.2).

Motivated by the paper of Castillo [8], we study the above linear nonhomogeneous system (1.1) and nonlinear nonhomogeneous system (1.2) and get some sufficient conditions of the existence and uniqueness for asymptotically almost periodic solutions. Our results generalize the results in [8].

In order to study equation (1.1), we first study the linear inhomogeneous DEPCA

$$y'(t) = A(t)y(t) + B(t)y([t]) + f(t), \quad t \in R. \tag{1.4}$$

$A(\cdot)$  is an almost periodic matrix-valued function,  $B(\cdot)$  is an almost periodic matrix-valued function or an asymptotically almost periodic matrix-valued function,  $f$  is an asymptotically almost periodic function.

By the variation of constants formula, a solution  $y$  of equation (1.4) is defined on  $R$  and satisfies the following equation:

$$y(t) = \left[ \Phi(t, n) + \int_n^t \Phi(t, u)B(u) du \right] y(n) + \int_n^t \Phi(t, u)f(u) du, \tag{1.5}$$

where  $\Phi(t, s) = \Phi(t)\Phi^{-1}(s)$ , and  $\Phi(t)$  is a fundamental matrix of the following system:

$$x'(t) = A(t)x(t) \tag{1.6}$$

in  $[n, n + 1]$  for all  $n \in Z$ . Furthermore, it satisfies  $\Phi(0) = I$ , where  $I$  is an identity matrix.

The solution  $y$  is continuous on  $R$ , and by taking  $t \rightarrow (n + 1)^-$ , we get the difference system

$$y(n + 1) = C(n)y(n) + h(n), \quad n \in Z, \tag{1.7}$$

where

$$C(n) = \Phi(n + 1, n) + \int_n^{n+1} \Phi(n + 1, u)B(u) du,$$

$$h(n) = \int_n^{n+1} \Phi(n + 1, u)f(u) du.$$

By (1.5),  $y = y(t)$  is a solution of (1.4) defined on  $R$  if and only if the matrix

$$I + \int_\tau^t \Phi(\tau, u)B(u) du \tag{1.8}$$

is invertible for all  $n \in Z$  and  $t, \tau \in [n, n + 1]$ , where  $I$  is an identity matrix (see [4, 5, 35, 36]). We can obtain the following fundamental matrix:

$$Z(t, n) = \Phi(t, n) + \int_n^t \Phi(t, u)B(u) du,$$

which is also invertible for all  $n \in Z$  and  $t \in [n, n + 1]$ . Therefore

$$C(n) = Z(n + 1, n)$$

is also invertible.

Note that the discrete system

$$x(n + 1) = C(n)x(n), \quad n \in Z \tag{1.9}$$

can be obtained by the linear homogeneous system

$$x'(t) = A(t)x(t) + B(t)x([t]), \quad t \in R. \tag{1.10}$$

The discrete solution of (1.7) is the restriction on  $Z$  of the continuous solution of (1.4), so these two equations are closely related, which reflects the mixed characteristics of DEPCA. Papaschinopoulos [32, 33] studied the DEPCA and obtained the result on the discrete system with exponential dichotomy and the concept of the corresponding exponential dichotomy. This is a traditional method for studying almost periodic solution or asymptotically almost periodic solution of differential equation.

In this work, we consider a more general  $y_\gamma$  and emphasize the behavior of solutions on the points  $t_n$ . In this case, the concept of traditional exponential dichotomy cannot be directly extended to (1.3). Therefore, we can only define the concept of the corresponding exponential dichotomy of (1.3) by other methods. After that, we can further prove the existence and uniqueness of asymptotically almost periodic solution for linear inhomogeneous system (1.1) (see Theorem 3.3). In addition, by using exponential dichotomy and the Banach contraction fixed point theorem, some sufficient conditions for the existence and uniqueness of asymptotically almost periodic solution for nonlinear nonhomogeneous system (1.2) are obtained (see Theorem 3.5).

The rest of this article is organized as follows: Sect. 2 provides the main definitions, assumptions, propositions, and lemmas that will be used. Section 3 is devoted to the main results of this work, that is, the existence and uniqueness of asymptotically almost periodic solution for system (1.1) and system (1.2).

### 2 Some definitions and lemmas

In this section, we present some useful definitions, propositions, and lemmas. Before that, the main assumptions of this section are given:

- (H<sub>1</sub>)  $A$  and  $B$  are almost periodic functions.
- (H<sub>2</sub>)  $A$  is an almost periodic function and  $B$  is an asymptotically almost periodic function.
- (H<sub>3</sub>) Fix a real-valued sequence  $\{t_n\}_{n=-\infty}^{+\infty}$  such that  $t_n < t_{n+1}$  and  $t_n \rightarrow \pm\infty$  as  $n \rightarrow \pm\infty$ . And  $\{t_n^{(k)}\}_{n=-\infty}^{+\infty}$  is equipotentially almost periodic for all  $k \in Z$ , where  $t_n^{(k)} = t_{n+k} - t_n$  (see Definition 2.6).
- (H<sub>4</sub>)  $f$  is a piecewise asymptotically almost periodic function, namely

$$T(f, \varepsilon) = \left\{ \tau \in R : |f(t + \tau) - f(t)| \leq \varepsilon, \forall t \in R - \left( \bigcup_{n \in Z} [t_n - \varepsilon, t_n + \varepsilon] \right) \right\}$$

is relatively dense on  $R$  for all  $\varepsilon > 0$ . And there is  $\delta_\varepsilon > 0$  such that  $|f(t' + \tau') - f(t')| \leq \varepsilon$  if  $\tau' \in R : |\tau'| \leq \delta_\varepsilon$  and  $t', t' + \tau'$  is in one of the intervals  $[t_n, t_{n+1}]$ .

- (H<sub>5</sub>)  $F$  is uniformly almost periodic on  $W$  and satisfies the Lipschitz condition, that is, there is  $L > 0$  such that

$$|F(t, x_1, \dots, x_l) - F(t, y_1, \dots, y_l)| \leq L \sum_{j=1}^l |x_j - y_j| \tag{2.1}$$

for all  $t \in R$  and  $(x_1, \dots, x_l), (y_1, \dots, y_l) \in W$ , where  $W$  is a compact subset in  $R^q$ .

**Definition 2.1** ([20]) The set  $E \subseteq R$  is called relatively dense if there is a real number  $l > 0$  such that  $E \cap [m, m + l] \neq \emptyset$  for all  $m \in R$ .

**Definition 2.2** ([20]) The  $f : R^q \rightarrow R^q$  is said to be an almost periodic function if the  $\varepsilon$ -translation set of  $f$

$$T(f, \varepsilon) = \{ \tau \in R^q : |f(t + \tau) - f(t)| < \varepsilon, \forall t \in R^q \}$$

is relatively dense on  $R$  for all  $\varepsilon > 0$ , where  $\tau$  is  $\varepsilon$ -period of  $f$ . We use  $AP(R^q, R^q)$  to represent all of these functions.

We use  $C_0(R^q)$  to represent the following set:

$$C_0(R^q) = \left\{ \varphi \in C(R^q) : \lim_{|t| \rightarrow +\infty} \|\varphi(t)\| = 0 \right\}.$$

**Definition 2.3** ([65]) The  $f : R^q \rightarrow R^q$  is said to be an asymptotically almost periodic function if  $f = g + \varphi$ , where  $g \in AP(R^q, R^q)$ ,  $\varphi \in C_0(R^q)$ . We use  $AAP(R^q, R^q)$  to represent all of these functions.

We use  $C_0S(Z, R^q)$  to represent the following set:

$$C_0S(Z, R^q) = \left\{ \varphi(n) : \lim_{|n| \rightarrow +\infty} \|\varphi(n)\| = 0 \right\}.$$

**Definition 2.4** ([65]) A sequence  $x : Z \rightarrow R^q$  is said to be almost periodic if the  $\varepsilon$ -translation set of  $x$

$$T(x, \varepsilon) = \{ \tau \in Z : |x(n + \tau) - x(n)| < \varepsilon, \forall n \in Z \}$$

is relatively dense on  $Z$  for all  $\varepsilon > 0$ , where  $Z$  denotes the set of integers. We use  $APS(Z, R^q)$  to represent all of these sequences.

**Definition 2.5** ([65]) The bounded sequence  $x : Z \rightarrow R^q$  is said to be asymptotically almost periodic if  $x = x_1 + x_2$ , where  $x_1 \in APS(Z, R^q)$ ,  $x_2 \in C_0S(Z, R^q)$ . We use  $AAPS(Z, R^q)$  to represent all of these sequences.

**Definition 2.6** ([8]) We say that  $\{t_n^{(k)}\}_{n=-\infty}^{+\infty}$  is equipotentially almost periodic for all  $k \in Z$  if the set

$$\bigcap_{k \in N} \{ T \in Z : |t_{T+n}^{(k)} - t_n^{(k)}| \leq \varepsilon, \text{ for all } n \in Z \}$$

is relatively dense for all  $\varepsilon > 0$ .

**Definition 2.7** ([32]) Let  $C(n)$  be a  $q \times q$  matrix and invertible, we say that the linear difference equation

$$y(n + 1) = C(n)y(n) \tag{2.2}$$

with exponential dichotomy on  $Z$  for all  $n \in Z$ . If there are positive constants  $K, \alpha > 0$  and projection  $P$  ( $P^2 = P$ ) such that

$$\begin{aligned} |Y(n)PY^{-1}(m)| &\leq Ke^{-\alpha(n-m)}, \quad n \geq m, \\ |Y(n)(I - P)Y^{-1}(m)| &\leq Ke^{-\alpha(m-n)}, \quad n < m, \end{aligned} \tag{2.3}$$

where  $Y(n)$  is a fundamental matrix of (2.2) and satisfies  $Y(0) = I$ .

**Lemma 2.1** ([64]) *Assume that  $A(t) \in AP(\mathbb{R}^q, \mathbb{R}^q)$ ,  $B(t) \in AAP(\mathbb{R}^q, \mathbb{R}^q)$ ,  $f(t) \in AAP(\mathbb{R}^q)$ , and the following equation*

$$y'(t) = A(t)y(t) + B(t)y([t]) + f(t), \quad t \in \mathbb{R}$$

*with exponential dichotomy holds. Then the equation has a unique solution  $y(t) \in AAP(\mathbb{R}^q)$ .*

**Lemma 2.2** ([64]) *If  $A(t), B(t), f(t)$  are almost periodic functions, then there is a positive number  $M > 0$  such that  $\max\{|A(t)|, |B(t)|, |f(t)|\} \leq M$ ,*

(1) *there exists  $k_0 > 0$  such that*

$$|X(t)X^{-1}(s)| \leq k_0, \quad 0 < t - s \leq 1;$$

(2) *if  $\tau \in T(A, \varepsilon)$ , then*

$$|X(t + \tau)X^{-1}(s + \tau) - X(t)X^{-1}(s)| \leq k_0\varepsilon e^M, \quad 0 < t - s \leq 1,$$

*where  $X(t)$  is a fundamental matrix of the equation*

$$x' = A(t)x$$

*and satisfies  $X(0) = I, A = A(t)$ .*

**Lemma 2.3** ([64]) *Let  $A(t)$  be an almost periodic function,  $X(t)$  is a fundamental matrix of the equation  $x' = A(t)x$ , then  $\{X(n + 1)X^{-1}(n) : n \in \mathbb{Z}\}$  is an almost periodic sequence.*

**Lemma 2.4** ([8]) *If  $\theta$  is defined as  $\theta = \sup_{n \in \mathbb{Z}}(t_{n+1} - t_n)$ , and  $K_0 = \exp(|A|_\infty \theta)$ , then  $|X(t, s)| \leq \sqrt{q}K_0$  for all  $t, s \in \mathbb{R}$  satisfying  $|s - t| \leq \theta$ .*

**Lemma 2.5** ([34]) *Assume that  $(H_3)$  holds. Let  $\varepsilon > 0, \Gamma \subseteq \Gamma_\varepsilon, \Gamma \neq \emptyset$  and  $P \subseteq \bigcup_{r \in \Gamma} P_r(\varepsilon)$  be such that  $P \cap P_r(\varepsilon) \neq \emptyset$  for all  $r \in \Gamma$ . Then the set  $\Gamma$  is relatively dense if and only if  $P$  is relatively dense.*

**Lemma 2.6** ([8])

(a) *If  $f_1, f_2$  are functions satisfying  $(H_4)$ , then given arbitrarily  $\varepsilon > 0$ ,*

*$\Gamma_\varepsilon \cap T(f_1, \varepsilon) \cap T(f_2, \varepsilon)$  is relatively dense.*

(b) *If  $\{g_1(n)\}_{n=-\infty}^{+\infty}$  and  $\{g_2(n)\}_{n=-\infty}^{+\infty}$  are almost periodic sequences, then given arbitrarily*

*$\varepsilon > 0, P_\varepsilon \cap T(g_1, \varepsilon) \cap T(g_2, \varepsilon)$  is relatively dense.*

**Lemma 2.7** ([8]) *Consider  $\theta$  defined in Lemma 2.4. Let  $\varepsilon > 0$ ,  $\tau \in \Gamma_\varepsilon \cap T(A, \varepsilon)$ , and  $p \in P_\tau(\varepsilon)$ . Then there is  $K' > 0$  such that, for all  $n \in Z$ ,*

- (a)  $|X(t_{n+p+1}, u + \tau) - X(t_{n+1}, u)| \leq K' \varepsilon$  for all  $u \in [t_n, t_{n+1}]$ ;
- (b)  $|X(t + \tau, t_{n+p}) - X(t, t_n)| \leq K' \varepsilon$  for all  $t \in [t_n, t_{n+1}]$ ;
- (c)  $|X(t + \tau, s + \tau) - X(t, s)| \leq K' \varepsilon$  for all  $s, t \in R: |t - s| \leq \theta$ ;
- (d)  $|X(t_{n+p+1}, t_{n+p}) - X(t_{n+1}, t_n)| \leq K' \varepsilon$ .

### 3 Main results

#### 3.1 The existence and uniqueness of the asymptotically almost periodic solution for system (1.1)

In this section, we consider a more general  $y_\gamma$ , where

$$y_\gamma(t) = (y(\gamma^{p_1}(t)), y(\gamma^{p_2}(t)), \dots, y(\gamma^{p_l}(t))),$$

$\gamma^{p_i}(t)$ ,  $i = 1, 2, \dots, l$ , denotes step functions. This definition of exponential dichotomy has been adapted from (1.3) (Definition 2.2) in the paper of Papashinopoulos [32], there  $\gamma = [\cdot]$ . Here, it is an exponential dichotomy for (2.2), which is not obvious to be extended for (1.3) in [32] in terms of  $Z(t, s)$  except for the cases where the projection for exponential dichotomy commutes with  $A(t)$  and  $B(t)$ . Therefore, we try to convert the exponential dichotomy of the corresponding (1.3) in [32] by other methods.

Next, we study a Cauchy operator for the linear part of (1.3).

Let  $X$  be a fundamental matrix of the following linear homogeneous system:

$$x' = A(t)x \tag{3.1}$$

and  $X(t, s) = X(t)X(s)^{-1}$ . Now we follow [4] to say what is the Cauchy matrix for (1.3).

For  $n \in Z$ ,  $t \in J_n$  satisfies  $t \geq s$ . Let  $Z_n(t) = X(t, t_n)J_n(t)$ , where

$$J_n(t) = I + \int_{t_n}^t X(t_n, u)B(u) du.$$

Assume that (A):  $J_n(t)$  is invertible for all  $n \in Z$  and  $t \in [t_n, t_{n+1}]$ . Let

$$H(n) = Z_n(t_{n+1}) \tag{3.2}$$

for all  $n \in Z$ . For  $\tau \in R$ , let  $k(\tau) \in Z$  such that  $\tau \in J_{k(\tau)}$ . Consider  $t > s$  such that  $k(t) > k(s)$ .

Then we define

$$\begin{aligned} Z(t, s) &= Z_{k(t)}(t) [H(k(t) - 1)H(k(t) - 2) \cdots H(k(s) + 1)] \\ &\quad \times H(k(s))^{-1} Z_{k(s)}(s)^{-1}. \end{aligned} \tag{3.3}$$

If  $t \leq s$ , by condition (A),  $Z(t, s) = Z(s, t)^{-1}$  is well defined. Therefore,  $Z(t, s)$  is the Cauchy matrix for (1.3) and bounded (see [2, 3, 36, 39, 46, 47, 49–51]).

In fact,

$$|Z(t, s)| \leq e^{|A|_\infty(t_{n+1}-t_n)} (1 + e^{|A|_\infty(t_{n+1}-t_n)} |B|_\infty(t_{n+1} - t_n))$$

for all  $t, s \in J_n$ . Consequently,  $Z(t, s)$  is bounded.

Consider the difference equation

$$\phi(n + 1) = H(n)\phi(n). \tag{3.4}$$

Notice that if  $z : R \rightarrow C$ , and  $z$  is a solution of (1.3), then  $\phi(n) = z(t_n)$  is a solution of (3.4).

( $H_6$ ) Assume that (3.4) has an exponential dichotomy.

According to Definition 2.1, assumption ( $H_6$ ) is equivalent to that there are a projection  $\Pi : R^q \rightarrow R^q$  and positive constants  $\rho, K > 0$  with  $\rho < 1$  such that

$$\begin{aligned} |G(n, k)| &\leq K\rho^{-(n-k-1)}, & \text{if } n \geq k + 1, \\ |G(n, k)| &\leq K\rho^{-(k+1-n)}, & \text{if } n < k + 1 \end{aligned} \tag{3.5}$$

for all  $n, k \in Z : \pm(n - k) \leq 0$ , where

$$G(n, k) = \begin{cases} \Phi(n)\Pi\Phi^{-1}(k + 1), & \text{if } n \geq k + 1 \\ -\Phi(n)(I - \Pi)\Phi^{-1}(k + 1), & \text{if } n < k + 1, \end{cases} \tag{3.6}$$

and  $\Pi$  is a projection operator ( $\Pi = \Pi^2$ ),  $\Phi$  is a fundamental matrix for system (3.4). In particular it will be said that system (3.4) is exponentially stable as  $n \rightarrow +\infty$  if it has an exponential dichotomy with  $\Pi = I$ .

If  $c$  is the bounded solution of the discrete system

$$c(n + 1) = H(n)c(n) + h(n), \tag{3.7}$$

then

$$c(n) = \sum_{k=-\infty}^{+\infty} G(n, k)h(k), \tag{3.8}$$

where the Green matrix  $G(n, k)$  is given by (3.6),  $h$  is given by

$$h(n) = \int_{t_n}^{t_{n+1}} X(t_{n+1}, u)f(u) du. \tag{3.9}$$

By the variation of constants formula (see [4, 36]) and (3.8), (3.9), we obtain that

$$\begin{aligned} y(t) &= Z(t, k(t))c(k(t)) + \int_{\gamma^0(t)}^t X(t, u)f(u) du \\ &= Z_{k(t)}(t) \times \left( \sum_{k=-\infty}^{+\infty} G(k(t), k) \int_{t_k}^{t_{k+1}} X(t_{k+1}, u)f(u) du \right) \\ &\quad + \int_{\gamma^0(t)}^t X(t, u)f(u) du \end{aligned} \tag{3.10}$$

for all  $t \in R$ , where  $c$  is the solution of discrete system (3.7). And (3.10) is a unique bounded solution of (1.1).

**Theorem 3.1** *Assume that  $(H_1)$ ,  $(H_3)$ , and  $(H_4)$  hold. Then the sequence  $H = \{H(n)\}_{n=-\infty}^{+\infty}$  given by (3.2) and the sequence  $h = \{h(n)\}_{n=-\infty}^{+\infty}$  given by (3.9) are asymptotically almost periodic.*

*Proof* Firstly, we prove that  $\{h(n)\}_{n=-\infty}^{+\infty}$  is asymptotically almost periodic.

By  $(H_4)$ ,  $f(t) \in \text{AAP}(\mathbb{R}^q)$ , let  $f(t) = f_1(t) + f_2(t)$ , where

$$f_1(t) \in \text{AP}(\mathbb{R}^q), \quad f_2(t) \in C_0(\mathbb{R}^q).$$

Then

$$\begin{aligned} h(n) &= \int_{t_n}^{t_{n+1}} X(t_{n+1}, u) f(u) \, du \\ &= \int_{t_n}^{t_{n+1}} X(t_{n+1}, u) (f_1(u) + f_2(u)) \, du \\ &= \int_{t_n}^{t_{n+1}} X(t_{n+1}, u) f_1(u) \, du + \int_{t_n}^{t_{n+1}} X(t_{n+1}, u) f_2(u) \, du. \end{aligned}$$

Now, we prove that

$$X(t_{n+1}, u) \in \text{APS}(Z, \mathbb{R}^q), \quad \int_{t_n}^{t_{n+1}} X(t_{n+1}, u) f_1(u) \, du \in \text{APS}(Z, \mathbb{R}^q). \tag{3.11}$$

In fact, by Lemma 2.7(a), we obtain that  $\{X(t_{n+1}, u)\}$  is an almost periodic sequence. Set

$$h_1(n) = \int_{t_n}^{t_{n+1}} X(t_{n+1}, u) f_1(u) \, du.$$

By Lemma 2.6,  $\Gamma = T(A, \varepsilon) \cap T(B, \varepsilon) \cap \Gamma_\varepsilon$  is relatively dense for any  $\varepsilon > 0$ . Let  $p \in P = \bigcup_{\tau \in \Gamma} P_\tau(\varepsilon)$ , where  $P_\tau(\varepsilon) = \{k \in \mathbb{Z} \mid \sup_{n \in \mathbb{Z}} |t_n^{(k)} - \tau| \leq \varepsilon\}$ .

Consequently, there is  $\tau \in \Gamma$  such that  $p \in P_\tau(\varepsilon)$ . Then we have

$$\begin{aligned} &h_1(n+p) - h_1(n) \\ &= \int_{t_{n+p}}^{t_{n+p+1}} X(t_{n+p+1}, u) f_1(u) \, du - \int_{t_n}^{t_{n+1}} X(t_{n+1}, u) f_1(u) \, du \\ &= \int_{t_{n+p}}^{t_{n+p+1}} X(t_{n+p+1}, u) f_1(u) \, du - \int_{t_n+\tau}^{t_{n+p+1}} X(t_{n+p+1}, u) f_1(u) \, du \\ &\quad + \int_{t_n+\tau}^{t_{n+p+1}} X(t_{n+p+1}, u) f_1(u) \, du - \int_{t_n}^{t_{n+1}} X(t_{n+p+1}, u + \tau) f_1(u + \tau) \, du \\ &\quad + \int_{t_n}^{t_{n+1}} X(t_{n+p+1}, u + \tau) f_1(u + \tau) \, du - \int_{t_n}^{t_{n+1}} X(t_{n+1}, u) f_1(u) \, du \\ &= \int_{t_{n+p}}^{t_n+\tau} X(t_{n+p+1}, u) f_1(u) \, du + \int_{t_{n+1}+\tau}^{t_{n+p+1}} X(t_{n+p+1}, u) f_1(u) \, du \\ &\quad + \int_{t_n}^{t_{n+1}} [X(t_{n+p+1}, u + \tau) f_1(u + \tau) - X(t_{n+1}, u) f_1(u)] \, du \end{aligned}$$

for all  $n \in \mathbb{Z}$ .

By Lemma 2.7, there are positive constants  $C$  and  $K'$  such that

$$\begin{aligned} \left| \int_{t_{n+p}}^{t_{n+p+\tau}} X(t_{n+p+1}, u) f_1(u) \, du \right| &\leq C |t_n^{(p)} - \tau|, \\ \left| \int_{t_{n+1+\tau}}^{t_{n+p+1}} X(t_{n+p+1}, u) f_1(u) \, du \right| &\leq C |t_{n+1}^{(p)} - \tau|, \\ \left| \int_{t_n}^{t_{n+1}} [X(t_{n+p+1}, u + \tau) f_1(u + \tau) - X(t_{n+1}, u) f_1(u)] \, du \right| &\leq K' \varepsilon. \end{aligned}$$

Therefore,

$$\begin{aligned} &|h_1(n + p) - h_1(n)| \\ &\leq |t_n^{(p)} - \tau| C + |t_{n+1}^{(p)} - \tau| C + K' \varepsilon \\ &\leq [2C + K'] \varepsilon \end{aligned}$$

for all  $n \in Z$ .

Hence,  $p \in T(h_1, [2C + K']\varepsilon)$ . Since  $p$  is taken arbitrarily in  $P$ , so  $P \subseteq T(h_1, [2C + K']\varepsilon)$ , by Lemma 2.5,  $P$  is relatively dense. Consequently,  $T(h_1, [2C + K']\varepsilon)$  is also relatively dense. Because  $\varepsilon > 0$  is arbitrary, hence  $h_1 \in \text{APS}(Z, R^q)$ .

According to Lemmas 2.2–2.4, we have

$$\begin{aligned} &\left| \int_{t_n}^{t_{n+1}} X(t_{n+1}, u) f_2(u) \, du \right| \\ &\leq \int_{t_n}^{t_{n+1}} |X(t_{n+1}, u) f_2(u)| \, du \\ &\leq k_0 \int_{t_n}^{t_{n+1}} |f_2(u)| \, du. \end{aligned}$$

And because  $f_2(t) \in C_0(R^q)$ , that is, as  $n \rightarrow \infty$ , one has  $u \rightarrow \infty, f_2(u) \rightarrow 0$ .

Hence,

$$\int_{t_n}^{t_{n+1}} X(t_{n+1}, u) f_2(u) \, du \in C_0S(Z, R^q).$$

Then

$$\{h(n)\}_{n=-\infty}^{+\infty} \in \text{AAPS}(Z, R^q).$$

Notice that

$$H(n) = X(t_{n+1}, t_n) + \int_{t_n}^{t_{n+1}} X(t_{n+1}, u) B(u) \, du$$

for all  $n \in Z$ . Lemma 2.3 implies that  $X(t_{n+1}, t_n) \in \text{APS}(Z, R^q)$ , and using a method similar to the method of proving  $\{h(n)\}_{n=-\infty}^{+\infty}$ , we get

$$\int_{t_n}^{t_{n+1}} X(t_{n+1}, u) B(u) \, du \in \text{AAPS}(Z, R^q).$$

From all the above, we have  $\{H(n)\}_{n=-\infty}^{+\infty} \in \text{AAPS}(Z, R^q)$ . □

**Theorem 3.2** *Assume that  $(H_1)$ ,  $(H_3)$ ,  $(H_4)$ , and  $(H_6)$  hold. Namely, we have linear difference equation (3.4) with exponential dichotomy on  $Z$ . Then the solution for linear inhomogeneous difference system (3.7) is an asymptotically almost periodic sequence.*

*Proof* We know that the solution of equation (3.7) is

$$c(n) = \sum_{n \in Z} G(n, k)h(k).$$

In terms of Theorem 3.1, we obtain that  $h(k)$  is an asymptotically almost periodic sequence. Thus, let  $h(k) = h_1(k) + h_2(k)$ , where  $h_1(k) \in \text{APS}(Z, R^q)$ ,  $h_2(k) \in C_0S(Z, R^q)$ . Then

$$c(n) = \sum_{n \in Z} G(n, k)(h_1(k) + h_2(k)).$$

Set

$$I_1 = \sum_{n \in Z} G(n, k)h_1(k), \quad I_2 = \sum_{n \in Z} G(n, k)h_2(k).$$

Notice that,  $\forall \tau \in T(h_1, \varepsilon)$ , we have

$$\begin{aligned} & \left| \sum_{n \in Z} G(n, k)h_1(k + \tau) - \sum_{n \in Z} G(n, k)h_1(k) \right| \\ &= \left| \sum_{n \in Z} G(n, k) \right| |h_1(k + \tau) - h_1(k)| \\ &\leq K \sum_{n \in Z} \rho^{-|n-k-1|} \varepsilon \\ &= K(1 + \rho^{-1})(1 - \rho^{-1})^{-1} \varepsilon, \end{aligned}$$

where  $K > 0$ ,  $\rho < 1$ . Therefore,  $I_1 \in \text{APS}(Z, R^q)$ .

Next, we just need to prove that  $I_2 \in C_0S(Z, R^q)$ . First, it will be proved that  $\lim_{n \rightarrow +\infty} I_2 = 0$ .

Notice that

$$\begin{aligned} I_2 &= \sum_{k \leq n-1} \Phi(n) \Pi \Phi^{-1}(k+1)h_2(k) \\ &\quad - \sum_{k \geq n} \Phi(n)(I - \Pi)\Phi^{-1}(k+1)h_2(k). \end{aligned}$$

Due to  $\lim_{n \rightarrow +\infty} \rho^{-(n-1)} = 0$ , then  $\forall \varepsilon > 0$ , there exists  $N_1 > 0$  such that  $|\rho^{-(n-1)}| < \varepsilon$  as  $n > N_1$ . And because  $\lim_{n \rightarrow +\infty} h_2(n) = 0$ , that is, for the above  $\varepsilon > 0$ , there is  $N_2 > 0$  such

that  $|h_2(n)| < \varepsilon$  as  $n > N_2$ . By taking  $N = \max\{N_1, N_2\}$ , as  $n > N$ , one has

$$|I_2| \leq \sum_{k \leq n-1} |\Phi(n)\Pi\Phi^{-1}(k+1)h_2(k)| + \sum_{k \geq n} |\Phi(n)(I - \Pi)\Phi^{-1}(k+1)h_2(k)|.$$

We estimate the first part of the above expression:

$$\begin{aligned} & \sum_{k \leq n-1} |\Phi(n)\Pi\Phi^{-1}(k+1)h_2(k)| \\ & \leq \sum_{k \leq n-1} K\rho^{-(n-k-1)}|h_2(k)| \\ & \leq \frac{K}{1 - \rho^{-1}}\varepsilon. \end{aligned}$$

Then we estimate the second part:

$$\begin{aligned} & \sum_{k \geq n} |\Phi(n)(I - \Pi)\Phi^{-1}(k+1)h_2(k)| \\ & = \sum_{k \geq n} K\rho^{-(n-k-1)}|h_2(k)| \\ & < \varepsilon \sum_{k \geq n} K\rho^{-(n-k-1)} \\ & = \frac{K\rho}{1 - \rho^{-1}}\varepsilon. \end{aligned}$$

Hence,  $\lim_{n \rightarrow +\infty} I_2 = 0$ . In a similar way,  $\lim_{n \rightarrow -\infty} I_2 = 0$ . In conclusion,  $\lim_{|n| \rightarrow \infty} I_2 = 0$ ; in other words,  $I_2 \in C_0S(Z, R^q)$ .

From all the above,  $c(n) \in \text{AAPS}(Z, R^q)$ . □

**Theorem 3.3** *If  $(H_1)$ ,  $(H_3)$ ,  $(H_4)$ , and  $(H_6)$  hold, then equation (1.1) has a unique asymptotically almost periodic solution.*

*Proof* The solution of equation (1.1) is

$$y(t) = \left[ X(t, t_n) + \int_{t_n}^t X(t, u)B(u) du \right] y(t_n) + \int_{t_n}^t X(t, u)f(u) du, \tag{3.12}$$

where  $t \in R$ ,  $t_n < t < t_{n+1}$ . Obviously,  $\{y(t_n) : n \in Z\}$  satisfies inhomogeneous difference equation (3.7). By Theorem 3.2, inhomogeneous difference equation (3.7) has a solution  $\{y_0(t_n) : n \in Z\} \in \text{AAPS}(Z, R^q)$  satisfying  $|y_0(t_n)| \leq \beta$  for all  $n \in Z$ , and the unique for  $\{y_0(t_n) : n \in Z\}$  ensures the solution  $y(t)$  of equation (1.1) satisfying  $y(t_n) = y_0(t_n)$  for all  $n \in Z$  (see Lemma 2.1).

The following proof shows that  $y(t)$  is an asymptotically almost periodic solution for equation (1.1).

Obviously,  $y(t)$  is a bounded continuous function. Next, we will prove that

$$y(t) \in \text{AAP}(\mathbb{R}^q).$$

By  $(H_1)$  and  $(H_4)$ , we have

$$A(t), B(t) \in \text{AP}(\mathbb{R}^q, \mathbb{R}^q), \quad f(t) \in \text{AAP}(\mathbb{R}^q).$$

Then let

$$\begin{aligned} f(t) &= f_1(t) + f_2(t), \quad f_1(t) \in \text{AP}(\mathbb{R}^q), f_2(t) \in C_0(\mathbb{R}^q), \\ y(t_n) &= y_1(t_n) + y_2(t_n), \quad y_1(t_n) \in \text{APS}(Z, \mathbb{R}^q), y_2(t_n) \in C_0S(Z, \mathbb{R}^q). \end{aligned}$$

Thus,

$$\begin{aligned} y(t) &= \left[ X(t, t_n) + \int_{t_n}^t X(t, u)B(u) \, du \right] y(t_n) + \int_{t_n}^t X(t, u)f(u) \, du \\ &= \left[ X(t, t_n) + \int_{t_n}^t X(t, u)B(u) \, du \right] [y_1(t_n) + y_2(t_n)] \\ &\quad + \int_{t_n}^t X(t, u)[f_1(u) + f_2(u)] \, du \\ &= \left[ X(t, t_n) + \int_{t_n}^t X(t, u)B(u) \, du \right] y_1(t_n) \\ &\quad + \left[ X(t, t_n) + \int_{t_n}^t X(t, u)B(u) \, du \right] y_2(t_n) \\ &\quad + \int_{t_n}^t X(t, u)f_1(u) \, du + \int_{t_n}^t X(t, u)f_2(u) \, du. \end{aligned}$$

Provided that

$$z(t) = \left[ X(t, t_n) + \int_{t_n}^t X(t, u)B(u) \, du \right] y_1(t_n) + \int_{t_n}^t X(t, u)f_1(u) \, du.$$

From Lemma 2.3, similar to the method of proving (3.11), one has

$$X(t, t_n) \in \text{APS}(Z, \mathbb{R}^q), \quad \int_{t_n}^t X(t, u)f_1(u) \, du \in \text{APS}(Z, \mathbb{R}^q).$$

Let

$$\sup_{n \in Z} |X(t, t_n)| = M_1, \quad \sup_{n \in Z} |y_1(t_n)| = M_2.$$

Taking  $\tau \in T(X(t, t_n), \frac{\varepsilon}{2}) \cap T(y_1(t_n), \frac{\varepsilon}{2})$ , for all  $n \in Z$ , we get

$$\begin{aligned} &|X(t + \tau, t_n + \tau)y_1(t_n + \tau) - X(t, t_n)y_1(t_n)| \\ &\leq |y_1(t_n + \tau)| |X(t + \tau, t_n + \tau) - X(t, t_n)| \end{aligned}$$

$$\begin{aligned}
 &+ |X(t, t_n)| |y_1(t_n + \tau) - y_1(t_n)| \\
 &\leq (M_1 + M_2)\varepsilon \\
 &= \varepsilon_1.
 \end{aligned}$$

Therefore,  $\tau \in T(X(t, t_n)y_1(t_n), \varepsilon_1)$ , where  $T(X(t, t_n)y_1(t_n), \varepsilon_1)$  is relatively dense on  $Z$  and  $X(t, t_n)y_1(t_n)$  is almost periodic. In the same way,  $\int_{t_n}^t X(t, u)B(u) du y_1(t_n)$  is also almost periodic. Hence,  $z(t)$  is almost periodic.

Now, we prove that the following function

$$\left[ X(t, t_n) + \int_{t_n}^t X(t, u)B(u) du \right] y_2(t_n) + \int_{t_n}^t X(t, u)f_2(u) du$$

is continuous on  $R^q$ ; we will proceed as in the proof of the continuity of  $y(t)$ .

According to  $|B(t)| \leq M$  (by condition  $(H_1)$ ,  $B(t) \in AP(R^q, R^q)$ ), we know that if

$$\Pi(t) = \left[ X(t, t_n) + \int_{t_n}^t X(t, u)B(u) du \right] y_2(t_n) + \int_{t_n}^t X(t, u)f_2(u) du,$$

then

$$\begin{aligned}
 \Pi(t) &\leq \left| \left[ X(t, t_n) + \int_{t_n}^t X(t, u)B(u) du \right] y_2(t_n) \right| \\
 &\quad + \left| \int_{t_n}^t X(t, u)f_2(u) du \right| \\
 &= \Pi_1(t) + \Pi_2(t).
 \end{aligned}$$

By Lemma 2.2, we have

$$\begin{aligned}
 \Pi_1(t) &\leq \left( |X(t, t_n)| + \int_{t_n}^t |X(t, u)||B(u)| du \right) y_2(t_n) \\
 &\leq (k_0 + k_0M)y_2(t_n)
 \end{aligned}$$

as  $|n| \rightarrow \infty$ .

And because  $y_2(t_n) \in C_0S(Z, R^q)$ , then  $\Pi_1(t) \rightarrow 0$ , so  $\Pi_1(t) \in C_0(R^q)$ . On the other hand, for  $f_2(t) \in C_0(R^q)$ , then  $\forall \varepsilon > 0, \exists t_1 > 0$ , we have  $|f_2(t)| < \varepsilon$  as  $|t| > t_1$ . Hence,

$$\Pi_2(t) = \int_{t_n}^t X(t, u)f_2(u) du \leq k_0 \int_{t_n}^t f_2(u) du \leq k_0\varepsilon.$$

That is,  $\Pi_2(t) \in C_0(R^q)$ . Thus,  $\Pi(t) \in C_0(R^q)$ .

Consequently,  $y(t) = z(t) + \Pi(t)$  is obtained if  $A(t), B(t) \in AP(R^q, R^q), f(t) \in AAP(R^q)$ . On the basis of  $z(t) \in AP(R^q), \Pi(t) \in C_0(R^q)$ , so  $y(t) \in AAP(R^q)$ .

Because the uniqueness of  $y(n)$  implies that  $y(t)$  is unique, hence  $y(t)$  is a unique asymptotically almost periodic solution of equation (1.1). □

*Remark 3.1* If  $(H_2)$  holds, in other words, if  $B(t) \in AAP(R^q, R^q)$  with the other conditions unchanged, then the conclusion remains true. The method of proving this conclusion is similar to the previous processes for proving Theorem 3.3, so it is omitted.

### 3.2 The existence and uniqueness of asymptotically almost periodic solution for system (1.2)

In order to study the existence of asymptotically almost periodic solution for (1.2), by  $(H_5)$ ,  $W$  is an arbitrary nonempty compact subset on  $R^q$ , and the set

$$T(F, \varepsilon, W) = \{ \tau \in R : |F(t + \tau, \omega) - F(t, \omega)| \leq \varepsilon, \forall (t, \omega) \in R \times W \}$$

is relatively dense for all  $\varepsilon > 0$ .

**Theorem 3.4** *Let  $y : R \rightarrow R^q$  be an asymptotically almost periodic solution of (1.1). Assume that  $(H_3)$  holds and  $F$  satisfies  $(H_5)$ . Then  $F(t, y_\gamma(t))$  satisfies  $(H_4)$ , where*

$$y_\gamma(t) = (y(\gamma^{p_1}(t)), y(\gamma^{p_2}(t)), \dots, y(\gamma^{p_l}(t))),$$

$\gamma^{p_i}(t), i = 1, 2, \dots, l$ , denotes step functions.

*Proof* Since  $y$  is asymptotically almost periodic, so for the almost periodic part of  $y$ , one has that  $\forall \varepsilon > 0, \tau \in T(y, \varepsilon) \cap T(F, \varepsilon, W)$  and  $F$  is uniformly continuous. Thus, there is  $\delta > 0$  such that  $|s - t| \leq \delta$  for all  $s, t \in R$ , we know that  $|y(t) - y(s)| \leq \varepsilon$ . In terms of  $P_\tau(\delta) = \{k \in Z | \sup_{n \in Z} |t_n^{(k)} - \tau| \leq \delta\}$  for all  $k \in Z$ , so  $|\gamma^{p_j}(t + \tau) - (\gamma^{p_j}(t) + \tau)| \leq \delta$  for all  $j = 1, \dots, l$ . Moreover,

$$\begin{aligned} &|F(t + \tau, y_\gamma(t + \tau)) - F(t, y_\gamma(t))| \\ &\leq |F(t + \tau, y_\gamma(t + \tau)) - F(t, y_\gamma(t + \tau))| \\ &\quad + |F(t, y_\gamma(t + \tau)) - F(t, y_\gamma(t))| \\ &\leq \varepsilon + Ll\varepsilon. \end{aligned}$$

Since  $\varepsilon > 0$  is taken arbitrarily, hence  $F(t, y_\gamma(t))$  satisfies  $(H_4)$ . □

**Theorem 3.5** *Let  $(H_1), (H_3), (H_4)$ , and  $(H_6)$  hold. Suppose that  $F$  satisfies  $(H_5)$ . If*

$$2 \frac{KLl}{1 - \rho} < 1, \tag{3.13}$$

*then equation (1.2) has a unique asymptotically almost periodic solution.*

*Proof* Set

$$(Tc)(n) = \sum_{k=-\infty}^{+\infty} G(n, k)h(k, \hat{c}(k)), \tag{3.14}$$

where

$$h(n, \hat{c}(n)) = \int_{t_n}^{t_{n+1}} X(t_{n+1}, s)F(s, \hat{c}(n)) ds$$

and  $G(n, k)$  is given in (3.6) and  $\hat{c}(n) = (c(n - p_1), \dots, c(n - p_l))$ .

If  $c$  is a fixed point of the operator defined by (3.14), from Theorem 3.2, we know that  $c$  is an asymptotically almost periodic solution of the following difference equation:

$$c(n + 1) = H(n)c(n) + h(n, \hat{c}(n)). \tag{3.15}$$

In what follows, we prove it in three steps: for  $c \in \text{AAPS}(Z, R^q)$ , one has  $Tc \in \text{AAPS}(Z, R^q)$ , that is,  $T : \text{AAPS}(Z, R^q) \rightarrow \text{AAPS}(Z, R^q)$ .

Firstly, we prove  $F(s, \hat{c}(n)) \in \text{AAPS}(Z, R^q)$ .

According to  $\hat{c}(n) \in \text{AAPS}(Z, R^q)$ , provided that  $\hat{c}(n) = \hat{c}_{\text{ap}}(n) + \hat{c}_{c_0}(n)$ , where

$$\hat{c}_{\text{ap}}(n) \in \text{APS}(Z, R^q), \quad \hat{c}_{c_0}(n) \in C_0S(Z, R^q)$$

and

$$F(s, \hat{c}(n)) = F(s, \hat{c}_{\text{ap}}(n)) + [F(s, \hat{c}(n)) - F(s, \hat{c}_{\text{ap}}(n))].$$

Then from  $(H_5)$ ,  $F$  satisfies the Lipschitz condition, we have

$$F(s, \hat{c}(n)) - F(s, \hat{c}_{\text{ap}}(n)) \leq L\hat{c}_{c_0}(n) \rightarrow 0.$$

Therefore,

$$F(s, \hat{c}(n)) - F(s, \hat{c}_{\text{ap}}(n)) \in C_0S(Z, R^q).$$

Meanwhile, for  $F$  satisfies the Lipschitz condition, hence  $F(s, \hat{c}_{\text{ap}}(n)) \in \text{APS}(Z, R^q)$ . Consequently,

$$F(s, \hat{c}(n)) \in \text{AAPS}(Z, R^q).$$

Secondly, we prove  $h(n, \hat{c}(n)) \in \text{AAPS}(Z, R^q)$ .

Because  $F(s, \hat{c}(n)) \in \text{AAPS}(Z, R^q)$ , let

$$F(s, \hat{c}(n)) = F_1(s, \hat{c}(n)) + F_2(s, \hat{c}(n)),$$

where

$$F_1(s, \hat{c}(n)) \in \text{APS}(Z, R^q), \quad F_2(s, \hat{c}(n)) \in C_0S(Z, R^q).$$

Then

$$\begin{aligned} h(n, \hat{c}(n)) &= \int_{t_n}^{t_{n+1}} X(t_{n+1}, s)F(s, \hat{c}(n)) \, ds \\ &= \int_{t_n}^{t_{n+1}} X(t_{n+1}, s)(F_1(s, \hat{c}(n)) + F_2(s, \hat{c}(n))) \, ds \\ &= \int_{t_n}^{t_{n+1}} X(t_{n+1}, s)F_1(s, \hat{c}(n)) \, ds + \int_{t_n}^{t_{n+1}} X(t_{n+1}, s)F_2(s, \hat{c}(n)) \, ds. \end{aligned}$$

Now, we prove

$$X(t_{n+1}, s) \in \text{APS}(Z, R^q), \quad \forall s \in R,$$

$$\int_{t_n}^{t_{n+1}} X(t_{n+1}, s)F_1(s, \hat{c}(n)) ds \in \text{APS}(Z, R^q).$$

In fact, by Lemma 2.7(a), we know that  $\{X(t_{n+1}, s)\} (s \in R)$  is an almost periodic sequence. Set

$$h_1(n, \hat{c}(n)) = \int_{t_n}^{t_{n+1}} X(t_{n+1}, s)F_1(s, \hat{c}(n)) ds.$$

By Lemma 2.6,  $\Gamma = T(A, \varepsilon) \cap T(B, \varepsilon) \cap \Gamma_\varepsilon$  is relatively dense for any  $\varepsilon > 0$ , where

$$\Gamma_\varepsilon = \left\{ r \in R \mid \exists k \in Z, \text{ such that } \sup_{n \in Z} |t_n^{(k)} - r| \leq \varepsilon \right\}.$$

Suppose that  $p \in P = \bigcup_{\tau \in \Gamma} P_\tau(\varepsilon)$ , where

$$P_\tau(\varepsilon) = \left\{ k \in Z \mid \sup_{n \in Z} |t_n^{(k)} - \tau| \leq \varepsilon \right\}.$$

Thus, there is  $\tau \in \Gamma$  such that  $p \in P_\tau(\varepsilon)$ , for all  $n \in Z$ , we have

$$\begin{aligned} & h_1(n + p, \hat{c}(n + p)) - h_1(n, \hat{c}(n)) \\ &= \int_{t_{n+p}}^{t_{n+p+1}} X(t_{n+p+1}, s)F_1(s, \hat{c}(n + p)) ds - \int_{t_n}^{t_{n+1}} X(t_{n+1}, s)F_1(s, \hat{c}(n)) ds \\ &= \int_{t_{n+p}}^{t_{n+p+1}} X(t_{n+p+1}, s)F_1(s, \hat{c}(n + p)) ds - \int_{t_{n+\tau}}^{t_{n+p+1}} X(t_{n+p+1}, s)F_1(s, \hat{c}(n + p)) ds \\ &\quad + \int_{t_{n+\tau}}^{t_{n+p+1}} X(t_{n+p+1}, s)F_1(s, \hat{c}(n + p)) ds \\ &\quad - \int_{t_n}^{t_{n+1}} X(t_{n+p+1}, s + \tau)F_1(s + \tau, \hat{c}(n + p)) ds \\ &\quad + \int_{t_n}^{t_{n+1}} X(t_{n+p+1}, s + \tau)F_1(s + \tau, \hat{c}(n + p)) ds \\ &\quad - \int_{t_n}^{t_{n+1}} X(t_{n+p+1}, s + \tau)F_1(s + \tau, \hat{c}(n)) ds \\ &\quad + \int_{t_n}^{t_{n+1}} X(t_{n+p+1}, s + \tau)F_1(s + \tau, \hat{c}(n)) ds - \int_{t_n}^{t_{n+1}} X(t_{n+1}, s)F_1(s, \hat{c}(n)) ds \\ &= \int_{t_{n+p}}^{t_{n+\tau}} X(t_{n+p+1}, s)F_1(s, \hat{c}(n + p)) ds + \int_{t_{n+1+\tau}}^{t_{n+p+1}} X(t_{n+p+1}, s)F_1(s, \hat{c}(n + p)) ds \\ &\quad + \int_{t_n}^{t_{n+1}} X(t_{n+p+1}, s + \tau)[F_1(s + \tau, \hat{c}(n + p)) - F_1(s + \tau, \hat{c}(n))] ds \\ &\quad + \int_{t_n}^{t_{n+1}} [X(t_{n+p+1}, s + \tau)F_1(s + \tau, \hat{c}(n)) - X(t_{n+1}, s)F_1(s, \hat{c}(n))] ds. \end{aligned}$$

For Lemma 2.7, there are positive constants  $C$  and  $K'$  such that

$$\begin{aligned} & \left| \int_{t_{n+p}}^{t_{n+p}+\tau} X(t_{n+p+1}, s)F_1(s, \hat{c}(n+p)) ds \right| \leq C|t_n^{(p)} - \tau|, \\ & \left| \int_{t_{n+1}+\tau}^{t_{n+p+1}} X(t_{n+p+1}, s)F_1(s, \hat{c}(n+p)) ds \right| \leq C|t_{n+1}^{(p)} - \tau|, \\ & \left| \int_{t_n}^{t_{n+1}} X(t_{n+p+1}, s+\tau)[F_1(s+\tau, \hat{c}(n+p)) - F_1(s+\tau, \hat{c}(n))] ds \right| \leq K'\varepsilon, \\ & \left| \int_{t_n}^{t_{n+1}} [X(t_{n+p+1}, s+\tau)F_1(s+\tau, \hat{c}(n)) - X(t_{n+1}, s)F_1(s, \hat{c}(n))] ds \right| \leq K'\varepsilon. \end{aligned}$$

Then, for all  $n \in Z$ , we have

$$\begin{aligned} & |h_1(n+p, \hat{c}(n+p)) - h_1(n, \hat{c}(n))| \\ & \leq |t_n^{(p)} - \tau|C + |t_{n+1}^{(p)} - \tau|C + 2K'\varepsilon \\ & \leq 2[C + K']\varepsilon. \end{aligned}$$

Therefore,  $p \in T(h_1, 2[C + K']\varepsilon)$ . And because  $p$  is taken arbitrarily in  $P$ , where

$$P = \bigcup_{\tau \in \Gamma} P_\tau(\varepsilon), P_\tau(\varepsilon) = \left\{ k \in Z \mid \sup_{n \in Z} |t_n^{(k)} - \tau| \leq \varepsilon \right\},$$

we get  $P \subseteq T(h_1, 2[C + K']\varepsilon)$ . By Lemma 2.5,  $P$  is relatively dense, thus,  $T(h_1, 2[C + K']\varepsilon)$  is also relatively dense. Since  $\varepsilon > 0$  is arbitrary, so  $h_1 \in \text{APS}(Z, R^q)$ .

By Lemmas 2.2–2.4, we know that  $X(t_{n+1}, s)$  is bounded. Therefore,

$$\begin{aligned} & \left| \int_{t_n}^{t_{n+1}} X(t_{n+1}, s)F_2(s, \hat{c}(n)) ds \right| \\ & \leq \int_{t_n}^{t_{n+1}} |X(t_{n+1}, s)F_2(s, \hat{c}(n))| ds \\ & \leq k_0 \int_{t_n}^{t_{n+1}} |F_2(s, \hat{c}(n))| ds. \end{aligned}$$

And since  $F_2(s, \hat{c}(n)) \in C_0S(Z, R^q)$ , we have that

$$\int_{t_n}^{t_{n+1}} X(t_{n+1}, s)F_2(s, \hat{c}(n)) ds \in C_0S(Z, R^q).$$

Hence,  $h(n, \hat{c}(n)) \in \text{AAPS}(Z, R^q)$ .

Thirdly, we prove  $T(c) \in \text{AAPS}(Z, R^q)$ .

Since  $h(n, \hat{c}(n)) \in \text{AAPS}(Z, R^q)$ , let

$$h(k, \hat{c}(k)) = f(k, \hat{c}(k)) + g(k, \hat{c}(k)),$$

where  $f(k, \hat{c}(k)) \in \text{APS}(Z, R^q), g(k, \hat{c}(k)) \in C_0S(Z, R^q)$ . Then

$$\begin{aligned} (T\hat{c})(n) &= \sum_{k=-\infty}^{+\infty} G(n, k)h(k, \hat{c}(k)) \\ &= \sum_{k=-\infty}^{+\infty} G(n, k)[f(k, \hat{c}(k)) + g(k, \hat{c}(k))] \\ &= \sum_{k=-\infty}^{+\infty} G(n, k)f(k, \hat{c}(k)) + \sum_{k=-\infty}^{+\infty} G(n, k)g(k, \hat{c}(k)). \end{aligned}$$

Set

$$I_1 = \sum_{k=-\infty}^{+\infty} G(n, k)f(k, \hat{c}(k)), \quad I_2 = \sum_{k=-\infty}^{+\infty} G(n, k)g(k, \hat{c}(k)).$$

$\forall \tau \in T(f, \frac{\varepsilon}{2})$ , one has

$$\begin{aligned} &\left| \sum_{k=-\infty}^{+\infty} G(n, k)f(k + \tau, \hat{c}(k + \tau)) - \sum_{k=-\infty}^{+\infty} G(n, k)f(k, \hat{c}(k)) \right| \\ &\leq \left| \sum_{k=-\infty}^{+\infty} G(n, k) \left( |f(k + \tau, \hat{c}(k + \tau)) - f(k, \hat{c}(k + \tau))| \right. \right. \\ &\quad \left. \left. + |f(k, \hat{c}(k + \tau)) - f(k, \hat{c}(k))| \right) \right| \\ &\leq K \sum_{k=-\infty}^{+\infty} \rho^{-|n-k-1|} \varepsilon \\ &= K(1 + \rho^{-1})(1 - \rho^{-1})^{-1} \varepsilon, \end{aligned}$$

where  $K > 0, \rho < 1$ . Consequently,  $I_1 \in \text{APS}(Z, R^q)$ .

In the following, we prove that  $I_2 \in C_0S(Z, R^q)$ . First of all, we prove  $\lim_{k \rightarrow +\infty} I_2 = 0$ .

$$\begin{aligned} I_2 &= \sum_{k \leq n-1} \Phi(n) \Pi \Phi^{-1}(k + 1)g(k, \hat{c}(k)) \\ &\quad - \sum_{k \geq n} \Phi(n)(I - \Pi)\Phi^{-1}(k + 1)g(k, \hat{c}(k)). \end{aligned}$$

Because  $\lim_{k \rightarrow +\infty} \rho^{-(n-k-1)} = 0$ , then for any  $\varepsilon > 0$  there exists  $N_1 > 0$ , one has  $|\rho^{-(n-k-1)}| < \varepsilon$  as  $k > N_1$ , and because  $\lim_{k \rightarrow +\infty} g(k, \hat{c}(k)) = 0$ , namely, for the above  $\varepsilon > 0$ , there is  $N_2 > 0$ , we get  $|g(k, \hat{c}(k))| < \varepsilon$  as  $k > N_2$ . By taking  $N = \max\{N_1, N_2\}$  as  $k > N$ , we have

$$\begin{aligned} |I_2| &\leq \sum_{k \leq n-1} |\Phi(n) \Pi \Phi^{-1}(k + 1)g(k, \hat{c}(k))| \\ &\quad + \sum_{k \geq n} |\Phi(n)(I - \Pi)\Phi^{-1}(k + 1)g(k, \hat{c}(k))|. \end{aligned}$$

Because the two part estimations of  $I_2$  in Theorem 3.5 (that is,  $\lim_{|k| \rightarrow \infty} I_2 = 0$ ) are similar to the two part estimations of  $I_2$  in Theorem 3.2 (that is,  $\lim_{|n| \rightarrow \infty} I_2 = 0$ ), we just need to replace  $h_2(k)$  with  $g(k, \hat{c}(k))$ .

In conclusion,  $\lim_{k \rightarrow +\infty} I_2 = 0$  is obtained; similarly,  $\lim_{k \rightarrow -\infty} I_2 = 0$ . Therefore,  $\lim_{|k| \rightarrow \infty} I_2 = 0$ , that is,  $I_2 \in C_0S(Z, R^q)$ .

From all the above, we have  $T : AAPS(Z, R^q) \rightarrow AAPS(Z, R^q)$ .

Moreover,

$$\begin{aligned} & |(Tc_1)(n) - (Tc_2)(n)| \\ &= \left| \sum_{k=-\infty}^{+\infty} G(n, k)h(k, \hat{c}_1(k)) - \sum_{k=-\infty}^{+\infty} G(n, k)h(k, \hat{c}_2(k)) \right| \\ &= \sum_{k=-\infty}^{+\infty} |G(n, k)| |h(k, \hat{c}_1(k)) - h(k, \hat{c}_2(k))| \\ &= \sum_{k=-\infty}^{+\infty} |G(n, k)| \left| \int_{t_k}^{t_{k+1}} X(t_{k+1}, s)F(s, \hat{c}_1(k)) ds \right. \\ &\quad \left. - \int_{t_k}^{t_{k+1}} X(t_{k+1}, s)F(s, \hat{c}_2(k)) ds \right| \\ &= \left| \sum_{n \geq k} G(n, k) + \sum_{n < k} G(n, k) \right| \left| \int_{t_k}^{t_{k+1}} X(t_{k+1}, s)[F(s, \hat{c}_1(k)) - F(s, \hat{c}_2(k))] ds \right| \\ &\leq KL \left| \sum_{n \geq k} \rho^{-(n-k-1)} + \sum_{n < k} \rho^{-(k+1-n)} \right| |c_1 - c_2|_\infty \\ &\leq 2 \frac{KL}{1 - \rho} |c_1 - c_2|_\infty. \end{aligned}$$

If (3.13) holds, then  $T : AAPS(Z, R^q) \rightarrow AAPS(Z, R^q)$  is a contracting mapping. By the Banach contraction fixed point theorem, there is  $c \in AAPS(Z, R^q)$ , which is a unique fixed point for  $T$ . Therefore, equation (3.15) has an asymptotically almost periodic solution  $c$ . Similar to Theorem 3.3, we can construct a solution of (1.2):

$$y(t) = \left[ X(t, t_n) + \int_{t_n}^t X(t, s)B(s) ds \right] y(t_n) + \int_{t_n}^t X(t, s)F(s, \hat{c}(n)) ds.$$

Moreover, we can prove that  $y(t)$  is a unique asymptotically almost periodic solution of equation (1.2). □

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### Authors' contributions

All authors have equal contribution in this article. All authors read and approved the final manuscript.

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