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Blow-up phenomena for a viscoelastic wave equation with strong damping and logarithmic nonlinearity

Tae Gab Ha[1] and Sun-Hye Park[2*]

*Correspondence:
sh-park@pusan.ac.kr
[2]Office for Education Accreditation, Pusan National University, Busan, South Korea
Full list of author information is available at the end of the article

Abstract

In this paper we consider the initial boundary value problem for a viscoelastic wave equation with strong damping and logarithmic nonlinearity of the form

$$u_{tt}(x,t) - \Delta u(x,t) + \int_0^t g(t-s)\Delta u(x,s)\,ds - \Delta u_t(x,t) = \left|u(x,t)\right|^{p-2}u(x,t)\ln\left|u(x,t)\right|$$

in a bounded domain $\Omega \subset \mathbb{R}^n$, where g is a nonincreasing positive function. Firstly, we prove the existence and uniqueness of local weak solutions by using Faedo–Galerkin's method and contraction mapping principle. Then we establish a finite time blow-up result for the solution with positive initial energy as well as nonpositive initial energy.

MSC: 35L05; 35L70; 35L71; 35B44; 35B40

Keywords: Viscoelastic wave equation; Logarithmic nonlinearity; Local existence; Finite time blow-up

1 Introduction

In this paper, we are concerned with the following viscoelastic wave equation with strong damping and logarithmic nonlinearity source:

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)\,ds - \Delta u_t = |u|^{p-2}u\ln|u| \quad \text{in } \Omega \times (0,T), \tag{1.1}$$

$$u = 0 \quad \text{on } \partial\Omega \times (0,T), \tag{1.2}$$

$$u(0) = u_0, \qquad u_t(0) = u_1 \quad \text{on } \Omega, \tag{1.3}$$

where $\Omega \subset \mathbb{R}^n$, $n \geq 1$, is a bounded domain with smooth boundary $\partial\Omega$. This type of equation is related to viscoelastic mechanics, quantum mechanics theory, nuclear physics, optics, geophysics and so on. For instance, the logarithmic nonlinearity arises in the inflation cosmology and super-symmetric fields in the quantum field theory. In the case $n = 1, 2$, Eq. (1.1) describes the transversal vibrations of a homogeneous viscous string and the longitudinal vibrations of a homogeneous bar, respectively. For the physical point of view, we refer to [1–3] and the references therein.

During the past decades, the strongly damped wave equations with source effect

$$u_{tt} - \Delta u - \omega \Delta u_t + \mu u_t = f(u) \tag{1.4}$$

have been studied extensively on existence, nonexistence, stability, and blow-up of solutions. In the case of power nonlinearity $f(u) = |u|^{p-2}u$, Sattinger [4] firstly considered the existence of local as well as global solutions for equation (1.4) with $\omega = \mu = 0$ by introducing the concepts of stable and unstable sets. Since then, the potential well method has become an important theory to the study of the existence and nonexistence of solutions [5–15]. Ikehata [8] gave properties of decay estimates and blow-up of solutions to (1.4) with linear damping ($\omega = 0$ and $\mu > 0$). Gazzola and Squassina [6] proved the global existence and finite time blow-up of solutions to problem (1.4) with weak and strong damping ($\omega > 0$). Liu [11] considered a viscoelastic version of (1.4). He investigated decay estimates for global solutions when the initial data enter the stable set and showed finite blow-up results when the initial data enter the unstable set.

In the case of logarithmic nonlinearity $f(u) = u \ln |u|^k$, Ma and Fang [16] proved the existence of global solutions and infinite time blow-up to problem (1.4) with $\omega = 1$, $\mu = 0$, and $k = 2$. Lian and Xu [17] investigated global existence, energy decay and infinite time blow-up when $\omega \geq 0$ and $\mu > -\omega\lambda_1$, where λ_1 is the first eigenvalue of the operator $-\Delta$ under homogeneous Dirichlet boundary conditions. The results of [16, 17] were obtained by use of the potential well method and the logarithmic Sobolev inequality.

By the way, there is not much literature for strongly damped wave equations with the logarithmic nonlinear source $|u|^{p-2}u \ln |u|$. Recently, Di et al. [18] considered problem (1.1)–(1.3) when the kernel function $g = 0$. The presence of the Laplacian operator $-\Delta u$ and the logarithmic nonlinearity $|u|^{p-2}u \ln |u|$ causes some difficulty so that one cannot apply the logarithmic Sobolev inequality [19]. Thus, they discussed the global existence, uniqueness, energy decay estimates and finite time blow-up of solutions by modifying the potential well method. We also refer to [20, 21] and the references therein for problems with logarithmic nonlinearity.

Motivated by these results, we study the existence and finite time blow-up of weak solutions for problem (1.1)–(1.3) in the present work by applying the ideas in [11, 18]. To the best our knowledge, this is the first work in the literature that takes into account a viscoelastic wave equation with strong damping and logarithmic nonlinearity in a bounded domain $\Omega \subset \mathbb{R}^n$.

The outline of this paper is as follows. In Sect. 2, we give materials needed for our work. In Sect. 3, we prove the local existence of solutions for problem (1.1)–(1.3) using Faedo–Galerkin's method and contraction mapping principle. In Sect. 4, we establish a finite time blow-up result.

2 Preliminaries

In this section we give notations, hypotheses, and some lemmas needed in our main results.

For a Banach space X, $\|\cdot\|_X$ denotes the norm of X. As usual, $(\cdot,\cdot)$ and $\langle\cdot,\cdot\rangle$ denote the inner product in the space $L^2(\Omega)$ and the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$, respectively. $\|\cdot\|_q$ denotes the norm of the space $L^q(\Omega)$. For brevity, we denote $\|\cdot\|_2$ by $\|\cdot\|$. Let c_q be the best constants in the Poincaré type inequality

$$\|v\|_q \le c_q\|\nabla v\|^2 \quad \text{for } v \in H_0^1(\Omega),$$

where

$$2 \le q < \infty, \quad \text{if } n = 1,2; \qquad 2 \le q \le \frac{2n}{n-2}, \quad \text{if } n \ge 3.$$

We need the following lemma.

Lemma 2.1 *For each $q > 0$,*

$$\left|s^q \ln s\right| \le \frac{1}{eq} \quad \textit{for } 0 < s < 1 \quad \textit{and} \quad 0 \le s^{-q}\ln s \le \frac{1}{eq} \quad \textit{for } s \ge 1.$$

Proof We can easily show this from simple calculation. So, we omit it here. □

Lemma 2.2 ([9]) *Let $L(t)$ be a positive, twice differentiable function satisfying the inequality*

$$L(t)L''(t) - (1+\delta)\big(L'(t)\big)^2 \ge 0 \quad \textit{for } t > 0,$$

with some $\delta > 0$. If $L(0) > 0$ and $L'(0) > 0$, then there exists a time $T_* \le \frac{L(0)}{\delta L'(0)}$ such that

$$\lim_{t\to T_*^-} L(t) = +\infty.$$

With regard to problem (1.1)–(1.3), we impose the following assumptions:

(H_1) *Hypotheses on p.*

The exponent p satisfies

$$2 < p < \infty, \quad \text{if } n = 1,2; \qquad 2 < p < \frac{2(n-1)}{n-2}, \quad \text{if } n \ge 3. \tag{2.1}$$

(H_2) *Hypotheses on g.*

The kernel function $g : [0,\infty) \to [0,\infty)$ is a nonincreasing and differentiable function satisfying

$$1 - \int_0^\infty g(s)\,ds := l > 0. \tag{2.2}$$

Definition 2.1 *Let $T > 0$. We say that a function u is a weak solution of problem* (1.1)–(1.3) *if*

$$u \in C\big([0,T];H_0^1(\Omega)\big) \cap C^1\big([0,T];L^2(\Omega)\big) \cap C^2\big([0,T];H^{-1}(\Omega)\big),$$

leads to

$$\begin{aligned}&\langle u_{tt}(t),w\rangle+\big(\nabla u(t),\nabla w\big)-\int_0^t g(t-s)\big(\nabla u(s),\nabla w\big)\,ds+\big(\nabla u_t(t),\nabla w\big)\\&\quad=\int_\Omega \big|u(x,t)\big|^{p-2}u(x,t)\ln\big|u(x,t)\big|w\,dx\end{aligned}\tag{2.3}$$

for any $w\in H_0^1(\Omega)$ *and* $t\in(0,T)$, *and*

$$u(0)=u_0\quad in\ H_0^1(\Omega),\qquad u_t(0)=u_1\quad in\ L^2(\Omega).$$

3 Local existence of solutions

In this section we prove the local existence of solutions making use of the Faedo–Galerkin method and the contraction mapping principle. For a fixed $T>0$, we consider the space

$$\mathcal{H}=C\big([0,T];H_0^1(\Omega)\big)\cap C^1\big([0,T];L^2(\Omega)\big)$$

with the norm

$$\|v\|_{\mathcal{H}}^2=\max_{0\le t\le T}\big(\big\|v_t(t)\big\|^2+l\big\|\nabla v(t)\big\|^2\big).$$

To show the existence and uniqueness of local solution to problem (1.1)–(1.3), we firstly establish the following result.

Lemma 3.1 *Assume that* (H_1) *and* (H_2) *hold. Then, for every* $u_0\in H_0^1(\Omega)$, $u_1\in L^2(\Omega)$, $v\in\mathcal{H}$, *there exists a unique*

$$u\in C\big([0,T];H_0^1(\Omega)\big)\cap C^1\big([0,T];L^2(\Omega)\big)\cap C^2\big([0,T];H^{-1}(\Omega)\big)$$

such that $u_t\in L^2([0,T];H_0^1(\Omega))$ *and*

$$u_{tt}-\Delta u+\int_0^t g(t-s)\Delta u(s)\,ds-\Delta u_t=|v|^{p-2}v\ln|v|\quad in\ \Omega\times(0,T),\tag{3.1}$$

$$u=0\quad on\ \partial\Omega\times(0,T),\tag{3.2}$$

$$u(0)=u_0,\qquad u_t(0)=u_1\quad on\ \Omega.\tag{3.3}$$

Proof Existence. Let $\{w_j\}_{j\in\mathbb{N}}$ be an orthogonal basis of $H_0^1(\Omega)$ which is orthonormal in $L^2(\Omega)$ and $W_m=\operatorname{span}\{w_1,w_2,\dots,w_m\}$, then there exist subsequences $u_0^m\in W_m$ and $u_1^m\in W_m$ such that $u_0^m\to u_0$ in $H_0^1(\Omega)$ and $u_1^m\to u_1$ in $L^2(\Omega)$, respectively. We will seek an approximate solution

$$u^m(x,t)=\sum_{j=1}^m h_j^m(t)w_j(x)$$

satisfying

$$\begin{aligned}&\big(u_{tt}^m(t),w\big)+\big(\nabla u^m(t),\nabla w\big)-\int_0^t g(t-s)\big(\nabla u^m(s),\nabla w\big)\,ds+\big(\nabla u_t^m(t),\nabla w\big)\\&\quad=\int_\Omega \big|v(x,t)\big|^{p-2}v(x,t)\ln\big|v(x,t)\big|w(x)\,dx\quad \text{for } w\in W_m\end{aligned}\tag{3.4}$$

and the initial conditions

$$u^m(0)=u_0^m,\qquad u_t^m(0)=u_1^m.\tag{3.5}$$

Since (3.4)–(3.5) is a normal system of ordinary differential equations, there exists a solution u^m on the interval $[0,t_m)\subset[0,T]$. We obtain an a priori estimate for the solution u^m so that it can be extended to the whole interval $[0,T]$ according to the extension theorem.

Step 1. A priori estimate. Replacing w by $u_t^m(t)$ in (3.4) and using the relation

$$\begin{aligned}\int_0^t g(t-s)\big(\nabla u^m(s),\nabla u_t^m(t)\big)\,ds={}&-\frac{g(t)}{2}\big\|\nabla u^m(t)\big\|^2+\frac12\big(g'\circ\nabla u^m\big)(t)\\&-\frac12\frac{d}{dt}\bigg(\big(g\circ\nabla u^m\big)(t)-\int_0^t g(s)\,ds\big\|\nabla u^m(t)\big\|^2\bigg),\end{aligned}$$

where

$$(g\circ\phi)(t)=\int_0^t g(t-s)\big\|\phi(t)-\phi(s)\big\|^2\,ds,$$

we have

$$\begin{aligned}&\frac{d}{dt}\bigg\{\big\|u_t^m(t)\big\|^2+\bigg(1-\int_0^t g(s)\,ds\bigg)\big\|\nabla u^m(t)\big\|^2+\big(g\circ\nabla u^m\big)(t)\bigg\}+2\big\|\nabla u_t^m(t)\big\|^2\\&\quad=\big(g'\circ\nabla u^m\big)(t)-g(t)\big\|\nabla u^m(t)\big\|^2+2\int_\Omega\big|v(x,t)\big|^{p-2}v(x,t)\ln\big|v(x,t)\big|u_t^m(x,t)\,dx.\end{aligned}$$

Integrating this over $(0,t)$ and making use of (H_2),

$$\begin{aligned}&\big\|u_t^m(t)\big\|^2+l\big\|\nabla u^m(t)\big\|^2+\big(g\circ\nabla u^m\big)(t)+2\int_0^t\big\|\nabla u_t^m(s)\big\|^2\,ds\\&\quad\le\big\|u_1^m\big\|^2+\big\|\nabla u_0^m\big\|^2+2\int_0^t\big\|\big|v(s)\big|^{p-2}v(s)\ln\big|v(s)\big|\big\|_{\frac{p}{p-1}}\big\|u_t^m(s)\big\|_p\,ds.\end{aligned}\tag{3.6}$$

In order to estimate the last term in the right hand side of (3.6), we let

$$\Omega_1=\big\{x\in\Omega:\big|u^m(x,t)\big|<1\big\}\quad\text{and}\quad\Omega_2=\big\{x\in\Omega:\big|u^m(x,t)\big|\ge1\big\}.$$

Since $2<p<\frac{2n}{n-2}$, we can take $\mu>0$ such that $2<p+\frac{\mu p}{p-1}<\frac{2n}{n-2}$. Applying Lemma 2.1, we infer that

$$\big\||v|^{p-2}v\ln|v|\big\|_{\frac{p}{p-1}}^{\frac{p}{p-1}}=\int_{\Omega_1}\big(\big||v|^{p-1}\ln|v|\big|\big)^{\frac{p}{p-1}}\,dx+\int_{\Omega_2}\big(\big||v|^{-\mu+(p-1+\mu)}\ln|v|\big|\big)^{\frac{p}{p-1}}\,dx$$

$$
\begin{aligned}
&\le \left(\frac{1}{e(p-1)}\right)^{\frac{p}{p-1}}|\Omega_1| + \int_{\Omega_2}\left(|v|^{-\mu}\ln|v|\right)^{\frac{p}{p-1}}|v|^{\frac{p(p-1+\mu)}{p-1}}\,dx \\
&\le \left(\frac{1}{e(p-1)}\right)^{\frac{p}{p-1}}|\Omega_1| + \left(\frac{1}{e\mu}\right)^{\frac{p}{p-1}}\int_{\Omega_2}|v|^{\frac{p(p-1+\mu)}{p-1}}\,dx \\
&\le \left(\frac{1}{e(p-1)}\right)^{\frac{p}{p-1}}|\Omega_1| + c_{\frac{p(p-1+\mu)}{p-1}}^{\frac{p(p-1+\mu)}{p-1}}\left(\frac{1}{e\mu}\right)^{\frac{p}{p-1}}\|\nabla v\|^{\frac{p(p-1+\mu)}{p-1}} \\
&\le C,
\end{aligned}
\tag{3.7}
$$

we used the fact that $v \in \mathcal{H}$ in the last inequality. Here and in the sequel, C denotes a generic positive constant independent of m and t and different from line to line or even in the same line.

From (3.7), we see that

$$
\begin{aligned}
2\int_0^t \left\||v|^{p-2}v\ln|v|\right\|_{\frac{p}{p-1}}\left\|u_t^m\right\|_p\,ds &\le C\int_0^t\left\|u_t^m(s)\right\|_p\,ds \\
&\le C\int_0^t\left\|\nabla u_t^m(s)\right\|\,ds \le CT + \int_0^t\left\|\nabla u_t^m(s)\right\|^2\,ds.
\end{aligned}
$$

Adapting this to (3.6), we get

$$
\begin{aligned}
\left\|u_t^m(t)\right\|^2 + l\left\|\nabla u^m(t)\right\|^2 + \left(g\circ\nabla u^m\right)(t) + \int_0^t\left\|\nabla u_t^m(s)\right\|^2\,ds &\le \left\|u_1^m\right\|^2 + \left\|\nabla u_0^m\right\|^2 + CT \\
&\le C.
\end{aligned}
$$

Step 2. Passage to the limit. So, there exists a subsequence of $\{u^m\}$, which we still denote by $\{u^m\}$, such that

$$
u^m \to u \quad \text{weakly star in } L^\infty\left(0,T;H_0^1(\Omega)\right), \tag{3.8}
$$

$$
u_t^m \to u_t \quad \text{weakly star in } L^\infty\left(0,T;L^2(\Omega)\right), \tag{3.9}
$$

$$
u_t^m \to u_t \quad \text{weakly in } L^2\left(0,T;H_0^1(\Omega)\right). \tag{3.10}
$$

Now, we integrate (3.4) over $(0,t)$ to get

$$
\begin{aligned}
&\left(u_t^m(t),w\right) - \left(u_1^m,w\right) + \int_0^t\left(\nabla u^m(s),\nabla w\right)ds - \int_0^t\int_0^\tau g(\tau-s)\left(\nabla u^m(s),\nabla w\right)ds\,d\tau \\
&\quad + \left(\nabla u^m(t),\nabla w\right) - \left(\nabla u_0^m,\nabla w\right) = \int_0^t\int_\Omega\left|v(x,s)\right|^{p-2}v(x,s)\ln\left|v(x,s)\right|w(x)\,dx\,ds.
\end{aligned}
$$

Taking the limit $m\to\infty$ in this, we have from (3.8) and (3.9) that

$$
\begin{aligned}
&\left(u_t(t),w\right) - (u_1,w) + \int_0^t\left(\nabla u(s),\nabla w\right)ds - \int_0^t\int_0^\tau g(\tau-s)\left(\nabla u(s),\nabla w\right)ds\,d\tau \\
&\quad + \left(\nabla u(t),\nabla w\right) - (\nabla u_0,\nabla w) = \int_0^t\int_\Omega\left|v(x,s)\right|^{p-2}v(x,s)\ln\left|v(x,s)\right|w(x)\,dx\,ds.
\end{aligned}
\tag{3.11}
$$

This remains valid for all $w \in H_0^1(\Omega)$. Differentiating (3.11) with respect to t, we have

$$\begin{aligned}&\langle u_{tt}(t), w\rangle + (\nabla u(t), \nabla w) - \int_0^t g(t-s)(\nabla u(s), \nabla w)\,ds + (\nabla u_t(t), \nabla w)\\&\quad = \int_\Omega |v(x,t)|^{p-2} v(x,t)\ln|v(x,t)|w(x)\,dx \quad \text{for } w \in H_0^1(\Omega).\end{aligned} \tag{3.12}$$

Now, we are left with verifying that the limit function u satisfies the initial conditions, that is,

$$u(0) = u_0 \quad \text{in } H_0^1(\Omega), \qquad u_t(0) = u_1 \quad \text{in } L^2(\Omega).$$

From (3.8), (3.9), and Lion's lemma [22], we get

$$u^m \to u \quad \text{in } C([0,T]; L^2(\Omega)). \tag{3.13}$$

Thus, $u^m(0) \to u(0)$ in $L^2(\Omega)$. Since $u^m(0) = u_0^m \to u_0$ in $H_0^1(\Omega)$, we observe that

$$u(0) = u_0 \quad \text{in } H_0^1(\Omega). \tag{3.14}$$

Next, multiplying (3.4) by $\phi \in C_0^\infty(0,T)$ and integrating it over $(0,T)$, we find

$$\begin{aligned}&-\int_0^T (u_t^m(t), w\phi'(t))\,dt + \int_0^T (\nabla u^m(t), \nabla w\phi(t))\,dt\\&\quad - \int_0^T \int_0^t g(t-s)(\nabla u^m(\tau), \nabla w\phi(t))\,d\tau\,dt\\&\quad - \int_0^T (\nabla u^m(t), \nabla w\phi'(t))\,dt = \int_0^T (|v(t)|^{p-2}v(t)\ln|v(t)|, w\phi(t))\,dt \quad \text{for } w \in W_m.\end{aligned}$$

Letting $m \to \infty$, we get

$$\begin{aligned}&-\int_0^T (u_t(t), w\phi'(t))\,dt + \int_0^T (\nabla u(t), \nabla w\phi(t))\,dt\\&\quad - \int_0^T \int_0^t g(t-s)(\nabla u(\tau), \nabla w\phi(t))\,d\tau\,dt\\&\quad - \int_0^T (\nabla u(t), \nabla w\phi'(t))\,dt = \int_0^T (|v(t)|^{p-2}v(t)\ln|v(t)|, w\phi(t))\,dt \quad \text{for } w \in H_0^1(\Omega).\end{aligned}$$

This yields $u_{tt} \in L^2(0,T; H^{-1}(\Omega))$. This and the fact that $u_t \in L^2(0,T; H_0^1(\Omega))$ imply that

$$u_t \in C([0,T]; H^{-1}(\Omega)).$$

Thus, $u_t^m(0) \to u_t(0)$ in $H^{-1}(\Omega)$. Owing to $u_t^m(0) = u_1^m \to u_1$ in $L^2(\Omega)$, we conclude

$$u_t(0) = u_1 \quad \text{in } L^2(\Omega).$$

Uniqueness. Let u and $\tilde{u}$ be the solutions of the linearized problem (3.1)–(3.3) and $w = u - \tilde{u}$. Then w satisfies

$$w_{tt} - \Delta w + \int_0^t g(t-s)\Delta w(s)\,ds - \Delta w_t = 0 \quad \text{in } \Omega \times (0,T),$$
$$w = 0 \quad \text{on } \partial\Omega \times (0,T),$$
$$w(0) = 0, \qquad w_t(0) = 0 \quad \text{on } \Omega.$$

By the same arguments of (3.6), we observe

$$\left\|w_t(t)\right\|^2 + l\left\|\nabla w(t)\right\|^2 + (g \circ \nabla w)(t) + 2\int_0^t \left\|\nabla w_t(s)\right\|^2\,ds \le 0,$$

and hence $w \equiv 0$. This completes the proof. □

Now, we are ready to prove the local existence of problem (1.1)–(1.3).

Theorem 3.1 *Assume that* (H_1) *and* (H_2) *hold. Then, for the initial data* $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$, *there exists a unique solution* u *of problem* (1.1)–(1.3).

Proof *Existence.* For $M > 0$ large enough and $T > 0$, we let

$$\mathcal{M}_T = \left\{u \in \mathcal{H} : \|u\|_{\mathcal{H}} \le M\right\}.$$

For a given $v \in \mathcal{H}$, there exists a unique solution u of problem (3.1)–(3.3). So, we can define a map $S : \mathcal{M}_T \to \mathcal{H}$ by $S(v) = u$. We will show that the map S is a contraction mapping on $\mathcal{M}_T$. By a similar computation to that of (3.6), we find

$$\begin{aligned}
&\left\|u_t(t)\right\|^2 + l\left\|\nabla u(t)\right\|^2 + (g \circ \nabla u)(t) + 2\int_0^t \left\|\nabla u_t(s)\right\|^2\,ds \\
&\quad\le \|u_1\|^2 + \|\nabla u_0\|^2 + 2\int_0^t \left\|\left|v(t)\right|^{p-2}v(t)\ln\left|v(t)\right|\right\|_{\frac{p}{p-1}} \left\|u_t(t)\right\|_p\,ds \\
&\quad\le \|u_1\|^2 + \|\nabla u_0\|^2 + 2\int_0^t \left\|\nabla u_t(s)\right\|\,ds \\
&\qquad + 2\int_0^t \left\{\left(\frac{1}{e(p-1)}\right)^{\frac{p}{p-1}} |\Omega_1| + c_{\frac{p(p-1+\mu)}{p-1}}^{\frac{p(p-1+\mu)}{p-1}} \left(\frac{1}{e\mu}\right)^{\frac{p}{p-1}} \left\|\nabla v(s)\right\|^{\frac{p(p-1+\mu)}{p-1}}\right\}^{\frac{2(p-1)}{p}}\,ds \\
&\quad\le \|u_1\|^2 + \|\nabla u_0\|^2 + 2\int_0^t \left\|\nabla u_t(s)\right\|\,ds + CT\left(1 + M^{2(p-1+\mu)}\right),
\end{aligned}$$

we used $v \in \mathcal{M}_T$ in the last inequality. Thus, we see

$$\left\|u_t(t)\right\|^2 + l\left\|\nabla u(t)\right\|^2 + (g \circ \nabla u)(t) \le \|u_1\|^2 + \|\nabla u_0\|^2 + CT\left(1 + M^{2(p-1+\mu)}\right). \tag{3.15}$$

We take $M > 0$ large enough so that

$$\|u_1\|^2 + \|\nabla u_0\|^2 \le \frac{M^2}{2},$$

then we choose $T > 0$ sufficiently small so that

$$CT\left(1 + M^{2(p-1+\mu)}\right) \leq \frac{M^2}{2}.$$

From (3.15), we have $\|u\|_{\mathcal{H}} \leq M$, that is,

$$S(\mathcal{M}_T) \subset \mathcal{M}_T.$$

It remains to show that S is a contraction mapping. Let $v_1, v_2 \in \mathcal{M}_T$, $u = S(v_1)$, $w = S(v_2)$ and $z = u - w$. Then z satisfies

$$\begin{aligned} & z_{tt} - \Delta z + \int_0^t g(t-s)\Delta z(s)\,ds - \Delta z_t \\ &\quad = |v_1|^{p-2}v_1 \ln|v_1| - |v_2|^{p-2}v_2 \ln|v_2| \quad \text{in } \Omega \times (0,T), && (3.16) \\ & z = 0 \quad \text{on } \partial\Omega \times (0,T), && (3.17) \\ & z(0) = 0, \qquad z_t(0) = 0 \quad \text{on } \Omega. && (3.18) \end{aligned}$$

Multiplying z_t in (3.16) and integrating it over $(0,t)$,

$$\begin{aligned} & \|z_t\|^2 + l\|\nabla z\|^2 + (g \circ \nabla z) + 2\int_0^t \left\|\nabla z_t(s)\right\|^2 ds \\ &\quad \leq 2\int_0^t \int_\Omega \left(|v_1|^{p-2}v_1 \ln|v_1| - |v_2|^{p-2}v_2 \ln|v_2|\right) z_t\,dx\,ds \\ &\quad = 2\int_0^t \int_\Omega \left((p-1)|\zeta|^{p-2}\ln|\zeta| + |\zeta|^{p-2}\right)(v_1 - v_2) z_t\,dx\,ds \\ &\quad = 2\int_0^t \int_\Omega |\zeta|^{p-2}(v_1 - v_2) z_t\,dx\,ds + 2(p-1)\int_0^t \int_\Omega |\zeta|^{p-2}\ln|\zeta|(v_1 - v_2) z_t\,dx\,ds \\ &\quad := \Xi_1 + \Xi_2, && (3.19) \end{aligned}$$

where $\zeta = \theta v_1 + (1-\theta)v_2$, here $0 < \theta < 1$. Young's inequality yields

$$\begin{aligned} \Xi_1 &\leq 2\int_0^t \|\zeta\|_{n(p-2)}^{p-2} \|v_1 - v_2\|_{\frac{2n}{n-2}} \|z_t\|\,ds \\ &\leq 2c_{\frac{2n}{n-2}} c_{n(p-2)}^{p-2} \int_0^t \|\nabla\zeta\|^{p-2} \|\nabla v_1 - \nabla v_2\| \|z_t\|\,ds \\ &\leq C\int_0^t \|\nabla\zeta\|^{2(p-2)} \|\nabla v_1 - \nabla v_2\|^2\,ds + \int_0^t \|\nabla z_t\|^2\,ds \\ &\leq CM^{2(p-2)}T\|v_1 - v_2\|_{\mathcal{H}}^2 + \int_0^t \|\nabla z_t\|^2\,ds && (3.20) \end{aligned}$$

and

$$\begin{aligned} \Xi_2 &\leq 2(p-1)\int_0^t \left\||\zeta|^{p-2}\ln|\zeta|\right\|_n \|v_1 - v_2\|_{\frac{2n}{n-2}} \|z_t\|\,ds \\ &\leq 2(p-1)c_{\frac{2n}{n-2}} \int_0^t \left\||\zeta|^{p-2}\ln|\zeta|\right\|_n \|\nabla v_1 - \nabla v_2\| \|z_t\|\,ds. && (3.21) \end{aligned}$$

Since $p-2<\frac{2}{n-2}$, there exists $\eta>0$ such that $n(p-2+\eta)<\frac{2n}{n-2}$. By similar arguments to (3.7), we deduce

$$
\begin{aligned}
\big\||\zeta|^{p-2}\ln|\zeta|\big\|_n^n &\le \left(\frac{1}{e(p-1)}\right)^n|\Omega_1| + \int_\Omega \big(|\zeta|^{-\eta}\ln|\zeta|\big)^n|\zeta|^{n(p-2+\eta)}\,dx \\
&\le \left(\frac{1}{e(p-1)}\right)^n|\Omega_1| + c_{n(p-2+\eta)}^{n(p-2+\eta)}\left(\frac{1}{e\eta}\right)^n\|\nabla\zeta\|^{n(p-2+\eta)} \\
&\le C\big(1+M^{n(p-2+\eta)}\big).
\end{aligned}
$$

Applying this to (3.21), we get

$$
\begin{aligned}
\Xi_2 &\le C\big(1+M^{(p-2+\eta)}\big)\int_0^t \|v_1-v_2\|_{\mathcal{H}}\|\nabla z_t\|\,ds \\
&\le CT\big(1+M^{(p-2+\eta)}\big)\|v_1-v_2\|_{\mathcal{H}}^2 + \int_0^t \|\nabla z_t\|^2\,ds.
\end{aligned}
\tag{3.22}
$$

Collecting (3.19), (3.20), (3.22), we arrive at

$$
\big\|z_t(t)\big\|^2 + l\big\|\nabla z(t)\big\|^2 + (g\circ\nabla z)(t) \le CT\big(1+M^{2(p-2)}+M^{(p-2+\eta)}\big)\|v_1-v_2\|_{\mathcal{H}}^2. \tag{3.23}
$$

Taking $T>0$ sufficiently small so that $CT(1+M^{2(p-2)}+M^{(p-2+\eta)})<1$, we conclude

$$
\big\|S(v_1)-S(v_2)\big\|_{\mathcal{H}} < \|v_1-v_2\|_{\mathcal{H}}.
$$

Thus, the contraction mapping principle ensures the existence of weak solutions.

Uniqueness. Let w and z be the solutions of problem (1.1)–(1.3) and $U=w-z$. Then U satisfies

$$
\begin{aligned}
&U_{tt}-\Delta U+\int_0^t g(t-s)\Delta U(s)\,ds-\Delta U_t = |w|^{p-2}w\ln|w| - |z|^{p-2}z\ln|z| \quad \text{in } \Omega\times(0,T), \\
&U=0 \quad \text{on } \partial\Omega\times(0,T), \\
&U(0)=0, \qquad U_t(0)=0 \quad \text{on } \Omega.
\end{aligned}
$$

By the same arguments as of (3.19), (3.20) and (3.21), we observe

$$
\big\|U_t(t)\big\|^2 + \big\|\nabla U(t)\big\|^2 \le C\int_0^t \big(\big\|U_t(s)\big\|^2+\big\|\nabla U(s)\big\|^2\big)\,ds.
$$

Gronwall's inequality gives $U\equiv 0$. This completes the proof. □

4 Finite time blow-up of solutions

In this section we establish the blow-up of the weak solution for problem (1.1)–(1.3). For this purpose, we set the following functionals:

$$
J(v) = \frac{1}{2}\left(1-\int_0^\infty g(s)\,ds\right)\|\nabla v\|^2 - \frac{1}{p}\int_\Omega \big|v(x)\big|^p\ln\big|v(x)\big|\,dx + \frac{1}{p^2}\|v\|_p^p, \tag{4.1}
$$

$$I(v) = \left(1 - \int_0^\infty g(s)\,ds\right)\|\nabla v\|^2 - \int_\Omega \left|v(x)\right|^p \ln\left|v(x)\right|\,dx, \tag{4.2}$$

then

$$J(v) = \left(\frac{1}{2} - \frac{1}{p}\right)\left(1 - \int_0^\infty g(s)\,ds\right)\|\nabla v\|^2 + \frac{1}{p}I(v) + \frac{1}{p^2}\|v\|_p^p. \tag{4.3}$$

Define the potential depth as

$$d = \inf_{v\in H_0^1(\Omega)\setminus\{0\}} \sup_{\lambda>0} J(\lambda v), \tag{4.4}$$

then, see e.g. [23–25],

$$0 < d = \inf_{v\in\mathcal{N}} J(v), \tag{4.5}$$

where $\mathcal{N}$ is the well-known Nehari manifold given by

$$\mathcal{N} = \left\{v \in H_0^1(\Omega) \setminus \{0\} \mid I(v) = 0\right\}.$$

Lemma 4.1 *For any $v \in H_0^1(\Omega) \setminus \{0\}$, there exists a unique $\lambda_* > 0$ such that*

$$I(\lambda v) = \lambda \frac{\partial J(\lambda v)}{\partial \lambda} \begin{cases} > 0, & 0 < \lambda < \lambda_*, \\ = 0, & \lambda = \lambda_*, \\ < 0, & \lambda > \lambda_*. \end{cases} \tag{4.6}$$

Proof For $\lambda > 0$, we have

$$\begin{aligned} \frac{\partial}{\partial\lambda}J(\lambda v) &= \lambda\left\{\left(1 - \int_0^\infty g(s)\,ds\right)\|\nabla v\|^2 - \lambda^{p-2}\int_\Omega \left|v(x)\right|^p \ln\left|v(x)\right|\,dx - \lambda^{p-2}\ln\lambda\|v\|_p^p\right\} \\ &:= \lambda K(\lambda v). \end{aligned} \tag{4.7}$$

By simple calculation, we also get

$$\begin{aligned} \frac{\partial}{\partial\lambda}K(\lambda v) &= -\lambda^{p-3}\left\{(p-2)\int_\Omega \left|v(x)\right|^p \ln\left|v(x)\right|\,dx + (p-2)\ln\lambda\|v\|_p^p + \|v\|_p^p\right\} \\ &\begin{cases} > 0, & 0 < \lambda < \lambda_1, \\ = 0, & \lambda = \lambda_1, \\ < 0, & \lambda > \lambda_1, \end{cases} \end{aligned}$$

where

$$\lambda_1 = \exp\left(\frac{(p-2)\int_\Omega |v(x)|^p \ln|v(x)|\,dx + \|v\|_p^p}{(2-p)\|v\|_p^p}\right) < 1. \tag{4.8}$$

Since $\lim_{\lambda\to 0^+} K(\lambda v) = (1 - \int_0^\infty g(s)\,ds)\|\nabla v\|^2 \geq 0$ and $\lim_{\lambda\to +\infty} K(\lambda v) = -\infty$, there exists a unique $\lambda_* > \lambda_1$ such that $K(\lambda_* v) = 0$. From this and (4.7), we have

$$\frac{\partial J(\lambda v)}{\partial \lambda} \begin{cases} > 0, & 0 < \lambda < \lambda_*, \\ = 0, & \lambda = \lambda_*, \\ < 0, & \lambda > \lambda_*. \end{cases}$$

Noting that $I(\lambda v) = \lambda \frac{\partial J(\lambda v)}{\partial \lambda}$, which is verified by a direct computation, we complete the proof. □

Now, we define the energy for problem (1.1)–(1.3) by

$$\begin{aligned} E(t) = {} & \frac{1}{2}\|u_t(t)\|^2 + \frac{1}{2}\left(1 - \int_0^t g(s)\,ds\right)\|\nabla u(t)\|^2 + \frac{1}{2}(g \circ \nabla u)(t) \\ & - \frac{1}{p}\int_\Omega |u(x,t)|^p \ln|u(x,t)|\,dx + \frac{1}{p^2}\|u(t)\|_p^p, \end{aligned} \tag{4.9}$$

then

$$E(t) \geq J(u(t)) + \frac{1}{2}\|u_t(t)\|^2 + \frac{1}{2}(g \circ \nabla u)(t) \geq J(u(t)). \tag{4.10}$$

Replacing w in (2.3) by $u_t(t)$ and using (H_2), one sees

$$\frac{d}{dt}E(t) + \|\nabla u_t(t)\|^2 = \frac{1}{2}(g' \circ \nabla u(t))(t) - \frac{g(t)}{2}\|\nabla u(t)\|^2 \leq 0$$

and hence

$$E(t) + \int_0^t \|\nabla u_t(s)\|^2\,ds \leq E(0) \quad \text{for } 0 \leq t < T_{\max}, \tag{4.11}$$

where $T_{\max}$ is the maximal existence time of the solution u of problem (1.1)–(1.3).

Lemma 4.2 *Let (H_1) and (H_2) hold. If $I(u_0) < 0$ and $E(0) < d$, then the solution u of problem (1.1)–(1.3) satisfies*

$$I(u(t)) < 0 \quad \textit{and} \quad E(t) < d \quad \textit{for } t \in [0, T_{\max}). \tag{4.12}$$

Proof From (4.11), it is clear that $E(t) < d$. Since $I(u_0) < 0$ and u is continuous on $[0, T_{\max})$,

$$I(u(t)) < 0 \quad \text{for some interval } [0, t_1) \subset [0, T_{\max}). \tag{4.13}$$

Let t_0 be the maximal time satisfying (4.13). Suppose $t_0 < T_{\max}$, then $I(u(t_0)) = 0$, that is,

$$u(t_0) \in \mathcal{N}.$$

Thus, we have from (4.5)

$$J(u(t_0)) \geq \inf_{v\in\mathcal{N}} J(v) = d.$$

But this is a contradiction for

$$J\big(u(t_0)\big) \le E(t_0) \le E(0) < d. \qquad \square$$

Theorem 4.1 *Let the conditions* (H_1) *and* (H_2) *hold. Assume that* $I(u_0) < 0$, $E(0) = \alpha d$, *where* $\alpha < 1$, *and the kernel function* g *satisfies*

$$\int_0^\infty g(s)\,ds \le \frac{p-2}{p-2+\frac{1}{(1-\hat{\alpha})^2 p + 2\hat{\alpha}(1-\hat{\alpha})}}, \tag{4.14}$$

where $\hat{\alpha} = \max\{0, \alpha\}$. *Moreover, suppose that* $(u_0, u_1) > 0$ *when* $E(0) = 0$. *Then the solution* u *of problem* (1.1)–(1.3) *blows up in finite time*.

Proof By contradiction, suppose that the solution u is global. For any $T > 0$, we consider $L : [0, T] \to \mathbb{R}^+$ defined by

$$L(t) = \big\|u(t)\big\|^2 + \int_0^t \big\|\nabla u(s)\big\|^2\,ds + (T-t)\|\nabla u_0\|^2 + b(t+T_0)^2, \tag{4.15}$$

where $T_0 > 0$ and $b \ge 0$, which are specified later. Then

$$L(t) > 0 \quad \text{for } t \in [0, T], \tag{4.16}$$

$$\begin{aligned} L'(t) &= 2\big(u(t), u_t(t)\big) + \big\|\nabla u(t)\big\|^2 - \|\nabla u_0\|^2 + 2b(t+T_0) \\ &= 2\big(u(t), u_t(t)\big) + 2\int_0^t \big(\nabla u(s), \nabla u_t(s)\big)\,ds + 2b(t+T_0), \end{aligned} \tag{4.17}$$

and, from (1.1),

$$\begin{aligned} L''(t) = {}& 2\big\|u_t(t)\big\|^2 - 2\big\|\nabla u(t)\big\|^2 + 2\int_0^t g(t-s)\big(\nabla u(t), \nabla u(s)\big)\,ds \\ & + 2\int_\Omega \big|u(x,t)\big|^p \ln\big|u(x,t)\big|\,dx + 2b \\ = {}& 2\big\|u_t(t)\big\|^2 - 2\bigg(1 - \int_0^t g(s)\,ds\bigg)\big\|\nabla u(t)\big\|^2 \\ & - 2\int_0^t g(t-s)\big(\nabla u(t), \nabla u(t) - \nabla u(s)\big)\,ds + 2\int_\Omega \big|u(x,t)\big|^p \ln\big|u(x,t)\big|\,dx \\ & + 2b, \end{aligned} \tag{4.18}$$

for almost every $t \in [0, T]$. By the Cauchy–Schwartz inequality and (4.15), we see that

$$\begin{aligned} \frac{(L'(t))^2}{4} &= \bigg(\big(u(t), u_t(t)\big) + \int_0^t \big(\nabla u(s), \nabla u_t(s)\big)\,ds + b(t+T_0)\bigg)^2 \\ &\le \bigg(\big\|u(t)\big\|^2 + \int_0^t \big\|\nabla u(s)\big\|^2\,ds + b(t+T_0)^2\bigg)\bigg(\big\|u_t(t)\big\|^2 + \int_0^t \big\|\nabla u_t(s)\big\|^2\,ds + b\bigg) \\ &= \big(L(t) - (T-t)\|\nabla u_0\|^2\big)\bigg(\big\|u_t(t)\big\|^2 + \int_0^t \big\|\nabla u_t(s)\big\|^2\,ds + b\bigg) \\ &\le L(t)\bigg(\big\|u_t(t)\big\|^2 + \int_0^t \big\|\nabla u_t(s)\big\|^2\,ds + b\bigg). \end{aligned} \tag{4.19}$$

Thus, we have from (4.18) and (4.19) that

$$L(t)L''(t) - \frac{p+2}{4}\big(L'(t)\big)^2 \geq L(t)F(t), \tag{4.20}$$

where

$$\begin{aligned} F(t) &= -p\|u_t(t)\|^2 - 2\left(1 - \int_0^t g(s)\,ds\right)\|\nabla u(t)\|^2 + 2\int_\Omega |u(x,t)|^p \ln|u(x,t)|\,dx \\ &\quad - 2\int_0^t g(t-s)\big(\nabla u(t), \nabla u(t) - \nabla u(s)\big)\,ds - (p+2)\int_0^t \|\nabla u_t(s)\|^2\,ds - pb. \end{aligned} \tag{4.21}$$

Applying (4.9) to this and using (4.11) and Young's inequality, we get

$$\begin{aligned} F(t) &= -2pE(t) + (p-2)\left(1 - \int_0^t g(s)\,ds\right)\|\nabla u(t)\|^2 + p(g\circ\nabla u)(t) + \frac{2}{p}\|u(t)\|_p^p \\ &\quad - (p+2)\int_0^t \|\nabla u_t(s)\|^2\,ds - 2\int_0^t g(t-s)\big(\nabla u(t), \nabla u(t) - \nabla u(s)\big)\,ds - pb \\ &\geq -2pE(0) + (p-2)\left(1 - \int_0^t g(s)\,ds\right)\|\nabla u(t)\|^2 + p(g\circ\nabla u)(t) + \frac{2}{p}\|u(t)\|_p^p \\ &\quad + (p-2)\int_0^t \|\nabla u_t(s)\|^2\,ds - 2\int_0^t g(t-s)\big(\nabla u(t), \nabla u(t) - \nabla u(s)\big)\,ds - pb \\ &\geq -2pE(0) + \left\{(p-2) - \left(p - 2 + \frac{1}{\epsilon}\right)\int_0^t g(s)\,ds\right\}\|\nabla u(t)\|^2 + (p-\epsilon)(g\circ\nabla u)(t) \\ &\quad + \frac{2}{p}\|u(t)\|_p^p + (p-2)\int_0^t \|\nabla u_t(s)\|^2\,ds - pb, \end{aligned} \tag{4.22}$$

where $\epsilon > 0$. We now consider the initial energy $E(0)$ divided into three cases: $E(0) < 0$, $E(0) = 0$, and $0 < E(0) < d$.

Case 1: $\alpha < 0$, *i.e.* $E(0) < 0$.

Taking $\epsilon = p$ in (4.22) and choosing $0 < b \leq -2E(0)$, we have from (4.14)

$$\begin{aligned} F(t) &\geq p\big(-2E(0) - b\big) + \left\{(p-2) - \left(p - 2 + \frac{1}{p}\right)\int_0^t g(s)\,ds\right\}\|\nabla u(t)\|^2 \\ &\quad + \frac{2}{p}\|u(t)\|_p^p + (p-2)\int_0^t \|\nabla u_t(s)\|^2\,ds \geq 0. \end{aligned} \tag{4.23}$$

Case 2: $\alpha = 0$, *i.e.* $E(0) = 0$.

Taking $\epsilon = p$ in (4.22) and $b = 0$, we see from (4.14) that

$$\begin{aligned} F(t) &\geq \left\{(p-2) - \left(p - 2 + \frac{1}{p}\right)\int_0^t g(s)\,ds\right\}\|\nabla u(t)\|^2 \\ &\quad + \frac{2}{p}\|u(t)\|_p^p + (p-2)\int_0^t \|\nabla u_t(s)\|^2\,ds \geq 0. \end{aligned} \tag{4.24}$$

Case 3: $0 < \alpha < 1$, *i.e.* $0 < E(0) < d$.

Taking $\epsilon = (1-\alpha)p + 2\alpha$ in (4.22), we find

$$F(t) \geq -2pE(0) + \left\{(p-2) - \left(p-2+\frac{1}{(1-\alpha)p+2\alpha}\right)\int_0^t g(s)\,ds\right\}\|\nabla u(t)\|^2 + \alpha(p-2)(g\circ\nabla u)(t) + \frac{2}{p}\|u(t)\|_p^p + (p-2)\int_0^t \|\nabla u_t(s)\|^2\,ds - pb. \tag{4.25}$$

Due to the condition (4.14), it follows that

$$(p-2) - \left(p-2+\frac{1}{(1-\alpha)p+2\alpha}\right)\int_0^t g(s)\,ds \geq \alpha(p-2)\left(1-\int_0^\infty g(s)\,ds\right), \tag{4.26}$$

and hence

$$\begin{aligned} F(t) &\geq -2pE(0) + \alpha(p-2)\left\{\left(1-\int_0^\infty g(s)\,ds\right)\|\nabla u(t)\|^2 + (g\circ\nabla u)(t)\right\} \\ &\quad + \frac{2}{p}\|u(t)\|_p^p + (p-2)\int_0^t\|\nabla u_t(s)\|^2\,ds - pb \\ &\geq -2pE(0) + \alpha(p-2)\left(1-\int_0^\infty g(s)\,ds\right)\|\nabla u(t)\|^2 + \frac{2\alpha}{p}\|u(t)\|_p^p - pb. \end{aligned} \tag{4.27}$$

On the other hand, it is noted that $I(u(t)) < 0$ for all $t \in [0,T]$ from Lemma 4.2. So, Lemma 4.1 ensures that the existence of $\lambda_* \in (0,1)$ satisfying $I(\lambda_* u(t)) = 0$. Hence, from (4.3) and (4.5)

$$\begin{aligned} d \leq J(\lambda_* u(t)) &= \left(\frac{1}{2}-\frac{1}{p}\right)\left(1-\int_0^\infty g(s)\,ds\right)\lambda_*^2\|\nabla u(t)\|^2 + \frac{\lambda_*^p}{p^2}\|u(t)\|_p^p \\ &< \frac{p-2}{2p}\left(1-\int_0^\infty g(s)\,ds\right)\|\nabla u(t)\|^2 + \frac{1}{p^2}\|u(t)\|_p^p. \end{aligned} \tag{4.28}$$

Since u is continuous on $[0,T]$, there exists $\kappa > 0$ such that

$$d+\kappa < \frac{p-2}{2p}\left(1-\int_0^\infty g(s)\,ds\right)\|\nabla u(t)\|^2 + \frac{1}{p^2}\|u(t)\|_p^p \quad \text{for all } t\in[0,T].$$

From this and (4.27), we get

$$\begin{aligned} F(t) &\geq -2p\alpha d + 2\alpha p\left\{\frac{p-2}{2p}\left(1-\int_0^\infty g(s)\,ds\right)\|\nabla u(t)\|^2 + \frac{1}{p^2}\|u(t)\|_p^p\right\} - pb \\ &> 2\alpha p\kappa - pb. \end{aligned} \tag{4.29}$$

Choosing $b > 0$ sufficiently small so that $2\alpha p\kappa - pb \geq 0$, we obtain

$$F(t) \geq 0. \tag{4.30}$$

Adapting (4.23), (4.24), (4.30) to (4.20), we infer

$$L(t)L''(t) - \frac{p+2}{4}\left(L'(t)\right)^2 \geq 0. \tag{4.31}$$

Now it remains to show $L'(0) > 0$. In the case of $E(0) = 0$, the condition $(u_0, u_1) > 0$ gives

$$L'(0) = 2(u_0, u_1) > 0.$$

For the cases of $E(0) < 0$ and $0 < E(0) < d$, we choose T_0 large enough so that

$$L'(0) = 2(u_0, u_1) + 2bT_0 > 0.$$

Thus, we conclude from Lemma 2.2 that

$$\lim_{t\to T_*^-} L(t) = +\infty \tag{4.32}$$

for

$$T_* \le \frac{4L(0)}{(p-2)L'(0)} = \frac{2\|u_0\|^2 + 2T\|\nabla u_0\|^2 + 2bT_0^2}{(p-2)((u_0, u_1) + bT_0)}.$$

Thus, we deduce that

$$T_* \le \frac{2\|u_0\|^2 + 2bT_0^2}{(p-2)(u_0, u_1) + (p-2)bT_0 - 2\|\nabla u_0\|^2}. \tag{4.33}$$

From (4.15), (4.32) and (4.33), we have

$$\lim_{t\to T_*^-} \left(\|u(t)\| + \int_0^t \|\nabla u(s)\|^2 \, ds \right) = +\infty.$$

This contradicts our assumption that the weak solution is global. Thus, we conclude that the weak solution u to problem (1.1)–(1.3) blows up in finite time. □

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Author details
[1]Department of Mathematics, and Institute of Pure and Applied Mathematics, Jeonbuk National University, Jeonju, South Korea. [2]Office for Education Accreditation, Pusan National University, Busan, South Korea.

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References

1. Barrow, J., Parsons, P.: Inflationary models with logarithmic potentials. Phys. Rev. D **52**, 5576–5587 (1995)
2. Gorka, P.: Logarithmic Klein–Gordon equation. Acta Phys. Pol. B **40**, 59–66 (2009)
3. Pata, V., Zelik, S.: Smooth attractors for strongly damped wave equations. Nonlinearity **19**, 1495–1506 (2006)
4. Sattinger, D.H.: On global solution of nonlinear hyperbolic equations. Arch. Ration. Mech. Anal. **30**, 148–172 (1968)
5. Cavalcanti, M.M., Domingos Cavalcanti, V.N., Martinez, P.: Existence and decay rate estimates for the wave equation with nonlinear boundary damping and source term. J. Differ. Equ. **203**, 119–158 (2004)
6. Gazzola, F., Squassina, M.: Global solutions and finite time blow up for damped semilinear wave equations. Ann. Inst. Henri Poincaré, Anal. Non Linéaire **23**, 185–207 (2006)
7. Ha, T.G.: On viscoelastic wave equation with nonlinear boundary damping and source term. Commun. Pure Appl. Anal. **9**, 1543–1576 (2010)
8. Ikehata, R.: Some remarks on the wave equations with nonlinear damping and source terms. Nonlinear Anal. **27**, 1165–1175 (1996)
9. Levine, H.A.: Instability and nonexistence of global solutions of nonlinear wave equation of the form $Putt = Au + F(u)$. Trans. Am. Math. Soc. **192**, 1–21 (1974)
10. Li, G., Yu, J., Liu, W.: Global existence, exponential decay and finite time blow-up of solutions for a class of semilinear pseudo-parabolic equations with conical degeneration. J. Pseudo-Differ. Oper. Appl. **8**, 629–660 (2017)
11. Liu, W.: Global existence, asymptotic behavior and blow-up of solutions for a viscoelastic equation with strong damping and nonlinear source. Topol. Methods Nonlinear Anal. **36**, 153–178 (2010)
12. Liu, W., Zhuang, H.: Global existence, asymptotic behavior and blow-up of solutions for a suspension bridge equation with nonlinear damping and source terms. NoDEA Nonlinear Differ. Equ. Appl. **24**, Article ID 67 (2017)
13. Liu, W., Zhu, B., Li, G.: Upper and lower bounds for the blow-up time for a viscoelastic wave equation with dynamic boundary conditions. Quaest. Math. https://doi.org/10.2989/16073606.2019.1595768
14. Sun, F., Liu, L., Wu, Y.: Blow-up of solutions for a nonlinear viscoelastic wave equation with initial data at arbitrary energy level. Appl. Anal. **98**, 2308–2327 (2019)
15. Wu, S.-T.: Blow-up of solution for a viscoelastic wave equation with delay. Acta Math. Sci. Ser. B **39**, 329–338 (2019)
16. Ma, L., Fang, Z.B.: Eenrgy decay estimates and infinite blow-up phenomena for a strongly damped semilinear wave equation with logarithmic nonlinear source. Math. Methods Appl. Sci. **41**, 2639–2653 (2018)
17. Lian, W., Xu, R.: Global well-posedness of nonlinear wave equation with weak and strong damping terms and logarithmic source term. Adv. Nonlinear Anal. **9**, 613–632 (2020)
18. Di, H., Shang, Y., Song, Z.: Initial boundary value problems for a class of strongly damped semilinear wave equations with logarithmic nonlinearity. Nonlinear Anal., Real World Appl. **51**, Article ID 102968 (2020)
19. Gross, L.: Logarithmic Sobolev inequalities. Am. J. Math. **97**, 1061–1083 (1975)
20. Cazenave, T., Haraux, A.: Equations d'evolution avec non-lineaire logarithmique. Ann. Fac. Sci. Toulouse Math. **2**, 21–51 (1980)
21. Chen, H., Luo, P., Liu, G.: Global solution and blow-up of a semilinear heat equation with logarithmic nonlinearity. J. Math. Anal. Appl. **442**, 84–98 (2015)
22. Lions, J.L.: Quelques Méthodes de Résolution des Problèmses aux Limites Non Linéaires. Dunod, Paris (1969)
23. Chen, H., Liu, G.: Global existence and nonexistence for semilinear parabolic equations with conical degeneration. J. Pseudo-Differ. Oper. Appl. **3**, 329–349 (2012)
24. Liu, Y.: On potential wells and applications to semilinear hyperbolic equations and parabolic equations. Nonlinear Anal. **64**, 2665–2687 (2006)
25. Payne, L., Sattinger, D.: Saddle points and instability of nonlinear hyperbolic equation. Isr. J. Math. **226**, 273–303 (1975)