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Some Hermite–Jensen–Mercer type inequalities for *k*-Caputo-fractional derivatives and related results

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Abstract

In this paper, certain Hermite–Hadamard–Mercer type inequalities are proved via *k*-Caputo fractional derivatives. We established some new *k*-Caputo fractional derivatives inequalities with Hermite–Hadamard–Mercer type inequalities for differentiable mapping $\psi^{(n)}$ whose derivatives in the absolute values are convex.

Keywords: Convex functions; Hermite–Hadamard inequalities; Jensen inequality; Jensen–Mercer inequality; *k*-Caputo fractional derivatives

1 Introduction

Let $0 < u_1 \le u_2 \le \cdots \le u_n$ and let $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ be non-negative weights such that $\sum_{i=1}^{n} \mu_i = 1$. The famous Jensen inequality [1] states that if ψ is a convex function on the interval $[\theta_1, \theta_2]$, then

$$\psi\left(\sum_{i=1}^{n} \mu_i u_i\right) \le \left(\sum_{i=1}^{n} \mu_i \psi(u_i)\right) \tag{1}$$

for all $u_i \in [\theta_1, \theta_2]$ and $\mu_i \in [0,1]$ $(i = 1, 2, \ldots, n)$.

In 1883, the Hermite–Hadamard (H-H) inequality was considered the most useful inequality in mathematical analysis. It is also known as the classical H-H inequality.

The Hermite–Hadamard inequality asserts that if $\psi : J \subseteq R \to R$ is a convex function defined on J and $\theta_1, \theta_2 \in J$ such that $\theta_1 < \theta_2$, then

$$\psi\left(\frac{\theta_1 + \theta_2}{2}\right) \le \frac{1}{\theta_2 - \theta_1} \int_{\theta_1}^{\theta_2} \psi(\lambda)\, d\lambda \le \frac{\psi(\theta_1) + \psi(\theta_2)}{2}.$$

For recent results related with the Jensen–Mercer inequality, see [1–4].

Theorem 1 *If ψ is a convex function on $[\theta_1, \theta_2]$, then*

$$\psi\left(\theta_1 + \theta_2 - \sum_{i=1}^{n} \mu_i u_i\right) \leq \psi(\theta_1) + \psi(\theta_2) - \sum_{i=1}^{n} \mu_i \psi(u_i), \tag{2}$$

$\forall u_i \in [\theta_1, \theta_2]$ *and all* $\mu_i \in [0,1]$ $(i = 1, 2, \ldots, n)$.

Inequality (2) is known as the Jensen–Mercer inequality. Recently, inequality (2) has been studied and generalized in [5–7].

Fractional calculus was generally a study kept for the best minds in mathematics. The early era of fractional calculus is as old as the history of differential calculus. One generalized the differential operators and ordinary integrals. However, the fractional derivatives have some more basic properties than the corresponding classical ones. On the other hand, besides the smooth requirement, the Caputo derivative does not coincide with the classical derivative [8]. It was introduced in 1967.

In the following, we give the definition of Caputo fractional derivatives (see [9–11] and the references therein).

Definition 1 Let $\alpha > 0$ and $\alpha \notin \{1, 2, 3, \ldots\}$, $n = [\alpha] + 1$, $\psi \in C^n[\theta_1, \theta_2]$. The Caputo fractional derivatives of order α are defined as follows:

$$\left({}^{c}D^{\alpha}_{\theta_1^+}\psi\right)(u) = \frac{1}{\Gamma(n-\alpha)} \int_{\theta_1}^{u} \frac{\psi^{(n)}(\lambda)}{(u-\lambda)^{\alpha-n+1}}\, d\lambda; \quad u > \theta_1,$$

and

$$\left({}^{c}D^{\alpha}_{\theta_2^-}\psi\right)(u) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_{u}^{\theta_2} \frac{\psi^{(n)}(\lambda)}{(\lambda-u)^{\alpha-n+1}}\, d\lambda; \quad u < \theta_2.$$

If $\alpha = n \in \{1, 2, 3, \ldots\}$ and the usual derivatives of ψ of order n exist, then the Caputo fractional derivatives $({}^{c}D^{\alpha}_{\theta_1^+}\psi)(u)$ coincide with $\psi^{(n)}(u)$.

In particular, we have

$$\left({}^{c}D^{0}_{\theta_1^+}\psi\right)(u) = \left({}^{c}D^{0}_{\theta_2^-}\psi\right)(u) = \psi(u),$$

where $n = 1$ and $\alpha = 0$.

Definition 2 (See [12]) Diaz and Parigun have defined the k-Gamma function Γ_k, a generalization of the classical Gamma function, which is given by the following formula:

$$\Gamma_k(x) = \lim_{n-\infty} \frac{n! k^n (nk)^{\frac{x}{k}} - 1}{(x)_{n,k}} k > 0.$$

It is shown that the Mellin transform of the exponential function $e^{-\frac{t^k}{k}}$ is the k-Gamma function given by

$$\Gamma_k(\alpha) = \int_0^{\infty} e^{-\frac{t^k}{k}} t^{\alpha-1}\, dt.$$

Obviously, $\Gamma_k(x + k) = x\Gamma_k(x)$, $\Gamma(x) = \lim_{k\to 1} \Gamma_k(x)$ and $\Gamma_k(x) = k^{\frac{x}{k}-1}\Gamma(\frac{x}{k})$.

Definition 3 ([13]) Let $\alpha > 0$, $k \geq 1$ and $\alpha \notin \{1,2,3,\dots\}$, $n = [\alpha] + 1$, $\psi \in C^n[\theta_1,\theta_2]$. The right-sided and left-sided Caputo k-fractional derivatives of order α are defined as follows:

$$\left({}^cD^{\alpha,k}_{\theta_1^+}\psi\right)(u) = \frac{1}{k\Gamma_k(n-\frac{\alpha}{k})}\int_{\theta_1}^{u}\frac{\psi^{(n)}(\lambda)}{(u-\lambda)^{\frac{\alpha}{k}-n+1}}\,d\lambda; \quad u > \theta_1 \tag{3}$$

and

$$\left({}^cD^{\alpha,k}_{\theta_2^-}\psi\right)(v) = \frac{(-1)^n}{k\Gamma_k(n-\frac{\alpha}{k})}\int_{v}^{\theta_2}\frac{\psi^{(n)}(\lambda)}{(\lambda-v)^{\frac{\alpha}{k}-n+1}}\,d\lambda; \quad v < \theta_2. \tag{4}$$

For $k = 1$, Caputo k-fractional derivatives give the definition of Caputo fractional derivatives.

In this article, by using the Jensen–Mercer inequality, we prove Hermite–Hadamard inequalities for fractional integrals and we establish some new Caputo k-fractional derivatives connected with the left and right sides of Hermite–Hadamard type inequalities for differentiable mappings whose derivatives in absolute values are convex.

Throughout the paper, we need the following assumptions.

$A_1 = \forall u,v \in [\theta_1,\theta_2]$, $\alpha > 0$, $k \geq 1$ and $\Gamma_k(\cdot)$ is the k-Gamma function.

2 Hermite–Hadamard–Mercer type inequalities for Caputo k-fractional derivatives

By using the Jensen–Mercer inequality, Hermite–Hadamard type inequalities can be expressed in Caputo k-fractional derivative form as follows.

Theorem 2 *Suppose that if $\psi : [\theta_1,\theta_2] \to R$ is a positive function with $0 \leq \theta_1 < \theta_2$ and $\psi \in C^n[\theta_1,\theta_2]$. If $\psi^{(n)}$ is a convex function on $[\theta_1,\theta_2]$ along with the assumptions in A_1, then the following inequalities for Caputo k-fractional derivatives hold*:

$$\begin{aligned}
&\psi^{(n)}\left(\theta_1+\theta_2-\frac{u+v}{2}\right)\\
&\quad\leq \psi^{(n)}(\theta_1)+\psi^{(n)}(\theta_2)-\frac{\Gamma_k(n-\frac{\alpha}{k}+k)}{2(v-u)^{n-\frac{\alpha}{k}}}\left\{\left({}^cD^{\alpha,k}_{u^+}\psi\right)(v)+(-1)^n\left({}^cD^{\alpha,k}_{v^-}\psi\right)(u)\right\}\\
&\quad\leq \psi^{(n)}(\theta_1)+\psi^{(n)}(\theta_2)-\psi^{(n)}\left(\frac{u+v}{2}\right).
\end{aligned} \tag{5}$$

Proof Using the Jensen–Mercer inequality, we have

$$\psi^{(n)}\left(\theta_1+\theta_2-\frac{w+z}{2}\right) \leq \psi^{(n)}(\theta_1)+\psi^{(n)}(\theta_2)-\frac{\psi^{(n)}(w)+\psi^{(n)}(z)}{2} \tag{6}$$

for all $w,z \in [\theta_1,\theta_2]$.

Now by change of variables $w = \lambda u + (1-\lambda)v$ and $z = (1-\lambda)u + \lambda v$, for all $u,v \in [\theta_1,\theta_2]$ and $\lambda \in [0,1]$ in (6), we have

$$\begin{aligned}
&\psi^{(n)}\left(\theta_1+\theta_2-\frac{u+v}{2}\right)\\
&\quad\leq \psi^{(n)}(\theta_1)+\psi^{(n)}(\theta_2)-\frac{\psi^{(n)}(\lambda u+(1-\lambda)v)+\psi^{(n)}((1-\lambda)u+\lambda v)}{2}.
\end{aligned}$$

Multiplying both sides by $\lambda^{n-\frac{\alpha}{k}-1}$ above and then integrating the resulting inequality with respect to λ over $[0,1]$, we have

$$\begin{aligned}
&\frac{1}{(n-\frac{\alpha}{k})}\psi^{(n)}\left(\theta_1+\theta_2-\frac{u+v}{2}\right)\\
&\quad\le \frac{1}{(n-\frac{\alpha}{k})}\left\{\psi^{(n)}(\theta_1)+\psi^{(n)}(\theta_2)\right\}\\
&\qquad-\frac{1}{2}\left\{\int_0^1 \lambda^{n-\frac{\alpha}{k}-1}\left(\psi^{(n)}\left(\lambda u+(1-\lambda)v\right)+\psi^{(n)}\left((1-\lambda)u+\lambda v\right)\right)d\lambda\right\},
\end{aligned}$$

hence

$$\begin{aligned}
&\psi^{(n)}\left(\theta_1+\theta_2-\frac{u+v}{2}\right)\\
&\quad\le \psi^{(n)}(\theta_1)+\psi^{(n)}(\theta_2)\\
&\qquad-\frac{\Gamma_k(n-\frac{\alpha}{k}+k)}{2(v-u)^{n-\frac{\alpha}{k}}}\left\{\left({}^cD_{u^+}^{\alpha,k}\psi\right)(v)+(-1)^n\left({}^cD_{v^-}^{\alpha,k}\psi\right)(u)\right\}
\end{aligned}$$

and so the first inequality of (5) is proved.

Now for the proof of second inequality of (5), we first note that if $\psi^{(n)}$ is a convex function, then for $\lambda\in[0,1]$, it gives

$$\begin{aligned}
\psi^{(n)}\left(\frac{u+v}{2}\right)&=\psi^{(n)}\left(\frac{\lambda u+(1-\lambda)v+(1-\lambda)u+\lambda v}{2}\right)\\
&\le\frac{\psi^{(n)}(\lambda u+(1-\lambda)v)+\psi^{(n)}((1-\lambda)u+\lambda v)}{2}.
\end{aligned}$$

Multiplying both sides by $\lambda^{n-\frac{\alpha}{k}-1}$ above and then integrating the resulting inequality with respect to λ over $[0,1]$, we have

$$\begin{aligned}
&\frac{1}{n-\frac{\alpha}{k}}\psi^{(n)}\left(\frac{u+v}{2}\right)\\
&\quad\le\frac{1}{2}\left\{\int_0^1 \lambda^{n-\frac{\alpha}{k}-1}\left(\psi^{(n)}\left(\lambda u+(1-\lambda)v\right)+\psi^{(n)}\left((1-\lambda)u+\lambda v\right)\right)d\lambda\right\},
\end{aligned}$$

hence

$$\psi^{(n)}\left(\frac{u+v}{2}\right)\le\frac{\Gamma_k(n-\frac{\alpha}{k}+k)}{2(v-u)^{n-\frac{\alpha}{k}}}\left\{\left({}^cD_{u^+}^{\alpha,k}\psi\right)(v)+(-1)^n\left({}^cD_{v^-}^{\alpha,k}\psi\right)(u)\right\}.$$

Multiplying by (-1) on both sides, we have

$$-\frac{\Gamma_k(n-\frac{\alpha}{k}+k)}{2(v-u)^{n-\frac{\alpha}{k}}}\left\{\left({}^cD_{u^+}^{\alpha,k}\psi\right)(v)+(-1)^n\left({}^cD_{v^-}^{\alpha,k}\psi\right)(u)\right\}\le-\psi^{(n)}\left(\frac{u+v}{2}\right). \tag{7}$$

Adding $\psi^{(n)}(\theta_1)+\psi^{(n)}(\theta_2)$ in both sides in (7), we get the second inequality of (5). □

Remark 1 If we take $k=1$ in Theorem 2, then it reduces to Theorem 2 in [14].

Theorem 3 *Suppose that if $\psi : [\theta_1, \theta_2] \to R$ is a positive function with $0 \le \theta_1 < \theta_2$ and $\psi \in C^n[\theta_1, \theta_2]$. If $\psi^{(n)}$ is a convex function on $[\theta_1, \theta_2]$ along with the assumptions in A_1, then the following inequalities for the Caputo k-fractional derivatives hold*:

$$\begin{aligned}
&\psi^{(n)}\left(\theta_1 + \theta_2 - \frac{u+v}{2}\right) \\
&\quad \le \frac{2^{n-\frac{\alpha}{k}-1}\Gamma_k(n-\frac{\alpha}{k}+k)}{(v-u)^{n-\frac{\alpha}{k}}}\left\{\left({}^cD^{\alpha,k}_{(\theta_1+\theta_2-\frac{u+v}{2})^+}\psi\right)(\theta_1+\theta_2-u)\right. \\
&\quad\quad \left. + (-1)^n\left({}^cD^{\alpha,k}_{(\theta_1+\theta_2-\frac{u+v}{2})^-}\psi\right)(\theta_1+\theta_2-v)\right\} \\
&\quad \le \psi^{(n)}(\theta_1) + \psi^{(n)}(\theta_2) - \left(\frac{\psi^{(n)}(u)+\psi^{(n)}(v)}{2}\right).
\end{aligned} \tag{8}$$

Proof To prove the first part of the inequality, we use the convexity of $\psi^{(n)}$,

$$\begin{aligned}
\psi^{(n)}\left(\theta_1 + \theta_2 - \frac{u_1+v_1}{2}\right) &= \psi^{(n)}\left(\frac{\theta_1+\theta_2-u_1+\theta_1+\theta_2-v_1}{2}\right) \\
&\le \frac{\psi^{(n)}(\theta_1+\theta_2-u_1)+\psi^{(n)}(\theta_1+\theta_2-v_1)}{2}
\end{aligned}$$

for all $u_1, v_1 \in [\theta_1, \theta_2]$. Now by writing the variables $u_1 = \frac{\lambda}{2}u + \frac{2-\lambda}{2}v$ and $v_1 = \frac{2-\lambda}{2}u + \frac{\lambda}{2}v$, for $u, v \in [\theta_1, \theta_2]$ and $\lambda \in [0, 1]$, we get

$$\begin{aligned}
&2\psi^{(n)}\left(\theta_1 + \theta_2 - \frac{u+v}{2}\right) \\
&\quad \le \psi^{(n)}\left(\theta_1+\theta_2-\left(\frac{\lambda}{2}u+\frac{2-\lambda}{2}v\right)\right) + \psi^{(n)}\left(\theta_1+\theta_2-\left(\frac{2-\lambda}{2}u+\frac{\lambda}{2}v\right)\right).
\end{aligned}$$

Multiplying both sides by $\lambda^{n-\frac{\alpha}{k}-1}$ above and then integrating the resulting inequality with respect to λ over $[0, 1]$, we have

$$\begin{aligned}
&2\psi^{(n)}\left(\theta_1 + \theta_2 - \frac{u+v}{2}\right)\int_0^1 \lambda^{n-\frac{\alpha}{k}-1}\,d\lambda \\
&\quad \le \int_0^1 \lambda^{n-\frac{\alpha}{k}-1}\left(\psi^{(n)}\left(\theta_1+\theta_2-\left(\frac{\lambda}{2}u+\frac{2-\lambda}{2}v\right)\right)\right. \\
&\quad\quad \left. + \psi^{(n)}\left(\theta_1+\theta_2-\left(\frac{2-\lambda}{2}u+\frac{\lambda}{2}v\right)\right)\right)d\lambda,
\end{aligned}$$

hence

$$\begin{aligned}
&\psi^{(n)}\left(\theta_1 + \theta_2 - \frac{u+v}{2}\right) \\
&\quad \le \frac{2^{n-\frac{\alpha}{k}-1}\Gamma_k(n-\frac{\alpha}{k}+k)}{(v-u)^{n-\frac{\alpha}{k}}}\left\{\left({}^cD^{\alpha,k}_{(\theta_1+\theta_2-\frac{u+v}{2})^+}\psi\right)(\theta_1+\theta_2-u)\right. \\
&\quad\quad \left. + (-1)^n\left({}^cD^{\alpha,k}_{(\theta_1+\theta_2-\frac{u+v}{2})^-}\psi\right)(\theta_1+\theta_2-v)\right\}
\end{aligned} \tag{9}$$

and so the first inequality of (8) is proved.

Now for the proof of second inequality of (5), we first note that if $\psi^{(n)}$ is a convex function, then for $\lambda \in [0,1]$, it yields

$$\begin{aligned} &\psi^{(n)}\left(\theta_1+\theta_2-\left(\frac{\lambda}{2}u+\frac{2-\lambda}{2}\nu\right)\right) \\ &\quad \le \psi^{(n)}(\theta_1)+\psi^{n}(\theta_2)-\left[\frac{\lambda}{2}\psi^{(n)}(u)+\frac{2-\lambda}{2}\psi^{(n)}(\nu)\right] \end{aligned} \tag{10}$$

and

$$\begin{aligned} &\psi^{(n)}\left(\theta_1+\theta_2-\left(\frac{2-\lambda}{2}u+\frac{\lambda}{2}\nu\right)\right) \\ &\quad \le \psi^{(n)}(\theta_1)+\psi^{(n)}(\theta_2)-\left[\frac{2-\lambda}{2}\psi^{(n)}(u)+\frac{\lambda}{2}\psi^{(n)}(\nu)\right]. \end{aligned} \tag{11}$$

By adding the inequalities of (10) and (11), we have

$$\begin{aligned} &\psi^{(n)}\left(\theta_1+\theta_2-\left(\frac{\lambda}{2}u+\frac{2-\lambda}{2}\nu\right)\right)+\psi^{(n)}\left(\theta_1+\theta_2-\left(\frac{2-\lambda}{2}u+\frac{\lambda}{2}\nu\right)\right) \\ &\quad \le 2\left(\psi^{(n)}(\theta_1)+\psi^{(n)}(\theta_2)\right)-\left(\psi^{(n)}(u)+\psi^{(n)}(\nu)\right). \end{aligned}$$

Multiplying both sides by $\lambda^{n-\frac{\alpha}{k}-1}$ in above and then integrating the resulting inequality with respect to λ over $[0,1]$, we have

$$\begin{aligned} &\int_0^1 \lambda^{n-\frac{\alpha}{k}-1}\left(\psi^{(n)}\left(\theta_1+\theta_2-\left(\frac{\lambda}{2}u+\frac{2-\lambda}{2}\nu\right)\right)+\psi^{(n)}\left(\theta_1+\theta_2-\left(\frac{2-\lambda}{2}u+\frac{\lambda}{2}\nu\right)\right)\right)d\lambda \\ &\quad \le \left\{2\left(\psi^{(n)}(\theta_1)+\psi^{n}(\theta_2)\right)-\left(\psi^{(n)}(u)+\psi^{(n)}(\nu)\right)\right\}\int_0^1 \lambda^{n-\frac{\alpha}{k}-1}\,d\lambda. \end{aligned}$$

This implies

$$\begin{aligned} &\frac{2^{n-\frac{\alpha}{k}}\Gamma_k(n-\frac{\alpha}{k})}{(\nu-u)^{n-\frac{\alpha}{k}}}\left\{\left({}^{c}D^{\alpha,k}_{(\theta_1+\theta_2-\frac{u+\nu}{2})^{+}}\psi\right)(\theta_1+\theta_1-u)\right. \\ &\qquad \left.+(-1)^{n}\left({}^{c}D^{\alpha,k}_{(\theta_1+\theta_2-\frac{u+\nu}{2})^{-}}\psi\right)(\theta_1+\theta_2-\nu)\right\} \\ &\quad \le \left\{2\left(\psi^{(n)}(\theta_1)+\psi^{(n)}(\theta_2)\right)-\left(\psi^{(n)}(u)+\psi^{(n)}(\nu)\right)\right\}\frac{1}{n-\frac{\alpha}{k}}. \end{aligned} \tag{12}$$

Multiplying (12) by $\frac{(n-\frac{\alpha}{k})}{2}$,

$$\begin{aligned} &\frac{2^{n-\frac{\alpha}{k}-1}\Gamma_k(n-\frac{\alpha}{k}+k)}{(\nu-u)^{n-\frac{\alpha}{k}}}\left\{\left({}^{c}D^{\alpha,k}_{(\theta_1+\theta_2-\frac{u+\nu}{2})^{+}}\psi\right)(\theta_1+\theta_2-u)\right. \\ &\qquad \left.+(-1)^{n}\left({}^{c}D^{\alpha,k}_{(\theta_1+\theta_2-\frac{u+\nu}{2})^{-}}\psi\right)(\theta_1+\theta_2-\nu)\right\} \\ &\quad \le \left(\psi^{(n)}(\theta_1)+\psi^{(n)}(\theta_2)\right)-\frac{\psi^{(n)}(u)+\psi^{(n)}(\nu)}{2}. \end{aligned} \tag{13}$$

Combining (9) and (13), we get (8). □

Remark 2 If we take $k = 1$ in Theorem 3, then it reduces to Theorem 3 in [14].

Lemma 1 *Suppose that $\psi : [\theta_1, \theta_2] \to R$ is a differentiable mapping on (θ_1, θ_2) with $0 \le \theta_1 < \theta_2$ and $\psi \in C^{n+1}[\theta_1, \theta_2]$ along with the assumptions in A_1, then the following equality for Caputo k-fractional derivatives holds*:

$$\begin{aligned}&\frac{\psi^{(n)}(\theta_1+\theta_2-u)+\psi^{(n)}(\theta_1+\theta_2-v)}{2}-\frac{\Gamma_k(n-\frac{\alpha}{k}+k)}{2(v-u)^{n-\frac{\alpha}{k}}}\\&\quad\times\bigl\{\bigl({}^cD^{\alpha,k}_{(\theta_1+\theta_2-v)^+}\psi\bigr)(\theta_1+\theta_2-u)+(-1)^n\bigl({}^cD^{\alpha,k}_{(\theta_1+\theta_2-u)^-}\psi\bigr)(\theta_1+\theta_2-v)\bigr\}\\&\quad=\frac{v-u}{2}\int_0^1\bigl(\lambda^{n-\frac{\alpha}{k}}-(1-\lambda)^{n-\frac{\alpha}{k}}\bigr)\psi^{(n+1)}\bigl(\theta_1+\theta_2-\bigl(\lambda u+(1-\lambda)v\bigr)\bigr)\,d\lambda.\end{aligned}\tag{14}$$

Proof It suffices to note that

$$I=\frac{v-u}{2}\{I_1-I_2\},\tag{15}$$

where

$$\begin{aligned}I_1&=\int_0^1\bigl(\lambda^{n-\frac{\alpha}{k}}\bigr)\psi^{(n+1)}\bigl(\theta_1+\theta_2-\bigl(\lambda u+(1-\lambda)v\bigr)\bigr)\,d\lambda\\&=\frac{\psi^{(n)}(\theta_1+\theta_2-u)}{v-u}\\&\quad-\frac{n-\frac{\alpha}{k}}{v-u}\int_0^1\lambda^{n-\frac{\alpha}{k}-1}\psi^{(n)}\bigl(\theta_1+\theta_2-\bigl(\lambda u+(1-\lambda)v\bigr)\bigr)\,d\lambda\\&=\frac{\psi^{(n)}(\theta_1+\theta_2-u)}{v-u}\\&\quad-\frac{\Gamma_k(n-\frac{\alpha}{k}+k)}{(v-u)^{n-\frac{\alpha}{k}+1}}\bigl\{(-1)^n\bigl({}^cD^{\alpha,k}_{(\theta_1+\theta_2-u)^-}\psi\bigr)(\theta_1+\theta_2-v)\bigr\}\end{aligned}\tag{16}$$

and

$$\begin{aligned}I_2&=\int_0^1(1-\lambda)^{n-\frac{\alpha}{k}}\psi^{(n+1)}\bigl(\theta_1+\theta_2-\bigl(\lambda u+(1-\lambda)v\bigr)\bigr)\,d\lambda\\&=-\frac{\psi^{(n)}(\theta_1+\theta_2-v)}{v-u}\\&\quad+\frac{n-\frac{\alpha}{k}}{v-u}\int_0^1(1-\lambda)^{n-\frac{\alpha}{k}-1}\psi^{(n)}\bigl(\theta_1+\theta_2-\bigl(\lambda u+(1-t\lambda)v\bigr)\bigr)\,d\lambda\\&=-\frac{\psi^{(n)}(\theta_1+\theta_2-v)}{v-u}\\&\quad+\frac{\Gamma_k(n-\frac{\alpha}{k}+k)}{(v-u)^{n-\frac{\alpha}{k}+1}}\bigl\{\bigl({}^cD^{\alpha,k}_{(\theta_1+\theta_2-v)^+}\psi\bigr)(\theta_1+\theta_2-u)\bigr\}.\end{aligned}\tag{17}$$

Combining (16) and (17) with (15) and get (14). □

Remark 3 If we take $k = 1$ in Lemma 1, then it reduces to Lemma 1 in [14].

Remark 4 If we take $u = a$ and $v = b$ in Lemma 1, then it reduces to Remark 2.5 in [11].

Lemma 2 *Suppose that $\psi : [\theta_1, \theta_2] \to R$ is a differentiable mapping on (θ_1, θ_2) with $0 \leq \theta_1 < \theta_2$ and $\psi \in C^{n+1}[\theta_1, \theta_2]$ along with the assumptions in A_1, then the following equality for Caputo k-fractional derivatives holds*:

$$\begin{aligned}
&\psi^{(n)}\left(\theta_1 + \theta_2 - \frac{u+v}{2}\right) - \frac{2^{n-\frac{\alpha}{k}-1}\Gamma_k(n - \frac{\alpha}{k} + k)}{(v-u)^{n-\frac{\alpha}{k}}} \\
&\quad \times \left\{\left({}^cD^{\alpha,k}_{(\theta_1+\theta_2-\frac{u+v}{2})^+}\psi\right)(\theta_1 + \theta_2 - u) + (-1)^n\left({}^cD^{\alpha,k}_{(\theta_1+\theta_2-\frac{u+v}{2})^-}\psi\right)(\theta_1 + \theta_2 - v)\right\} \\
&\quad = \frac{v-u}{4}\left[\int_0^1 \lambda^{n-\frac{\alpha}{k}}\psi^{(n+1)}\left(\theta_1 + \theta_2 - \left(\frac{\lambda}{2}u + \frac{2-\lambda}{2}v\right)\right)d\lambda\right. \\
&\quad \left. - \int_0^1 \lambda^{n-\frac{\alpha}{k}}\psi^{(n+1)}\left(\theta_1 + \theta_2 - \left(\frac{2-\lambda}{2}u + \frac{\lambda}{2}v\right)\right)d\lambda\right].
\end{aligned} \tag{18}$$

Proof It suffices to note that

$$I = \frac{v-u}{4}\{I_1 - I_2\}, \tag{19}$$

where

$$\begin{aligned}
I_1 &= \int_0^1 \lambda^{n-\frac{\alpha}{k}}\psi^{(n+1)}\left(\theta_1 + \theta_2 - \left(\frac{\lambda}{2}u + \frac{2-\lambda}{2}v\right)\right)d\lambda \\
&= \frac{2}{v-u}\psi^{(n)}\left(\theta_1 + \theta_2 - \frac{u+v}{2}\right) \\
&\quad - \frac{2(n-\frac{\alpha}{k})}{v-u}\int_0^1 \lambda^{n-\frac{\alpha}{k}-1}\psi^{(n)}\left(\theta_1 + \theta_2 - \left(\frac{\lambda}{2}u + \frac{2-\lambda}{2}v\right)\right)d\lambda \\
&= \frac{2}{v-u}\psi^{(n)}\left(\theta_1 + \theta_2 - \frac{u+v}{2}\right) \\
&\quad - (-1)^n\frac{2^{n-\frac{\alpha}{k}+1}\Gamma_k(n - \frac{\alpha}{k} + k)}{(v-u)^{n-\frac{\alpha}{k}+1}}\left({}^cD^{\alpha,k}_{(\theta_1+\theta_2-\frac{u+v}{2})^-}\psi\right)(\theta_1 + \theta_2 - v)
\end{aligned} \tag{20}$$

and

$$\begin{aligned}
I_2 &= \int_0^1 \lambda^{n-\frac{\alpha}{k}}\psi^{(n+1)}\left(\theta_1 + \theta_2 - \left(\frac{2-\lambda}{2}u + \frac{\lambda}{2}v\right)\right)d\lambda \\
&= -\frac{2}{v-u}\psi^{(n)}\left(\theta_1 + \theta_2 - \frac{u+v}{2}\right) \\
&\quad + \frac{2(n-\alpha)}{v-u}\int_0^1 \lambda^{n-\frac{\alpha}{k}-1}\psi^{(n)}\left(\theta_1 + \theta_2 - \left(\frac{2-\lambda}{2}u + \frac{\lambda}{2}v\right)\right)d\lambda \\
&= -\frac{2}{v-u}\psi^{(n)}\left(\theta_1 + \theta_2 - \frac{u+v}{2}\right) \\
&\quad + \frac{2^{n-\frac{\alpha}{k}+1}\Gamma_k(n - \frac{\alpha}{k} + k)}{(v-u)^{n-\frac{\alpha}{k}+1}}\left({}^cD^{\alpha,k}_{(\theta_1+\theta_2-\frac{u+v}{2})^+}\psi\right)(\theta_1 + \theta_2 - u).
\end{aligned} \tag{21}$$

Combining (20) and (21) with (19), we get (18). □

Remark 5 If we take $k = 1$ in Lemma 2, then it reduces to Lemma 2 in [14].

Remark 6 If we take $u = a$ and $v = b$ in Lemma 2, then it reduces to Lemma 2 in [10].

Theorem 4 *Suppose that $\psi : [\theta_1, \theta_2] \to R$ is a differentiable mapping on (θ_1, θ_2) with $0 \le \theta_1 < \theta_2$ and $\psi \in C^{n+1}[\theta_1, \theta_2]$. If $|\psi^{(n+1)}|$ is a convex function on $[\theta_1, \theta_2]$ along with the assumptions in A_1, then the following inequality for Caputo k-fractional derivatives holds*:

$$\begin{aligned}
&\biggl| \frac{\psi^{(n)}(\theta_1+\theta_2-u)+\psi^{(n)}(\theta_1+\theta_2-v)}{2} - \frac{\Gamma_k(n-\frac{\alpha}{k}+k)}{2(v-u)^{n-\frac{\alpha}{k}}} \\
&\quad\times \bigl(\bigl({}^cD^{\alpha,k}_{(\theta_1+\theta_1-v)^+}\psi\bigr)(\theta_1+\theta_2-u) + (-1)^n \bigl({}^cD^{\alpha,k}_{(\theta_1+\theta_2-u)^-}\psi\bigr)(\theta_1+\theta_2-v)\bigr) \biggr| \\
&\quad\le \frac{v-u}{n-\frac{\alpha}{k}+1}\left(1-\frac{1}{2^{n-\frac{\alpha}{k}}}\right)\Bigl\{ \bigl|\psi^{(n+1)}(\theta_1)\bigr| + \bigl|\psi^{(n+1)}(\theta_2)\bigr| \\
&\quad - \left(\frac{|\psi^{(n+1)}(u)| + |\psi^{(n+1)}(v)|}{2} \right) \Bigr\}.
\end{aligned} \tag{22}$$

Proof By using Lemma 1 and the Jensen–Mercer inequality, we have

$$\begin{aligned}
&\biggl| \frac{\psi^{(n)}(\theta_1+\theta_2-u)+\psi^{(n)}(\theta_1+\theta_2-v)}{2} - \frac{\Gamma_k(n-\frac{\alpha}{k}+k)}{2(v-u)^{n-\frac{\alpha}{k}}} \\
&\quad\times \bigl(\bigl({}^cD^{\alpha,k}_{(\theta_1+\theta_2-v)^+}\psi\bigr)(\theta_1+\theta_2-u) + (-1)^n \bigl({}^cD^{\alpha,k}_{(\theta_1+\theta_2-u)^-}\psi\bigr)(\theta_1+\theta_2-v)\bigr) \biggr| \\
&\quad\le \frac{v-u}{2}\int_0^1 \bigl|\lambda^{n-\frac{\alpha}{k}} - (1-\lambda)^{n-\frac{\alpha}{k}}\bigr| \bigl|\psi^{(n+1)}\bigl(\theta_1+\theta_2 \\
&\quad - (\lambda u + (1-\lambda)v)\bigr)\bigr| \,d\lambda \bigl|\psi^{(n+1)}(\theta_1)\bigr| \\
&\quad\le \frac{v-u}{2}\int_0^1 \Bigl\{ \bigl|\lambda^{n-\frac{\alpha}{k}} - (1-\lambda)^{n-\frac{\alpha}{k}}\bigr| \\
&\quad + \bigl|\psi^{(n+1)}(\theta_2)\bigr| - \bigl(\lambda\bigl|\psi^{(n+1)}(u)\bigr| + (1-\lambda)\bigl|\psi^{(n+1)}(v)\bigr|\bigr) \Bigr\}\,d\lambda \\
&\quad\le \frac{v-u}{2}[I_1+I_2],
\end{aligned} \tag{23}$$

where

$$\begin{aligned}
I_1 &= \int_0^{\frac{1}{2}} \bigl((1-\lambda)^{n-\frac{\alpha}{k}} - \lambda^{n-\frac{\alpha}{k}}\bigr) \\
&\quad\times \bigl\{ \bigl|\psi^{(n+1)}(\theta_1)\bigr| + \bigl|\psi^{(n+1)}(\theta_2)\bigr| - \bigl(\lambda\bigl|\psi^{(n+1)}(u)\bigr| + (1-\lambda)\bigl|\psi^{(n+1)}(v)\bigr|\bigr)\bigr\}\,d\lambda \\
&= \bigl(\bigl|\psi^{(n+1)}(\theta_1)\bigr| + \bigl|\psi^{(n+1)}(\theta_2)\bigr|\bigr)\left(\frac{1}{n-\frac{\alpha}{k}+1} - \frac{2^{-n+\frac{\alpha}{k}}}{n-\frac{\alpha}{k}+1}\right) \\
&\quad - \biggl\{ \bigl|\psi^{(n+1)}(u)\bigr| \left(\frac{1}{(n-\frac{\alpha}{k}+1)(n-\frac{\alpha}{k}+2)} - \frac{2^{-n+\frac{\alpha}{k}-1}}{n-\frac{\alpha}{k}+1} \right) \\
&\quad + \bigl|\psi^{(n+1)}(v)\bigr| \left(\frac{1}{(n-\frac{\alpha}{k}+2)} - \frac{2^{-n+\frac{\alpha}{k}-1}}{n-\frac{\alpha}{k}+1} \right) \biggr\}
\end{aligned} \tag{24}$$

and

$$
\begin{aligned}
I_2 &= \int_{\frac{1}{2}}^{1} \left(\lambda^{n-\frac{\alpha}{k}} - (1-\lambda)^{n-\frac{\alpha}{k}}\right) \\
&\quad \times \left\{\left|\psi^{(n+1)}(\theta_1)\right| + \left|\psi^{(n+1)}(\theta_2)\right| - \left(\lambda\left|\psi^{(n+1)}(u)\right| + (1-\lambda)\left|\psi^{(n+1)}(v)\right|\right)\right\} d\lambda \\
&= \left(\left|\psi^{(n+1)}(\theta_1)\right| + \left|\psi^{(n+1)}(\theta_2)\right|\right)\left(\frac{1}{(n-\frac{\alpha}{k}+1)} - \frac{2^{-n+\frac{\alpha}{k}}}{n-\frac{\alpha}{k}+1}\right) \\
&\quad - \left\{\left|\psi^{(n+1)}(u)\right|\left(\frac{1}{(n-\frac{\alpha}{k}+2)} - \frac{2^{-n+\frac{\alpha}{k}-1}}{n-\frac{\alpha}{k}+1}\right)\right. \\
&\quad \left. + \left|\psi^{(n+1)}(v)\right|\left(\frac{1}{(n-\frac{\alpha}{k}+1)(n-\frac{\alpha}{k}+2)} - \frac{2^{-n+\frac{\alpha}{k}-1}}{n-\frac{\alpha}{k}+1}\right)\right\}.
\end{aligned}
\tag{25}
$$

Combining (24) and (25) with (23) and we get (22). This completes the proof. □

Remark 7 If we take $k = 1$ in Theorem 4, then it reduces to Theorem 4 in [14].

Remark 8 If we take $u = a$ and $v = b$ in Theorem 4, then it reduces to Corollary 2.7 in [11].

Theorem 5 *Suppose that $\psi : [\theta_1, \theta_2] \to R$ is a differentiable mapping on (θ_1, θ_2) with $0 \le \theta_1 < \theta_2$ and $\psi \in C^{n+1}[\theta_1, \theta_2]$. If $|\psi^{(n+1)}|$ is a convex function on $[\theta_1, \theta_2]$ along with the assumptions in A_1, then the following inequality for Caputo k-fractional derivatives holds*:

$$
\begin{aligned}
&\left|\psi^{(n)}\left(\theta_1+\theta_2-\frac{u+v}{2}\right) - \frac{2^{n-\frac{\alpha}{k}-1}\Gamma_k(n-\frac{\alpha}{k}+k)}{(v-u)^{n-\frac{\alpha}{k}}}\right. \\
&\quad \left. \times \left[\left({}^{c}D^{\alpha,k}_{(\theta_1+\theta_2-\frac{u+v}{2})^+}\psi\right)(\theta_1+\theta_2-u) + (-1)^n\left({}^{c}D^{\alpha,k}_{(\theta_1+\theta_2-\frac{u+v}{2})^-}\psi\right)(\theta_1+\theta_2-v)\right]\right| \\
&\le \frac{v-u}{2(n-\frac{\alpha}{k}+1)}\left\{\left|\psi^{(n+1)}(\theta_1)\right| + \left|\psi^{(n+1)}(\theta_2)\right|\right. \\
&\quad \left. - \left(\frac{|\psi^{(n+1)}(u)| + |\psi^{(n+1)}(v)|}{2}\right)\right\}.
\end{aligned}
\tag{26}
$$

Proof By using Lemma 2 and the Jensen–Mercer inequality, we have

$$
\begin{aligned}
&\left|\psi^{(n)}\left(\theta_1+\theta_2-\frac{u+v}{2}\right) - \frac{2^{n-\frac{\alpha}{k}-1}\Gamma(n-\frac{\alpha}{k}+1)}{(v-u)^{n-\frac{\alpha}{k}}}\right. \\
&\quad \left. \times \left[\left({}^{c}D^{\alpha,k}_{(\theta_1+\theta_2-\frac{u+v}{2})^+}\psi\right)(\theta_1+\theta_2-u) + (-1)^n\left({}^{c}D^{\alpha,k}_{(\theta_1+\theta_2-\frac{u+v}{2})^-}\psi\right)(\theta_1+\theta_2-v)\right]\right| \\
&\le \frac{v-u}{4}\left[\int_0^1 \lambda^{n-\frac{\alpha}{k}}\left|\psi^{(n+1)}\left(\theta_1+\theta_2-\left(\frac{\lambda}{2}u+\frac{2-\lambda}{2}v\right)\right)\right| d\lambda\right. \\
&\quad \left. - \int_0^1 \lambda^{n-\frac{\alpha}{k}}\left|\psi^{(n+1)}\left(\theta_1+\theta_2-\left(\frac{2-\lambda}{2}u+\frac{\lambda}{2}v\right)\right)\right| d\lambda\right] \\
&\le \frac{v-u}{4}\left[\int_0^1 \lambda^{n-\frac{\alpha}{k}}\left\{\left|\psi^{(n+1)}(\theta_1)\right|\right.\right. \\
&\quad \left. + \left|\psi^{(n+1)}(\theta_2)\right| - \left(\frac{\lambda}{2}\left|\psi^{(n+1)}(u)\right| + \frac{(2-\lambda)}{2}\left|\psi^{(n+1)}(v)\right|\right)\right\} d\lambda
\end{aligned}
$$

$$+ \int_0^1 \lambda^{n-\frac{\alpha}{k}} \left\{ \left|\psi^{(n+1)}(\theta_1)\right| + \left|\psi^{(n+1)}(\theta_2)\right| - \left(\frac{(2-\lambda)}{2} \left|\psi^{(n+1)}(u)\right| + \frac{\lambda}{2} \left|\psi^{(n+1)}(\nu)\right| \right) \right\} d\lambda \Bigg]$$

by using calculus tools, we obtain

$$\leq \frac{\nu - u}{2(n - \frac{\alpha}{k} + 1)} \left\{ \left|\psi^{(n+1)}(\theta_1)\right| + \left|\psi^{(n+1)}(\theta_2)\right| - \left(\frac{|\psi^{(n+1)}(u)| + |\psi^{(n+1)}(\nu)|}{2} \right) \right\}.$$

This completes the proof. □

Remark 9 If we take $k = 1$ in Theorem 5, then it reduces to Theorem 5 in [14].

Theorem 6 *Suppose that $\psi : [\theta_1, \theta_2] \to R$ is a differentiable mapping on (θ_1, θ_2) with $0 \leq \theta_1 < \theta_2$ and $\psi \in C^{n+1}[\theta_1, \theta_2]$. If $|\psi^{(n+1)}|^q$ is a convex function on $[\theta_1, \theta_2]$, $q > 1$ and along with the assumptions in A_1, then the following inequality for Caputo k-fractional derivatives holds*:

$$\begin{aligned} &\Bigg| \psi^{(n)}\left(\theta_1 + \theta_2 - \frac{u+\nu}{2}\right) - \frac{2^{n-\frac{\alpha}{k}-1} \Gamma_k(n - \frac{\alpha}{k} + k)}{(\nu - u)^{n-\frac{\alpha}{k}}} \\ &\quad \times \left[\left({}^cD^{\alpha,k}_{(\theta_1+\theta_2-\frac{u+\nu}{2})^+} \psi\right)(\theta_1 + \theta_2 - u) + (-1)^n \left({}^cD^{\alpha,k}_{(\theta_1+\theta_2-\frac{u+\nu}{2})^-} \psi\right)(\theta_1 + 2 - \nu) \right] \Bigg| \\ &\leq \frac{\nu - u}{4} \left(\frac{1}{np - \frac{\alpha}{k}p + 1} \right)^{\frac{1}{p}} \Bigg[\Bigg(\left|\psi^{(n+1)}(\theta_1)\right|^q + \left|\psi^{(n+1)}(\theta_2)\right|^q \\ &\quad - \left(\frac{|\psi^{(n+1)}(u)|^q + 3|\psi^{(n+1)}(\nu)|^q}{4} \right) \Bigg)^{\frac{1}{q}} + \Bigg(\left|\psi^{(n+1)}(\theta_1)\right|^q + \left|\psi^{(n+1)}(\theta_2)\right|^q \\ &\quad - \left(\frac{3|\psi^{(n+1)}(u)|^q + |\psi^{(n+1)}(\nu)|^q}{4} \right) \Bigg)^{\frac{1}{q}} \Bigg]. \end{aligned} \tag{27}$$

Proof By using Lemma 2 and applying the Hölder integral inequality, we have

$$\begin{aligned} &\Bigg| \psi^{(n)}\left(\theta_1 + \theta_2 - \frac{u+\nu}{2}\right) - \frac{2^{n-\frac{\alpha}{k}-1} \Gamma_k(n - \frac{\alpha}{k} + k)}{(\nu - u)^{n-\frac{\alpha}{k}}} \\ &\quad \times \left[\left({}^cD^{\alpha,k}_{(\theta_1+\theta_2-\frac{u+\nu}{2})^+} \psi\right)(\theta_1 + \theta_2 - u) + (-1)^n \left({}^cD^{\alpha,k}_{(\theta_1+\theta_2-\frac{u+\nu}{2})^-} \psi\right)(\theta_1 + \theta_2 - \nu) \right] \Bigg| \\ &\leq \frac{\nu - u}{4} \Bigg[\left(\int_0^1 \lambda^{(n-\frac{\alpha}{k})p} \, d\lambda \right)^{\frac{1}{p}} \left(\int_0^1 \left| \psi^{(n+1)}\left(\theta_1 + \theta_2 - \left(\frac{\lambda}{2} u + \frac{2-\lambda}{2} \nu \right) \right) \right|^q d\lambda \right)^{\frac{1}{q}} \\ &\quad + \left(\int_0^1 \lambda^{(n-\frac{\alpha}{k})p} \, d\lambda \right)^{\frac{1}{p}} \left(\int_0^1 \left| \psi^{(n+1)}\left(\theta_1 + \theta_2 - \left(\frac{2-\lambda}{2} u + \frac{\lambda}{2} \nu \right) \right) \right|^q d\lambda \right)^{\frac{1}{q}} \Bigg]. \end{aligned}$$

By the convexity of $|\psi^{(n+1)}|^q$, we have

$$\begin{aligned}
&\leq \frac{\nu-u}{4}\left(\frac{1}{np-\frac{\alpha}{k}p+1}\right)^{\frac{1}{p}}\left[\left\{\int_0^1\left(\left|\psi^{(n+1)}(\theta_1)\right|^q+\left|\psi^{(n+1)}(\theta_2)\right|^q\right.\right.\right.\\
&\quad\left.\left.-\left(\frac{\lambda}{2}\left|\psi^{(n+1)}(u)\right|^q+\frac{2-\lambda}{2}\left|\psi^{(n+1)}(\nu)\right|^q\right)\right)d\lambda\right\}^{\frac{1}{q}}+\left\{\int_0^1\left(\left|\psi^{(n+1)}(\theta_1)\right|^q\right.\right.\\
&\quad\left.\left.\left.+\left|\psi^{(n+1)}(\theta_2)\right|^q-\left(\frac{2-\lambda}{2}\left|\psi^{(n+1)}(u)\right|^q+\frac{\lambda}{2}\left|\psi^{(n+1)}(\nu)\right|^q\right)\right)d\lambda\right\}^{\frac{1}{q}}\right]\\
&\leq \frac{\nu-u}{4}\left(\frac{1}{np-\frac{\alpha}{k}p+1}\right)^{\frac{1}{p}}\left[\left(\left|\psi^{(n+1)}(\theta_1)\right|^q+\left|\psi^{(n+1)}(\theta_2)\right|^q\right.\right.\\
&\quad\left.-\left(\frac{|\psi^{n+1}(u)|^q+3|\psi^{(n+1)}(\nu)|^q}{4}\right)\right)^{\frac{1}{q}}\\
&\quad\left.+\left(\left|\psi^{(n+1)}(\theta_1)\right|^q+\left|\psi^{(n+1)}(\theta_2)\right|^q-\left(\frac{3|\psi^{(n+1)}(u)|^q+|\psi^{(n+1)}(\nu)|^q}{4}\right)\right)^{\frac{1}{q}}\right].
\end{aligned}$$

This completes the proof. □

Remark 10 If we take $k=1$ in Theorem 6, then it reduces to Theorem 6 in [14].

3 New Hölder and improved İşcan inequalities

Theorem 7 *Suppose that $\psi:[\theta_1,\theta_2]\to R$ is a differentiable mapping on (θ_1,θ_2) with $0\leq\theta_1<\theta_2$ and $\psi\in C^{n+1}[\theta_1,\theta_2]$. If $|\psi^{(n+1)}|^q$ is a convex function on $[\theta_1,\theta_2]$, $q>1$ and along with the assumptions in A_1, then the following inequality for Caputo k-fractional derivatives holds*:

$$\begin{aligned}
&\left|\psi^{(n)}\left(\theta_1+\theta_2-\frac{u+\nu}{2}\right)-\frac{2^{n-\frac{\alpha}{k}-1}\Gamma_k(n-\frac{\alpha}{k}+k)}{(\nu-u)^{n-\frac{\alpha}{k}}}\right.\\
&\quad\left.\times\left[\left({}^cD^{\alpha,k}_{(\theta_1+\theta_2-\frac{u+\nu}{2})^+}\psi\right)(\theta_1+\theta_2-u)+(-1)^n\left({}^cD^{\alpha,k}_{(\theta_1+\theta_2-\frac{u+\nu}{2})^-}\psi\right)(\theta_1+\theta_2-\nu)\right]\right|\\
&\quad\leq\frac{\nu-u}{4}\left[\left\{\left(\frac{1}{((n-\frac{\alpha}{k})p+1)((n-\frac{\alpha}{k})p+2)}\right)^{\frac{1}{p}}\left(\frac{1}{2}\left(\left|\psi^{(n+1)}(\theta_1)\right|^q\right.\right.\right.\right.\\
&\qquad\left.\left.+\left|\psi^{(n+1)}(\theta_2)\right|^q\right)-\left(\frac{1}{12}\left|\psi^{(n+1)}(u)\right|^q+\frac{5}{12}\left|\psi^{(n+1)}(\nu)\right|^q\right)\right)^{\frac{1}{q}}\\
&\qquad+\left(\frac{1}{((n-\frac{\alpha}{k})p+2)}\right)^{\frac{1}{p}}\left(\frac{1}{2}\left(\left|\psi^{(n+1)}(\theta_1)\right|^q+\left|\psi^{(n+1)}(\theta_2)\right|^q\right)\right.\\
&\qquad\left.\left.-\left(\frac{1}{6}\left|\psi^{(n+1)}(u)\right|^q+\frac{1}{3}\left|\psi^{(n+1)}(\nu)\right|^q\right)\right)^{\frac{1}{q}}\right\}\\
&\qquad+\left\{\left(\frac{1}{((n-\frac{\alpha}{k})p+1)((n-\frac{\alpha}{k})p+2)}\right)^{\frac{1}{p}}\right.\\
&\qquad\times\left(\frac{1}{2}\left(\left|\psi^{(n+1)}(\theta_1)\right|^q+\left|\psi^{(n+1)}(\theta_2)\right|^q\right)-\left(\frac{5}{12}\left|\psi^{(n+1)}(u)\right|^q+\frac{1}{12}\left|\psi^{(n+1)}(\nu)\right|^q\right)\right)^{\frac{1}{q}}
\end{aligned}$$

$$+\left(\frac{1}{((n-\frac{\alpha}{k})p+2)}\right)^{\frac{1}{p}}\left(\frac{1}{2}\left(\left|\psi^{(n+1)}(\theta_1)\right|^q+\left|\psi^{n+1}(\theta_2)\right|^q\right)\right.$$
$$\left.-\left(\frac{1}{3}\left|\psi^{(n+1)}(u)\right|^q+\frac{1}{6}\left|\psi^{(n+1)}(v)\right|^q\right)\right)^{\frac{1}{q}}\Bigg\}\Bigg]. \tag{28}$$

Proof By using Lemma 2 with Jensen–Mercer inequality and applying the Hölder–İşcan integral inequality [Theorem 1.4, [15]], we have

$$\begin{aligned}
&\left|\psi^{(n)}\left(\theta_1+\theta_2-\frac{u+v}{2}\right)-\frac{2^{n-\frac{\alpha}{k}-1}\Gamma_k(n-\frac{\alpha}{k}+k)}{(v-u)^{n-\frac{\alpha}{k}}}\right.\\
&\quad\left.\times\left[\left({}^cD^{\alpha,k}_{(\theta_1+\theta_2-\frac{u+v}{2})^+}\psi\right)(\theta_1+\theta_2-u)+(-1)^n\left({}^cD^{\alpha,k}_{(\theta_1+\theta_2-\frac{u+v}{2})^-}\psi\right)(\theta_1+\theta_2-v)\right]\right|\\
&\leq\frac{v-u}{4}\left\{\left(\int_0^1(1-\lambda)\lambda^{np-\frac{\alpha}{k}p}\,d\lambda\right)^{\frac{1}{p}}\right.\\
&\quad\times\left(\int_0^1(1-\lambda)\left|\psi^{(n+1)}\left(\theta_1+\theta_2-\left(\frac{\lambda}{2}u+\frac{2-\lambda}{2}v\right)\right)\right|^q d\lambda\right)^{\frac{1}{q}}\\
&\quad\left.+\left(\int_0^1\lambda^{np-\frac{\alpha}{k}p+1}\,dt\right)^{\frac{1}{p}}\left(\int_0^1\lambda\left|\psi^{(n+1)}\left(\theta_1+\theta_2-\left(\frac{\lambda}{2}u+\frac{2-\lambda}{2}v\right)\right)\right|^q d\lambda\right)^{\frac{1}{q}}\right\}\\
&\quad+\left\{\left(\int_0^1(1-\lambda)\lambda^{np-\frac{\alpha}{k}p}\,d\lambda\right)^{\frac{1}{p}}\left(\int_0^1(1-\lambda)\left|\psi^{(n+1)}\left(\theta_1+\theta_2\right.\right.\right.\right.\\
&\quad\left.\left.\left.-\left(\frac{2-\lambda}{2}u+\frac{\lambda}{2}v\right)\right)\right|^q d\lambda\right)^{\frac{1}{q}}\\
&\quad\left.+\left(\int_0^1\lambda^{np-\frac{\alpha}{k}p+1}\,d\lambda\right)^{\frac{1}{p}}\left(\int_0^1\lambda\left|\psi^{(n+1)}\left(\theta_1+\theta_2-\left(\frac{2-\lambda}{2}u+\frac{\lambda}{2}v\right)\right)\right|^q d\lambda\right)^{\frac{1}{q}}\right\}.
\end{aligned}\tag{29}$$

By the convexity of $|\psi^{(n+1)}|^q$

$$\begin{aligned}
&\left|\psi^{(n+1)}\left(\theta_1+\theta_2-\left(\frac{\lambda}{2}u+\frac{2-\lambda}{2}v\right)\right)\right|^q\\
&\quad\leq\left|\psi^{(n+1)}(\theta_1)\right|^q+\left|\psi^{(n+1)}(\theta_2)\right|^q-\left(\frac{\lambda}{2}\left|\psi^{(n+1)}(u)\right|^q+\frac{2-\lambda}{2}\left|\psi^{(n+1)}(v)\right|^q\right).
\end{aligned}\tag{30}$$

It is easy to see that

$$\int_0^1(1-\lambda)\lambda^{np-\frac{\alpha}{k}p}\,d\lambda=\frac{1}{((n-\frac{\alpha}{k})p+1)((n-\frac{\alpha}{k})p+2)}\tag{31}$$

and

$$\begin{aligned}
&\int_0^1(1-\lambda)\left|\psi^{(n+1)}\left(\theta_1+\theta_2-\left(\frac{\lambda}{2}u+\frac{2-\lambda}{2}v\right)\right)\right|^q d\lambda\\
&\quad=\frac{1}{2}\left(\left|\psi^{(n+1)}(\theta_1)\right|^q+\left|\psi^{(n+1)}(\theta_2)\right|^q\right)\\
&\qquad-\left(\frac{1}{12}\left|\psi^{(n+1)}(u)\right|^q+\frac{5}{12}\left|\psi^{(n+1)}(v)\right|^q\right)
\end{aligned}\tag{32}$$

and

$$\int_0^1 \lambda^{np-\frac{\alpha}{k}p+1}\,d\lambda = \frac{1}{((n-\frac{\alpha}{k})p+2)} \tag{33}$$

and

$$\begin{aligned}&\int_0^1 \lambda\left|\psi^{(n+1)}\left(\theta_1+\theta_2-\left(\frac{\lambda}{2}u+\frac{2-\lambda}{2}v\right)\right)\right|^q d\lambda\\&\quad=\frac{1}{2}\left(\left|\psi^{(n+1)}(\theta_1)\right|^q+\left|\psi^{(n+1)}(\theta_2)\right|^q\right)-\left(\frac{1}{6}\left|\psi^{(n+1)}(u)\right|^q+\frac{1}{3}\left|\psi^{(n+1)}(v)\right|^q\right)\end{aligned} \tag{34}$$

and

$$\begin{aligned}&\int_0^1 (1-\lambda)\left|\psi^{(n+1)}\left(\theta_1+\theta_2-\left(\frac{2-\lambda}{2}u+\frac{\lambda}{2}v\right)\right)\right|^q d\lambda\\&\quad=\frac{1}{2}\left(\left|\psi^{(n+1)}(\theta_1)\right|^q+\left|\psi^{(n+1)}(\theta_2)\right|^q\right)-\left(\frac{5}{12}\left|\psi^{(n+1)}(u)\right|^q+\frac{1}{12}\left|\psi^{(n+1)}(v)\right|^q\right)\end{aligned} \tag{35}$$

and

$$\begin{aligned}&\int_0^1 \lambda\left|\psi^{(n+1)}\left(\theta_1+\theta_2-\left(\frac{2-\lambda}{2}u+\frac{\lambda}{2}v\right)\right)\right|^q d\lambda\\&\quad=\frac{1}{2}\left(\left|\psi^{(n+1)}(\theta_1)\right|^q+\left|\psi^{n+1}(\theta_2)\right|^q\right)-\left(\frac{1}{3}\left|\psi^{(n+1)}(u)\right|^q+\frac{1}{6}\left|\psi^{(n+1)}(v)\right|^q\right)\end{aligned} \tag{36}$$

By combining (31), (32), (33), (34), (35), (36), with (29) we get (28).
This completes the proof. □

Remark 11 If we take $k=1$ in Theorem 7, then it reduces to Theorem 7 in [14].

Theorem 8 *Suppose that $\psi:[\theta_1,\theta_2]\to R$ is a differentiable mapping on (θ_1,θ_2) with $\theta_1<\theta_2$ and $\psi\in C^{n+1}[\theta_1,\theta_2]$. If $|\psi^{(n+1)}|^q$ is a convex function on $[\theta_1,\theta_2]$, $q\geq 1$ and along with the assumptions in A_1, then the following inequality for Caputo k-fractional derivatives holds*:

$$\begin{aligned}&\Bigg|\psi^{(n)}\left(\theta_1+\theta_2-\frac{u+v}{2}\right)-\frac{2^{n-\frac{\alpha}{k}-1}\Gamma_k(n-\frac{\alpha}{k}+k)}{(v-u)^{n-\frac{\alpha}{k}}}\\&\qquad\times\left[\left({}^cD^{\alpha,k}_{(\theta_1+\theta_2-\frac{u+v}{2})^+}\psi\right)(\theta_1+\theta_2-u)+(-1)^n\left({}^cD^{\alpha,k}_{(\theta_1+\theta_2-\frac{u+v}{2})^-}\psi\right)(\theta_1+\theta_2-v)\right]\Bigg|\\&\quad\leq\frac{v-u}{4}\left\{\left\{\left(\frac{1}{(n-\frac{\alpha}{k}+1)(n-\frac{\alpha}{k}+2)}\right)^{1-\frac{1}{q}}\right.\right.\\&\qquad\times\left(\frac{|\psi^{n+1}(\theta_1)|^q+|\psi^{n+1}(\theta_2)|^q}{(n-\frac{\alpha}{k}+1)(n-\frac{\alpha}{k}+2)}-\left(\frac{|\psi^{(n+1)}(u)|^q}{2(n-\frac{\alpha}{k}+2)(n-\frac{\alpha}{k}+3)}\right.\right.\\&\qquad\left.\left.+\frac{(n-\frac{\alpha}{k}+5)|\psi^{(n+1)}(v)|^q}{2(n-\frac{\alpha}{k}+1)(n-\frac{\alpha}{k}+2)(n-\frac{\alpha}{k}+3)}\right)\right)^{\frac{1}{q}}\\&\qquad+\left(\frac{1}{(n-\frac{\alpha}{k}+2)}\right)^{1-\frac{1}{q}}\left(\frac{|\psi^{(n+1)}(\theta_1)|^q+|\psi^{(n+1)}(\theta_2)|^q}{(n-\frac{\alpha}{k}+2)}\right.\end{aligned}$$

$$
\begin{aligned}
&-\left(\frac{|\psi^{(n+1)}(u)|^q}{2(n-\frac{\alpha}{k}+3)}+\frac{|\psi^{(n+1)}(v)|^q}{2(n-\frac{\alpha}{k}+2)(n-\frac{\alpha}{k}+3)}\right)\Bigg)^{\frac{1}{q}}\Bigg\} \\
&+\Bigg\{\left(\frac{1}{(n-\frac{\alpha}{k}+1)(n-\frac{\alpha}{k}+2)}\right)^{1-\frac{1}{q}} \\
&\times\Bigg(\frac{|\psi^{(n+1)}(\theta_1)|^q+|\psi^{(n+1)}(\theta_2)|^q}{(n-\frac{\alpha}{k}+1)(n-\frac{\alpha}{k}+2)}-\Bigg(\frac{(n-\frac{\alpha}{k}+5)|\psi^{n+1}(u)|^q}{2(n-\frac{\alpha}{k}+1)(n-\frac{\alpha}{k}+2)(n-\frac{\alpha}{k}+3)} \\
&+\frac{|\psi^{(n+1)}(v)|^q}{2(n-\frac{\alpha}{k}+2)(n-\frac{\alpha}{k}+3)}\Bigg)\Bigg)^{\frac{1}{q}}+\left(\frac{1}{(n-\frac{\alpha}{k}+2)}\right)^{1-\frac{1}{q}} \\
&\times\Bigg(\frac{|\psi^{(n+1)}(\theta_1)|^q+|\psi^{(n+1)}(\theta_2)|^q}{(n-\frac{\alpha}{k}+2)} \\
&-\left(\frac{(n-\frac{\alpha}{k}+4)|\psi^{(n+1)}(u)|^q}{2(n-\frac{\alpha}{k}+2)(n-\frac{\alpha}{k}+3)}+\frac{|\psi^{(n+1)}(v)|^q}{2(n-\frac{\alpha}{k}+3)}\right)\Bigg)^{\frac{1}{q}}\Bigg\}\Bigg\}.
\end{aligned}
\tag{37}
$$

Proof By using Lemma 2 with the Jensen–Mercer inequality and applying the improved power-mean integral inequality [Theorem1.5, [15]], we have

$$
\begin{aligned}
&\Bigg|\psi^{(n)}\left(\theta_1+\theta_2-\frac{u+v}{2}\right)-\frac{2^{n-\frac{\alpha}{k}-1}\Gamma_k(n-\frac{\alpha}{k}+1)}{(v-u)^{n-\frac{\alpha}{k}}} \\
&\quad\times\left[\left({}^cD^{\alpha,k}_{(\theta_1+\theta_2-\frac{u+v}{2})^+}\psi\right)(\theta_1+\theta_2-u)+(-1)^n\left({}^cD^{\alpha,k}_{(\theta_1+\theta_2-\frac{u+v}{2})^-}\psi\right)(\theta_1+\theta_2-v)\right]\Bigg| \\
&\leq\frac{v-u}{4}\Bigg\{\left(\int_0^1(1-\lambda)\lambda^{n-\frac{\alpha}{k}}\,d\lambda\right)^{1-\frac{1}{q}} \\
&\quad\times\left(\int_0^1(1-\lambda)\lambda^{n-\frac{\alpha}{k}}\left|\psi^{(n+1)}\left(\theta_1+\theta_2-\left(\frac{\lambda}{2}u+\frac{2-\lambda}{2}v\right)\right)\right|^q d\lambda\right)^{\frac{1}{q}} \\
&\quad+\left(\int_0^1\lambda^{n-\frac{\alpha}{k}+1}\,d\lambda\right)^{1-\frac{1}{q}}\Bigg(\int_0^1\lambda^{n-\frac{\alpha}{k}+1}\Bigg|\psi^{(n+1)}\Bigg(\theta_1+\theta_2 \\
&\quad-\left(\frac{\lambda}{2}u+\frac{2-\lambda}{2}v\right)\Bigg)\Bigg|^q d\lambda\Bigg)^{\frac{1}{q}}\Bigg\} \\
&\quad+\Bigg\{\left(\int_0^1(1-\lambda)\lambda^{n-\frac{\alpha}{k}}\,d\lambda\right)^{1-\frac{1}{q}}\Bigg(\int_0^1(1-\lambda)\lambda^{n-\frac{\alpha}{k}}\Bigg|\psi^{(n+1)}\Bigg(\theta_1+\theta_2 \\
&\quad-\left(\frac{\lambda}{2}v+\frac{2-\lambda}{2}u\right)\Bigg)\Bigg|^q d\lambda\Bigg)^{\frac{1}{q}}+\left(\int_0^1\lambda^{n-\frac{\alpha}{k}+1}\,d\lambda\right)^{1-\frac{1}{q}} \\
&\quad\times\left(\int_0^1\lambda^{n-\frac{\alpha}{k}+1}\left|\psi^{(n+1)}\left(\theta_1+\theta_2-\left(\frac{\lambda}{2}v+\frac{2-\lambda}{2}u\right)\right)\right|^q dt\right)^{\frac{1}{q}}\Bigg\}.
\end{aligned}
\tag{38}
$$

It is easy to see that

$$
\int_0^1(1-\lambda)\lambda^{n-\frac{\alpha}{k}}\,d\lambda=\frac{1}{(n-\frac{\alpha}{k}+1)(n-\frac{\alpha}{k}+2)}
\tag{39}
$$

and

$$\int_0^1 (1-\lambda)\lambda^{n-\frac{\alpha}{k}} \left|\psi^{n+1}\left(\theta_1+\theta_2-\left(\frac{\lambda}{2}u+\frac{2-\lambda}{2}v\right)\right)\right|^q d\lambda$$
$$= \frac{(|\psi^{(n+1)}(\theta_1)|^q + |\psi^{(n+1)}(\theta_2)|^q)}{(n-\frac{\alpha}{k}+1)(n-\frac{\alpha}{k}+2)}$$
$$- \left(\frac{|\psi^{(n+1)}(u)|^q}{2(n-\frac{\alpha}{k}+2)(n-\frac{\alpha}{k}+3)} + \frac{(n-\frac{\alpha}{k}+5)|\psi^{(n+1)}(v)|^q}{2(n-\frac{\alpha}{k}+1)(n-\frac{\alpha}{k}+2)(n-\frac{\alpha}{k}+3)}\right) \tag{40}$$

and

$$\int_0^1 \lambda^{n-\frac{\alpha}{k}+1}\, d\lambda = \frac{1}{(n-\frac{\alpha}{k}+2)} \tag{41}$$

and

$$\int_0^1 \lambda^{n-\frac{\alpha}{k}+1} \left|\psi^{(n+1)}\left(\theta_1+\theta_2-\left(\frac{\lambda}{2}u+\frac{2-\lambda}{2}v\right)\right)\right|^q d\lambda$$
$$= \frac{(|\psi^{(n+1)}(\theta_1)|^q + |\psi^{(n+1)}(\theta_2)|^q)}{(n-\frac{\alpha}{k}+2)}$$
$$- \left(\frac{|\psi^{(n+1)}(u)|^q}{2(n-\frac{\alpha}{k}+3)} + \frac{|\psi^{(n+1)}(v)|^q}{2(n-\frac{\alpha}{k}+2)(n-\frac{\alpha}{k}+3)}\right) \tag{42}$$

and

$$\int_0^1 (1-\lambda)\lambda^{n-\frac{\alpha}{k}} \left|\psi^{(n+1)}\left(\theta_1+\theta_2-\left(\frac{2-\lambda}{2}u+\frac{\lambda}{2}v\right)\right)\right|^q d\lambda$$
$$= \frac{(|\psi^{(n+1)}(\theta_1)|^q + |\psi^{(n+1)}(\theta_2)|^q)}{(n-\frac{\alpha}{k}+1)(n-\frac{\alpha}{k}+2)}$$
$$- \left(\frac{(n-\frac{\alpha}{k}+5)|\psi^{(n+1)}(u)|^q}{2(n-\frac{\alpha}{k}+1)(n-\frac{\alpha}{k}+2)(n-\frac{\alpha}{k}+3)} + \frac{|\psi^{(n+1)}(v)|^q}{2(n-\frac{\alpha}{k}+2)(n-\frac{\alpha}{k}+3)}\right) \tag{43}$$

and

$$\int_0^1 \lambda^{n-\frac{\alpha}{k}+1} \left|\psi^{(n+1)}\left(\theta_1+\theta_2-\left(\frac{2-\lambda}{2}u+\frac{\lambda}{2}v\right)\right)\right|^q d\lambda$$
$$= \left(\frac{|\psi^{(n+1)}(\theta_1)|^q + |\psi^{(n+1)}(\theta_2)|^q}{(n-\frac{\alpha}{k}+2)}\right.$$
$$\left. - \left(\frac{(n-\frac{\alpha}{k}+4)|\psi^{(n+1)}(u)|^q}{2(n-\frac{\alpha}{k}+2)(n-\frac{\alpha}{k}+3)} + \frac{|\psi^{(n+1)}(v)|^q}{2(n-\frac{\alpha}{k}+3)}\right)\right). \tag{44}$$

By combining (39), (40), (41), /(42), (43), (44) with (38) we get (37), which completes the proof. □

4 Conclusion

In this article, we show Hermite–Hadamard type inequalities can be expressed in Caputo k-fractional derivative form by employing the Jensen–Mercer inequality. New Hermite–Jensen–Mercer type inequalities using Caputo k-fractional derivatives are established for differentiable mappings whose derivatives in absolute values are convex. Some known results are recaptured as special cases of our results. We hope that our new idea and technique may inspire many researcher in this fascinating field.

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