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A coupled system of generalized Sturm–Liouville problems and Langevin fractional differential equations in the framework of nonlocal and nonsingular derivatives

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Abstract

In this paper, we study a coupled system of generalized Sturm–Liouville problems and Langevin fractional differential equations described by Atangana–Baleanu–Caputo (ABC for short) derivatives whose formulations are based on the notable Mittag-Leffler kernel. Prior to the main results, the equivalence of the coupled system to a nonlinear system of integral equations is proved. Once that has been done, we show in detail the existence–uniqueness and Ulam stability by the aid of fixed point theorems. Further, the continuous dependence of the solutions is extensively discussed. Some examples are given to illustrate the obtained results.

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1 Introduction

The subject of fractional calculus is a generalization of ordinary differentiation and integration to an arbitrary order, which might be noninteger. Very recently it was recognized that fractional calculus arise naturally in various fields of science and engineering. Today we witness an increasing number of proposals for operators, both in the form of derivatives and integrals [1, 2] with the extension of fractional calculus. In consequence, there are several contributions focusing on different definitions of fractional derivatives such as the Riemann–Liouville (RL), Hadamard, Grünwald–Letnikov, Riesz, Caputo, Marchaud, Weyl, and Hilfer derivatives; see [3–12] for some detailed information. All these derivatives are known to contain singular kernels and some generalized fractional derivatives are novel such as conformable fractional derivative [13], beta-derivative [14], or we have a new definition [15, 16]. Generally, various definitions differ from one another in choosing special kernels and some form of differential operator. For example, for the kernel $k(t,s) = t-s$ and the differential operator d/dt, we obtain the Riemann–Liouville definition.

In the recent contribution, Caputo and Fabrizio [17] proposed a new formulation involving a fractional derivative whose kernel is an exponential function. Motivated by [17], Atangana and Baleanu in [18], introduced a new definition of the fractional derivative to answer some outstanding questions that were posed by many researchers within the field of fractional calculus based on fractional operators with Mittag-Leffler, nonsingular smooth kernel. Their derivative has a nonsingular and nonlocal kernel and accepts all properties of fractional derivatives. This new derivative has gained widely attention and attracted a large number of scientists in different scientific fields for the exploration of diverse topics. Afterward, many articles on this subject have been published in order to generalize the results of the fractional derivative without a singular kernel in many directions. To the best of our knowledge, few contributions associated with ABC-fractional derivatives have been published; see [19–22] and the references therein.

In addition, the Sturm–Liouville problem plays an important role in different areas of applied sciences and engineering; for example see [23]. A standard form of the linear Sturm–Liouville differential equation of second order is defined by

$$-\mathrm{D}_t\big[p(\tau)\mathrm{D}_\tau[u]\big] = f\big(t,u(t)\big), \quad t \in (a,b), \mathrm{D}_t \equiv \frac{d}{dt}, \tag{1.1}$$

with appropriate initial conditions, where the functions $p(t)$ and $u(t)$ are continuous on the interval $[a,b]$ such that $p(t) > 0$ and $u(t) > 0$. D is the usual derivative and $f : [a,b] \times \mathbb{R} \to \mathbb{R}^+$ is a continuous function. The fractional Sturm–Liouville problems were developed by some researchers in theory and application; see [24].

On the other hand, in [25] Langevin introduced the classical Langevin equation as follows:

$$\mathrm{D}_t\big[\mathrm{D}_\tau[u] + \lambda u(\tau)\big] = f\big(t,u(t)\big), \quad t \in (a,b), \lambda > 0. \tag{1.2}$$

The classical Langevin equation with various boundary conditions has been studied by many authors; see [26] and the references therein. Various generalizations of the Langevin equation have been offered to describe dynamical processes in a fractal medium. This gives rise to the study of the fractional Langevin equation; see [27]. The fractional Langevin equation was introduced by Mainardi and collaborators in the earlier 1990s. Several types of fractional Langevin equation were studied in [28–33].

Meanwhile, in the same year, research into fractional order systems has become a subject of focus because of many advantages of fractional derivatives. For more papers on fractional order systems, see [34–48] and the references therein.

More recently, the study of fractional Langevin equation in frame of Caputo derivative has comparably been of small scale; see [49, 50] in which the authors discussed Sturm–Liouville and Langevin equations via Hadamard fractional derivatives and systems of fractional Langevin equations of Riemann–Liouville and Caputo types, respctively However, to the best of our knowledge, few of the relevant studies on coupled systems of fractional differential equations have been briefly reviewed for further information on this topic.

To conclude this introductory section, we introduce the coupled system involving ABC differential operators with nonsingular kernel, which are discussed throughout this paper, which take the form

$$\begin{cases} \mathbf{D}_t^{\alpha_i}[p_i(\tau)\mathbf{D}_\tau^{\beta_i}[u_i] + q_i(\tau)u_i(\tau)] = f_i(t, u_1(t), u_2(t)), & t \in (0, T), T > 0, \\ u_i(0) = 0, \qquad p_i(T)\mathbf{D}_T^{\beta_i}[u_i] + q_i(T)u_i(T) = 0, & i = 1, 2, \end{cases} \tag{1.3}$$

where $\mathbf{D}_t^{\circ}$ denotes the ABC-fractional derivative with $(^\circ) \in \{\alpha_i, \beta_i\}$ and $0 < \alpha_i, \beta_i \leq 1$, $J = [0, T]$, $p_i \in C(J, \mathbb{R} \setminus \{0\})$; $q_i \in C(J, \mathbb{R})$ and $f_i \in (J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ are some continuous functions.

Note that system (1.3) is a generalization of Sturm–Liouville and Langevin fractional differential systems. In the special case if $q_i(t) \equiv 0$ then (1.3) is reduced to the Sturm–Liouville fractional differential equations. For the case $p_i(t) \equiv 1$ and $q_i(t) \equiv \lambda_i$ system (1.3) is reduced to the Langevin fractional differential equations. However, the theorems we present include and extend some previous results.

We arrange this paper as follows: In Sect. 2, we introduce some notations, properties, lemmas, definitions of fractional calculus. We present a slight generalization for the Ulam–Hyers theorem which was used in studying the stability. Section 3 contains main results and is divided into 6 subsections. In Sect. 3.1 we first solve the corresponding linear problem and show the equivalence between the nonlinear problem (1.3) and integral equation. In Sect. 3.2, we adopt Banach's contraction mapping principle In Sect. 3.3, we use Krasnoselskii's fixed point theorem to prove the existence and uniqueness of solutions for problem (1.3). Section 3.4 is devoted to the stable solution of the fractional coupled systems (1.3) which is provided by using the classical technique of nonlinear functional analysis investigated by Ulam. In Sect. 3.5, we look at the question as to how the solution u varies when we change the order of the ABC-differential operator or the initial values and the dependence on parameters of nonlinear term f is also established. Illustrative examples are presented in the last subsection. Finally, the paper is concluded in Sect. 4.

2 Preliminaries

In this subsection, we introduce some notations, definitions, properties and lemmas of fractional calculus, we present briefly the so-called operators with nonsingular kernel. and present preliminary results needed in our proofs later.

Definition 2.1 Let $s \in [1, \infty)$ and (a, b) be an open subset of $\mathbb{R}$, the space $\mathbb{H}^s(a, b)$ is defined by

$$\mathbb{H}^s(a, b) = \{f(t) \in L^2(a, b) : \mathbf{D}_t^{\beta}[f] \in L^2(a, b), \text{for all } |\beta| \leq s\}, \quad b > a \geq 0.$$

The left-sided RL-fractional derivative of order $\alpha \in (n-1, n]$, of a continuous function $f : [0, \infty) \longrightarrow \mathbb{R}$ is given by

$$\mathfrak{D}_t^{\alpha}[f] := \mathfrak{D}_t^{\alpha}[f(\tau)] = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t-\tau)^{n-\alpha-1} f(\tau)\, d\tau, \tag{2.1}$$

provided that the right side is pointwise defined on $\mathbb{R}^+$.

The corresponding left-sided RL-integral operator of order $0 < \alpha \leq 1$, of a continuous function $f : [0, \infty) \longrightarrow \mathbb{R}$ is given by

$$\mathfrak{I}_t^{\alpha}[f] := \mathfrak{I}_t^{\alpha}[f(\tau)] = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau, \tag{2.2}$$

provided that the right side is pointwise defined on $\mathbb{R}^+$.

Let us recall the well-known definition of the Caputo fractional derivative [3]. Given $b > a, f \in \mathbb{H}^1(a,b)$ and $0 < \alpha < 1$, the Caputo fractional derivative of f of order α is given by

$$^c\mathfrak{D}_t^\alpha[f] = \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-\tau)^{-\alpha} \mathrm{D}_\tau[f]\,d\tau. \tag{2.3}$$

By changing the kernel $(t-\tau)^{-\alpha}$ by the function

$$E_\alpha\left[-\frac{\alpha}{1-\alpha}(t-\tau)^\alpha\right]$$

and $1/\Gamma(1-\alpha)$ by $\mathbf{B}(\alpha)/(1-\alpha)$, one obtains the new ABC-fractional derivative of order $0 < \alpha < 1$,

$$\mathbf{D}_t^\alpha[f] = \frac{\mathbf{B}(\alpha)}{(1-\alpha)} \int_a^t E_\alpha\left[-\frac{\alpha}{1-\alpha}(t-\tau)^\alpha\right] \mathrm{D}_\tau[f]\,d\tau, \tag{2.4}$$

where $f \in \mathbb{H}^1(0,1)$, $0 < \alpha < 1$ and $\mathbf{B}(\alpha)$ is the known normalized positive function satisfying the properties $\mathbf{B}(0) = 1$, $\mathbf{B}(1) = 1$ and

$$\mathbf{B}(\alpha) = 1 - \alpha + \frac{\alpha}{\Gamma(\alpha)}.$$

According to the ABC derivative, it is clear that, if f is a constant function, then $\mathbf{D}_t^\alpha f(t) = 0$ as in the usual Caputo derivative. The main difference between the usual Caputo derivative and ABC-derivative is that, contrary to the usual Caputo definition, the new kernel has no singularity for $t = \tau$. This ABC-fractional derivative $\mathbf{D}_t^\alpha$ is less affected by the past, compared with the $^c\mathfrak{D}_t^\alpha$ which shows a slow stabilization. The term E_α can be expressed as a single- or two- parameter Mittag-Leffler function defined by power series expansions

$$E_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k + \beta)}, \quad t \in \mathbb{C}, \tag{2.5}$$

where $\alpha > 0$ and $\beta \in \mathbb{C}$. When $\beta = 1$, we shortly write $E_{\alpha,1}(t) = E_\alpha(t)$.

The fractional integral associated to the ABC-fractional derivative with no-singular and non-local kernel is defined by

$$\mathbf{I}_t^\alpha[f] = \frac{(1-\alpha)}{\mathbf{B}(\alpha)} f(t) + \frac{\alpha}{\mathbf{B}(\alpha)} \mathfrak{I}_t^\alpha[f], \quad 0 < \alpha < 1, \tag{2.6}$$

where $\mathfrak{I}_t^\alpha$ is the left RL-fractional integral given in (2.2).

We shall state some properties of the fractional integral and fractional differential operators.

Property 2.2 Let $f(t) \in \mathbb{H}^1(a,b)$.

(i) The RL-fractional integral operators $\mathfrak{I}_\tau^\alpha$ satisfy the semigroup property

$$\mathfrak{I}_t^\alpha\left[\mathfrak{I}_\tau^\beta[f]\right] = \mathfrak{I}_t^{\alpha+\beta}[f], \quad \alpha \geq 0, \beta \geq 0.$$

(ii) The ABC-fractional derivative and ABC-fractional integral of a function f fulfill the semigroup property [51],

$$\mathbf{I}_t^{\alpha}\big[\mathbf{D}_\tau^{\alpha}[f]\big] = f(t) - f(a), \quad 0 < \alpha < 1.$$

(iii) The following statement holds:

$$\mathbf{I}_t^{\alpha}\big[\mathbf{I}_\tau^{\beta}[f]\big] = \frac{1}{\mathbf{B}(\alpha)\mathbf{B}(\beta)} \times \big[(1-\alpha)(1-\beta)f(t) + (1-\beta)\alpha\mathfrak{I}_t^{\alpha}[f] + (1-\alpha)\beta\mathfrak{I}_t^{\beta}[f] + \alpha\beta\mathfrak{I}_t^{\alpha+\beta}[f]\big].$$

Property 2.3 Let $f(t) \in L^1(a,b)$. The following statements hold:

(i) For any $\alpha \geq 0$ and $\beta > 0$,

$$\mathfrak{I}_t^{\alpha}\big[(\tau-a)^{\beta-1}\big] = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(t-a)^{\beta+\alpha-1}.$$

For $j = 1, 2, \ldots, [\beta] + 1$,

$$\mathfrak{D}_t^{\alpha}\big[(\tau-a)^{\beta-j}\big] = 0.$$

(ii) The RL-fractional integral and ABC-fractional integral of a function f fulfill the semigroup property

$$\mathfrak{I}_t^{\beta}\big[\mathbf{I}_\tau^{\alpha}[1]\big] = \frac{((1-\alpha) + \alpha\mathfrak{I}_t^{\alpha}[1])}{\mathbf{B}(\alpha)}\mathfrak{I}_t^{\beta}[1].$$

In this paper, we take $X = \mathcal{C}(J,\mathbb{R})$ to be the Banach space of all continuous functions defined on J and endowed with the usual supremum norm. Obviously, the product space $(X \times X, \|(\cdot,\cdot)\|)$ is also a Banach space with the norm

$$\big\|(u_1,u_2)\big\| = \max\big\{\|u_1\|, \|u_2\|\big\}.$$

Let Υ, Υ_1, $\Upsilon_2 : X \times X \to X \times X$ be three operators such that

$$\Upsilon(u_1,u_2)(t) = \big(\Upsilon_1(u_1,u_2)(t), \Upsilon_2(u_1,u_2)(t)\big), \quad \forall (u_1,u_2) \in X \times X, \tag{2.7}$$

with

$$\big\|\Upsilon(u_1,u_2)\big\| = \max\big\{\big\|\Upsilon_1(u_1,u_2)\big\|, \big\|\Upsilon_2(u_1,u_2)\big\|\big\}. \tag{2.8}$$

For completeness, we state the fixed point theorems and Ulam–Hyers stability theorem that will be employed therein.

Theorem 2.4 ([52]) *Let $\mathcal{B}_r$ be the closed ball of radius $r > 0$, centred at zero, in a Banach space X with $\Upsilon : \mathcal{B}_r \to X$ a contraction and $\Upsilon(\partial\mathcal{B}_r) \subseteq \mathcal{B}_r$. Then Υ has a unique fixed point in $\mathcal{B}_r$.*

Theorem 2.5 ([52]) *Let $\mathcal{M}$ be a closed, convex, non-empty subset of a Banach space $X \times X$. Suppose that $\mathbb{E}$ and $\mathbb{F}$ map $\mathcal{M}$ into X and that*

(i) $\mathbb{E}u + \mathbb{F}v \in \mathcal{M}$ *for all* $u, v \in \mathcal{M}$;

(ii) $\mathbb{E}$ *is compact and continuous*;

(iii) $\mathbb{F}$ *is a contraction mapping.*

Then there exists $w \in \mathcal{M}$ such that $\mathbb{E}w + \mathbb{F}w = w$, where $w = (u, v) \in X \times X$.

Definition 2.6 ([53]) Let X be a Banach space and $\Upsilon_1, \Upsilon_2 : X \times X \to X \times X$ be two operators. Then the operational equations system provided by

$$\begin{cases} v_1(t) = \Upsilon_1(v_1, v_2)(t), \\ v_2(t) = \Upsilon_2(v_1, v_2)(t), \end{cases} \tag{2.9}$$

is called Ulam–Hyers stable if we can find $\sigma_j > 0$, $j = 1, \dots, 4$, such that, for each $\varepsilon_1, \varepsilon_2 > 0$, and each solution-pair $(v_1^*, v_2^*) \in X \times X$ of the inequalities

$$\begin{cases} \|v_1^* - \Upsilon_1(v_1^*, v_2^*)\| \le \varepsilon_1, \\ \|v_2^* - \Upsilon_2(v_1^*, v_2^*)\| \le \varepsilon_2, \end{cases} \tag{2.10}$$

there exists a solution $(u_1^*, u_2^*) \in X \times X$ of system (2.9) such that

$$\begin{cases} \|v_1^* - u_1^*\| \le \sigma_1 \varepsilon_1 + \sigma_2 \varepsilon_2, \\ \|v_2^* - u_2^*\| \le \sigma_3 \varepsilon_1 + \sigma_4 \varepsilon_2. \end{cases} \tag{2.11}$$

Theorem 2.7 ([53]) *Let X be a Banach space, $\Upsilon_1, \Upsilon_2 : X \times X \to X \times X$ be two operators such that*

$$\begin{cases} \|\Upsilon_1(v_1, v_2) - \Upsilon_1(v_1^*, v_2^*)\| \le \sigma_1 \|v_1 - v_1^*\| + \sigma_2 \|v_2 - v_2^*\|, \\ \|\Upsilon_2(v_1, v_2) - \Upsilon_2(v_1^*, v_2^*)\| \le \sigma_3 \|v_1 - v_1^*\| + \sigma_4 \|v_2 - v_2^*\|, \end{cases} \tag{2.12}$$

for all $(v_1, v_2), (v_1^*, v_2^*) \in X \times X$ and if the matrix

$$\mathcal{H}_\sigma = \begin{pmatrix} \sigma_1 & \sigma_2 \\ \sigma_3 & \sigma_4 \end{pmatrix} \tag{2.13}$$

converges to zero. Then the operational equations system (2.12) is Ulam–Hyers stable.

3 Main results

This section contains our main results.

3.1 Fractional coupled system (1.3)

In order to study the nonlinear fractional coupled system (1.3), we first consider the associated linear problem and obtain its solution.

3.1.1 Linear fractional coupled system

In this subsection, we consider now the linear coupled system

$$\begin{cases} \mathbf{D}_t^{\alpha_i}[p_i(\tau)\mathbf{D}_\tau^{\beta_i}[u_i] + q_i(\tau)u_i(\tau)] = x_i(t), & t \in J, \\ u_i(0) = 0, \qquad p_i(T)\mathbf{D}_T^{\beta_i}[u_i] + q_i(T)u_i(T) = 0, & i = 1, 2. \end{cases} \tag{3.1}$$

Lemma 3.1 *Considering the first equation of system (3.1), we assume that $x_i \in C(J, \mathbb{R}) \cap L^1(J)$. Then, problem (3.1) is equivalent to the integral equation*

$$\begin{aligned} u_i(t) = {} & \frac{(1-\beta_i)}{\mathbf{B}(\beta_i)p_i(0)}\left(\mathbf{I}_T^{\alpha_i}[x_i] - \frac{1-\alpha_i}{\mathbf{B}(\alpha_i)}x_i(0)\right) + \mathbf{I}_t^{\beta_i}\left[\frac{1}{p_i}\mathbf{I}_\tau^{\alpha_i}[x_i]\right] \\ & - \mathbf{I}_T^{\alpha_i}[x_i] \times \mathbf{I}_t^{\beta_i}\left[\frac{1}{p_i}\right] - \mathbf{I}_t^{\beta_i}\left[\frac{q_i}{p_i}u_i\right], \quad t \in J, i = 1, 2. \end{aligned} \tag{3.2}$$

Proof Assume $u_i(t)$ satisfies (3.1). By applying the fractional integral operators $\mathbf{I}^{\alpha_i}$ and $\mathbf{I}^{\beta_i}$ successively to (3.1), we obtain

$$\mathbf{D}_t^{\beta_i}[u_i] = \frac{c_1}{p_i(t)} + \frac{1}{p_i(t)}\mathbf{I}_t^{\alpha_i}[x_i] - \frac{q_i(t)}{p_i(t)}u_i(t), \tag{3.3}$$

$$u_i(t) = c_2 + c_1\mathbf{I}_t^{\beta_i}\left[\frac{1}{p_i}\right] + \mathbf{I}_t^{\beta_i}\left[\frac{1}{p_i}\mathbf{I}_\tau^{\alpha_i}[x_i]\right] - \mathbf{I}_t^{\beta_i}\left[\frac{q_i}{p_i}u_i\right], \tag{3.4}$$

for some real constants c_1 and c_2. Using the first boundary condition $u_i(0) = 0$ in (3.4), we have

$$c_2 + \frac{(1-\beta_i)}{\mathbf{B}(\beta_i)p_i(0)}\left[c_1 + \frac{1-\alpha_i}{\mathbf{B}(\beta_i)}x_i(0)\right] = 0. \tag{3.5}$$

Using the second boundary condition in (3.3), we have

$$c_1 = -\mathbf{I}_T^{\alpha_i}[x_i]. \tag{3.6}$$

Substituting the value of c_1 in (3.5), we obtain

$$c_2 = \frac{(1-\beta_i)}{\mathbf{B}(\beta_i)p_i(0)}\left(\mathbf{I}_T^{\alpha_i}[x_i] - \frac{(1-\alpha_i)}{\mathbf{B}(\alpha_i)}x_i(0)\right). \tag{3.7}$$

Substituting the values of c_1 and c_2 from (3.6) and (3.7), respectively, in (3.4), we end up with (3.2).

Conversely, it can be easily shown by direct computation that the integral equation (3.2) satisfies the boundary value problem (3.1). The proof is complete.

By a solution of problem (3.1) we mean a pair of functions $(u_1, u_2) \in X \times X$ satisfying (3.2) for all $t \in J$, $i = 1, 2$. □

Lemma 3.2 *Let $x_i \in \mathcal{C}(J, \mathbb{R}) \cap L^1(J)$, $i = 1, 2$. Then the integral solution for the linear system of fractional differential equations (3.1) is given by the pair of functions $(u_1, u_2) \in X \times X$, with (3.2).*

3.1.2 Nonlinear fractional coupled system

In this subsection, we consider a nonlinear coupled system of the form (1.3).

From problem (3.1) we get the fractional integral system

$$\begin{aligned} u_i(t) = {}& \frac{(1-\beta_i)}{\mathbf{B}(\beta_i)p_i(0)}\left(\mathbf{I}_T^{\alpha_i}[f_i] - \frac{1-\alpha_i}{\mathbf{B}(\alpha_i)}f_i(0,0,0)\right) + \mathbf{I}_t^{\beta_i}\left[\frac{1}{p_i}\mathbf{I}_\tau^{\alpha_i}[f_i]\right] \\ & - \mathbf{I}_T^{\alpha_i}[f_i] \times \mathbf{I}_t^{\beta_i}\left[\frac{1}{p_i}\right] - \mathbf{I}_t^{\beta_i}\left[\frac{q_i}{p_i}u_i\right], \quad t \in J, i = 1,2, \end{aligned} \tag{3.8}$$

which is equivalent to the initial value problem (1.3).

By virtue of Lemma 3.2, we get the following.

Lemma 3.3 *Suppose that* $f_1, f_2 : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *are continuous functions. Then* $(u_1,u_2) \in X \times X$ *is a solution of* (1.3) *if and only if* $(u_1,u_2) \in X \times X$ *is a solution of system* (3.8).

Proof The proof is immediate from Lemma 3.1, so we omit it. □

Since problem (1.3) and Eq. (3.8) are equivalent, it is enough to prove that there exists only one solution to (3.8).

In this paper, a closed ball with radius r centered on the zero function in $X \times X$ is defined by

$$\mathcal{B}_r = \left\{(u_1,u_2) \in X \times X : \left\|(u_1,u_2)\right\| \le r\right\}.$$

We define an operator $\Psi : X \times X \to X \times X$ by

$$(\Psi u)(t) = \Psi(u_1,u_2)(t) = \left(\Psi_1(u_1,u_2)(t), \Psi_2(u_1,u_2)(t)\right), \quad \forall (u_1,u_2) \in X \times X, \tag{3.9}$$

where

$$\begin{aligned} \Psi_i(u_1,u_2)(t) = {}& \frac{(1-\beta_i)}{\mathbf{B}(\beta_i)p_i(0)}\left(\mathbf{I}_T^{\alpha_i}[f_i] - \frac{1-\alpha_i}{\mathbf{B}(\alpha_i)}f_i(0)\right) + \mathbf{I}^{\beta_i}\left[\frac{1}{p_i}\mathbf{I}_\tau^{\alpha_i}[f_i]\right] \\ & - \mathbf{I}_T^{\alpha_i}[f_i] \times \mathbf{I}_t^{\beta_i}\left[\frac{1}{p_i}\right] - \mathbf{I}_t^{\beta_i}\left[\frac{q_i}{p_i}u_i\right], \quad t \in J, i = 1,2. \end{aligned} \tag{3.10}$$

Observe that problem (3.8) has solutions if and only if the operator equation $\Psi u = u$ has fixed points.

We make use of the following notations: for $i = 1,2$

$$p_i^* = \inf\left\{\left|p_i(t)\right| : t \in J\right\}, \qquad q_i^* = \sup\left\{\left|q_i(t)\right| : t \in J\right\}$$

and

$$\gamma_{j,\beta_i}^* = \sup\left\{\left|\gamma_{j,\beta_i}^*(t)\right| : t \in J\right\} = \gamma_{j,\beta_i}^*(T), \qquad \mu_i^* = \sup\left\{\left|\mu_i(t)\right| : t \in J\right\} = \mu_i(T), \tag{3.11}$$

where

$$\gamma_{j,\beta_i}(t) := \mathbf{I}_t^{\beta_i}[1] = \frac{1}{\mathbf{B}(\beta_i)}\left((1-\beta_i) + j\beta_i\mathfrak{I}_t^{\beta_i}[1]\right), \quad i = 1,2, j \in \mathbb{N}, \tag{3.12}$$

and

$$\mu_i(t) := \Im_{1,i}(t) + \Im_{2,i}(t) + \Im_{3,i}(t) = \left(\frac{(1-\beta_i)}{\mathbf{B}(\beta_i)} + 2\gamma_{\beta_i}(t)\right)\gamma_{1,\alpha_i}(t), \tag{3.13}$$

where

$$\begin{aligned}
&\Im_{1,i}(t) := \frac{(1-\beta_i)}{\mathbf{B}(\beta_i)}\mathbf{I}_t^{\alpha_i}[1] = \frac{(1-\beta_i)}{\mathbf{B}(\beta_i)}\gamma_{1,\alpha_i}(t),\\
&\Im_{2,i}(t) := \mathbf{I}_t^{\beta_i}\big[\mathbf{I}_\tau^{\alpha_i}[1]\big] = \gamma_{1,\alpha_i}(t)\gamma^*_{1,\beta_1}(t),\\
&\Im_{3,i}(t) := \mathbf{I}_t^{\alpha_i}[1] \times \mathbf{I}_t^{\beta_i}[1] = \gamma^*_{1,\beta_1}(t)\gamma_{1,\alpha_i}(t), \quad \text{and}\\
&\Im_{4,i}(t) := \mathbf{I}_t^{\beta_i}[1] = \gamma^*_{1,\beta_1}(t).
\end{aligned} \tag{3.14}$$

Throughout the remaining part of this paper, we assume the following conditions hold.

($\mathbf{A}_1$) Assume that $f_i : J \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous functions and there exist constants $M_i > 0$ such that, for all $t \in J$ and $u_i, v_i \in \mathbb{R}$, $i = 1, 2$, we have

$$\big|f_i(t, u_1, u_2) - f_i(t, v_1, v_2)\big| \le M_i\big(|u_1 - v_1| + |u_2 - v_2|\big).$$

($\mathbf{A}_2$) Assume that there exist real constants $N_i > 0$ such that

$$\big|f_i(t, u_1, u_2)\big| \le N_i, \quad i = 1, 2,$$

for all $(t, u_1, u_2) \in J \times \mathbb{R} \times \mathbb{R}$. Also, let

$$a_i = \max_{t\in J}\big|f_i(t, 0, 0)\big| < \infty, \quad i = 1, 2.$$

By our assumption, for $(t, u_1, u_2) \in J \times \mathbb{R} \times \mathbb{R}$, we have

$$\big|f_i(t, u_1, u_2)\big| \le \big|f_i(t, u_1, u_2) - f_i(t, 0, 0)\big| + \big|f_i(t, 0, 0)\big| \le M_i\big(\|u_1\| + \|u_2\|\big) + a_i. \tag{3.15}$$

Let us introduce the notation

$$f_i := (f_i)(t) \equiv f_{i,u} = f_i(t, u) = f_i\big(t, u_1(t), u_2(t)\big), \qquad f_i(t, 0) := f_i(t, 0, 0) \tag{3.16}$$

and

$$\eta_i = p_i(0)\frac{(1-\alpha_i)(1-\beta_i)}{\mathbf{B}(\alpha_i)\mathbf{B}(\beta_i)}f_i(0, 0, 0). \tag{3.17}$$

3.2 Existence and uniqueness of the solution of (3.8)

In this subsection, we apply Banach's fixed point theorem to establish existence and uniqueness of solutions of (3.8).

Theorem 3.4 *Assume* ($\mathbf{A}_1$) *and* $0 < p_i^*(2M_i\mu_i^* + q_i^*\gamma^*_{1,\beta_1}) < 1$, *for* $i = 1, 2$, *hold. If we choose*

$$r \ge \max\left\{\frac{p_1^*\mu_1^*a_1 + \eta_1^*}{1 - p_1^*(2M_1\mu_1^* + q_1^*\gamma^*_{1,\beta_1})}, \frac{p_2^*\mu_2^*a_2 + \eta_2^*}{1 - p_2^*(2M_2\mu_2^* + q_2^*\gamma^*_{1,\beta_2})}\right\}, \tag{3.18}$$

then problem (1.3) *has a unique solution* $u \in \mathcal{B}_r$.

Proof *Step* i. We show that $\Psi(\mathcal{B}_r) \subseteq \mathcal{B}_r$. To see this, for $u = (u_1, u_2) \in \mathcal{B}_r$, $t \in J$, $i = 1, 2$, we have

$$\begin{aligned}\left|\Psi_i(u_1,u_2)(t)\right| &\le \frac{(1-\beta_i)}{\mathbf{B}(\beta_i)p_i(0)}\left|\mathbf{I}_T^{\alpha_i}[f_i] - \frac{1-\alpha_i}{\mathbf{B}(\alpha_i)}f_i(0)\right| + \left|\mathbf{I}_t^{\beta_i}\left[\frac{1}{p_i}\mathbf{I}_\tau^{\alpha_i}[f_i]\right]\right| \\ &\quad + \left|\mathbf{I}_T^{\alpha_i}[f_i] \times \mathbf{I}_t^{\beta_i}\left[\frac{1}{p_i}\right]\right| + \left|\mathbf{I}_t^{\beta_i}\left[\frac{q_i}{p_i}u_i\right]\right|. \end{aligned} \tag{3.19}$$

We used the fact that

$$\begin{aligned}&\left|\mathbf{I}_t^{\alpha_i}[f_i]\right| \\ &\quad = \frac{1}{\mathbf{B}(\alpha_i)}\left|(1-\alpha_i)f_i\big(t,u_1(t),u_2(t)\big) + \alpha_i \Im_t^{\alpha_i}[f_i]\right| \\ &\quad \le \frac{(1-\alpha_i)[|f_i(t,u) - f_i(t,0)| + |f_i(t,0)|] + \alpha_i(\Im_t^{\alpha_i}[|f_i(\tau,u) - f_i(\tau,0)|] + \Im_t^{\alpha_i}[|f_i(\tau,0)|])}{\mathbf{B}(\alpha_i)} \\ &\quad \le \frac{(1-\alpha_i)(M_i(\|u_1\| + \|u_2\|) + a_i) + \alpha_i(M_i(\|u_1\| + \|u_2\|) + a_i) \times \Im_t^{\alpha_i}[1]}{\mathbf{B}(\alpha_i)}.\end{aligned}$$

These imply that

$$\left|\mathbf{I}_T^{\alpha_i}[f_i]\right| \le \gamma_{1,\alpha_i}(T)\big(M_i\big(\|u_1\| + \|u_2\|\big) + a_i\big). \tag{3.20}$$

Thus, we have

$$\left|\frac{(1-\beta_i)}{\mathbf{B}(\beta_i)|p_i(0)|}\mathbf{I}_T^{\alpha_i}[f_i] - \eta_i\right| \le p_i^* \Im_{1,i}(t)\big(M_i\big(\|u_1\| + \|u_2\|\big) + a_i\big) + \eta_i. \tag{3.21}$$

From (3.15) and (3.20), we obtain

$$\mathbf{I}_t^{\beta_i}\left[\frac{1}{p_i}\mathbf{I}_\tau^{\alpha_i}[f_i]\right] \le p_i^* \Im_{2,i}(t)\big(M_i\big(\|u_1\| + \|u_2\|\big) + a_i\big), \tag{3.22}$$

again from (3.15) and (3.20), one has

$$\left|\mathbf{I}_T^{\alpha_i}[f_i] \times \mathbf{I}_t^{\beta_i}\left[\frac{1}{p_i}\right]\right| \le p_i^* \Im_{3,i}(t)\big(M_i\big(\|u_1\| + \|u_2\|\big) + a_i\big). \tag{3.23}$$

In view of (3.20), we have

$$\left|\mathbf{I}_t^{\beta_i}\left[\frac{q_i}{p_i}u_i\right]\right| \le \frac{(1-\beta_i)}{\mathbf{B}(\beta_i)}\left|\frac{q_i(t)}{p_i(t)}u_i(t)\right| + \frac{\beta_i}{\mathbf{B}(\beta_i)}\Im_t^{\beta_i}\left[\frac{q_i}{p_i}u_i\right] \le q_i^* p_i^* \Im_{4,i}(t)\|u_i\|. \tag{3.24}$$

Using the above estimate in inequality (3.19), we obtain

$$\left|\Psi_i(u_1,u_2)(t)\right| \le p_i^*\left|\mu_i(t)\right|\big(M_i\big(\|u_1\| + \|u_2\|\big) + a_i\big) + q_i^* p_i^*\left|\Im_{4,i}(t)\right|\|u_i\| + \eta_i, \tag{3.25}$$

where $\mu_i(t)$ and $\Im_{4,i}(t)$ are given by (3.13) and (3.14), respectively.

Taking the maximum on both sides of the inequality (3.25), the following can be obtained:

$$\begin{aligned}\left\|\Psi_i(u_1,u_2)(t)\right\| &\leq p_i^*\mu_i^*\left(2M_i\|u\|+a_i\right)+q_i^*p_i^*\gamma_{1,\beta_1}^*\|u\|+\eta_i\\ &\leq \left(2M_ip_i^*\mu_i^*+q_i^*p_i^*\gamma_{1,\beta_1}^*\right)\|u\|+p_i^*\mu_i^*a_i+\eta_i\\ &\leq p_i^*\left(2M_i\mu_i^*+q_i^*\gamma_{1,\beta_1}^*\right)r+p_i^*\mu_i^*a_i+\eta_i\leq r.\end{aligned}\tag{3.26}$$

Choose a real constant $r>0$ such that

$$r\geq\max\left\{\frac{p_1^*\mu_1^*a_1+\eta_1^*}{1-p_1^*(2M_1\mu_1^*+q_1^*\gamma_{1,\beta_1}^*)},\frac{p_2^*\mu_2^*a_2+\eta_2^*}{1-p_2^*(2M_2\mu_2^*+q_2^*\gamma_{1,\beta_2}^*)}\right\},\tag{3.27}$$

with

$$0<p_i^*\left(2M_i\mu_i^*+q_i^*\gamma_{1,\beta_1}^*\right)<1,\quad \text{for } i=1,2,\tag{3.28}$$

and taking into account that (3.28), we conclude that (3.27) holds.

Step ii. Next, we show that Ψ is a contraction mapping. To see this, let $u=(u_1,u_2)$, $v=(v_1,v_2)\in\mathcal{B}_r$ and for any $t\in J$, we get

$$\begin{aligned}\left|(\Psi_iv)(t)-(\Psi_iu)(t)\right| &= \frac{(1-\beta_i)}{\mathbf{B}(\beta_i)p_i(0)}\left|\mathbf{I}_T^{\alpha_i}[f_i](v)-\mathbf{I}_T^{\alpha_i}[f_i](u)\right|\\ &\quad+\left|\mathbf{I}_t^{\beta_i}\left[\frac{1}{p_i}\mathbf{I}_\tau^{\alpha_i}[f_i]\right](v)-\mathbf{I}_t^{\beta_i}\left[\frac{1}{p_i}\mathbf{I}_\tau^{\alpha_i}[f_i]\right](u)\right|\\ &\quad+\left|\mathbf{I}_T^{\alpha_i}[f_i](v)-\mathbf{I}_T^{\alpha_i}[f_i](u)\right|\times\left|\mathbf{I}_t^{\beta_i}\left[\frac{1}{p_i}\right]\right|\\ &\quad+\left|\mathbf{I}_t^{\beta_i}\left[\frac{q_i}{p_i}v_i\right]-\mathbf{I}_t^{\beta_i}\left[\frac{q_i}{p_i}u_i\right]\right|,\quad t\in J, i=1,2.\end{aligned}\tag{3.29}$$

In view of (3.20), we have

$$\left|\mathbf{I}_T^{\alpha_i}[f_i](v)-\mathbf{I}_T^{\alpha_i}[f_i](u)\right|\leq\left|\Im_{1,i}(t)\right|M_i\left(\|u_1-v_1\|+\|u_2-v_2\|\right).\tag{3.30}$$

Similarly to the above argument, we can also obtain

$$\begin{aligned}&\left|\mathbf{I}_t^{\beta_i}\left[\frac{1}{p_i}\mathbf{I}_t^{\alpha_i}[f_i]\right](v)-\mathbf{I}_t^{\beta_i}\left[\frac{1}{p_i}\mathbf{I}_t^{\alpha_i}[f_i]\right](u)\right|\\ &\quad\leq\frac{(1-\beta_i)}{\mathbf{B}(\beta_i)}\left|\frac{1}{p_i}\right|\left|\mathbf{I}_t^{\alpha_i}[f_i](v)-\mathbf{I}_t^{\alpha_i}[f_i](u)\right|+\frac{\beta_i}{\mathbf{B}(\beta_i)}\left|\Im_t^{\beta_i}\left(\frac{1}{p_i}\mathbf{I}_t^{\alpha_i}[f_i](v)\right)-\Im_t^{\beta_i}\left(\frac{1}{p_i}\mathbf{I}_t^{\alpha_i}[f_i](u)\right)\right|\\ &\quad\leq\frac{(1-\beta_i)}{\mathbf{B}(\beta_i)}\left\|\frac{1}{p_i}\right\|\left|\mathbf{I}_t^{\alpha_i}[f_i](v)-\mathbf{I}_t^{\alpha_i}[f_i](u)\right|+\frac{\beta_i}{\mathbf{B}(\beta_i)}\Im_t^{\beta_i}\left\|\frac{1}{p_i}\right\|\left|\mathbf{I}_t^{\alpha_i}[f_i](v)-\mathbf{I}_t^{\alpha_i}[f_i](u)\right|,\end{aligned}\tag{3.31}$$

again from (3.31), we have

$$\left|\mathbf{I}_t^{\beta_i}\left[\frac{1}{p_i}\mathbf{I}_t^{\alpha_i}[f_i]\right](v)-\mathbf{I}_t^{\beta_i}\left[\frac{1}{p_i}\mathbf{I}_t^{\alpha_i}[f_i]\right](u)\right|\leq p_i^*\left|\Im_{2,i}(t)\right|M_i\left(\|u_1-v_1\|+\|u_2-v_2\|\right).\tag{3.32}$$

In the same way, we obtain

$$\left|\mathbf{I}_T^{\alpha_i}[f_i](v)-\mathbf{I}_T^{\alpha_i}[f_i](u)\right|\times\left|\mathbf{I}_t^{\beta_i}\left[\frac{1}{p_i}\right]\right|\le p_i^*\left|\Im_{3,i}(t)\right|M_i\left(\|u_1-v_1\|+\|u_2-v_2\|\right),\tag{3.33}$$

$$\left|\mathbf{I}_t^{\beta_i}\left[\frac{q_i}{p_i}v_i\right]-\mathbf{I}_t^{\beta_i}\left[\frac{q_i}{p_i}u_i\right]\right|\le q_i^*p_i^*\left|\Im_{4,i}(t)\right|\|u_i-v_i\|.\tag{3.34}$$

Using (3.29)–(3.34), we obtain

$$\left|(\Psi_i u)(t)-(\Psi_i v)(t)\right|\le p_i^*\left|\mu_i(t)\right|\left(M_i\|u_1-v_1\|+\|u_2-v_2\|\right)+q_i^*p_i^*\left|\Im_{4,i}(t)\right|\|u_i-v_i\|,\tag{3.35}$$

with $\Im_{4,i}(t)$ as in (3.14),

$$\left\|(\Psi_i u)(t)-(\Psi_i v)(t)\right\|\le 2p_i^*\left|\mu_i(t)\right|M_i\|u-v\|+q_i^*p_i^*\left|\gamma_{1,\beta_1}^*(t)\right|\|u-v\|,\tag{3.36}$$

where $\gamma_{1,\beta_1}^*(t)$ and $\mu_i(t)$ are given by (3.12) and (3.13) respectively.

Furthermore, for any $t\in J$, from inequality (3.36), we obtain

$$\left\|\Psi(u_1,u_2)-\Psi(v_1,v_2)\right\|\le L\|u-v\|,\tag{3.37}$$

with

$$0<p_i^*\left(2M_i\mu_i^*+q_i^*\gamma_{1,\beta_1}^*\right)<1,\quad\text{for } i=1,2,\tag{3.38}$$

implying that (3.37) holds, where

$$L=\max\left\{p_1^*\left(2M_1\mu_1^*+q_1^*\gamma_{1,\beta_1}^*\right),p_2^*\left(2M_2\mu_2^*+q_2^*\gamma_{1,\beta_2}^*\right)\right\}.\tag{3.39}$$

Since $L<1$, therefore, the operator Ψ is a contraction. Thus, by Theorem 2.4, problem (1.3) has a unique solution $u\in B_r$. This completes the proof. □

3.3 Existence of solutions of (3.8)

In this subsection, define the following operators: $\mathbb{E},\mathbb{F}:\mathcal{B}_r\to X\times X$ and $\mathbb{T}:\mathcal{B}_r\to X\times X$ by $\mathbb{E}=(\mathbb{E}_1,\mathbb{E}_2)$, $\mathbb{F}=(\mathbb{F}_1,\mathbb{F}_2)$ and $\mathbb{T}=\mathbb{E}+\mathbb{F}$, with

$$(\mathbb{E}u)(t)=\left(\mathbb{E}_1(u_1,u_2),\mathbb{E}_2(u_1,u_2)\right)(t)\quad\text{and}\quad(\mathbb{F}u)(t)=\left(\mathbb{F}_1(u_1,u_2),\mathbb{F}_2(u_1,u_2)\right)(t),\tag{3.40}$$

where the operators $\mathbb{E}_i:X\times X\longrightarrow X$ and $\mathbb{F}_i:X\times X\longrightarrow X$ are defined, respectively, by

$$\begin{aligned}&(\mathbb{E}_1u_1)(t)=\mathbb{E}_1(u_1,u_2)(t)\quad\text{and}\quad(\mathbb{E}_2u_2)(t)=\mathbb{E}_2(u_1,u_2)(t),\\&(\mathbb{F}_1u_1)(t)=\mathbb{F}_1(u_1,u_2)(t)\quad\text{and}\quad(\mathbb{F}_2u_2)(t)=\mathbb{F}_2(u_1,u_2)(t),\\&(\mathbb{E}_iu_i)(t)=\frac{\beta_i}{\mathbf{B}(\beta_i)}\times\Im_t^{\beta_i}\left[\frac{q_i}{p_i}u_i\right],\quad t\in J,i=1,2,\end{aligned}\tag{3.41}$$

and

$$(\mathbb{F}_i u_i)(t) = \frac{(1-\beta_i)}{\mathbf{B}(\beta_i)p_i(0)}\left(\mathbf{I}_T^{\alpha_i}[f_i] - \frac{1-\alpha_i}{\mathbf{B}(\alpha_i)}f_i(0)\right) + \mathbf{I}_t^{\beta_i}\left[\frac{1}{p_i}\mathbf{I}_\tau^{\alpha_i}[f_i]\right]$$
$$- \mathbf{I}_T^{\alpha_i}[f_i] \times \mathbf{I}_t^{\beta_i}\left[\frac{1}{p_i}\right] - \frac{(1-\beta_i)}{\mathbf{B}(\beta_i)}\left(\frac{q_i(t)}{p_i(t)}u_i(t)\right) - \frac{2\beta_i}{\mathbf{B}(\beta_i)} \times \mathfrak{I}_t^{\beta_i}\left[\frac{q_i}{p_i}u_i\right], \tag{3.42}$$

with

$$\|\mathbb{E}u\| = \max\{\|\mathbb{E}_1 u_1\|, \|\mathbb{E}_2 u_2\|\} \quad \text{and} \quad \|\mathbb{F}u\| = \max\{\|\mathbb{F}_1 u_1\|, \|\mathbb{F}_2 u_2\|\}$$

and

$$\mathbb{T}(u_1, u_2)(t) = \mathbb{E}(u_1, u_2)(t) + \mathbb{F}(u_1, u_2)(t). \tag{3.43}$$

The operator $\mathbb{T}$ is well defined as f_1 and f_2 are continuous functions. Then the system of integral equations (3.8) can be written as an operator equation of the form

$$(u_1, u_2)(t) = \mathbb{T}(u_1, u_2)(t) \tag{3.44}$$

and solutions of problem (3.44) mean solutions of the operator equation, that is, fixed points of $\mathbb{T}$.

We apply Krasnoselskii's fixed point theorem to establish the existence of solutions of system (1.3).

Theorem 3.5 *Assume* $(\mathbf{A}_1)$, $(\mathbf{A}_2)$ *and* $0 < q_i^* p_i^* \gamma_{3,\beta_i}^* < 1$, *for* $i = 1, 2$ *hold. If we choose*

$$r \geq \max\left\{\frac{\mu_1^* N_1 + \eta_1}{1 - q_1^* p_1^* \gamma_{3,\beta_1}^*}, \frac{\mu_2^* N_2 + \eta_2}{1 - q_2^* p_2^* \gamma_{3,\beta_2}^*}\right\}, \tag{3.45}$$

then the boundary value problem (1.3) *has at least one solution* $u \in \mathcal{B}_r$.

Proof We will prove the theorem in several steps. Clearly, $\mathcal{B}_r$ is a closed, convex, nonempty subset of $X \times X$.

Step 1: *The first condition of Theorem* 2.5 *holds.*

That is,

$$\mathbb{E}u + \mathbb{F}v \in \mathcal{B}_r, \quad \forall u, v \in \mathcal{B}_r. \tag{3.46}$$

For this purpose, take $u = (u_1, u_2)$ and $v = (v_1, v_2)$ in $\mathcal{B}_r$, $t \in J$, and consider

$$\left|(\mathbb{E}_i u_i)(t)\right| \leq \frac{q_i^* p_i^* \beta_i \times \mathfrak{I}_T^{\beta_i}[1]}{\mathbf{B}(\beta_i)}\|u_i\|, \quad i = 1, 2. \tag{3.47}$$

Now taking the maximum on both sides of the inequality (3.47), we obtain

$$\|\mathbb{E}_i u_i\| \leq \frac{q_i^* p_i^* \beta_i \times \mathfrak{I}_T^{\beta_i}[1]}{\mathbf{B}(\beta_i)} r, \quad i = 1, 2. \tag{3.48}$$

Analogously, we obtain

$$\frac{(1-\beta_i)(\mathbf{I}_T^{\alpha_i}[f_i] - \frac{1-\alpha_i}{\mathbf{B}(\alpha_i)}f_i(0))}{\mathbf{B}(\beta_i)p_i(0)} \le p_i^* \left|\Im_{1,i}(t)\right| N_i + \eta_i,$$

$$\mathbf{I}_t^{\beta_i}\left[\frac{1}{p_i}\mathbf{I}_\tau^{\alpha_i}[f_i]\right] - \mathbf{I}_T^{\alpha_i}[f_i] \times \mathbf{I}_t^{\beta_i}\left[\frac{1}{p_i}\right] \le p_i^* \left|\Im_{2,i}(t) + \Im_{3,i}(t)\right| N_i, \tag{3.49}$$

$$\left|\frac{(1-\beta_i)}{\mathbf{B}(\beta_i)}\left(\frac{q_i(t)}{p_i(t)}u_i(t)\right) + \frac{2\beta_i}{\mathbf{B}(\beta_i)}\Im_t^{\beta_i}\left[\frac{q_i}{p_i}u_i\right]\right| \le \frac{q_i^* p_i^*((1-\beta_i) + 2\beta_i \Im_t^{\beta_i}[1])}{\mathbf{B}(\beta_i)}\|u_i\|. \tag{3.50}$$

Therefore, from (3.42), (3.49) and (3.50), we get

$$\left|(\mathbb{F}_i v_i)(t)\right| \le p_i^* \left|\mu_i(t)\right| N_i + q_i^* p_i^* \left|\gamma_{2,\beta_i}(t)\right| \|v_i\| + \eta_i, \quad i = 1, 2. \tag{3.51}$$

Similarly, taking the maximum on both sides of the inequality (3.51), the following can be obtained:

$$\|\mathbb{F}_i v_i\| \le p_i^* \mu_i^* N_i + \eta_i + q_i^* p_i^* \gamma_{2,\beta_i}^* r, \quad i = 1, 2. \tag{3.52}$$

where

$$\gamma_{2,\beta_i}(t) = \frac{1}{\mathbf{B}(\beta_i)}\big((1-\beta_i) + 2\beta_i \Im_t^{\beta_i}[1]\big), \quad i = 1, 2, \tag{3.53}$$

Consequently,

$$\|\mathbb{F}_i u_i + \mathbb{E}_i v_i\| \le p_i^* \mu_i^* N_i + \eta_i + q_i^* p_i^* \gamma_{3,\beta_i}^* r, \quad i = 1, 2. \tag{3.54}$$

Hence, using (3.48) and (3.53), we can conclude that

$$\|\mathbb{E}u + \mathbb{F}v\| \le \|\mathbb{E}u\| + \|\mathbb{F}v\| \le p_i^* \big(\mu_i^* N_i + q_i^* \gamma_{3,\beta_i}^* r\big) + \eta_i \le r, \tag{3.55}$$

where

$$\gamma_{j,\beta_i}^* = \frac{((1-\beta_i) + j\beta_i \Im_T^{\beta_i}[1])}{\mathbf{B}(\beta_i)}, \quad i = 1, 2, j \in \mathbb{N}. \tag{3.56}$$

Choose a real constant $r > 0$ such that

$$r \ge \max\left\{\frac{p_1^* \mu_1^* N_1 + \eta_1}{1 - q_1^* p_1^* \gamma_{3,\beta_1}^*}, \frac{p_2^* \mu_2^* N_2 + \eta_2}{1 - q_2^* p_2^* \gamma_{3,\beta_2}^*}\right\}, \tag{3.57}$$

with

$$0 < q_i^* p_i^* \gamma_{3,\beta_i}^* < 1, \quad \text{for } i = 1, 2. \tag{3.58}$$

Thus, $\|\mathbb{E}u + \mathbb{F}v\| \le r$, this implying that (3.45) holds.

Step 2: $\mathbb{F}$ *is a contraction mapping.*

To see this, let $u = (u_1, u_2)$ and $v = (v_1, v_2) \in \mathcal{B}_r$. Following the proof of Theorem 2.4, we have

$$\begin{aligned} \big|(\mathbb{F}_i u)(t) - (\mathbb{F}_i v)(t)\big| &\leq p_i^* \big|\Im_{1,i}(T) + \Im_{2,i}(T) + \Im_{3,i}(T) \\ &\quad \times \big|M_i\big(\|u_1 - v_1\| + \|u_2 - v_2\|\big) + q_i^* p_i^* \big|\gamma^*_{2,\beta_i}(T)\big| \|u_i - v_i\|. \end{aligned} \tag{3.59}$$

Taking the maximum on both sides of the inequality (3.59), we obtain

$$\big|(\mathbb{F}_i u)(t) - (\mathbb{F}_i v)(t)\big| \leq 2p_i^* \mu_i^* M_i \|u - v\| + q_i^* p_i^* \gamma^*_{2,\beta_i} \|u - v\| \leq L_i \|u - v\|, \tag{3.60}$$

where $\mu_i(t)$ is defined in (3.13). So, from (3.60), we get

$$\|\mathbb{F}u - \mathbb{F}v\| \leq L\|u - v\|, \tag{3.61}$$

where $L = \max\{L_1, L_2\}$, with

$$L_1 = p_1^*\big(2\mu_1^* M_1 + q_1^* \gamma^*_{2,\beta_1}\big) \quad \text{and} \quad L_2 = p_2^*\big(2\mu_2^* M_2 + q_2^* \gamma^*_{3,\beta_2}\big). \tag{3.62}$$

When $L < 1$, the operator $\mathbb{F}$ is a contraction.

Step 3: $\mathbb{E}$ *is continuous in* $X \times X$.

Let $\{(u_{1,n}, u_{2,n})\}$ be a sequence of a bounded set

$$U_r = \big\{(u_1, u_2) \in X \times X : \big\|(u_1, u_2)\big\| \leq r\big\}$$

such that $(u_{1,n}, u_{2,n}) \longrightarrow (u_1, u_2)$ as $n \longrightarrow \infty$ in U_r,

$$\begin{aligned} \big|\mathbb{E}_i u_{i,n}(t) - \mathbb{E}_i u_i(t)\big| &= \frac{\beta_i}{\mathbf{B}(\beta_i)} \times \Im_t^{\beta_i}\left[\frac{q_i}{p_i} u_{i,n}\right] - \frac{\beta_i}{\mathbf{B}(\beta_i)} \times \Im_t^{\beta_i}\left[\frac{q_i}{p_i} u_i\right] \\ &\leq \frac{q_i^* p_i^* \beta_i}{\mathbf{B}(\beta_i)} \times \Im_t^{\beta_i}\big[|u_{i,n} - u_i|\big], \quad \text{for } i = 1, 2. \end{aligned} \tag{3.63}$$

Now taking the maximum on both sides of the inequality (3.63), we obtain

$$\|\mathbb{E}_i u_{i,n} - \mathbb{E}_i u_i\| \leq \frac{q_i^* p_i^* \beta_i \times |\Im_{4,i}(t)|}{\mathbf{B}(\beta_i)} \|u_{i,n} - u_i\|, \tag{3.64}$$

which implies that

$$\mathbb{E}(u_{1,n}, u_{2,n}) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.$$

Clearly, $\mathbb{E}$ is continuous in view of the continuity of u_1 and u_2.

Step 4: $\mathbb{E}$ *is equicontinuous*.

For this purpose, take $(u_1, u_2) \in \mathcal{B}_r$, $t_1, t_2 \in J$ such that $t_1 < t_2$. Then we have

$$\begin{aligned} &\big|(\mathbb{E}_i u)(t_2) - (\mathbb{E}_i u)(t_1)\big| \\ &\quad \leq \frac{\beta_i}{\mathbf{B}(\beta_i)} \left(\mathbf{I}^{\beta_i}_{0,t_1}\left[\frac{q_i}{p_i} u_i\right] + \mathbf{I}^{\beta_i}_{t_1,t_2}\left[\frac{q_i}{p_i} u_i\right] - \mathbf{I}^{\beta_i}_{0,t_1}\left[\frac{q_i}{p_i} u_i\right]\right) \end{aligned}$$

$$\begin{aligned}
&\le \frac{\beta_i}{\mathbf{B}(\beta_i)}\frac{1}{\Gamma(\beta_i)}\int_0^{t_1}\left[(t_2-\tau)^{\beta_i-1}-(t_1-\tau)^{\beta_i-1}\right]\left|\frac{q_i(\tau)}{p_i(\tau)}u_i(\tau)\right|d\tau \\
&\quad + \frac{\beta_i}{\mathbf{B}(\beta_i)}\frac{1}{\Gamma(\beta_i)}\int_{t_1}^{t_2}(t_2-\tau)^{\beta_i-1}\left|\frac{q_i(\tau)}{p_i(\tau)}u_i(\tau)\right|d\tau \\
&\le \frac{\beta_i q_i^* p_i^*}{\mathbf{B}(\beta_i)}\frac{1}{\Gamma(\beta_i)}\left[\int_0^{t_1}\left[(t_2-\tau)^{\beta_i-1}-(t_1-\tau)^{\beta_i-1}\right]d\tau+\int_{t_1}^{t_2}(t_2-\tau)^{\beta_i-1}\,d\tau\right] \\
&\le \frac{\beta_i q_i^* p_i^*}{\mathbf{B}(\beta_i)}\frac{1}{\Gamma(\beta_i+1)}\left({t_2}^{\beta_i}-{t_1}^{\beta_i}+2(t_2-t_1)^{\beta_i}\right), \qquad (3.65)
\end{aligned}$$

$$\left|\mathbb{E}_i(u_1,u_2)(t_2)-\mathbb{E}_i(u_1,u_2)(t_1)\right| \le \frac{2\beta_i q_i^* p_i^*}{\mathbf{B}(\beta_i)}\frac{1}{\Gamma(\beta_i+1)}(t_2-t_1)^{\beta_i}<\epsilon, \qquad (3.66)$$

provided

$$|t_2-t_1|<\delta^{\beta_i}=\left(\frac{2\beta_i q_i^* p_i^*}{\mathbf{B}(\beta_i)}\frac{1}{\Gamma(\beta_i+1)}\right)^{-1}\times\epsilon,$$

proving the claim. Observe that $|\mathbb{E}_i(u_1,u_2)(t_2)-\mathbb{E}_i(u_1,u_2)(t_1)|\to 0$ as $t_1\to t_2$, implying that $\mathbb{E}(u_1,u_2)$ is equicontinuous and thus the operator $\mathbb{E}(u_1,u_2)$ is completely continuous.

Step 5: $\mathbb{E}$ *is uniformly bounded.*

It follows from (3.51) that $\mathbb{E}$ is uniformly bounded. Therefore, by the Arzelà–Ascoli theorem, we conclude that $\mathbb{E}$ is a compact operator. Thus, all the conditions of Theorem 2.5 are fulfilled. Hence, system (1.3) has at least one solution $u\in\mathcal{B}_r$. The proof is complete. □

3.4 Ulam-type stability of solutions of (3.8)

In this subsection, we use Urs's [53] approach to establishing the Ulam–Hyers stability of solutions of (1.3). Thanks to Definition 2.6 and Theorem 2.7, the respective results are obtained.

Theorem 3.6 *Assume* $(\mathbf{A}_1)$ *and* $0<p_i^*(2M_i\mu_i^*+q_i^*\gamma_{1,\beta_i}^*)<1$ *for* $i=1,2$, *hold. Choose*

$$r\ge\max\left\{\frac{p_1^*\mu_1^*a_1+\eta_1}{1-p_1^*(2M_1\mu_1^*+q_1^*\gamma_{1,\beta_1}^*)},\frac{p_2^*\mu_2^*a_2+\eta_2}{1-p_2^*(2M_2\mu_2^*+q_2^*\gamma_{1,\beta_2}^*)}\right\}. \qquad (3.67)$$

Further, assume the spectral radius of matrix $\tilde{\mathcal{H}}_\sigma$ *is less than one. Then the solutions of* (1.3) *are Ulam–Hyers stable.*

Proof In view of Theorem 2.4, we have

$$\begin{cases}\|\Psi_1(u_1,u_2)-\Psi_1(v_1,v_2)\|\le\tilde{\sigma}_1\|u_1-v_1\|+\tilde{\sigma}_2\|u_2-v_2\|,\\ \|\Psi_2(u_1,u_2)-\Psi_2(v_1,v_2)\|\le\tilde{\sigma}_3\|u_1-v_1\|+\tilde{\sigma}_4\|u_2-v_2\|,\end{cases} \qquad (3.68)$$

which implies that

$$\left\|\Psi(u_1,u_2)-\Psi(v_1,v_2)\right\|\le\tilde{\mathcal{H}}_\sigma\begin{pmatrix}\|u_1-v_1\|\\ \|u_2-v_2\|\end{pmatrix}, \qquad (3.69)$$

where

$$\tilde{\mathcal{H}}_\sigma = \begin{pmatrix} \tilde{\sigma}_1 & \tilde{\sigma}_2 \\ \tilde{\sigma}_3 & \tilde{\sigma}_4 \end{pmatrix} \equiv \begin{pmatrix} p_1^*(\mu_1^* M_1 + q_1^* \gamma_{1,\beta_1}^*) & p_1^* \mu_1^* M_1 \\ p_2^* \mu_2^* M_2 & p_2^*(\mu_2^* M_2 + q_2^* \gamma_{1,\beta_2}^*) \end{pmatrix}. \tag{3.70}$$

Since the spectral radius of $\tilde{\mathcal{H}}_\sigma$ is less than one, the solution of (1.3) is Ulam–Hyers stable. □

3.5 Dependence of solution on the parameters

For f_i Lipschitz in the second variables, the solution's dependence on the order of the differential operator, the boundary values, and the nonlinear term f_i are also discussed.

In the following, for any $u_i \in X$, we let

$$f_i^\epsilon := (f_i^\epsilon)(t) = f_i(t, u_1^\epsilon(t), u_2^\epsilon(t)), \quad t \in (0,T). \tag{3.71}$$

3.5.1 The dependence on parameters of the left-hand side of (3.8)

In this subsection, we show that the solutions of two equations with neighboring orders will (under suitable conditions on their right-hand sides f_i) lie close to one another.

Theorem 3.7 *Suppose that the conditions of Theorem 2.5 hold. Let $u(t) = (u_1(t), u_2(t))$ and $u^\epsilon(t) = (u_1^\epsilon(t), u_2^\epsilon(t))$ be the solutions, respectively, of problems (1.3) and*

$$\mathbf{D}^{\alpha_i - \epsilon}(p_i(t)\mathbf{D}^{\beta_i} + q_i(t))u_i(t) = f_i(t, u_1(t), u_2(t)), \quad t \in (0,T), i = 1,2, \tag{3.72}$$

with the boundary conditions (1.3), where $0 < \alpha_i - \epsilon < \alpha_i \le 1$. Then $\|u^\epsilon - u\| = \mathcal{O}(\epsilon)$, for ϵ sufficiently small.

Proof By the above theorems, we can obtain the following results. Let

$$\begin{aligned} u_i^\epsilon(t) = {} & \frac{(1-\beta_i)}{\mathbf{B}(\beta_i)p_i(0)} \mathbf{I}_T^{\alpha_i - \epsilon}[f_i^\epsilon] - \eta_i + \mathbf{I}_t^{\beta_i}\left[\frac{1}{p_i}\mathbf{I}_\tau^{\alpha_i-\epsilon}[f_i^\epsilon]\right] \\ & - \mathbf{I}_T^{\alpha_i-\epsilon}[f_i^\epsilon] \times \mathbf{I}_t^{\beta_i}\left[\frac{1}{p_i}\right] - \mathbf{I}_t^{\beta_i}\left[\frac{q_i}{p_i}u_i^\epsilon\right], \quad t \in J, i = 1,2. \end{aligned} \tag{3.73}$$

On the one hand, from (3.8) and (3.73)

$$\begin{aligned} \left|u_i^\epsilon(t) - u_i(t)\right| = {} & \frac{(1-\beta_i)}{\mathbf{B}(\beta_i)p_i(0)}\left|\mathbf{I}_T^{\alpha_i-\epsilon}[f_i^\epsilon] - \mathbf{I}_T^{\alpha_i}[f_i]\right| \\ & + \left|\mathbf{I}_t^{\beta_i}\left[\frac{1}{p_i}\mathbf{I}_\tau^{\alpha_i-\epsilon}[f_i^\epsilon]\right] - \mathbf{I}_t^{\beta_i}\left[\frac{1}{p_i}\mathbf{I}_\tau^{\alpha_i}[f_i]\right]\right| \\ & + \left|\mathbf{I}_T^{\alpha_i-\epsilon}[f_i^\epsilon] \times \mathbf{I}_t^{\beta_i}\left[\frac{1}{p_i}\right] - \mathbf{I}_T^{\alpha_i}[f_i] \times \mathbf{I}_t^{\beta_i}\left[\frac{1}{p_i}\right]\right| \\ & + \left|\mathbf{I}_t^{\beta_i}\left[\frac{q_i}{p_i}u_i^\epsilon\right] - \mathbf{I}_t^{\beta_i}\left[\frac{q_i}{p_i}u_i\right]\right|. \end{aligned} \tag{3.74}$$

From (3.74)

$$\begin{aligned}\left|\mathbf{I}_T^{\alpha_i-\epsilon}\left[f_i^\epsilon\right]-\mathbf{I}_T^{\alpha_i}[f_i]\right| &= \left|\mathbf{I}_T^{\alpha_i-\epsilon}\left[f_i^\epsilon\right]-\mathbf{I}_T^{\alpha_i-\epsilon}[f_i]\right|+\left|\mathbf{I}_T^{\alpha_i-\epsilon}[f_i]-\mathbf{I}_T^{\alpha_i}[f_i]\right| \\ &= \mathbf{I}_T^{\alpha_i-\epsilon}\left[\left|f_i^\epsilon-f_i\right|\right]+\left|\mathbf{I}_T^{\alpha_i-\epsilon}[1]-\mathbf{I}_T^{\alpha_i}[1]\right||f_i|. \end{aligned} \tag{3.75}$$

Repeating arguments similar to that above we can arrive at

$$\begin{aligned}&\left|\mathbf{I}_t^{\beta_i}\left[\frac{1}{p_i}\mathbf{I}_\tau^{\alpha_i-\epsilon}f_i^\epsilon\right]-\mathbf{I}_t^{\beta_i}\left[\frac{1}{p_i}\mathbf{I}_\tau^{\alpha_i}f_i\right]\right| \\ &\quad= \mathbf{I}_t^{\beta_i}\left[\frac{1}{p_i}\mathbf{I}_\tau^{\alpha_i-\epsilon}\right]\left[\left|f_i^\epsilon-f_i\right|\right]+\mathbf{I}_t^{\beta_i}\left[\frac{1}{p_i}\left|\mathbf{I}_\tau^{\alpha_i-\epsilon}[1]-\mathbf{I}_t^{\alpha_i}[1]\right|\right]|f_i|, \end{aligned} \tag{3.76}$$

$$\begin{aligned}&\left|\mathbf{I}_t^{\alpha_i-\epsilon}\left[f_i^\epsilon\right]\times\mathbf{I}_t^{\beta_i}\left[\frac{1}{p_i}\right]-\mathbf{I}_t^{\alpha_i}[f_i]\times\mathbf{I}_t^{\beta_i}\left[\frac{1}{p_i}\right]\right| \\ &\quad= \mathbf{I}_t^{\beta_i}\left[\frac{1}{p_i}\right]\times\mathbf{I}_t^{\alpha_i-\epsilon}\left[\left|f_i^\epsilon-f_i\right|\right]+\left|\mathbf{I}_t^{\alpha_i-\epsilon}[1]-\mathbf{I}_t^{\alpha_i}[1]\right||f_i|, \end{aligned} \tag{3.77}$$

$$\left|\mathbf{I}_t^{\beta_i}\left[\frac{q_i}{p_i}u_i^\epsilon\right]-\mathbf{I}_t^{\beta_i}\left[\frac{q_i}{p_i}u_i\right]\right| = \mathbf{I}_t^{\beta_i}\left[\frac{q_i}{p_i}\right]\left|u_i^\epsilon-u_i\right|. \tag{3.78}$$

From (3.74)–(3.78), we can get

$$\begin{aligned}\left|u_i^\epsilon(t)-u_i(t)\right| &\le p_i^* m_{1,i}(t)\left|f_i^\epsilon-f_i\right|+p_i^* n_{1,i}(t)|f_i| \\ &\quad+q_i^* p_i^* l_{1,i}(t)\left|u_i^\epsilon(t)-u_i(t)\right|, \quad i=1,2, \end{aligned} \tag{3.79}$$

where

$$m_{1,i}(t) = \left(\frac{(1-\beta_i)}{\mathbf{B}(\beta_i)}\left(\mathbf{I}_T^{\alpha_i-\epsilon}[1]\right)+\mathbf{I}_t^{\beta_i}\left[\mathbf{I}_\tau^{\alpha_i-\epsilon}\right][1]+\mathbf{I}_t^{\beta_i}[1]\times\mathbf{I}_t^{\alpha_i-\epsilon}[1]\right), \tag{3.80}$$

$$n_{1,i}(t) = \left(\frac{(1-\beta_i)}{\mathbf{B}(\beta_i)}\left(\left|\mathbf{I}_T^{\alpha_i-\epsilon}[1]-\mathbf{I}_T^{\alpha_i}[1]\right|\right)+\mathbf{I}_t^{\beta_i}\left[\left|\mathbf{I}_\tau^{\alpha_i-\epsilon}[1]-\mathbf{I}_t^{\alpha_i}[1]\right|\right]+\left|\mathbf{I}_t^{\alpha_i-\epsilon}[1]-\mathbf{I}_t^{\alpha_i}[1]\right|\right), \tag{3.81}$$

and

$$l_{1,i}(t) = \mathbf{I}_t^{\beta_i}[1]. \tag{3.82}$$

From (3.79) and $(\mathbf{A}_1)$ we have

$$\begin{aligned}&\left|u_i^\epsilon(t)-u_i(t)\right| \le \frac{p_i^* n_{1,i}(t)|f_i|}{1-p_i^*(2m_{1,i}(t)M_i+q_i^* l_{1,i}(t))}, \\ &\quad\text{with } 0<p_i^*\left(2m_{1,i}(t)M_i+q_i^* l_{1,i}(t)\right)<1, \end{aligned} \tag{3.83}$$

as a result, we obtain the following:

$$\left\|u_i^\epsilon-u_i\right\| \le \frac{p_i^* n_i^* \||f_i\||}{1-\mathcal{L}_i}, \quad \text{with } 0<\mathcal{L}_i=p_i^*\left[2m_i^* M_i+q_i^* l_i^*\right]<1, i=1,2, \tag{3.84}$$

where

$$\begin{aligned}
&|||f_i||| = \sup\{\max|f_i(t,u_1(t),u_2(t))| : t \in (0,T)\},\\
&m_i^* = \sup\{|m_{1,i}(t)| : t \in J\}, \qquad n_i^* = \sup\{|n_{1,i}(t)| : t \in J\},\\
&l_i^* = \sup\{|l_{1,i}(t)| : t \in J\}, \quad i = 1,2.
\end{aligned}$$

Consequently, from (3.84), we obtain

$$\|u^\epsilon - u\| \le \frac{p^* n^* |||f^*|||}{1-\mathcal{L}} \quad \text{with } \mathcal{L} = \max\{\mathcal{L}_1, \mathcal{L}_2\}, \tag{3.85}$$

where

$$p^* = \max\{p_1^*, p_2^*\}, \qquad f^* = \max\{f_1, f_2\},$$

and

$$m^* = \max\{m_1^*, m_2^*\}, \qquad n^* = \max\{n_1^*, n_2^*\}, \qquad l^* = \max\{l_1^*, l_2^*\},$$

Thus, in accordance with (3.85), we obtain $\|u^\epsilon - u\| = O(\epsilon)$. □

Theorem 3.8 *Suppose that the conditions of Theorem 2.5 hold. Let $u(t)$, $u^\epsilon(t)$ be the solutions, respectively, of problems (1.3) and*

$$\mathbf{D}^{\alpha_i-\epsilon}(p_i(t)\mathbf{D}^{\beta_i-\epsilon} + q_i(t))u_i(t) = f_i(t,u_1(t),u_2(t)), \quad t \in J, i = 1,2, \tag{3.86}$$

with the boundary conditions

$$u_i(0) = 0, \qquad p_i(T)\mathbf{D}^{\beta_i-\epsilon}u_i(T) + q_i(T)u_i(T) = 0, \tag{3.87}$$

where $0 < \alpha_i - \epsilon < \alpha_i \le 1$ and $0 < \beta_i - \epsilon < \beta_i \le 1$. Then $\|u^\epsilon - u\| = \mathcal{O}(\epsilon)$, for ϵ sufficiently small.

Proof Let $u(t)$ and $u^\epsilon(t)$ be the solutions of (1.3) and (3.86)–(3.87), respectively. Hence, by the above theorems, we can obtain the following results. Let

$$\begin{aligned}
u_i^\epsilon(t) = {} & \frac{(1-\beta_i-\epsilon)}{B(\beta_i-\epsilon)p_i(0)}\left(\mathbf{I}_T^{\alpha_i-\epsilon}[f_i^\epsilon] - \eta_i\right) + \mathbf{I}_t^{\beta_i-\epsilon}\left[\frac{1}{p_i}\mathbf{I}_\tau^{\alpha_i-\epsilon}[f_i^\epsilon]\right]\\
& - \mathbf{I}_T^{\alpha_i-\epsilon}[f_i^\epsilon] \times \mathbf{I}_t^{\beta_i-\epsilon}\left[\frac{1}{p_i}\right] - \mathbf{I}_t^{\beta_i-\epsilon}\left[\frac{q_i}{p_i}u_i^\epsilon\right], \quad t \in J, i = 1,2.
\end{aligned} \tag{3.88}$$

be the solution of (3.86)–(3.87).

On the one hand, from (3.8) and (3.88)

$$\begin{aligned}
|u_i^\epsilon(t) - u_i(t)| = {} & \left|\frac{(1-\beta_i-\epsilon)}{B(\beta_i-\epsilon)p_i(0)}\mathbf{I}_T^{\alpha_i-\epsilon}[f_i^\epsilon] - \frac{(1-\beta_i)}{B(\beta_i)p_i(0)}\mathbf{I}_T^{\alpha_i}[f_i]\right|\\
& + \left|\mathbf{I}_t^{\beta_i-\epsilon}\left[\frac{1}{p_i}\mathbf{I}_\tau^{\alpha_i-\epsilon}[f_i^\epsilon]\right] - \mathbf{I}_t^{\beta_i}\left[\frac{1}{p_i}\mathbf{I}_\tau^{\alpha_i}[f_i]\right]\right|
\end{aligned}$$

$$+\left|\mathbf{I}_T^{\alpha_i-\epsilon}\left[f_i^\epsilon\right]\times\mathbf{I}_t^{\beta_i-\epsilon}\left[\frac{1}{p_i}\right]-\mathbf{I}_T^{\alpha_i}[f_i]\times\mathbf{I}_t^{\beta_i}\left[\frac{1}{p_i}\right]\right| + \left|\mathbf{I}_t^{\beta_i-\epsilon}\left[\frac{q_i}{p_i}u_i^\epsilon\right]-\mathbf{I}_t^{\beta_i}\left[\frac{q_i}{p_i}u_i\right]\right|. \tag{3.89}$$

Similar to the above, we can obtain

$$\begin{aligned}&\left|\frac{(1-\beta_i-\epsilon)}{B(\beta_i-\epsilon)p_i(0)}\mathbf{I}_T^{\alpha_i-\epsilon}\left[f_i^\epsilon\right]-\frac{(1-\beta_i)}{B(\beta_i)p_i(0)}\mathbf{I}_T^{\alpha_i}[f_i]\right|\\&\quad=\frac{(1-\beta_i-\epsilon)}{B(\beta_i-\epsilon)p_i(0)}\mathbf{I}_T^{\alpha_i-\epsilon}\left[\left|f_i^\epsilon-f_i\right|\right]\\&\qquad+\left(\frac{(1-\beta_i-\epsilon)}{B(\beta_i-\epsilon)p_i(0)}-\frac{(1-\beta_i)}{B(\beta_i)p_i(0)}\right)\left|\mathbf{I}_T^{\alpha_i-\epsilon}[1]-\mathbf{I}_T^{\alpha_i}[1]\right||f_i|.\end{aligned} \tag{3.90}$$

Analogously, we have

$$\begin{aligned}&\left|\mathbf{I}_t^{\beta_i-\epsilon}\left[\frac{1}{p_i}\mathbf{I}_\tau^{\alpha_i-\epsilon}\left[f_i^\epsilon\right]\right]-\mathbf{I}_t^{\beta_i}\left[\frac{1}{p_i}\mathbf{I}_\tau^{\alpha_i}[f_i]\right]\right|\\&\quad=\mathbf{I}_t^{\beta_i-\epsilon}\left[\frac{1}{p_i}\mathbf{I}_\tau^{\alpha_i-\epsilon}\left[f_i^\epsilon-f_i\right]\right]+\left|\mathbf{I}_t^{\beta_i-\epsilon}\left[\frac{1}{p_i}\mathbf{I}_\tau^{\alpha_i-\epsilon}[1]\right]-\mathbf{I}_t^{\beta_i}\left[\frac{1}{p_i}\mathbf{I}_\tau^{\alpha_i}[1]\right]\right||f_i|,\end{aligned} \tag{3.91}$$

$$\begin{aligned}&\left|\mathbf{I}_T^{\alpha_i-\epsilon}\left[f_i^\epsilon\right]\times\mathbf{I}_t^{\beta_i-\epsilon}\left[\frac{1}{p_i}\right]-\mathbf{I}_T^{\alpha_i}[f_i]\times\mathbf{I}_t^{\beta_i}\left[\frac{1}{p_i}\right]\right|\\&\quad=\mathbf{I}_t^{\beta_i-\epsilon}\left[\frac{1}{p_i}\right]\times\mathbf{I}_T^{\alpha_i-\epsilon}[1]\left|f_i^\epsilon-f_i\right|\\&\qquad+\left|\mathbf{I}_T^{\alpha_i-\epsilon}[1]\times\mathbf{I}_t^{\beta_i-\epsilon}\left[\frac{1}{p_i}\right]-\mathbf{I}_T^{\alpha_i}[1]\times\mathbf{I}_t^{\beta_i}\left[\frac{1}{p_i}\right]\right||f_i|,\end{aligned} \tag{3.92}$$

$$\left|\mathbf{I}_t^{\beta_i-\epsilon}\left[\frac{q_i}{p_i}u_i^\epsilon\right]-\mathbf{I}_t^{\beta_i}\left[\frac{q_i}{p_i}u_i\right]\right|=\mathbf{I}_t^{\beta_i-\epsilon}\left[\frac{q_i}{p_i}\left|u_i^\epsilon-u_i\right|\right]+\left|\mathbf{I}_t^{\beta_i-\epsilon}\left[\frac{q_i}{p_i}\right]-\mathbf{I}_t^{\beta_i}\left[\frac{q_i}{p_i}\right]\right||u_i|. \tag{3.93}$$

Taking similar procedures to (3.74) to (3.89), we obtain

$$\begin{aligned}\left|u_i^\epsilon(t)-u_i(t)\right|&\le p_i^*m_{2,i}(t)\left|f_i^\epsilon-f_i\right|+p_i^*n_{2,i}(t)|f_i|\\&\quad+p_i^*q_i^*l_{2,i}(t)\left|u_i^\epsilon-u_i\right|+p_i^*q_i^*e_{2,i}(t)|u_i|,\quad i=1,2,\end{aligned} \tag{3.94}$$

where

$$m_{2,i}(t)=\left(\frac{(1-\beta_i-\epsilon)}{B(\beta_i-\epsilon)}\mathbf{I}_T^{\alpha_i-\epsilon}[1]+\mathbf{I}_t^{\beta_i-\epsilon}\left[\mathbf{I}_\tau^{\alpha_i-\epsilon}[1]\right]+\mathbf{I}_t^{\beta_i-\epsilon}[1]\times\mathbf{I}_T^{\alpha_i-\epsilon}[1]\right), \tag{3.95}$$

$$\begin{aligned}n_{2,i}(t)&=\left(\frac{(1-\beta_i-\epsilon)}{B(\beta_i-\epsilon)}-\frac{(1-\beta_i)}{B(\beta_i)}\right)\left|\mathbf{I}_T^{\alpha_i-\epsilon}[1]-\mathbf{I}_T^{\alpha_i}[1]\right|\\&\quad+\left|\mathbf{I}_t^{\beta_i-\epsilon}\left[\mathbf{I}_\tau^{\alpha_i-\epsilon}[1]\right]-\mathbf{I}_t^{\beta_i}\left[\mathbf{I}_\tau^{\alpha_i}[1]\right]\right|+\left|\mathbf{I}_T^{\alpha_i-\epsilon}[1]\times\mathbf{I}_t^{\beta_i-\epsilon}[1]-\mathbf{I}_T^{\alpha_i}[1]\times\mathbf{I}_t^{\beta_i}[1]\right|,\end{aligned} \tag{3.96}$$

and

$$l_{2,i}(t)=\mathbf{I}_t^{\beta_i-\epsilon}[1],\qquad e_{2,i}(t)=\mathbf{I}_t^{\beta_i-\epsilon}[1]-\mathbf{I}_t^{\beta_i}[1]. \tag{3.97}$$

From (3.94) with (3.15), we have

$$\begin{aligned}&\left|u_i^{\epsilon}(t)-u_i(t)\right| \le \frac{p_i^* n_{2,i}(t)|f_i| + p_i^* q_i^* e_{2,i}(t)}{1-p_i^*(2m_{2,i}(t)M_i + q_i^* l_{2,i}(t))},\\ &\quad\text{with } 0 < p_i^*\bigl(2m_{2,i}(t)M_i + q_i^* l_{2,i}(t)\bigr) < 1.\end{aligned} \tag{3.98}$$

Similarly, it can be shown that

$$\left\|u_i^{\epsilon}-u_i\right\| \le \frac{p_i^* n_i^* \||f_i\|| + p_i^* q_i^* e_i^*}{1-p_i^*(2m_i^* M_i + q_i^* l_i^*)}, \quad \text{with } 0 < p_i^*\bigl(2m_i^* M_i + q_i^* l_i^*\bigr) < 1, \tag{3.99}$$

where

$$\begin{aligned}&m_i^* = \sup\bigl\{\left|m_{2,i}(t)\right| : t \in J\bigr\}, \qquad n_i^* = \sup\bigl\{\left|n_{2,i}(t)\right| : t \in J\bigr\},\\ &l_i^* = \sup\bigl\{\left|l_{2,i}(t)\right| : t \in J\bigr\}, \quad i = 1,2,\\ &e_i^* = \sup\bigl\{\left|e_{2,i}(t)\right| : t \in J\bigr\}.\end{aligned}$$

Consequently, from (3.99), we obtain

$$\left\|u^{\epsilon}-u\right\| \le \frac{p^* n^* |\|f^*\|| + p^* q^* e^*}{1-\mathcal{L}} \quad \text{with } \mathcal{L} = \max\{\mathcal{L}_1, \mathcal{L}_2\}, \tag{3.100}$$

where

$$\begin{aligned}&p^* = \max\bigl\{p_1^*, p_2^*\bigr\}, \qquad q^* = \max\bigl\{q_1^*, q_2^*\bigr\}, \qquad f^* = \max\{f_1, f_2\}, \qquad e^* = \max\bigl\{e_1^*, e_2^*\bigr\},\\ &m^* = \max\bigl\{m_1^*, m_2^*\bigr\}, \qquad n^* = \max\bigl\{n_1^*, n_2^*\bigr\}, \qquad l^* = \max\bigl\{l_1^*, l_2^*\bigr\}.\end{aligned}$$

Thus, in accordance with (3.100), we obtain $\|u^{\epsilon}-u\| = O(\epsilon)$. □

Theorem 3.9 *Suppose that the conditions of Theorem 2.5 hold. Let $u(t)$, $u^{\epsilon}(t)$ be the solutions, respectively, of problems (1.3) and*

$$\mathbf{D}_t^{\alpha_i}\bigl[\bigl(p_i(\tau)\mathbf{D}_\tau^{\beta_i-\epsilon} + q_i(\tau)\bigr)u_i(\tau)\bigr] = f_i\bigl(t, u_1(t), u_2(t)\bigr), \quad t \in J, i = 1,2, \tag{3.101}$$

with the boundary conditions

$$u_i(0) = 0, \qquad p_i(T)\mathbf{D}_T^{\beta_i-\epsilon}[u_i] + q_i(T)u_i(T) = 0, \tag{3.102}$$

where $0 < \beta_i - \epsilon < \beta_i \le 1$. Then $\|u^{\epsilon}-u\| = \mathcal{O}(\epsilon)$, for ϵ sufficiently small.

Proof By the above theorems, we can obtain the following results. Let

$$\begin{aligned}u_i^{\epsilon}(t) = {}& \frac{(1-\beta_i-\epsilon)}{B(\beta_i-\epsilon)p_i(0)}\mathbf{I}_T^{\alpha_i}\bigl[f_i^{\epsilon}\bigr] + \mathbf{I}_t^{\beta_i-\epsilon}\left[\frac{1}{p_i}\mathbf{I}_t^{\alpha_i}\bigl[f_i^{\epsilon}\bigr]\right]\\ &- \mathbf{I}_T^{\alpha_i}\bigl[f_i^{\epsilon}\bigr] \times \mathbf{I}_t^{\beta_i-\epsilon}\left[\frac{1}{p_i}\right] - \mathbf{I}_t^{\beta_i-\epsilon}\left[\frac{q_i}{p_i}u_i^{\epsilon}\right], \quad t \in J, i = 1,2,\end{aligned} \tag{3.103}$$

be the solution of (3.101)–(3.102).

On the one hand, from (3.8) and (3.103)

$$\begin{aligned} \left|u_i^\epsilon(t)-u_i(t)\right| = &\left|\frac{(1-\beta_i-\epsilon)}{B(\beta_i-\epsilon)p_i(0)}\mathbf{I}_T^{\alpha_i}\left[f_i^\epsilon\right]-\frac{(1-\beta_i)}{B(\beta_i)p_i(0)}\mathbf{I}_T^{\alpha_i}[f_i]\right| \\ &+\left|\mathbf{I}_t^{\beta_i-\epsilon}\left[\frac{1}{p_i}\mathbf{I}_\tau^{\alpha_i}\left[f_i^\epsilon\right]\right]-\mathbf{I}_t^{\beta_i}\left[\frac{1}{p_i}\mathbf{I}_\tau^{\alpha_i}[f_i]\right]\right| \\ &+\left|\mathbf{I}_T^{\alpha_i}\left[f_i^\epsilon\right]\times\mathbf{I}_t^{\beta_i-\epsilon}\left[\frac{1}{p_i}\right]-\mathbf{I}_T^{\alpha_i}[f_i]\times\mathbf{I}_t^{\beta_i}\left[\frac{1}{p_i}\right]\right| \\ &+\left|\mathbf{I}_t^{\beta_i-\epsilon}\left[\frac{q_i}{p_i}u_i^\epsilon\right]-\mathbf{I}_t^{\beta_i}\left[\frac{q_i}{p_i}u_i\right]\right|, \end{aligned} \tag{3.104}$$

$$\begin{aligned} &\left|\frac{(1-\beta_i-\epsilon)}{B(\beta_i-\epsilon)p_i(0)}\mathbf{I}_T^{\alpha_i}\left[f_i^\epsilon\right]-\frac{(1-\beta_i)}{B(\beta_i)p_i(0)}\mathbf{I}_T^{\alpha_i}[f_i]\right| \\ &\quad=\frac{(1-\beta_i-\epsilon)}{B(\beta_i-\epsilon)p_i(0)}\mathbf{I}_T^{\alpha_i}\left[\left|f_i^\epsilon-f_i\right|\right]+\left(\frac{(1-\beta_i-\epsilon)}{B(\beta_i-\epsilon)p_i(0)}-\frac{(1-\beta_i)}{B(\beta_i)p_i(0)}\right)\left|\mathbf{I}_T^{\alpha_i}[1]\right||f_i|. \end{aligned} \tag{3.105}$$

In a similar manner, we can get

$$\begin{aligned} &\left|\mathbf{I}_t^{\beta_i-\epsilon}\left[\frac{1}{p_i}\mathbf{I}_t^{\alpha_i}\left[f_i^\epsilon\right]\right]-\mathbf{I}_t^{\beta_i}\left[\frac{1}{p_i}\mathbf{I}_t^{\alpha_i}[f_i]\right]\right| \\ &\quad=\mathbf{I}_t^{\beta_i-\epsilon}\left[\frac{1}{p_i}\mathbf{I}_t^{\alpha_i}\left[f_i^\epsilon-f_i\right]\right]+\left|\mathbf{I}_t^{\beta_i-\epsilon}\left[\frac{1}{p_i}\mathbf{I}_t^{\alpha_i-\epsilon}[1]\right]-\mathbf{I}_t^{\beta_i}\left[\frac{1}{p_i}\mathbf{I}_t^{\alpha_i}[1]\right]\right||f_i|, \end{aligned} \tag{3.106}$$

$$\begin{aligned} &\left|\mathbf{I}_T^{\alpha_i}\left[f_i^\epsilon\right]\times\mathbf{I}_t^{\beta_i-\epsilon}\left[\frac{1}{p_i}\right]-\mathbf{I}_T^{\alpha_i}[f_i]\times\mathbf{I}_t^{\beta_i}\left[\frac{1}{p_i}\right]\right| \\ &\quad=\mathbf{I}_t^{\beta_i-\epsilon}\left[\frac{1}{p_i}\right]\times\mathbf{I}_T^{\alpha_i}[1]\left|f_i^\epsilon-f_i\right|+\left|\mathbf{I}_T^{\alpha_i}[1]\times\left(\mathbf{I}_t^{\beta_i-\epsilon}\left[\frac{1}{p_i}\right]-\mathbf{I}_t^{\beta_i}\left[\frac{1}{p_i}\right]\right)\right||f_i|, \end{aligned} \tag{3.107}$$

$$\begin{aligned} &\left|\mathbf{I}_t^{\beta_i-\epsilon}\left[\frac{q_i}{p_i}u_i^\epsilon\right]-\mathbf{I}_t^{\beta_i}\left[\frac{q_i}{p_i}u_i\right]\right| \\ &\quad=\mathbf{I}_t^{\beta_i-\epsilon}\left[\frac{q_i}{p_i}\left|u_i^\epsilon-u_i\right|\right]+\left|\mathbf{I}_t^{\beta_i-\epsilon}\left[\frac{q_i}{p_i}\right]-\mathbf{I}_t^{\beta_i}\left[\frac{q_i}{p_i}\right]\right|\times|u_i|. \end{aligned} \tag{3.108}$$

Moreover, we have by (3.104)–(3.108)

$$\begin{aligned} \left|u_i^\epsilon(t)-u_i(t)\right| \le{}& p_i^* m_{3,i}(t)\left|f_i^\epsilon-f_i\right|+p_i^* n_{3,i}(t)|f_i| \\ &+p_i^* q_i^* l_{3,i}(t)\left|u_i^\epsilon(t)-u_i(t)\right|+p_i^* q_i^* e_{3,i}(t)\left|u_i(t)\right|, \quad i=1,2, \end{aligned} \tag{3.109}$$

where

$$m_{3,i}(t)=\left(\frac{(1-\beta_i-\epsilon)}{B(\beta_i-\epsilon)}\mathbf{I}_T^{\alpha_i}[1]+\mathbf{I}_t^{\beta_i-\epsilon}\left[\mathbf{I}_t^{\alpha_i}[1]\right]+\mathbf{I}_t^{\beta_i-\epsilon}[1]\times\mathbf{I}_T^{\alpha_i}[1]\right), \tag{3.110}$$

$$\begin{aligned} n_{3,i}(t)={}&\left(\frac{(1-\beta_i-\epsilon)}{B(\beta_i-\epsilon)}-\frac{(1-\beta_i)}{B(\beta_i)}\right)\left|\mathbf{I}_T^{\alpha_i}[1]\right|+\left|\mathbf{I}_t^{\beta_i-\epsilon}\left[\mathbf{I}_t^{\alpha_i-\epsilon}[1]\right]-\mathbf{I}_t^{\beta_i}\left[\mathbf{I}_t^{\alpha_i}[1]\right]\right| \\ &+\left|\mathbf{I}_T^{\alpha_i}[1]\times\left(\mathbf{I}_t^{\beta_i-\epsilon}[1]-\mathbf{I}_t^{\beta_i}[1]\right)\right|, \end{aligned} \tag{3.111}$$

$$l_{3,i}(t)=\mathbf{I}_t^{\beta_i-\epsilon}[1], \qquad e_{3,i}(t)=\left|\mathbf{I}_t^{\beta_i-\epsilon}[1]-\mathbf{I}_t^{\beta_i}[1]\right|. \tag{3.112}$$

Thus, from (3.109) with (3.15), we get

$$\left|u_i^\epsilon(t) - u_i(t)\right| \le \frac{p_i^* n_{3,i}(t)|f_i| + p_i^* q_i^* e_{3,i}(t)}{1 - (2p_i^* m_{3,i}(t)M_i + p_i^* q_i^* l_{3,i}(t))}, \quad i = 1, 2. \tag{3.113}$$

Consequently, we obtain

$$\begin{aligned} &\|u^\epsilon - u\| \le \frac{p^* n^* \||f^*\|| + p^* q^* e^*}{1 - \mathcal{L}} \\ &\quad \text{with } 0 < \mathcal{L}_i = p_i^*\left(2m_i^* M_i + q_i^* l^*\right) < 1, \qquad \mathcal{L} = \max\{\mathcal{L}_1, \mathcal{L}_2\}, \\ &p^* = \max\left\{p_1^*, p_2^*\right\}, \qquad q^* = \max\left\{q_1^*, q_2^*\right\}, \qquad f^* = \max\{f_1, f_2\}, \\ &e^* = \max\left\{e_{3,1}^*, e_{3,2}^*\right\}, \qquad n^* = \max\left\{n_{3,1}^*, n_{3,2}^*\right\}. \end{aligned} \tag{3.114}$$

Thus, in accordance with (3.114) we obtain $\|u^\epsilon - u\| = O(\epsilon)$. □

3.5.2 The dependence on parameters of the right-hand side of (3.8)

$$\begin{aligned} &\mathbf{D}^{\alpha_i}\left(p_i(t)\mathbf{D}^{\beta_i} + q_i(t)\right)u_i(t) \\ &\quad = f_i\left(t, u_1(t), u_2(t)\right) + \epsilon g_i\left(t, u_1(t), u_2(t)\right), \quad t \in J, i = 1, 2, \end{aligned} \tag{3.115}$$

$$u_i(0) = 0, \qquad p_i(T)\mathbf{D}_T^{\beta_i}[u_i] + q_i(T)u_i(T) = 0, \quad i = 1, 2. \tag{3.116}$$

Theorem 3.10 *Assume that the hypotheses in Theorem 2.5 hold. Let $u(t)$, $u^\epsilon(t)$ be the solutions, respectively, of problems (1.3) and*

$$D^{\alpha_i}\left(p_i(t)D^{\beta_i} + q_i(t)\right)u_i(t) = f_i\left(t, u_1(t), u_2(t)\right) + \epsilon g_i\left(t, u_1(t), u_2(t)\right), \quad t \in J, i = 1, 2, \tag{3.117}$$

with boundary conditions (1.3), where $1 < \alpha_i \le 2$ and $(g_i^\epsilon)(t) := g_i(t, u_1^\epsilon(t), u_2^\epsilon(t))$, $t \in (0, T)$. Then $\|u^\epsilon - u\| = O(\epsilon)$.

Proof In accordance with Lemma 3.2, we have

$$\begin{aligned} u_i^\epsilon(t) = {}& \frac{(1-\beta_i)}{\mathbf{B}(\beta_i)p(0)}\mathbf{I}_T^{\alpha_i}\left[f_i^\epsilon + \epsilon g_i^\epsilon\right] + \mathbf{I}_t^{\beta_i}\left(\frac{1}{p_i}\mathbf{I}_\tau^{\alpha_i}\left[f_i^\epsilon + \epsilon g_i^\epsilon\right]\right) \\ & - \mathbf{I}_T^{\alpha_i}\left[f_i^\epsilon + \epsilon g_i^\epsilon\right] \times \mathbf{I}_t^{\beta_i}\left[\frac{1}{p_i}\right] - \mathbf{I}_t^{\beta_i}\left[\frac{q_i}{p_i}u_i^\epsilon\right], \end{aligned} \tag{3.118}$$

$$\begin{aligned} &\left|u_i^\epsilon(t) - u_i(t)\right| \\ &\quad = \frac{(1-\beta_i)}{\mathbf{B}(\beta_i)p(0)}\left(\mathbf{I}_T^{\alpha_i}\left[f_i^\epsilon + \epsilon g_i^\epsilon\right] - \mathbf{I}_T^{\alpha_i}[f_i]\right) + \left(\mathbf{I}_t^{\beta_i}\left[\frac{1}{p_i}\mathbf{I}_\tau^{\alpha_i}\left[f_i^\epsilon + \epsilon g_i^\epsilon\right]\right] - \mathbf{I}_t^{\beta_i}\left[\frac{1}{p_i}\mathbf{I}_\tau^{\alpha_i}[f_i]\right]\right) \\ &\qquad - \left(\mathbf{I}_T^{\alpha_i}\left[f_i^\epsilon + \epsilon g_i^\epsilon\right] - \mathbf{I}_T^{\alpha_i}[f_i]\right) \times \mathbf{I}_t^{\beta_i}\left[\frac{1}{p_i}\right] \\ &\qquad - \left(\mathbf{I}_t^{\beta_i}\left[\frac{q_i}{p_i}u_i^\epsilon\right] - \mathbf{I}_t^{\beta_i}\left[\frac{q_i}{p_i}u_i\right]\right), \quad T \in J, \end{aligned} \tag{3.119}$$

$$\left|\mathbf{I}_T^{\alpha_i}\left[f_i^\epsilon + \epsilon g_i^\epsilon\right] - \mathbf{I}_T^{\alpha_i}[f_i]\right| = \mathbf{I}_T^{\alpha_i}\left[\left|f_i^\epsilon - f_i\right|\right] + \epsilon\mathbf{I}_T^{\alpha_i}\left[\left|g_i^\epsilon\right|\right]. \tag{3.120}$$

Similarly, it can be shown that

$$\begin{aligned} &\left|\mathbf{I}_t^{\beta_i}\left[\frac{1}{p_i}\mathbf{I}_t^{\alpha_i}[f_i^\epsilon + \epsilon g_i^\epsilon]\right] - \mathbf{I}_t^{\beta_i}\left[\frac{1}{p_i}\mathbf{I}_\tau^{\alpha_i}[f_i]\right]\right| \\ &\quad = \left|\mathbf{I}_t^{\beta_i}\left[\frac{1}{p_i}\mathbf{I}_\tau^{\alpha_i}[f_i^\epsilon - f_i]\right] + \epsilon \mathbf{I}_t^{\beta_i}\left[\frac{1}{p_i}\mathbf{I}_\tau^{\alpha_i}\left[\left|g_i^\epsilon\right|\right]\right]\right|, \end{aligned} \tag{3.121}$$

$$\left(\mathbf{I}_T^{\alpha_i}[f_i^\epsilon + \epsilon g_i^\epsilon] - \mathbf{I}_T^{\alpha_i}[f_i]\right) \times \mathbf{I}_t^{\beta_i}\left[\frac{1}{p_i}\right] = \left(\mathbf{I}_T^{\alpha_i}[f_i^\epsilon - f_i] + \epsilon \mathbf{I}_T^{\alpha_i}[g_i^\epsilon]\right) \times \mathbf{I}_t^{\beta_i}\left[\left|\frac{1}{p_i}\right|\right], \tag{3.122}$$

$$\left|\mathbf{I}_t^{\beta_i}\left[\frac{q_i}{p_i}u_i^\epsilon\right] - \mathbf{I}_t^{\beta_i}\left[\frac{q_i}{p_i}u_i\right]\right| = \left|\frac{q_i}{p_i}\right|\mathbf{I}_t^{\beta_i}\left[\left|u_i^\epsilon - u_i\right|\right]. \tag{3.123}$$

Rewriting (3.119) as

$$\left|u_i^\epsilon(t) - u_i(t)\right| \le p_i^* m_{4,i}(t)\left|f_i^\epsilon - f_i\right| + p_i^* q_i^* l_{4,i}(t)\left|u_i^\epsilon - u_i\right| + p_i^* d_{4,i}(t)\left|g_i^\epsilon\right|, \quad i = 1, 2, \tag{3.124}$$

where

$$m_{4,i}(t) = \frac{(1-\beta_i)}{\mathbf{B}(\beta_i)}\mathbf{I}_T^{\alpha_i}[1] + \mathbf{I}_t^{\beta_i}\left[\mathbf{I}_\tau^{\alpha_i}[1]\right] + \mathbf{I}_T^{\alpha_i}[1] \times \mathbf{I}_t^{\beta_i}[1], \tag{3.125}$$

$$d_{4,i}(t) = \epsilon\left(\frac{1}{p_i^*}\mathbf{I}_T^{\alpha_i}[1] + \mathbf{I}_t^{\beta_i}\left[\mathbf{I}_\tau^{\alpha_i}[1]\right] + \mathbf{I}_T^{\alpha_i}[1] \times \mathbf{I}_t^{\beta_i}[1]\right), \tag{3.126}$$

$$l_{4,i}(t) = \mathbf{I}_t^{\beta_i}[1]. \tag{3.127}$$

Hence, from (3.124) with (3.15), we have

$$\left|u_i^\epsilon(t) - u_i(t)\right| \le \frac{p_i^* d_{4,i}(t)|g_i^\epsilon|}{1 - p_i^*(2m_{4,i}(t)M_i + q_i^* l_{4,i}(t))}, \quad i = 1, 2, \tag{3.128}$$

again from (3.128), one has

$$\begin{aligned} &\left\|u^\epsilon - u\right\| \le \frac{p^* d^* \|g^*\|}{1 - \mathcal{L}} \\ &\quad \text{with } 0 < \mathcal{L}_i = p_i^*\left(2m_i^* M_i + q_i^* l^*\right) < 1, \qquad \mathcal{L} = \max\{\mathcal{L}_1, \mathcal{L}_2\}, \\ &p^* = \max\left\{p_1^*, p_2^*\right\}, \qquad d^* = \max\left\{d_{4,1}^*, g_{4,2}^*\right\}, \\ &m^* = \max\left\{md_{4,1}^*, m_{4,2}^*\right\}, \qquad g^* = \max\left\{g_1^*, g_2^*\right\}. \end{aligned} \tag{3.129}$$

Then we have $d^* \longrightarrow 0$ as $\epsilon \longrightarrow 0$, implies $\|u^\epsilon - u\| = O(\epsilon)$ as desired. □

3.5.3 The dependence on parameters of initial conditions of (1.3)

The following theorem investigates the continuous dependence of the solutions of system (1.3) on the initial value and the functions f_i. For this purpose, we introduce small changes in the initial conditions of (1.3) and consider (1.3-a) with boundary conditions

$$u_i(0) = 0, \qquad p_i(T)\mathbf{D}^{\beta_i}u_i(T+\epsilon) + q_i(T+\epsilon)u_i(T+\epsilon) = 0. \tag{3.130}$$

Theorem 3.11 *Assume the conditions of Theorem* 2.5 *hold. Let* $u(t)$, $u^\epsilon(t)$ *be respective solutions of problems* (1.3) *and the boundary conditions* (1.3-*a*)– (3.114). *Then* $\|u^\epsilon - u\| = O(\epsilon)$.

Proof Let $u(t) = (u_1(t), u_2(t))$ and $u^\epsilon(t) = (u_1^\epsilon(t), u_2^\epsilon(t))$ be the solutions of (1.3) and (1.3-a)–(3.114), respectively. Hence

$$\begin{aligned} u_i^\epsilon(t) &= \frac{(1-\beta_i)}{\mathbf{B}(\beta_i)p(0)}\left(\mathbf{I}_{T+\epsilon}^{\alpha_i}\left[f_i^\epsilon\right] - \frac{1-\alpha_i}{\mathbf{B}(\alpha_i)}f_i^\epsilon(0)\right) + \mathbf{I}_t^{\beta_i}\left(\frac{1}{p_i}\mathbf{I}_\tau^{\alpha_i}\left[f_i^\epsilon\right]\right) \\ &\quad - \mathbf{I}_{T+\epsilon}^{\alpha_i}\left[f_i^\epsilon\right] \times \mathbf{I}_t^{\beta_i}\left[\frac{1}{p_i}\right] - \mathbf{I}_t^{\beta_i}\left[\frac{q_i}{p_i}u_i^\epsilon\right]. \end{aligned} \tag{3.131}$$

Now we derive from (3.8) and (3.131) that

$$\begin{aligned} \left|u_i^\epsilon(t) - u_i(t)\right| &= \frac{(1-\beta_i)}{\mathbf{B}(\beta_i)p(0)}\left(\left|\mathbf{I}_{T+\epsilon}^{\alpha_i}\left[f_i^\epsilon\right] - \mathbf{I}_T^{\alpha_i}[f_i]\right| - \frac{1-\alpha_i}{\mathbf{B}(\alpha_i)}\left|f_i^\epsilon(0) - f_i(0)\right|\right) \\ &\quad + \left|\mathbf{I}_t^{\beta_i}\left[\frac{1}{p_i}\mathbf{I}_\tau^{\alpha_i}\left[f_i^\epsilon\right]\right] - \mathbf{I}_t^{\beta_i}\left[\frac{1}{p_i}\mathbf{I}_\tau^{\alpha_i}[f_i]\right]\right| \\ &\quad - \left|\mathbf{I}_{T+\epsilon}^{\alpha_i}\left[f_i^\epsilon\right] \times \mathbf{I}_t^{\beta_i}\left[\frac{1}{p_i}\right] - \mathbf{I}_T^{\alpha_i}[f_i] \times \mathbf{I}_t^{\beta_i}\left[\frac{1}{p_i}\right]\right| \\ &\quad - \left|\mathbf{I}_t^{\beta_i}\left[\frac{q_i}{p_i}u_i^\epsilon\right] - \mathbf{I}_t^{\beta_i}\left[\frac{q_i}{p_i}u_i\right]\right|, \end{aligned} \tag{3.132}$$

$$\left|\mathbf{I}_{T+\epsilon}^{\alpha_i}\left[f_i^\epsilon\right] - \mathbf{I}_T^{\alpha_i}[f_i]\right| = \left|\mathbf{I}_{T+\epsilon}^{\alpha_i}\left[f_i^\epsilon - f_i\right]\right| + \left|\mathbf{I}_{T+\epsilon}^{\alpha_i}[1] - \mathbf{I}_T^{\alpha_i}[1]\right||f_i|. \tag{3.133}$$

Similarly to the above argument, we can also obtain

$$\left|\mathbf{I}_t^{\beta_i}\left[\frac{1}{p_i}\mathbf{I}_\tau^{\alpha_i}\left[f_i^\epsilon\right]\right] - \mathbf{I}_t^{\beta_i}\left[\frac{1}{p_i}\mathbf{I}_\tau^{\alpha_i}[f_i]\right]\right| = \left|\frac{1}{p_i}\right|\left|\mathbf{I}_\tau^{\alpha_i}\left[f_i^\epsilon - f_i\right]\right|\mathbf{I}_t^{\beta_i}[1], \tag{3.134}$$

$$\begin{aligned} &\left|\mathbf{I}_{T+\epsilon}^{\alpha_i}\left[f_i^\epsilon\right] \times \mathbf{I}_t^{\beta_i}\left[\frac{1}{p_i}\right] - \mathbf{I}_T^{\alpha_i}[f_i] \times \mathbf{I}_t^{\beta_i}\left[\frac{1}{p_i}\right]\right| \\ &\quad = \left|\frac{1}{p_i}\right|\left(\left|\mathbf{I}_{T+\epsilon}^{\alpha_i}\left[f_i^\epsilon - f_i\right]\right| + \left|\mathbf{I}_{T+\epsilon}^{\alpha_i}[1] - \mathbf{I}_T^{\alpha_i}[1]\right||f_i|\right) \times \mathbf{I}_t^{\beta_i}[1], \end{aligned} \tag{3.135}$$

$$\left|\mathbf{I}_t^{\beta_i}\left[\frac{q_i}{p_i}u_i^\epsilon\right] - \mathbf{I}_t^{\beta_i}\left[\frac{q_i}{p_i}u_i\right]\right| = \left|\frac{q_i}{p_i}\right|\left|\mathbf{I}_t^{\beta_i}[1]\right|\left|u_i^\epsilon - u_i\right|. \tag{3.136}$$

From (3.132)–(3.136), we derive that

$$\begin{aligned} \left|u_i^\epsilon(t) - u_i(t)\right| &\le p_i^* m_{5,i}(t)\left[\left|f_i^\epsilon - f_i\right|\right] + p_i^* n_{5,i}(t)|f_i| + p_i^* q_i^* l_{5,i}(t)\left|u_i^\epsilon - u_i\right| \\ &\quad + p_i^* e_{5,i}(t), \quad i = 1, 2, \end{aligned} \tag{3.137}$$

where

$$\begin{aligned} m_{5,i}(t) &= \frac{(1-\beta_i)}{\mathbf{B}(\beta_i)p(0)}\left(\left|\mathbf{I}_{T+\epsilon}^{\alpha_i}[1]\right| + \left|\mathbf{I}_\tau^{\alpha_i}[1]\right|\mathbf{I}_t^{\beta_i}[1] + \left(\left|\mathbf{I}_{T+\epsilon}^{\alpha_i}[1]\right|\right) \times \mathbf{I}_t^{\beta_i}[1]\right), \\ n_{5,i}(t) &= \frac{(1-\beta_i)}{\mathbf{B}(\beta_i)}\left(1 + \mathbf{I}_t^{\beta_i}[1]\right)\left(\left|\mathbf{I}_{T+\epsilon}^{\alpha_i}[1] - \mathbf{I}_T^{\alpha_i}[1]\right|\right), \\ l_{5,i}(t) &= \left|\mathbf{I}_t^{\beta_i}[1]\right|, \\ e_{5,i}(t) &= \frac{(1-\beta_i)}{\mathbf{B}(\beta_i)}\left(-\frac{1-\alpha_i}{\mathbf{B}(\alpha_i)}\left|f_i^\epsilon(0) - f_i(0)\right|\right). \end{aligned}$$

Combining (3.15) with (3.137), we have

$$\begin{aligned} \left|u_i^\epsilon(t)-u_i(t)\right| &= p_i^*\left(2m_{5,i}(t)M_i+q_i^*l_{5,i}(t)\right)\left|u_i^\epsilon(t)-u_i(t)\right|+p_i^*n_{5,i}(t)|f_i| \\ &\quad + p_i^*e_{5,i}(t), \quad i=1,2, \qquad (3.138) \\ \left|u_i^\epsilon(t)-u_i(t)\right| &= \frac{p_i^*n_{5,i}(t)|f_i|+p_i^*e_{5,i}(t)}{1-p_i^*(2m_{5,i}(t)M_i+q_i^*l_{5,i}(t))}, \quad i=1,2. \qquad (3.139) \end{aligned}$$

Taking the maximum on both sides of the inequality (3.139), the following can be obtained:

$$\left\|u_i^\epsilon-u_i\right\| \le \frac{p_i^*(n_i^*\||f_i\||+e_i^*)}{1-L_i}, \quad i=1,2, \tag{3.140}$$

$$\mathcal{L}_i = p_i^*\left(2m_i^*M_i+q_i^*l_i^*\right). \tag{3.141}$$

From the inequality (3.140) we have

$$\left\|u^\epsilon-u\right\| \le \frac{p^*(n^*\||f\||+e^*)}{1-\mathcal{L}}, \tag{3.142}$$

where

$$\begin{aligned} &\mathcal{L}=\max\{\mathcal{L}_1,\mathcal{L}_2\}, \qquad m^*=\max\left\{m_1^*,m_2^*\right\}, \\ &n^*=\max\left\{n_1^*,n_2^*\right\}, \qquad l^*=\max\left\{l_1^*,l_2^*\right\}. \end{aligned} \tag{3.143}$$

Then we have $n^*\||f\||+e^* \longrightarrow 0$ as $\epsilon \longrightarrow 0$, implies $\|u^\epsilon-u\|=O(\epsilon)$ as desired. □

3.6 Examples

In this subsection, we will give examples to illustrate our main result.

Example 3.12 Let us first consider system (1.3) with

$$f_1(t,u_1,u_2)=\frac{1/6}{1+|u_1(t)|+|u_2(t)|} \quad \text{and} \quad f_2(t,u_1,u_2)=\frac{5}{16}\left(\sin u_1(t)+\cos u_1(t)\right)+u_2(t).$$

It is easy to see that the function f_i satisfies condition (H_1).

From system (1.3) we take $\alpha_1=3/5$, $\beta_1=2/3$ and $\alpha_2=2/5$, $\beta_2=3/4$, $p_1=t^{3/2}+1/8$, $q_1=t^{2/7}-1$, $p_2=t^{5/3}+1/9$, $q_2=t^{3/10}-1$. By using the Maple program, we can find that

$$\begin{aligned} &0<p_1^*\left(2M_1\mu_1^*+q_1^*\gamma_{1,\beta_1}^*\right)<1 \quad \text{iff} \quad 0.0465<T<5.2691, \\ &0<p_2^*\left(2M_2\mu_2^*+q_2^*\gamma_{1,\beta_2}^*\right)<1 \quad \text{iff} \quad 0<T\le 2.2268. \end{aligned}$$

We see that $T_{\min}=0.0465<T\le T_{\max}=2.2268$, and all the conditions of Theorem 3.4 are satisfied. Thus, the coupled system (1.3) has at least one solution. For example, when $T\in\{T_{\min},1,2,2.2260,T_{\max}\}$, we have

T	r_1	r_2
0.0465	0.023148	0.047611
1	0.105130	0.283450
2	0.234970	2.493400
2.2260	0.276200	644.8100
2.2268	0.276370	4850.400

Then $r \geq \max\{r_1, r_2\} = 4850.400$. In this way, we have actually shown that the coupled system (1.3) has at least one solution and the solution lies in

$$\Omega = \left\{(u_1, u_2) \in X : \left\|(u_1, u_2)\right\| < 4850.400\right\}.$$

Example 3.13 Consider problem (1.3), with

$$f_1(t, u_1, u_2) = \frac{1/6}{1 + |u_1(t)| + |u_2(t)|} \quad \text{and} \quad f_2(t, u_1, u_2) = \frac{1}{64}\left(\sin u_1(t) + \cos u_1(t)\right) + u_2(t).$$

From system (1.3) we take $\alpha_1 = 3/5$, $\beta_1 = 2/3$ and $\alpha_2 = 2/5$, $\beta_2 = 3/4$, $p_1 = t^{3/2} + 1/8$, $q_1 = 0$, $p_2 = 1$, $q_2 = \frac{1}{7}$,

It is easy to see that the function f_i satisfies condition (H_1). Set $T = 2$, we can find that

$$\begin{aligned}
&M_1 = 1/6, \qquad p_1^* = 1/8, \qquad q_1^* = 0, \\
&\mu_1^* = 7.1531, \qquad a_1 = 1/6, \qquad \eta_1^* = 3.3522 \times 10^{-2}, \\
&M_2 = 1/32, \qquad p_2^* = 1, \qquad q_2^* = \frac{1}{7}, \\
&\mu_2^* = 6.2079, \qquad a_2 = \frac{1}{64}, \qquad \eta_2^* = 0.13937,
\end{aligned}$$

the assumptions of Theorem 3.4 are satisfied with

$$r \geq \max\{r_1, r_2\} = \max\{0.26005, 0.68882\} = 0.68882.$$

Further, we see that (3.8) holds.

Example 3.14 Consider problem (1.3), with

$$\begin{aligned}
f_1(t, u_1, u_2) &= \frac{t}{3} + \frac{t^3}{5}\sin\left|u_1(t)\right| + \frac{t^5}{7}\cos\left|u_2(t)\right|, \qquad N_1 = 718/105, \\
f_2(t, u_1, u_2) &= \frac{1}{2} + \frac{t^2}{4}\sin\left|u_1(t)\right| + \frac{t^4}{6}\cos\left|u_2(t)\right|, \qquad N_2 = 25/6.
\end{aligned}$$

For system (1.3) we take $\alpha_1 = 3/5$, $\beta_1 = 2/3$ and $\alpha_2 = 2/5$, $\beta_2 = 3/4$, $p_1 = t^{3/2} + 1/8$, $q_1 = t^{2/7} - 1$, $p_2 = t^{5/3} + 1/7$, $q_2 = t^{3/10} - 1$.

It is easy to see that the function f_i satisfies condition $(\mathbf{A}_2)$. Set $T = 2$, we can find that

$$p_1^* = 8, \qquad q_1^* = 0.21, \qquad \mu_1^* = 7.1531, \qquad \eta_1 = 0, \qquad \gamma_{3,\beta_1}^* = 2.826,$$

$$p_2^* = 7, \qquad q_2^* = 0.23, \qquad \mu_2^* = 6.2079, \qquad \eta_2 = 0.78047, \qquad \gamma_{3,\beta_2}^* = 2.9001.$$

On the other hand, we have

$$r_1 = \frac{\mu_1^* N_1 + \eta_1}{1 - q_1^* p_1^* \gamma_{3,\beta_1}^*} = 52.833 \quad \text{and} \quad r_2 = \frac{\mu_2^* N_2 + \eta_2}{1 - q_2^* p_2^* \gamma_{3,\beta_2}^*} = 29.453;$$

the assumptions of Theorem 3.5 are satisfied with $r \geq 52.833$.

Example 3.15 For system (1.3) we take $\alpha_1 = 3/5$, $\beta_1 = 2/3$ and $\alpha_2 = 2/5$, $\beta_2 = 3/4$, $p_1 = t^{3/2} + 1/8$, $q_1 = 0$, $p_2 = 1$, $q_2 = \frac{1}{7}$, with

$$f_1(t, u_1, u_2) = \frac{1/6}{1 + |u_1(t)| + |u_2(t)|}, \qquad f_2(t, u_1, u_2) = \frac{1}{64}\big(\sin u_1(t) + \cos u_1(t)\big) + u_2(t),$$

$$p_1^* = 1/8, \qquad q_1^* = 0, \qquad M_1 = 1/6, \qquad \mu_1^* = 7.1531,$$

$$p_2^* = 1, \qquad q_2^* = \frac{1}{7}, \qquad M_2 = 1/32, \qquad \mu_2^* = 6.2079.$$

Then, by the use of Theorem 3.6, we have

$$\tilde{\mathcal{H}}_\sigma = \begin{pmatrix} p_1^*(M_1 \mu_1^* + q_1^* \gamma_{1,\beta_1}^*) & p_2^* \mu_2^* M_2 \\ p_1^* \mu_1^* M_1 & p_2^*(\mu_2^* M_2 + q_2^* \gamma_{1,\beta_2}^*) \end{pmatrix} = \begin{pmatrix} 0.14902 & 0.19400 \\ 0.14902 & 0.46285 \end{pmatrix}.$$

Here, te characteristic polynomial is $\lambda^2 - 0.61187\lambda + 4.0064 \times 10^{-2}$, the spectral radius $\rho(\tilde{\mathcal{H}}_\sigma) = 0.53731 < 1$. Therefore, the matrix $\tilde{\mathcal{H}}_\sigma$ converges to zero, and hence the solutions of (1.3) are Hyers–Ulam stable by using Theorem 2.4.

4 Conclusions

The theory of fractional operators with nonsingular kernels is new and we need to study the qualitative properties of differential equations involving such operators. This paper is different from the ones presented in the previous literature and shows that it is possible to extend the analysis of the coupled system with the Sturm–Liouville problems and the nonlinear Langevin equation to the concepts of fractional differentiation, using the newly introduced notion of the ABC-fractional derivative with nonlocal and nonsingular kernel. ABC-fractional operators were therefore used in this work to present some results dealing with the existence and uniqueness of solutions for the coupled system. As a first step, the coupled system is transformed to a fixed point problem by applying the tools of ABC-fractional calculus. Based on this, the existence results are established by means of Krasnoselskii's fixed point theorem and Banach's contraction principle. The paper also presented a discussion of the Ulam–Hyers stability of the solution of the proposed problem. We also analyzed the continuous dependence of solutions as regards the right-hand side of the equations, initial value condition and the fractional order for the coupled system. We conclude that such a method is very powerful, effectual and suitable for the solution of coupled systems. The concerned theory has been enriched by providing suitable examples.

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