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On the solutions of a max-type system of difference equations of higher order

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Abstract

In this paper, we study the following max-type system of difference equations of higher order:

$$\begin{cases} x_n = \max\{A, \frac{y_{n-t}}{x_{n-s}}\}, \\ y_n = \max\{B, \frac{x_{n-t}}{y_{n-s}}\}, \end{cases} \quad n \in \{0,1,2,\ldots\},$$

where $A, B \in (0,+\infty)$, $t, s \in \{1,2,\ldots\}$ with $\gcd(s,t) = 1$, the initial values $x_{-d}, y_{-d}, x_{-d+1}, y_{-d+1}, \ldots, x_{-1}, y_{-1} \in (0,+\infty)$ and $d = \max\{t,s\}$.

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1 Introduction

Concrete nonlinear difference equations and systems have attracted some recent attention (see, e.g., [1–39]). One of the classes of such equations/systems are max-type difference equations/systems. For some results of solutions of many max-type difference equations and systems, such as eventual periodicity, the boundedness character and attractivity, see, e.g. [1–5, 7–9, 11–16, 18–25, 28–30, 32–36, 38, 39] and the references therein. Our purpose in this paper is to study the eventual periodicity of the following max-type system of difference equation of higher order:

$$\begin{cases} x_n = \max\{A, \frac{y_{n-t}}{x_{n-s}}\}, \\ y_n = \max\{B, \frac{x_{n-t}}{y_{n-s}}\}, \end{cases} \quad n \in \mathbf{N}_0 \equiv \{0,1,\ldots\}, \tag{1.1}$$

where $A, B \in \mathbf{R}_+ \equiv (0,+\infty)$, $t, s \in \mathbf{N} \equiv \{\mathbf{1}, \mathbf{2}, \ldots\}$ with $\gcd(s,t) = 1$, the initial values $x_{-d}, y_{-d}, x_{-d+1}, y_{-d+1}, \ldots, x_{-1}, y_{-1} \in \mathbf{R}_+$ and $d = \max\{t,s\}$.

When $t = 1$ and $s = 2$, (1.1) reduces to the max-type system of difference equations

$$\begin{cases} x_n = \max\{A, \frac{y_{n-1}}{x_{n-2}}\}, \\ y_n = \max\{B, \frac{x_{n-1}}{y_{n-2}}\}, \end{cases} \quad n \in \mathbf{N}_0. \tag{1.2}$$

Fotiades and Papaschinopoulos in [5] showed that every positive solution of (1.2) is eventually periodic.

In 2012, Stević [23] obtained in an elegant way the general solution to the following max-type system of difference equations:

$$\begin{cases} x_{n+1} = \max\{\frac{A}{x_n}, \frac{y_n}{x_n}\}, \\ y_{n+1} = \max\{\frac{A}{y_n}, \frac{x_n}{y_n}\}, \end{cases} \quad n \in \mathbf{N}_0, \tag{1.3}$$

for the case $x_0, y_0 \geq A > 0$ and $y_0/x_0 \geq \max\{A, 1/A\}$.

In [35], Sun and Xi studied the following max-type system of difference equations:

$$\begin{cases} x_n = \max\{\frac{1}{x_{n-m}}, \min\{1, \frac{A}{y_{n-r}}\}\}, \\ y_n = \max\{\frac{1}{y_{n-m}}, \min\{1, \frac{B}{x_{n-t}}\}\}, \end{cases} \quad n \in \mathbf{N}_0, \tag{1.4}$$

where $A, B \in \mathbf{R}_+$, $m, r, t \in \mathbf{N}$ and the initial values $x_{-d}, y_{-d}, x_{-d+1}, y_{-d+1}, \ldots, x_{-1}, y_{-1} \in \mathbf{R}_+$ with $d = \max\{m, r, t\}$ and showed that every positive solution of (1.4) is eventually periodic with period $2m$.

When $m = r = t = 1$ and $A = B$, (1.4) reduces to the max-type system of difference equations

$$\begin{cases} x_n = \max\{\frac{1}{x_{n-1}}, \min\{1, \frac{A}{y_{n-1}}\}\}, \\ y_n = \max\{\frac{1}{y_{n-1}}, \min\{1, \frac{A}{x_{n-1}}\}\}, \end{cases} \quad n \in \mathbf{N}_0. \tag{1.5}$$

Yazlik et al. [39] in 2015 obtained in an elegant way the general solution of (1.5).

In 2012, Stević [24] studied the following max-type system of difference equations:

$$\begin{cases} y_n^{(1)} = \max_{1 \leq i \leq m_1} \{f_{1i}(y^{(1)}_{n-k^{(1)}_{i,1}}, y^{(2)}_{n-k^{(1)}_{i,2}}, \ldots, y^{(l)}_{n-k^{(1)}_{i,l}}, n), y^{(1)}_{n-s}\}, \\ y_n^{(2)} = \max_{1 \leq i \leq m_2} \{f_{2i}(y^{(1)}_{n-k^{(2)}_{i,1}}, y^{(2)}_{n-k^{(2)}_{i,2}}, \ldots, y^{(l)}_{n-k^{(2)}_{i,l}}, n), y^{(2)}_{n-s}\}, \\ \ldots, \\ y_n^{(l)} = \max_{1 \leq i \leq m_l} \{f_{li}(y^{(1)}_{n-k^{(l)}_{i,1}}, y^{(2)}_{n-k^{(l)}_{i,2}}, \ldots, y^{(l)}_{n-k^{(l)}_{i,l}}, n), y^{(l)}_{n-s}\}, \end{cases} \quad n \in \mathbf{N}_0, \tag{1.6}$$

where $s, l, m_j, k^{(j)}_{i,t} \in \mathbf{N}$ $(j, t \in \{1, 2, \ldots, l\})$ and $f_{ji} : \mathbf{R}^l_+ \times \mathbf{N}_0 \longrightarrow \mathbf{R}_+$ $(j \in \{1, \ldots, l\}$ and $i \in \{1, \ldots, m_j\})$, and showed that every positive solution of (1.6) is eventually periodic with (not necessarily prime) period s if f_{ji} satisfy some conditions.

Moreover, Stević et al. [29] in 2014 investigated the following max-type system of difference equations:

$$\begin{cases} y_n^{(1)} = \max_{1 \leq i_1 \leq m_1} \{f_{1i_1}(y^{(1)}_{n-k^{(1)}_{i_1,1}}, y^{(2)}_{n-k^{(1)}_{i_1,2}}, \ldots, y^{(l)}_{n-k^{(1)}_{i_1,l}}, n), y^{(\sigma(1))}_{n-t_1 s}\}, \\ x_n^{(2)} = \max_{1 \leq i_2 \leq m_2} \{f_{2i_2}(y^{(1)}_{n-k^{(2)}_{i_2,1}}, y^{(2)}_{n-k^{(2)}_{i_2,2}}, \ldots, y^{(l)}_{n-k^{(2)}_{i_2,l}}, n), y^{(\sigma(2))}_{n-t_2 s}\}, \\ \ldots, \\ y_n^{(l)} = \max_{1 \leq i_l \leq m_l} \{f_{li_l}(y^{(1)}_{n-k^{(l)}_{i_l,1}}, y^{(2)}_{n-k^{(l)}_{i_l,2}}, \ldots, y^{(l)}_{n-k^{(l)}_{i_l,l}}, n), y^{(\sigma(l))}_{n-t_l s}\}, \end{cases} \quad n \in \mathbf{N}_0, \tag{1.7}$$

where $s, l, m_j, t_j, k_{i_j,h}^{(j)} \in \mathbf{N}$ ($j, h \in \{1, 2, \ldots, l\}$), $(\sigma(1), \ldots, \sigma(l))$ is a permutation of $(1, \ldots, l)$ and $f_{ji_j} : \mathbf{R}_+^l \times \mathbf{N}_0 \longrightarrow \mathbf{R}_+$ ($j \in \{1, \ldots, l\}$ and $i_j \in \{1, \ldots, m_j\}$). They showed that every positive solution of (1.7) is eventually periodic with period sT for some $T \in \mathbf{N}$ if f_{ji_j} satisfy some conditions.

2 Main results and proofs

In this section, we study the eventual periodicity of positive solutions of system (1.1). Let $\{(x_n, y_n)\}_{n\geq -d}$ be a solution of (1.1) with the initial values $x_{-d}, y_{-d}, x_{-d+1}, y_{-d+1}, \ldots, x_{-1}, y_{-1} \in \mathbf{R}_+$.

Lemma 2.1 *If $x_n = A$ eventually, then y_n is a periodic sequence with period $2s$ eventually. If $y_n = B$ eventually, then x_n is a periodic sequence with period $2s$ eventually.*

Proof Assume that $x_n = A$ eventually. By (1.1) we see

$$y_n = \max\left\{B, \frac{A}{y_{n-s}}\right\} \quad \text{eventually,} \tag{2.1}$$

which implies $y_n y_{n-s} \geq A$ eventually and

$$\begin{aligned} B \leq y_n &= \max\left\{B, \frac{A}{y_{n-s}}\right\} \\ &= \max\left\{B, \frac{Ay_{n-2s}}{y_{n-s}y_{n-2s}}\right\} \\ &\leq \max\{B, y_{n-2s}\} \leq y_{n-2s} \quad \text{eventually.} \end{aligned} \tag{2.2}$$

Then, for any $0 \leq i \leq 2s-1$, y_{2ns+i} is eventually nonincreasing.

We claim that, for every $0 \leq i \leq 2s-1$, y_{2ns+i} is a constant sequence eventually. Assume on the contrary that, for some $0 \leq i \leq 2s-1$, y_{2ns+i} is not a constant sequence eventually. Then there exists a sequence of positive integers $k_1 < k_2 < \cdots$ such that, for any $n \in \mathbf{N}$, we have

$$\begin{aligned} B < y_{2sk_{n+1}+i} &= \frac{A}{y_{2sk_{n+1}+i-s}} \\ < y_{2sk_n+i} &= \frac{A}{y_{2sk_n+i-s}}, \end{aligned} \tag{2.3}$$

which implies $y_{2sk_{n+1}+i-s} > y_{2sk_n+i-s}$ for any $n \in \mathbf{N}$. This is a contradiction. Thus y_n is a periodic sequence with period $2s$ eventually. The second case follows from the previously proved one by interchanging letters. The proof is complete. □

Lemma 2.2 *If $A \geq B \geq 1/A$, then $x_{2(n+1)t+i} \leq x_{2nt+i}$ for any $n \geq t+s$ and $i \in \mathbf{N}_0$. If $B \geq A \geq 1/B$, then $y_{2(n+1)t+i} \leq y_{2nt+i}$ for any $n \geq t+s$ and $i \in \mathbf{N}_0$.*

Proof Assume that $A \geq B \geq 1/A$. By (1.1) we see that $x_n \geq A$ and $y_n \geq B$ for any $n \in \mathbf{N}_0$, and

$$x_{2(n+1)t+i} = \max\left\{A, \frac{B}{x_{2(n+1)t+i-s}}, \frac{x_{2nt+i}}{x_{2(n+1)t+i-s}y_{2(n+1)t+i-t-s}}\right\}. \tag{2.4}$$

Since $B/x_{2(n+1)t+i-s} \le B/A \le 1 \le A$ and $x_{2(n+1)t+i-s}y_{2(n+1)t+i-t-s} \ge AB \ge 1$ for $2(n+1)t+i \ge t+s$, we obtain

$$x_{2(n+1)t+i} \le \max\{A, x_{2nt+i}\} \le x_{2nt+i}. \tag{2.5}$$

The second case follows from the previously proved one by interchanging letters. The proof is complete. □

Theorem 2.1 *Let $AB > 1$. If $A \ge B$, then $x_n = A$ eventually and y_n is a periodic sequence with period $2s$ eventually. If $B > A$, then $y_n = B$ eventually and x_n is a periodic sequence with period $2s$ eventually.*

Proof Assume that $A \ge B$. For any $0 \le i \le 2t-1$ and $n \in \mathbf{N}_0$, we have

$$A \le x_{2(n+1)t+i} = \max\left\{A, \frac{x_{2nt+i}}{x_{2(n+1)t-s+i}y_{2(n+1)t-s-t+i}}\right\}. \tag{2.6}$$

By Lemma 2.2 we may let $\lim_{n\longrightarrow\infty} x_{2nt+i} = A_i$. Note that

$$\lim_{n\longrightarrow\infty} \frac{x_{2nt+i}}{x_{2(n+1)t-s+i}y_{2(n+1)t-s-t+i}} \le \lim_{n\longrightarrow\infty} \frac{x_{2nt+i}}{AB} = \frac{A_i}{AB} < A_i \tag{2.7}$$

and

$$\lim_{n\longrightarrow\infty} x_{2(n+1)t+i} = A_i. \tag{2.8}$$

Thus we have $x_n = A$ eventually. By Lemma 2.1, we see that y_n is a periodic sequence with period $2s$ eventually. The second case follows from the previously proved one by interchanging letters. The proof is complete. □

In the following, we assume $AB = 1$. For any $i \in \mathbf{N}_0$, let

$$\lim_{n\longrightarrow\infty} x_{2nt+i} = A_i \quad \text{if } A \ge B \tag{2.9}$$

and

$$\lim_{n\longrightarrow\infty} y_{2nt+i} = B_i \quad \text{if } B \ge A. \tag{2.10}$$

Then $A_i \ge A$ and $B_i \ge B$.

Lemma 2.3 *If $A \ge B = 1/A$ and $A_i > A$ for some $i \in \mathbf{N}_0$, then, for any $k \in \mathbf{N}$, $x_{2nt+ks+i}$ and $y_{2nt-t+ks+i}$ are constant sequences eventually. If $B \ge A = 1/B$ and $B_i > B$ for some $i \in \mathbf{N}_0$, then, for any $k \in \mathbf{N}$, $y_{2nt+ks+i}$ and $x_{2nt-t+ks+i}$ are constant sequences eventually.*

Proof Assume that $A \geq B$ and $A_i > A$ for some $i \in \mathbf{N}_0$. Since $A_i > A$, it follows from Lemma 2.2 and (1.1) that

$$\begin{aligned} x_{2nt+i} &= \max\left\{A, \frac{x_{2(n-1)t+i}}{x_{2nt-s+i}y_{2nt-t-s+i}}\right\} \\ &= \frac{x_{2(n-1)t+i}}{x_{2nt-s+i}y_{2nt-t-s+i}} \quad \text{eventually}. \end{aligned} \tag{2.11}$$

From this we have

$$\begin{aligned} B &\leq \lim_{n\longrightarrow\infty} y_{2nt-t-s+i} \\ &= \lim_{n\longrightarrow\infty} \frac{x_{2(n-1)t+i}}{x_{2nt+i}x_{2nt-s+i}} \\ &= \frac{1}{A_{-s+i}} \leq \frac{1}{A} = B. \end{aligned} \tag{2.12}$$

This implies

$$\lim_{n\longrightarrow\infty} x_{2nt-s+i} = A \tag{2.13}$$

and

$$\lim_{n\longrightarrow\infty} y_{2nt-t-s+i} = B \tag{2.14}$$

and

$$\lim_{n\longrightarrow\infty} y_{2nt-t+i} = \lim_{n\longrightarrow\infty} x_{2nt+i}x_{2nt-s+i} = A_iA. \tag{2.15}$$

Since

$$x_{2nt+s+i} = \max\left\{A, \frac{x_{2(n-1)t+s+i}}{x_{2nt+i}y_{2nt-t+i}}\right\} \tag{2.16}$$

and

$$\lim_{n\longrightarrow\infty} \frac{x_{2(n-1)t+s+i}}{x_{2nt+i}y_{2nt-t+i}} = \frac{A_{s+i}}{A_i^2A} < A_{s+i}, \tag{2.17}$$

we see that $x_{2nt+s+i} = A$ eventually. Note that

$$\lim_{n\longrightarrow\infty} \frac{x_{2(n-1)t+s+i}}{y_{2nt-t+i}} = \frac{A}{AA_i} < B, \tag{2.18}$$

from which it follows that

$$\begin{aligned} y_{2nt-t+s+i} &= \max\left\{B, \frac{x_{2(n-1)t+s+i}}{y_{2nt-t+i}}\right\} \\ &= B \quad \text{eventually}, \end{aligned} \tag{2.19}$$

and

$$\begin{aligned} y_{2nt-t+2s+i} &= \max\left\{B, \frac{x_{2(n-1)t+2s+i}}{y_{2nt-t+s+i}}\right\} \\ &= \frac{x_{2(n-1)t+2s+i}}{B} \quad \text{eventually}, \end{aligned} \tag{2.20}$$

and

$$\begin{aligned} x_{2nt+2s+i} &= \max\left\{A, \frac{y_{2nt-t+2s+i}}{x_{2nt+s+i}}\right\} \\ &= \max\{A, x_{2(n-1)t+2s+i}\} \\ &= x_{2(n-1)t+2s+i} \quad \text{eventually}. \end{aligned} \tag{2.21}$$

If $x_{2nt+2s+i} > A$ eventually, then, in a similar fashion, we obtain:

(1) $x_{2nt+3s+i} = A$ eventually and $y_{2nt-t+3s+i} = B$ eventually.

(2) $x_{2nt+4s+i}$ and $y_{2nt-t+4s+i}$ are constant sequences eventually.

If $x_{2nt+2s+i} = A$ eventually, then $y_{2nt-t+2s+i} = A/B$ eventually, and

$$\begin{aligned} y_{2nt-t+3s+i} &= \max\left\{B, \frac{x_{2(n-1)t+3s+i}}{y_{2nt-t+2s+i}}\right\} \\ &= \max\left\{B, \frac{x_{2(n-1)t+3s+i}B}{A}\right\} \\ &= \frac{x_{2(n-1)t+3s+i}B}{A} \quad \text{eventually}, \end{aligned} \tag{2.22}$$

and

$$\begin{aligned} x_{2nt+3s+i} &= \max\left\{A, \frac{y_{2nt-t+3s+i}}{x_{2nt+2s+i}}\right\} \\ &= \max\left\{A, \frac{x_{2(n-1)t+3s+i}B}{A^2}\right\} \quad \text{eventually}. \end{aligned} \tag{2.23}$$

From this we see that if $A = B$, then

$$x_{2nt+3s+i} = x_{2(n-1)t+3s+i} \quad \text{eventually}, \tag{2.24}$$

and if $A > B$, then

$$x_{2nt+3s+i} = A \quad \text{eventually}, \tag{2.25}$$

since

$$\lim_{n\longrightarrow\infty} \frac{x_{2(n-1)t+3s+i}B}{A^2} = \frac{A_{3s+i}B}{A^2} < A_{3s+i}. \tag{2.26}$$

Using induction and arguments similar to the ones developed in the above given proof, we can show that, for any $k \in \mathbf{N}$, $x_{2nt+ks+i}$ and $y_{2nt-t+ks+i}$ are constant sequences eventually. The second case follows from the previously proved one by interchanging letters. The proof is complete. □

Lemma 2.4 *If $A = 1/B > B$ and for some $i \in \mathbf{N}_0$, $x_{2nt+i} > A$ eventually and $A_i = A$, then, for any $k \in \mathbf{N}$, $x_{2nt+ks+i}$ and $y_{2nt-t+ks+i}$ are constant sequences eventually. If $B = 1/A > A$ and for some $i \in \mathbf{N}_0$, $y_{2nt+i} > B$ eventually and $B_i = B$, then, for any $k \in \mathbf{N}$, $y_{2nt+ks+i}$ and $x_{2nt-t+ks+i}$ are constant sequences eventually.*

Proof Assume that $A = 1/B > B$ and for some $i \in \mathbf{N}_0$, $x_{2nt+i} > A$ eventually and $A_i = A$. By (1.1) we have

$$\begin{aligned} x_{2nt+i} &= \max\left\{A, \frac{y_{2nt-t+i}}{x_{2nt-s+i}}\right\} \\ &= \frac{y_{2nt-t+i}}{x_{2nt-s+i}} \quad \text{eventually,} \end{aligned} \tag{2.27}$$

and

$$x_{2nt+s+i} = \max\left\{A, \frac{x_{2(n-1)t+s+i}}{x_{2nt+i}y_{2nt-t+i}}\right\} \tag{2.28}$$

and

$$\begin{aligned} \lim_{n\longrightarrow\infty} \frac{x_{2(n-1)t+s+i}}{x_{2nt+i}y_{2nt-t+i}} &= \lim_{n\longrightarrow\infty} \frac{x_{2(n-1)t+s+i}}{x_{2nt+i}^2 x_{2nt-s+i}} \\ &\leq \frac{A_{s+i}}{A^3} < A_{s+i}. \end{aligned} \tag{2.29}$$

Then we see that $x_{2nt+s+i} = A$ eventually. From this and $y_{2nt-t+i} \geq A^2$ eventually it follows that

$$\begin{aligned} y_{2nt-t+s+i} &= \max\left\{B, \frac{x_{2(n-1)t+s+i}}{y_{2nt-t+i}}\right\} \\ &= \max\left\{B, \frac{A}{y_{2nt-t+i}}\right\} \\ &= B \quad \text{eventually,} \end{aligned} \tag{2.30}$$

and

$$\begin{aligned} y_{2nt-t+2s+i} &= \max\left\{B, \frac{x_{2(n-1)t+2s+i}}{y_{2nt-t+s+i}}\right\} \\ &= \frac{x_{2(n-1)t+2s+i}}{B} \quad \text{eventually,} \end{aligned} \tag{2.31}$$

and

$$\begin{aligned} x_{2nt+2s+i} &= \max\left\{A, \frac{y_{2nt-t+2s+i}}{x_{2nt+s+i}}\right\} \\ &= \max\{A, x_{2(n-1)t+2s+i}\} \\ &= x_{2(n-1)t+2s+i} \quad \text{eventually.} \end{aligned} \tag{2.32}$$

Thus $x_{2nt+2s+i}$ and $y_{2nt-t+2s+i}$ are constant sequences eventually. Using arguments similar to the ones developed in the proof of Lemma 2.3, we can show that, for any $k \in \mathbf{N}$, $x_{2nt+ks+i}$ and

$y_{2nt-t+ks+i}$ are constant sequences eventually. The second case follows from the previously proved one by interchanging letters. The proof is complete. □

Theorem 2.2

(1) *Assume $A = 1/B > B$. Then one of the following statements holds.*
   (i) *$x_n = A$ eventually and y_n is a periodic sequence with period $2s$ eventually.*
   (ii) *If s is odd, then x_n, y_n are periodic sequences with period $2t$ eventually.*
   (iii) *If s is even, then x_n is a periodic sequence with period $2t$ eventually and y_n is a periodic sequence with period $2st$ eventually.*

(2) *Assume $B = 1/A > A$. Then one of the following statements holds.*
   (i) *$y_n = B$ eventually and x_n is a periodic sequence with period $2s$ eventually.*
   (ii) *If s is odd, then x_n, y_n are periodic sequences with period $2t$ eventually.*
   (iii) *If s is even, then y_n is a periodic sequence with period $2t$ eventually and x_n is a periodic sequence with period $2st$ eventually.*

Proof Assume that $A = 1/B > B$. If $x_n = A$ eventually, then by Lemma 2.1 we see that y_n is a periodic sequence with period $2s$ eventually. Now we assume that $x_n \not\equiv A$ eventually. Then we have $A_i > A$ (or $x_{2nt+i} > A$ eventually and $A_i = A$) for some $0 \le i \le 2t - 1$.

If s is odd, then $\gcd(2t, s) = 1$. Thus, for every $j \in \{0, 1, 2, \dots, 2t - 1\}$, there exist some $1 \le i_j \le 2t$ and integer λ_j such that $i_j s = \lambda_j 2t + j$ since $\{rs : 0 \le r \le 2t - 1\} = \{0, 1, 2, \dots, 2t - 1\}$ (mod $2t$). By Lemma 2.3 and Lemma 2.4 we see that, for any $k \in \mathbf{N}$, $x_{2nt+ks+i}$ and $y_{2nt-t+ks+i}$ are constant sequences eventually. Thus, for any $0 \le r \le 2t - 1$, x_{2nt+r} and y_{2nt+r} are constant sequences eventually, which implies that x_n, y_n are periodic sequences with period $2t$ eventually.

In the following, we assume that s is even with $s = 2s'$. Then $\gcd(t, s') = 1$ and t is odd. Thus, for every $j \in \{0, 1, 2, \dots, t - 1\}$, there exist some $1 \le i_j \le t$ and integer λ_j such that $i_j s' = \lambda_j t + j$ and $i_j s = \lambda_j 2t + 2j$.

If $x_{2nt+i} \not\equiv A$ eventually for some $i \in \{0, 2, \dots\}$ and $x_{2nt+l} \not\equiv A$ eventually for some $l \in \{1, 3, \dots\}$, then by Lemma 2.3 and Lemma 2.4 we see that, for any $k \in \mathbf{N}$, $x_{2nt+ks+i}$, $y_{2nt-t+ks+i}$, $x_{2nt+ks+l}$ and $y_{2nt-t+ks+l}$ are constant sequences eventually. Thus, for any $0 \le r \le 2t - 1$, x_{2nt+r} and y_{2nt+r} are constant sequences eventually, which implies that x_n, y_n are periodic sequences with period $2t$ eventually.

If $x_{2nt+i} \not\equiv A$ eventually for some $i \in \{0, 2, \dots\}$ and $x_{2nt+l} = A$ eventually for any $l \in \{1, 3, \dots\}$, then by Lemma 2.3 and Lemma 2.4 we see that, for any $k \in \mathbf{N}$, $x_{2nt+ks+i}$ and $y_{2nt-t+ks+i}$ are constant sequences eventually. This implies that, for every $r \in \{0, 1, 2, \dots, 2t - 1\}$, x_{2nt+r} is constant sequence eventually and for every $l \in \{1, 3, \dots\}$, y_{2nst+l} is constant sequence eventually. By (1.1) we see that there exists $N \in \mathbf{N}$ such that, for any $n \ge N$ and $r \in \{0, 2, \dots\}$,

$$y_{2nt+r} = \max\left\{B, \frac{A}{y_{2nt+r-s}}\right\}. \tag{2.33}$$

Then we have $y_{2nt+r} y_{2nt+r-s} \ge A$. Thus, for any $n \ge N$ and $l \in \{1, 3, \dots\}$ and $k \in \mathbf{N}$,

$$\begin{aligned} B \le y_{2nt+t+2ks+l} &= \max\left\{B, \frac{x_{2nt+2ks+l}}{y_{2nt+t+2ks+l-s}}\right\} \\ &= \max\left\{B, \frac{A y_{2nt+t+2ks+l-2s}}{y_{2nt+t+2ks+l-s} y_{2nt+t+2ks+l-2s}}\right\} \end{aligned}$$

$$\begin{aligned}&\le \max\{B, y_{2nt+t+2ks-2s+l}\}\\&= y_{2nt+t+2ks-2s+l} \quad \text{eventually.}\end{aligned} \tag{2.34}$$

Then, for every $n \ge N$ and $l \in \{1,3,\ldots\}$, we have

$$B \le \cdots \le y_{2nt+t+2ks+l} \le y_{2nt+t+2ks-2s+l} \le y_{2nt+t+2s+l} \le y_{2nt+t+l}. \tag{2.35}$$

We claim that, for every $n \ge N$ and $l \in \{1,3,\ldots\}$, $\{y_{2nt+t+2ks+l}\}_{k\in\mathbf{N}}$ is a constant sequence eventually. Assume on the contrary that, for some $n \ge N$ and some $l \in \{1,3,\ldots\}$, $\{y_{2nt+t+2ks+l}\}_{k\in\mathbf{N}}$ is not a constant sequence eventually. Then there exists a sequence of positive integers $k_1 < k_2 < \cdots$ such that, for any $r \in \mathbf{N}$, we have

$$\begin{aligned}B < y_{2nt+t+2k_rs+l} &= \frac{A}{y_{2nt+t+2k_rs+l-s}}\\< y_{2nt+t+2k_{r-1}s+l} &= \frac{A}{y_{2nt+t+2k_{r-1}s+l-s}},\end{aligned} \tag{2.36}$$

which implies $y_{2nt+t+2k_rs+l-s} > y_{2nt+t+2k_{r-1}s+l-s}$ for any $r \in \mathbf{N}$. This is a contradiction. Take $ps > N$. Then $y_{2nst+t+l} = y_{2pst+t+2(nt-pt)s+l}$ is a constant sequence eventually for any $l \in \{1,3,\ldots\}$. From the above we see that y_n is a periodic sequence with period $2st$ eventually.

In a similar fashion, we can show that if $x_{2nt+i} = A$ eventually for any $i \in \{0,2,\ldots\}$ and $x_{2nt+l} \neq A$ eventually for some $l \in \{1,3,\ldots\}$, then also statement (1(iii)) holds.

The second case follows from the previously proved one by interchanging letters. The proof is complete. □

Now we assume that $A = B = 1$. Then, for any $0 \le i \le 2t-1$ and $n \in \mathbf{N}_0$, we have $1 \le x_{2(n+1)t+i} \le x_{2nt+i}$ eventually and $1 \le y_{2(n+1)t+i} \le y_{2nt+i}$ eventually.

Lemma 2.5 *Let $A = B = 1$ and $s \ge t$. Then the following statements hold.*

(1) *If $A_i = 1$, then $B_{t+i} = 1$. If $B_i = 1$, then $A_{t+i} = 1$.*

(2) *If $x_N = 1$ for some $N \in \mathbf{N}$ and $A_{2nt+N+ks} = 1$ for any $k, n \in \mathbf{N}$, then $x_{2nt+N+ks} = y_{2nt+t+ks+N} = 1$ for any $k, n \in \mathbf{N}$. If $y_N = 1$ for some $N \in \mathbf{N}$ and $B_{2nt+N+ks} = 1$ for any $k, n \in \mathbf{N}$, then $y_{2nt+N+ks} = x_{2nt+t+ks+N} = 1$ for any $k, n \in \mathbf{N}$.*

(3) *If s is even and $\gcd(s,t) = 1$, then $1 \in \{x_n : n \in \{0,2,\ldots\}\} \cup \{y_{t+n} : n \in \{0,2,\ldots\}\}$ and $1 \in \{x_n : n \in \{1,3,\ldots\}\} \cup \{y_{t+n} : n \in \{1,3,\ldots\}\}$.*

(4) $1 \in \{x_n : n \in \mathbf{N}\} \cup \{y_n : n \in \mathbf{N}\}$.

Proof (1) Assume that $A_i = 1$. Assume on the contrary that $B_{t+i} > 1$. It follows from (1.1) that

$$y_{2nt+t+i} = \frac{x_{2nt+i}}{y_{2nt+t-s+i}}. \tag{2.37}$$

This implies

$$1 \le \lim_{n\longrightarrow\infty} y_{2nt+t-s+i} = \frac{1}{B_{t+i}} < 1. \tag{2.38}$$

This is a contradiction. The second case follows from the previously proved one by interchanging letters.

(2) If $x_N = 1$ for some $N \in \mathbf{N}$, then $x_{2nt+N} = 1$ for any $n \in \mathbf{N}$. It follows from (1.1) that

$$\begin{aligned} y_{2nt+t+N} &= \max\left\{1, \frac{x_{2nt+N}}{y_{2nt+t-s+N}}\right\} \\ &= \max\left\{1, \frac{1}{y_{2nt+t-s+N}}\right\} = 1 \end{aligned} \tag{2.39}$$

and

$$\begin{aligned} y_{2nt+t+s+N} &= \max\left\{1, \frac{x_{2nt+s+N}}{y_{2nt+t+N}}\right\} \\ &= x_{2nt+s+N} \end{aligned} \tag{2.40}$$

and

$$\begin{aligned} x_{2(n+1)t+N+s} &= \max\left\{1, \frac{y_{2nt+t+s+N}}{x_{2(n+1)t+N}}\right\} \\ &= \max\{1, y_{2nt+t+s+N}\} \\ &= y_{2nt+t+s+N} \\ &= x_{2nt+N+s}. \end{aligned} \tag{2.41}$$

Thus $x_{2nt+N+s} = y_{2nt+t+s+N} = 1$ for any $n \in \mathbf{N}$. In a similar fashion, we can show that $x_{2nt+N+ks} = y_{2nt+t+ks+N} = 1$ for any $k, n \in \mathbf{N}$. The second case follows from the previously proved one by interchanging letters.

(3) If s is even and $\gcd(s,t) = 1$, then t is odd. Assume on the contrary that $1 \notin \{x_n : n \in \{0, 2, \ldots\}\} \cup \{y_{t+n} : n \in \{0, 2, \ldots\}\}$. Then it follows from (1.1) that, for any $n \in \mathbf{N}$,

$$\begin{aligned} y_{2nt+t} &= \max\left\{1, \frac{x_{2nt}}{y_{2nt+t-s}}\right\} \\ &= \frac{x_{2nt}}{y_{2nt+t-s}} > 1 \end{aligned} \tag{2.42}$$

and

$$\begin{aligned} x_{2nt+2t-s} &= \max\left\{1, \frac{y_{2nt+t-s}}{x_{2nt+2t-2s}}\right\} \\ &= \frac{y_{2nt+t-s}}{x_{2nt+2t-2s}} > 1. \end{aligned} \tag{2.43}$$

Thus

$$\begin{aligned} x_{2nt} &> x_{2nt-2(s-t)} \\ &> x_{2nt-4(s-t)} \\ &\cdots \\ &> x_{2t(n-t+s)}. \end{aligned} \tag{2.44}$$

This is a contradiction.

(4) Case (4) is treated similarly to case (3). The proof is complete. □

Theorem 2.3 *Let $A = B = 1$ and $s \geq t$. Then one of the following statements holds.*
(1) *$x_n = 1$ eventually and y_n is a periodic sequence with period $2s$ eventually.*
(2) *$y_n = 1$ eventually and x_n is a periodic sequence with period $2s$ eventually.*
(3) *x_n, y_n are periodic sequences with period $2t$ eventually.*

Proof If $x_n = 1$ (or $y_n = 1$) eventually, then by Lemma 2.1 we see that y_n (or x_n) is a periodic sequence with period $2s$ eventually. Now we assume that $x_n \neq 1$ eventually. Then we have $A_i > 1$ for some $0 \leq i \leq 2t - 1$ or $\lim_{n\longrightarrow\infty} x_n = 1$.

If s is odd, then $\gcd(2t, s) = 1$. Thus, for every $j \in \{0, 1, 2, \dots, 2t - 1\}$, there exist some $1 \leq i_j \leq 2t$ and integer λ_j such that $i_j s = \lambda_j 2t + j$. By Lemma 2.3 and Lemma 2.5 we see that, for any $k \in \mathbf{N}$, $x_{2nt+ks+i}$ and $y_{2nt-t+ks+i}$ are constant sequences eventually, or for some $N \in \mathbf{N}$, $x_{2nt+N+ks} = y_{2nt+t+ks+N} = 1$ for any $k, n \in \mathbf{N}$, or for some $N \in \mathbf{N}$, $y_{2nt+N+ks} = x_{2nt+t+ks+N} = 1$ for any $k, n \in \mathbf{N}$. Thus, for any $0 \leq r \leq 2t - 1$, x_{2nt+r} and y_{2nt+r} are constant sequences eventually, which implies that x_n, y_n are periodic sequences with period $2t$ eventually.

In the following, we assume that s is even with $s = 2s'$, then $\gcd(t, s') = 1$ and t is odd. Thus, for every $j \in \{0, 1, 2, \dots, t - 1\}$, there exist some $1 \leq i_j \leq t$ and integer λ_j such that $i_j s = \lambda_j 2t + 2j$.

If $A_i > 1$ for some $i \in \{0, 2, \dots\}$, then by Lemma 2.3 we see that, for any $k \in \mathbf{N}$, $x_{2nt+ks+i}$ and $y_{2nt-t+ks+i}$ are constant sequences eventually. If $A_i = 1$ for any $i \in \{0, 2, \dots\}$, then by Lemma 2.5 we have $B_{t+i} = 1$ for any $i \in \{0, 2, \dots\}$ and $x_{2nt+i+ks} = y_{2nt-t+ks+i} = 1$ for any $k \in \mathbf{N}$ eventually. In a similar fashion, also we can show that, for any $i \in \{1, 3, \dots\}$, $x_{2nt+ks+i}$ and $y_{2nt-t+ks+i}$ are constant sequences eventually for any $k \in \mathbf{N}$, or $x_{2nt+i+ks} = y_{2nt-t+ks+i} = 1$ for any $k \in \mathbf{N}$ eventually for any $i \in \{1, 3, \dots\}$ and $k \in \mathbf{N}$. Thus, for any $0 \leq r \leq 2t - 1$, x_{2nt+r} and y_{2nt+r} are constant sequences eventually. This implies that x_n, y_n are periodic sequences with period $2t$ eventually.

Using the previously proved one by interchanging letters, also we can show that if $y_n \neq 1$ eventually, then x_n, y_n are periodic sequences with period $2t$ eventually. The proof is complete. □

In Example 3.1 of [37], we showed that the equation

$$x_n = \frac{x_{n-t}}{x_{n-s}} (t > s) \tag{2.45}$$

has a positive solution z_n $(n \geq -t)$ with $1 < z_{n+1} < z_n$ for any $n \geq -t$ and $\lim_{n\longrightarrow\infty} z_n = 1$.

From Example 3.1 of [37], we obtain the following theorem.

Theorem 2.4 *Let $A \leq 1$ and $B \leq 1$ and $s < t$. Assume z_n $(n \geq -t)$ is a positive solution of (2.45) with $1 < z_{n+1} < z_n$ for any $n \geq -t$ and $\lim_{n\longrightarrow\infty} z_n = 1$. Then equation (1.1) have a solution (x_n, y_n) with $1 < x_{n+1} = y_{n+1} = z_{n+1} < x_n = y_n = z_n$ for any $n \geq -t$ and $\lim_{n\longrightarrow\infty} x_n = \lim_{n\longrightarrow\infty} y_n = 1$.*

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