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A maximum principle for fully coupled controlled forward–backward stochastic difference systems of mean-field type

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Abstract

In this paper, we consider the optimal control problem for fully coupled forward–backward stochastic difference equations of mean-field type under weak convexity assumption. By virtue of employing a suitable product rule and formulating a mean-field backward stochastic difference equation, we establish the stochastic maximum principle and also derive, under additional assumptions, that the stochastic maximum principle is also a sufficient condition. As an application, a Stackelberg game of mean-field backward stochastic difference equation is presented to demonstrate our results.

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1 Introduction

Let $T > 0$ be fixed, $(\Omega, \mathfrak{F}, \{\mathfrak{F}_t\}_{0\le t\le T}, \mathbb{P})$ be a filtered probability space, on which a martingale process W_t with independent increments is defined, and $\mathfrak{F}_t = \sigma\{W_l, l = 0, 1, \dots, t-1\} \vee \mathcal{N}_{\mathbb{P}}$ (the set of all $\mathbb{P}$-null subsets). Consider the following discrete-time fully coupled stochastic system:

$$\begin{cases} \Delta X_t = b(t, X_t, Y_t, Z_t, \mathbb{E}X_t, \mathbb{E}Y_t, \mathbb{E}Z_t, u_t) \\ \qquad\quad + \sigma(t, X_t, Y_t, Z_t, \mathbb{E}X_t, \mathbb{E}Y_t, \mathbb{E}Z_t, u_t)\Delta W_t, \\ -\Delta Y_t = f(t+1, X_{t+1}, Y_{t+1}, Z_{t+1}, \mathbb{E}X_{t+1}, \mathbb{E}Y_{t+1}, \mathbb{E}Z_{t+1}, u_{t+1}) - Z_t\Delta W_t - \Delta M_t, \\ X_0 = x_0, \qquad Y_T = l(X_T, \mathbb{E}X_T), \end{cases} \tag{1.1}$$

with the cost functional

$$J(u) = \mathbb{E}\left\{\sum_{t=0}^{T-1} g(t, X_t, Y_t, Z_t, \mathbb{E}X_t, \mathbb{E}Y_t, \mathbb{E}Z_t, u_t) + h(X_T, \mathbb{E}X_T)\right\}. \tag{1.2}$$

Here, we reserve the notation Δ for the backward difference operator $\Delta X_t = X_{t+1} - X_t$. W, M are the square integrable martingale processes and M is strongly orthogonal to W. $\mathbb{E}$ means the expectation operator and f, b, σ, g, h, l are given functions (satisfying some proper conditions to be elaborated later). Then we could present the following stochastic optimal control problem.

Problem $\mathcal{A}$. Find $\bar{u} \in \mathscr{U}_{\mathrm{ad}}$ (which shall be defined later) such that

$$J(\bar{u}) = \inf_{u \in \mathscr{U}_{\mathrm{ad}}} J(u).$$

For stochastic optimal control problems (see [1]), one of the main topics is to establish the stochastic maximum principle (SMP). A wide range of concerns have been given to different versions of SMP (see [2–5]), especially, to forward–backward stochastic control systems (see [6–9]). It is widely recognized that forward–backward stochastic differential equations (FBSDEs) are extensively studied and there are productive results (see [6, 10, 11]). Nevertheless, discrete-time optimal control problems are more relevant to economic, engineering, biomedical, operation research problems, optimizing complex technological systems, etc. As is known to all, Pontryagin maximum principle in continuous-time framework cannot be extended to discrete-time counterpart, except for some very special cases, due to the nature of admissible control variations. Naturally, it motivated us to formulate discrete analog and even some improper results were deduced. Butkovskii [12] clearly demonstrated some errors in the existing works. The intrinsic reason for the errors is that the significance of convexity has been ignored. Generally speaking, the discrete-time maximum principle fails unless a certain convexity precondition is imposed on the control system. Pshenichnyi [13] elaborated why discrete-time systems require a certain convexity assumption for the effectiveness of the necessary condition while continuous-time systems enjoy it automatically because of the so-called hidden convexity. To the best of our knowledge, the study on the SMP of forward–backward stochastic difference equations (FBSΔEs) is quite rare in the literature. To fill the gap, in this work, we are devoted to considering the SMP of the forward–backward stochastic difference systems.

As for the discrete-time framework, recently, Mahmudov [14] derived the first-order and second-order necessary optimality conditions for discrete-time stochastic optimal control problems by virtue of new discrete-time backward stochastic equation and backward stochastic matrix equation. Lin and Zhang [15] investigated the SMP where the state equation was just on a forward SΔE with the convex control domain. Xu et al. [16] considered the solvability of fully coupled FBSΔEs, in which the BSΔE was given as the conditional expectation form and the coefficients in the backward equation were degenerate. Some representative works in this direction include [17–20]. Very recently, Ji and Liu [21] first discussed the SMP for FBSΔEs under the convex control domain, which made substantial progresses in discrete-time forward–backward systems.

In 2009, Buckdahn et al. [22] investigated a special case of backward stochastic differential equations (BSDEs), the so-called mean-field BSDEs, which were derived by a limit of high dimensional FBSDEs, parallel to a large stochastic particle system. From then on, many authors discussed the mean-field system in different frameworks (see [4, 23]).

Motivated by the above discussions, our purpose of this paper is to derive the more general and constructive SMP for mean-field system (1.1)–(1.2) under weaker convexity assumption. From the perspective of the techniques adopted for discrete-time case, the obstacles encountered are twofold. The first issue entails choosing a suitable expression of the product rule

$$\Delta\langle X_t, Y_t\rangle = \langle X_{t+1}, \Delta Y_t\rangle + \langle \Delta X_t, Y_t\rangle = \langle X_t, \Delta Y_t\rangle + \langle \Delta X_t, Y_{t+1}\rangle.$$

In our setting, the Itô formula in continuous-time framework is invalid. In addition, most of the methods applied to discuss continuous-time systems cannot be directly adapted to discrete-time cases. Hence, it is necessary to employ a more characteristic and refined approach for investigating the discrete-time stochastic optimal control problems. The next issue entails formulating the discrete-time counterpart BSΔE as (1.1), which is distinctly different from the continuous-time BSDE. Lately, many authors have been devoted to considering BSΔE (see [24–26]). In general, there are two approaches to formulating BSΔE. One is driving by a finite state process (see [25]). In this work, we adopt another formulation as in [24], which is driven by a martingale with independent increments and the generator f in (1.1) relies on time $t+1$. Based on these arguments, we could obtain the dual principle. It is worth mentioning that our paper differs from [19] in the following aspects. Firstly, our work is based on a weaker convexity assumption. Secondly, our results are obtained in the mean-field framework. Thirdly, we not only establish the SMP, but also derive, under additional assumptions, the SMP, which turns to be a sufficient condition. Finally, as an application, we present a Stackelberg game of mean-field BSΔE to demonstrate our results. To sum up, this is the first paper to discuss the discrete-time forward–backward stochastic optimal control problems of mean-field type under weaker convexity assumption, enabling us to establish the more general and constructive SMP. Our work generalizes and enhances the previously known SMP of [19, 23]. Meanwhile, it extends the classical results of [17, 23] to the mean-field theory as well as forward–backward system. It is interesting to remark that the results of our work also remain for multi-dimensional driving process; in addition, we could also consider a more general system, in which the mean-field terms are allowed to depend on some functional of the law (see [27]). There is no essential difficulty.

The reminder of the paper is organized as follows. The next section states some preliminaries. Section 3 is devoted to considering MF-FBSΔE (1.1). In Sect. 4, we establish the SMP and the sufficient condition for Problem $\mathcal{A}$. In Sect. 5, a Stackelberg game of mean-field BSΔE is given to illustrate the theoretical results. Section 6 presents some perspectives and open problems.

2 Preliminaries

Let $\mathbb{T} = \{0, 1, \dots, T\}$. For a vector x, x' stands for its transpose. I represents the unit matrix with appropriate dimension. $\mathbb{E}_t$ means the conditional mathematical expectation $\mathbb{E}[\cdot|\mathfrak{F}_t]$. $\mathfrak{F}_0 = \{\emptyset, \Omega\}$ and $\mathfrak{F} = \mathfrak{F}_T$. For $t \in \{0, \dots, T-1\}$, $\mathbb{E}_t[\Delta W_t] = \mathbb{E}[\Delta W_t] = 0$, $\mathbb{E}[\Delta W_t \Delta W_t'] = I$. Now, we shall introduce some spaces to be used frequently in what follows.

$$\mathcal{L}^2(\mathfrak{F}_t; \mathbb{R}^n) = \{X_t : \Omega \to \mathbb{R}^n \mid X_t \text{ is } \mathfrak{F}_t\text{-measurable}, \mathbb{E}|X_t|^2 < \infty\},$$

$$\mathcal{S}^2(0, T; \mathbb{R}^n) = \{X : \{0, 1, \dots, T\} \times \Omega \to \mathbb{R}^n \mid X \text{ is } \{\mathfrak{F}_t\}\text{-adapted}, \mathbb{E}|X_t|^2 < \infty\},$$

$$\mathcal{M}^2[0, T] := \mathcal{S}^2(0, T; \mathbb{R}^n) \times \mathcal{S}^2(0, T; \mathbb{R}^n) \times \mathcal{S}^2(0, T-1; \mathbb{R}^n) \times \mathcal{S}^2(0, T; \mathbb{R}^n),$$

$$\mathcal{K}^2[0,T] := \mathcal{S}^2(0,T;\mathbb{R}^n) \times \mathcal{S}^2(0,T;\mathbb{R}^n) \times \mathcal{S}^2(0,T-1;\mathbb{R}^n),$$
$$\mathcal{H}^2[0,T] := \mathcal{S}^2(0,T;\mathbb{R}^n) \times \mathcal{S}^2(0,T-1;\mathbb{R}^n) \times \mathcal{S}^2(0,T;\mathbb{R}^n),$$
$$\mathcal{N}^2[0,T] := \mathcal{S}^2(0,T-1;\mathbb{R}^n) \times \mathcal{S}^2(0,T-1;\mathbb{R}^n) \times \mathcal{S}^2(1,T;\mathbb{R}^n).$$

In addition, we introduce the following admissible control set:

$$\mathscr{U}_{\text{ad}} = \{(u_0,u_1,\ldots,u_T) : u_t \in \mathcal{L}^2(\mathfrak{F}_t;\mathbb{R}^m) \text{ and } u_t \in U_t \subseteq \mathbb{R}^m\}.$$

Definition 2.1 ([28]) A point $\bar{y} \in S \subset \mathbb{R}^m$ is called a relative interior point of S along the straight line $l(\bar{y},\tilde{y}) := \{\tilde{y} \mid \tilde{y} = \bar{y} + \kappa(\tilde{y} - \bar{y}), \tilde{y} \in S \setminus \{\bar{y}\} \neq \emptyset, \kappa \in \mathbb{R}\}$ if there exists $\gamma = \gamma(\tilde{y}) \in (0,1]$ such that $\bar{y} + \varepsilon(\tilde{y} - \bar{y}) \in S$ holds for all $\varepsilon \in [-\gamma,\gamma]$. Besides, $\bar{y}$ is called a relative interior point of S in a broad sense if $\bar{y}$ is a relative interior point of S along every straight line in the set $\{l(\bar{y},y) : y \in S \setminus \{\bar{y}\}\}$. The totality of these points is called a relative interior of S in a broad sense and is denoted by $\operatorname{ri} S$. S is called relatively open in a broad sense if $\operatorname{ri} S = S$.

Definition 2.2 ([18]) A set $S \subset \mathbb{R}^m$ is called γ-convex relative to a point $y_0 \in S$ if, for each point $y \in S$, there exists $\gamma = \gamma(y) \in (0,1]$ such that, for all $\varepsilon \in [0,\gamma]$, $y_0 + \varepsilon(y - y_0) \in S$ holds. S is γ-convex if S is γ-convex relative to all of its points.

Definition 2.3 A set $S \subset \mathbb{R}^m$ is called $\pm\gamma$-convex relative to a point $y_0 \in S$ if, for each point $y \in S$, there exists $\gamma = \gamma(y) \in (0,1]$ such that, for all $\varepsilon \in [0,\gamma]$ or for all $\varepsilon \in [-\gamma,0]$, $y_0 + \varepsilon(y - y_0) \in S$ holds. S is $\pm\gamma$-convex if S is $\pm\gamma$-convex relative to all of its points.

Remark 2.1 It is obvious that a relatively open set in a broad sense, convex and open sets are a γ-convex set. Besides, a γ-convex set is a $\pm\gamma$-convex set. Nevertheless, the reverse does not always hold. For instance, $M_1 = [1,2) \cup (3,4]$ is γ-convex, but it is neither a convex set nor an open set. $M_2 = [1,2] \cup [3,4]$ is $\pm\gamma$-convex, but it is not γ-convex.

We proceed to introducing some notations and basic assumptions which shall be assumed throughout the paper. Denote the usual inner product by $\langle \cdot,\cdot \rangle$ and the norm by $|\cdot|$ of a Euclidean space. For $\Gamma = (x,y,z,\tilde{x},\tilde{y},\tilde{z})$, define $F(t,\Gamma,u) = (-f(t,\Gamma,u), b(t,\Gamma,u), \sigma(t,\Gamma,u))$ and $\Lambda = b,\sigma,f,g$.

(A1) $f(t,y,z,\tilde{y},\tilde{z})$ is uniformly Lipschitz continuous and independent of z, $\tilde{z}$ at $t = T$, i.e., for any $y,y_1,z,z_1 \in \mathbb{R}^n$, there exists a constant $c > 0$ such that

$$\begin{cases} |f(T,y,z,\tilde{y},\tilde{z}) - f(T,y_1,z_1,\tilde{y}_1,\tilde{z}_1)| \le c(|y-y_1| + |\tilde{y}-\tilde{y}_1|), \quad t = T, \mathbb{P}\text{-a.s.}, \\ |f(t,y,z,\tilde{y},\tilde{z}) - f(t,y_1,z_1,\tilde{y}_1,\tilde{z}_1)| \\ \quad \le c(|y-y_1| + |\tilde{y}-\tilde{y}_1| + |z-z_1| + |\tilde{z}-\tilde{z}_1|), \quad t \in \{1,2,\ldots,T-1\}, \mathbb{P}\text{-a.s.}, \\ f(t,0,0,0,0) \in \mathcal{L}^2(\mathfrak{F}_t;\mathbb{R}^n), \quad t \in \{1,2,\ldots,T\}, \mathbb{P}\text{-a.s.} \end{cases}$$

(A2) b,σ,f,g are uniformly Lipschitz continuous and differentiable on Γ, u; l, h are continuously differentiable on x, $\tilde{x}$, and all the derivatives are uniformly bounded. Moreover, f is independent of z, $\tilde{z}$ at $t = T$.

(A3) $\forall \Gamma$, $u \in \mathscr{U}_{\text{ad}}$, $\Lambda(\cdot,\Gamma,u)$ is a $\mathfrak{F}_t$-adapted process, $l(x,\tilde{x}) \in \mathcal{L}^2(\mathfrak{F};\mathbb{R}^n)$, $h(x,\tilde{x}) \in \mathcal{L}^2(\mathfrak{F};\mathbb{R})$, and $F(t,0,0) \in \mathcal{L}^2(\mathfrak{F}_t;\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$, $g(t,0,0) \in \mathcal{L}^2(\mathfrak{F}_t;\mathbb{R})$.

(A4) (Monotonic conditions)
For $t \in \{1,\dots,T-1\}$,

$$\begin{aligned}&\mathbb{E}\bigl\langle F(t,\Gamma,u)-F(t,\Gamma_1,u),\Theta-\Theta_1\bigr\rangle \le -\beta\mathbb{E}|\Theta-\Theta_1|^2, \quad \mathbb{P}\text{-a.s.},\\ &\quad\forall\Gamma=(x,y,z,\tilde{x},\tilde{y},\tilde{z}),\Gamma_1=(x_1,y_1,z_1,\tilde{x}_1,\tilde{y}_1,\tilde{z}_1),\Theta=(x,y,z),\Theta_1=(x_1,y_1,z_1).\end{aligned}$$

For $t=T$,

$$\mathbb{E}\bigl\langle -f(T,x,y,\tilde{x},\tilde{y},u)+f(T,x_1,y_1,\tilde{x}_1,\tilde{y}_1,u),x-x_1\bigr\rangle \le -\beta\mathbb{E}|x-x_1|^2, \quad \mathbb{P}\text{-a.s.}$$

For $t=0$,

$$\begin{aligned}&\mathbb{E}\bigl\langle b(0,\Gamma,u)-b(0,\Gamma_1,u),y-y_1\bigr\rangle+\mathbb{E}\bigl\langle\sigma(0,\Gamma,u)-\sigma(0,\Gamma_1,u),z-z_1\bigr\rangle\\ &\quad\le-\beta\mathbb{E}\bigl(|y-y_1|^2+|z-z_1|^2\bigr), \quad \mathbb{P}\text{-a.s.}\end{aligned}$$

Besides,

$$\mathbb{E}\bigl\langle l(x,\tilde{x})-l(x_1,\tilde{x}_1),x-x_1\bigr\rangle \ge c\mathbb{E}|x-x_1|^2, \mathbb{P}\text{-a.s.},$$

where c, β are nonnegative constants.

(A5) The set U_t ($t\in\mathbb{T}$) is $\pm\gamma$-convex.
Throughout the paper, we formally denote $f(0,\Gamma,u)=b(T,\Gamma,u)=\sigma(T,\Gamma,u)=g(T,\Gamma,u)\equiv 0$. Let $\pi\in\mathcal{L}^2(\mathfrak{F};\mathbb{R}^n)$, we consider the following MF-BSΔE:

$$\begin{cases}\Delta Y_t=-f(t+1,Y_{t+1},Z_{t+1},\mathbb{E}Y_{t+1},\mathbb{E}Z_{t+1})+Z_t\Delta W_t+\Delta M_t,\\ Y_T=\pi.\end{cases} \tag{2.1}$$

Definition 2.4 The triple of processes $(Y,Z,M)\in\mathcal{H}^2[0,T]$ is called a solution of MF-BSΔE (2.1) if it satisfies (2.1) for any $t\in\{0,1,\dots,T-1\}$ and M is a martingale process strongly orthogonal to W.

Theorem 2.1 *Assume that* (A1) *holds, then for any* $\pi\in\mathcal{L}^2(\mathfrak{F};\mathbb{R}^n)$, *MF-BS$\Delta$E* (2.1) *admits a unique adapted solution* (Y,Z,M).

Proof Firstly, we shall show the existence by using the backward induction method. From (A1) and $\pi\in\mathcal{L}^2(\mathfrak{F};\mathbb{R}^n)$, we have $f(T,\pi,\mathbb{E}\pi)\in\mathcal{L}^2(\mathfrak{F};\mathbb{R}^n)$. Then $\mathbb{E}\{|\mathbb{E}_{T-1}[\pi+f(T,\pi,\mathbb{E}\pi)]|^2\}<\infty$. Hence, $\pi+f(T,\pi,\mathbb{E}\pi)-\mathbb{E}_{T-1}[\pi+f(T,\pi,\mathbb{E}\pi)]$ is a square integrable martingale difference. Further, by the Galtchouk–Kunita–Watanabe decomposition in [29], there are $Z_{T-1}\in\mathfrak{F}_{T-1}$, $Z_{T-1}\Delta W_{T-1}\in\mathcal{L}^2(\mathfrak{F}_{T-1};\mathbb{R}^n)$, and $\Delta M_{T-1}\in\mathcal{L}^2(\mathfrak{F}_{T-1};\mathbb{R}^n)$ such that $\mathbb{E}_{T-1}[\Delta M_{T-1}]=\mathbb{E}_{T-1}[\Delta M_{T-1}\Delta W'_{T-1}]=0$ and

$$\pi+f(T,\pi,\mathbb{E}\pi)-\mathbb{E}_{T-1}\bigl[\pi+f(T,\pi,\mathbb{E}\pi)\bigr]=Z_{T-1}\Delta W_{T-1}+\Delta M_{T-1}. \tag{2.2}$$

Here, ΔM_{T-1} is uniquely determined in that decomposition. Multiplying (2.2) by $\Delta W'_{T-1}$ and then applying $\mathbb{E}_{T-1}$ to both sides, we derive

$$\mathbb{E}_{T-1}\big[\big(\pi + f(T,\pi,\mathbb{E}\pi)\big)\Delta W'_{T-1}\big] = Z_{T-1}.$$

We further obtain

$$\mathbb{E}|Z_{T-1}|^2 \le \mathbb{E}\big\{\mathbb{E}_{T-1}\big[\big|\pi + f(T,\pi,\mathbb{E}\pi)\big|^2\big]\mathbb{E}_{T-1}\big[\Delta W^2_{T-1}\big]\big\} < \infty.$$

This implies $Y_{T-1} = \mathbb{E}_{T-1}[\pi + f(T,\pi,\mathbb{E}\pi)] \in \mathcal{L}^2(\mathfrak{F}_{T-1};\mathbb{R}^n)$. Thus, we determine Y_{T-1}, Z_{T-1}, ΔM_{T-1}.

We continue this backward procedure. Assume that $Y_{t+1} \in \mathcal{L}^2(\mathfrak{F}_{t+1};\mathbb{R}^n)$, $t \in \{0,1,\dots,T-2\}$. Similarly to the above discussions, we know $Z_t \in \mathfrak{F}_t$, $Z_t\Delta W_t \in \mathcal{L}^2(\mathfrak{F}_t;\mathbb{R}^n)$, $\Delta M_t \in \mathcal{L}^2(\mathfrak{F}_t;\mathbb{R}^n)$ such that $\mathbb{E}_t[\Delta M_t] = \mathbb{E}_t[\Delta M_t \Delta W'_t] = 0$ and

$$\begin{cases} Y_{t+1} + f(t+1, Y_{t+1}, \mathbb{E}Y_{t+1}, Z_{t+1}, \mathbb{E}Z_{t+1}) - \mathbb{E}_t[Y_{t+1} + f(t+1, Y_{t+1}, \mathbb{E}Y_{t+1}, Z_{t+1}, \mathbb{E}Z_{t+1})] \\ \quad = Z_t\Delta W_t + \Delta M_t, \\ Z_t = \mathbb{E}_t[(Y_{t+1} + f(t+1, Y_{t+1}, \mathbb{E}Y_{t+1}, Z_{t+1}, \mathbb{E}Z_{t+1}))\Delta W'_t], \\ Y_t = \mathbb{E}_t[Y_{t+1} + f(t+1, Y_{t+1}, \mathbb{E}Y_{t+1}, Z_{t+1}, \mathbb{E}Z_{t+1})]. \end{cases}$$

In summary, we deduce $(Y_t, Z_t, \Delta M_t) \in \mathcal{L}^2(\mathfrak{F}_t;\mathbb{R}^n) \times \mathcal{L}^2(\mathfrak{F}_t;\mathbb{R}^n) \times \mathcal{L}^2(\mathfrak{F}_t;\mathbb{R}^n)$, $0 \le t \le T-2$. Without loss of generality, let $M_0 = 0$ and $M_t = M_0 + \sum_{s=0}^{t-1}\Delta M_s$, we see that (2.1) holds for $t \in \{0,1,\dots,T-1\}$. In addition, M is a square integrable martingale process. Furthermore, since

$$\mathbb{E}_{t-1}\big[M_t W'_t\big] = \sum_{s=0}^{t-2}\Delta M_s \mathbb{E}_{t-1}\big[W'_t\big] + \mathbb{E}_{t-1}\big[\Delta M_{t-1}(W_{t-1} + \Delta W_{t-1})'\big] = M_{t-1}W'_{t-1},$$

we get that M is strongly orthogonal to W. The existence is finished.

Next, we shall prove the uniqueness. Assume that there are two solutions $(Y^1_t, Z^1_t, \tilde{Y}^1_t, \tilde{Z}^1_t, M^1_t)$ and $(Y^2_t, Z^2_t, \tilde{Y}^2_t, \tilde{Z}^2_t, M^2_t)$ of MF-BSΔE (2.1). Then

$$\begin{aligned} Y^1_{T-1} - Y^2_{T-1} = {} & -f\big(T, Y_T, Z^1_T, \tilde{Y}_T, \tilde{Z}^1_T\big) + f\big(T, Y_T, Z^2_T, \tilde{Y}_T, \tilde{Z}^2_T\big) \\ & + Z^1_{T-1}\Delta W_{T-1} + \Delta M^1_{T-1} - Z^2_{T-1}\Delta W_{T-1} - \Delta M^2_{T-1}. \end{aligned} \tag{2.3}$$

Combining $Z^1_{T-1} = \mathbb{E}_{T-1}[(\pi + f(T,\pi,\mathbb{E}\pi))\Delta W'_{T-1}] = Z^2_{T-1}$ with (2.2), we have $\Delta M^1_{T-1} = \Delta M^2_{T-1}$. Thus, using (2.3), we can immediately get that $Y^1_{T-1} = Y^2_{T-1}$. The inductive method and $Z^1_0 = Z^2_0 = M^1_0 = M^2_0 = 0$ yield $(Y^1_t, Z^1_t, M^1_t) = (Y^2_t, Z^2_t, M^2_t)$ for $t \in \{0,1,\dots,T-1\}$. □

3 Controlled MF-FBSΔEs

In this section, we focus on the fully coupled MF-FBSΔE (1.1). Let $\bar{u}_t$ and $(\bar{X}_t, \bar{Y}_t, \bar{Z}_t)$ be the optimal control and optimal trajectory of Problem $\mathcal{A}$, respectively. Assume that (A2)–(A5) hold, firstly we define a multi-valued mapping:

$$\mathcal{I}(t, v_t) = \begin{cases} 1, & (t, v_t) \in \mathbb{T} \times U^+_t, \\ -1, & (t, v_t) \in \mathbb{T} \times U^-_t, \\ \pm 1, & (t, v_t) \in \mathbb{T} \times U^+_t \cap U^-_t, \end{cases} \tag{3.1}$$

where U_t^+, U_t^- represent that the set U_t is γ-convex and $-\gamma$-convex, respectively. Notice that the set U_t is $\pm\gamma$-convex, there exists $\gamma > 0$ such that, for all $\varepsilon \in (0,\gamma]$, we could find another admissible control

$$u_t^\varepsilon = \bar{u}_t + \alpha(t)\varepsilon(v_t - \bar{u}_t), \quad v_t \in U_t,$$

where $\alpha(t) \in \mathcal{I}(t, v_t)$. We construct a needle variation

$$\alpha(t)\varepsilon(v_t - \bar{u}_t) = \begin{cases} \alpha\varepsilon(v_\theta - \bar{u}_\theta), & t = \theta, \\ 0, & t \in \mathbb{T} \setminus \{\theta\}, \end{cases}$$

where $\alpha \in \mathcal{I}(\theta, v_\theta)$, $(\theta, v_\theta) \in \mathbb{T} \times U_\theta$, and $\varepsilon \in (0, \gamma^*]$ with $\gamma^* = \frac{\gamma(v_\theta)}{1+\alpha}$. Denote that $(X_t^\varepsilon, Y_t^\varepsilon, Z_t^\varepsilon)$ is the state trajectory corresponding to the admissible control u_t^ε. Now, we give the following existence and uniqueness theorem.

Theorem 3.1 *Assume that* (A2)–(A4) *hold, then there exists a unique adapted solution* $(X, Y, Z, M) \in \mathcal{M}^2[0, T]$ *for mean-field system* (1.1).

We shall apply the following two technical lemmas to prove the existence part of Theorem 3.1, and the proof of these lemmas shall be presented in the sequel.

Lemma 3.1 *Suppose* $(r, \phi, \varphi) \in \mathcal{N}^2[0, T]$, $\lambda \in \mathcal{L}^2(\mathfrak{F}; \mathbb{R}^n)$, *then the following linear MF-FBSΔE*

$$\begin{cases} \Delta X_t = -Y_t - \mathbb{E}Y_t + r_t + (-Z_t - \mathbb{E}Z_t + \phi_t)\Delta W_t, \\ \Delta Y_t = -X_{t+1} - \mathbb{E}X_{t+1} - \varphi_{t+1} + Z_t \Delta W_t + \Delta M_t, \\ X_0 = x_0, \qquad Y_T = X_T + \mathbb{E}X_T + \lambda, \end{cases} \tag{3.2}$$

has a unique solution $(X, Y, Z, M) \in \mathcal{M}^2[0, T]$.

Now, we define a family of MF-FBSΔEs parameterized by $\mu \in [0, 1]$ *as follows*:

$$\begin{cases} \Delta X_t = b^\mu(t, \Gamma_t, u_t) + r_t + [\sigma^\mu(t, \Gamma_t, u_t) + \phi_t]\Delta W_t, \\ \Delta Y_t = -f^\mu(t+1, \Gamma_{t+1}, u_{t+1}) - \varphi_{t+1} + Z_t \Delta W_t + \Delta M_t, \\ X_0 = x_0, \qquad Y_T = l^\mu(X_T, \mathbb{E}X_T) + \lambda, \end{cases} \tag{3.3}$$

where

$$\begin{cases} b^\mu(t, \Gamma_t, u_t) = \mu b(t, \Gamma_t, u_t) + (1-\mu)(-Y_t - \mathbb{E}Y_t), \\ \sigma^\mu(t, \Gamma_t, u_t) = \mu\sigma(t, \Gamma_t, u_t) + (1-\mu)(-Z_t - \mathbb{E}Z_t), \\ f^\mu(t, \Gamma_t, u_t) = \mu f(t, \Gamma_t, u_t) + (\mu - 1)(-X_t - \mathbb{E}X_t), \\ l^\mu(X_t, \mathbb{E}X_t) = \mu l(X_t, \mathbb{E}X_t) + (\mu - 1)(-X_t - \mathbb{E}X_t), \\ \Gamma_t = (X_t, Y_t, Z_t, \mathbb{E}X_t, \mathbb{E}Y_t, \mathbb{E}Z_t). \end{cases}$$

Lemma 3.2 *For a given* $\mu_0 \in [0, 1)$ *and any* $(r, \phi, \varphi) \in \mathcal{N}^2[0, T]$, $\lambda \in \mathcal{L}^2(\mathfrak{F}; \mathbb{R}^n)$, *MF-FBS$\Delta$Es* (3.3) *have a unique solution. Then there exists* $\delta_0 \in (0, 1)$ *such that, for any* $\mu \in [\mu_0, \mu_0 + \delta_0]$ *and* $(r, \phi, \varphi) \in \mathcal{N}^2[0, T]$, $\lambda \in \mathcal{L}^2(\mathfrak{F}; \mathbb{R}^n)$, *MF-FBS$\Delta$Es* (3.3) *have a unique solution.*

Proof of Theorem 3.1 Uniqueness. Suppose that (X,Y,Z,M) and $(\check{X},\check{Y},\check{Z},\check{M})$ are two solutions of (1.1), we denote

$$\begin{cases}\hat{\Gamma}=(\hat{X},\hat{Y},\hat{Z},\mathbb{E}\hat{X},\mathbb{E}\hat{Y},\mathbb{E}\hat{Z})=\Gamma-\check{\Gamma}\\ \quad=(X-\check{X},Y-\check{Y},Z-\check{Z},\mathbb{E}X-\mathbb{E}\check{X},\mathbb{E}Y-\mathbb{E}\check{Y},\mathbb{E}Z-\mathbb{E}\check{Z}),\\ \hat{b}(t)=b(t,\Gamma_t,u_t)-b(t,\check{\Gamma}_t,u_t),\qquad \hat{\sigma}(t)=\sigma(t,\Gamma_t,u_t)-\sigma(t,\check{\Gamma}_t,u_t),\\ \hat{f}(t)=f(t,\Gamma_t,u_t)-f(t,\check{\Gamma}_t,u_t),\\ \hat{\Theta}=(\hat{X},\hat{Y},\hat{Z})=\Theta-\check{\Theta}=(X-\check{X},Y-\check{Y},Z-\check{Z}),\qquad \hat{M}=M-\check{M}.\end{cases}$$

For $t\in\{0,1,\dots,T-1\}$, it yields that

$$\begin{aligned}\Delta\langle\hat{X}_t,\hat{Y}_t\rangle&=\langle\hat{X}_{t+1},\Delta\hat{Y}_t\rangle+\langle\Delta\hat{X}_t,\hat{Y}_t\rangle\\&=\langle\hat{b}(t),\hat{Y}_t\rangle-\langle\hat{X}_{t+1},\hat{f}(t+1)\rangle+\langle\hat{\sigma}(t)\Delta W_t,\hat{Z}_t\Delta W_t\rangle+\Psi_t,\end{aligned}$$

where

$$\Psi_t=\langle\hat{\sigma}(t)\Delta W_t,\hat{Y}_t\rangle+\langle\Delta\hat{X}_t+\hat{X}_t,\Delta\hat{M}_t\rangle+\langle\hat{X}_t+\hat{b}(t),\hat{Z}_t\Delta W_t\rangle.$$

Notice that W, M, $\check{M}$ are square integrable martingale processes and M, $\check{M}$ are strongly orthogonal to W, we obtain $\mathbb{E}[\Psi_t]=0$. Furthermore,

$$\begin{aligned}&\mathbb{E}\langle X_T-\check{X}_T,l(X_T,\mathbb{E}X_T)-l(\check{X}_T,\mathbb{E}\check{X}_T)\rangle\\&=\mathbb{E}\langle\hat{X}_T,\hat{Y}_T\rangle=\mathbb{E}\sum_{t=0}^{T-1}\Delta\langle\hat{X}_t,\hat{Y}_t\rangle=\mathbb{E}\sum_{t=0}^{T-1}\{\langle\hat{X}_{t+1},-\hat{f}(t+1)\rangle+\langle\hat{\sigma}(t),\hat{Z}_t\rangle+\langle\hat{b}(t),\hat{Y}_t\rangle\}\\&=\mathbb{E}\left\{\sum_{t=1}^{T-1}\langle F(t,\Gamma_t,u_t)-F(t,\check{\Gamma}_t,u_t),\Theta_t-\check{\Theta}_t\rangle+\langle\hat{b}(0),\hat{Y}_0\rangle+\langle\hat{\sigma}(0),\hat{Z}_0\rangle-\langle\hat{X}_T,\hat{f}(T)\rangle\right\}.\end{aligned}$$

Using the monotonic conditions, it follows that

$$\begin{aligned}&c\mathbb{E}|X_T-\check{X}_T|^2\\&\le\mathbb{E}\langle X_T-\check{X}_T,l(X_T,\mathbb{E}X_T)-l(\check{X}_T,\mathbb{E}\check{X}_T)\rangle\\&=\mathbb{E}\left\{\sum_{t=1}^{T-1}\langle F(t,\Gamma_t,u_t)-F(t,\check{\Gamma}_t,u_t),\Theta_t-\check{\Theta}_t\rangle+\langle\hat{b}(0),\hat{Y}_0\rangle+\langle\hat{\sigma}(0),\hat{Z}_0\rangle-\langle\hat{X}_T,\hat{f}(T)\rangle\right\}\\&\le-c_1\mathbb{E}\left\{\sum_{t=0}^{T}|X_t-\check{X}_t|^2+\sum_{t=0}^{T-1}|Y_t-\check{Y}_t|^2+\sum_{t=0}^{T-1}|Z_t-\check{Z}_t|^2\right\},\end{aligned}$$

which further implies

$$\mathbb{E}\left\{\sum_{t=0}^{T}|X_t-\check{X}_t|^2+\sum_{t=0}^{T-1}|Y_t-\check{Y}_t|^2+\sum_{t=0}^{T-1}|Z_t-\check{Z}_t|^2\right\}=0.$$

Besides, it is easy to see $\mathbb{E}|Y_T-\check{Y}_T|^2=0$ and $\mathbb{E}\sum_{t=0}^{T}|M_t-\check{M}_t|^2=0$. Thereby, $\Theta=\check{\Theta}$.

Existence. By Lemma 3.1, we can immediately get that, when $\mu = 0$, for any $(r,\phi,\varphi) \in \mathcal{N}^2[0,T]$, $\lambda \in \mathcal{L}^2(\mathfrak{F};\mathbb{R}^n)$, MF-FBS∆Es (3.3) have a unique solution. By Lemma 3.2, for any $(r,\phi,\varphi) \in \mathcal{N}^2[0,T]$, $\lambda \in \mathcal{L}^2(\mathfrak{F};\mathbb{R}^n)$, (3.3) can be solved successively for $\mu \in [0,\delta_0],[\delta_0,2\delta_0],\ldots$. Hence, we can deduce that when $\mu = 1$, for any $(r,\phi,\varphi) \in \mathcal{N}^2[0,T]$, $\lambda \in \mathcal{L}^2(\mathfrak{F};\mathbb{R}^n)$, MF-FBS∆Es (3.3) have a unique solution. Let $r_t = \phi_t = \varphi_t = \lambda = 0$, we conclude that MF-FBS∆E (1.1) has a solution. □

Proof of Lemma 3.1 We consider the following BS∆E:

$$\begin{cases} \Delta Y_t^* = Y_t^* + \mathbb{E}Y_t^* - r_t - \varphi_{t+1} + (2Z_t^* + \mathbb{E}Z_t^* - \phi_t)\Delta W_t + \Delta M_t^*, \\ Y_T^* = \lambda. \end{cases}$$

Using Theorem 2.1, the above equation admits a unique solution (Y^*,Z^*,M^*). Then we solve the following forward equation:

$$\begin{cases} \Delta X_t = -X_{t+1} - \mathbb{E}X_{t+1} - Y_t^* - \mathbb{E}Y_t^* + r_t + (-Z_t^* - \mathbb{E}Z_t^* + \phi_t)\Delta W_t, \\ X_0 = x_0. \end{cases}$$

Let $Y = Y^* + X$, $Z = Z^*$, and $M = M^*$, we can see that (X,Y,Z,M) is a solution of (3.2). Thus, the existence is finished. With regards to the uniqueness, it suffices to apply the method of the proof of uniqueness in Theorem 3.1; here, we omit it. □

Proof of Lemma 3.2 Notice that

$$\begin{cases} b^{\mu_0+\delta}(t,\Gamma_t,u_t) = b^{\mu_0}(t,\Gamma_t,u_t) + \delta[b(t,\Gamma_t,u_t) + Y_t + \mathbb{E}Y_t], \\ \sigma^{\mu_0+\delta}(t,\Gamma_t,u_t) = \sigma^{\mu_0}(t,\Gamma_t,u_t) + \delta[\sigma(t,\Gamma_t,u_t) + Z_t + \mathbb{E}Z_t], \\ f^{\mu_0+\delta}(t,\Gamma_t,u_t) = f^{\mu_0}(t,\Gamma_t,u_t) + \delta[f(t,\Gamma_t,u_t) - X_t - \mathbb{E}X_t], \\ l^{\mu_0+\delta}(X_t,\mathbb{E}X_t) = l^{\mu_0}(X_t,\mathbb{E}X_t) + \delta[l(X_t,\mathbb{E}X_t) - X_t - \mathbb{E}X_t]. \end{cases}$$

Set $\Lambda^i = (X^i,Y^i,\mathbb{E}X^i,\mathbb{E}Y^i)$, $\Theta^i = (X^i,Y^i,Z^i)$, $\Gamma^i = (X^i,Y^i,Z^i,\mathbb{E}X^i,\mathbb{E}Y^i,\mathbb{E}Z^i)$, and $\Gamma^0 = 0$ to solve iteratively the following equations:

$$\begin{cases} \Delta X_t^{i+1} = b^{\mu_0}(t,\Gamma_t^{i+1},u_t) + \delta\{b(t,\Gamma_t^i,u) + Y_t^i + \mathbb{E}Y_t^i\} + r_t \\ \qquad\qquad + \{\sigma^{\mu_0}(t,\Gamma_t^{i+1},u_t) + \delta[\sigma(t,\Gamma_t^i,u) + Z_t^i + \mathbb{E}Z_t^i] + \phi_t\}\Delta W_t, \\ \Delta Y_t^{i+1} = -f^{\mu_0}(t+1,\Gamma_{t+1}^{i+1},u_{t+1}) - \delta\{f(t+1,\Gamma_{t+1}^i,u_{t+1}) - X_{t+1}^i - \mathbb{E}X_{t+1}^i\} \\ \qquad\qquad - \varphi_{t+1} + Z_t^{i+1}\Delta W_t + \Delta M_t^{i+1}, \\ X_0 = x_0, Y_T^{i+1} = l^{\mu_0}(X_T^{i+1},\mathbb{E}X_T^{i+1}) + \delta\{l(X_T^i,\mathbb{E}X_T^i) - X_T^i - \mathbb{E}X_T^i\} + \lambda. \end{cases} \tag{3.4}$$

Then we apply the product rule to $\hat{X}_t^{i+1}\hat{Y}_t^{i+1}$ yielding

$$\begin{aligned} &\mathbb{E}\langle \hat{X}_T^{i+1}, l^{\mu_0}(X_T^{i+1},\mathbb{E}X_T^{i+1}) - l^{\mu_0}(X_T^i,\mathbb{E}X_T^i)\rangle + \mathbb{E}\langle \hat{X}_T^{i+1}, \delta[X_T^{i-1} - X_T^i + \mathbb{E}X_T^{i-1} - \mathbb{E}X_T^i]\rangle \\ &\quad + \mathbb{E}\langle \hat{X}_T^{i+1}, \delta[l(X_T^i,\mathbb{E}X_T^i) - l(X_T^{i-1},\mathbb{E}X_T^{i-1})]\rangle \\ &= \mathbb{E}\langle \hat{X}_T^{i+1}, \hat{Y}_T^{i+1}\rangle \end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}\sum_{t=1}^{T-1}\langle F^{\mu_0}(t,\Gamma_t^{i+1},u_t) - F^{\mu_0}(t,\Gamma_t^{i},u_t),\hat{\Theta}_t^{i+1}\rangle \\
&\quad + \mathbb{E}\sum_{t=1}^{T-1}\delta\langle F(t,\Gamma_t^{i},u_t) - F(t,\Gamma_t^{i-1},u_t) + \hat{\Theta}_t^{i},\hat{\Theta}_t^{i+1}\rangle \\
&\quad + \mathbb{E}\langle b^{\mu_0}(0,\Gamma_0^{i+1},u_0) - b^{\mu_0}(0,\Gamma_0^{i},u_0),\hat{Y}_0^{i+1}\rangle \\
&\quad + \mathbb{E}\delta\langle b(0,\Gamma_0^{i},u_0) - b(0,\Gamma_0^{i-1},u_0) + \hat{Y}_0^{i},\hat{Y}_0^{i+1}\rangle \\
&\quad + \mathbb{E}\langle \sigma^{\mu_0}(0,\Gamma_0^{i+1},u_0) - \sigma^{\mu_0}(0,\Gamma_0^{i},u_0),\hat{Z}_0^{i+1}\rangle \\
&\quad + \mathbb{E}\delta\langle \sigma(0,\Gamma_0^{i},u_0) - \sigma(0,\Gamma_0^{i-1},u_0) + \hat{Z}_0^{i},\hat{Z}_0^{i+1}\rangle \\
&\quad - \mathbb{E}\langle f^{\mu_0}(T,\Lambda_T^{i+1},u_T) - f^{\mu_0}(T,\Lambda_T^{i},u_T),\hat{X}_T^{i+1}\rangle \\
&\quad - \mathbb{E}\delta\langle f(T,\Lambda_T^{i},u_T) - f(T,\Lambda_T^{i-1},u_T) + \hat{X}_T^{i},\hat{X}_T^{i+1}\rangle,
\end{aligned}
$$

where $\hat{\Gamma}_t^i = \Gamma_t^i - \Gamma_t^{i-1}$ and $\hat{\Theta}_t^i = \Theta_t^i - \Theta_t^{i-1}$. Set $\beta_0 = \min\{1,\beta\}$, we claim that

$$
\mathbb{E}\left\{2|\hat{X}_T^{i+1}|^2 + \sum_{t=0}^{T-1}|\hat{\Theta}_t^{i+1}|^2\right\} \le \frac{\delta(1+c)}{\beta_0}\left\{2\mathbb{E}|\hat{X}_T^{i}||\hat{X}_T^{i+1}| + \mathbb{E}\sum_{t=0}^{T-1}|\hat{\Theta}_t^{i}||\hat{\Theta}_t^{i+1}|\right\}.
$$

Let $\varepsilon = \frac{\beta_0}{\delta(1+c)}$, by means of $ab \le \frac{a^2}{2\varepsilon} + \frac{\varepsilon b^2}{2}$, we have

$$
\begin{aligned}
\mathbb{E}\left\{2|\hat{X}_T^{i+1}|^2 + \sum_{t=0}^{T-1}|\hat{\Theta}_t^{i+1}|^2\right\} &\le \frac{1}{2}\left(\frac{\delta(1+c)}{\beta_0}\right)^2\left\{2\mathbb{E}|\hat{X}_T^{i}|^2 + \mathbb{E}\sum_{t=0}^{T-1}|\hat{\Theta}_t^{i}|^2\right\} \\
&\quad + \frac{1}{2}\left\{2\mathbb{E}|\hat{X}_T^{i+1}|^2 + \mathbb{E}\sum_{t=0}^{T-1}|\hat{\Theta}_t^{i+1}|\right\},
\end{aligned}
$$

which indicates

$$
2\mathbb{E}|\hat{X}_T^{i+1}|^2 + \mathbb{E}\sum_{t=0}^{T-1}|\hat{\Theta}_t^{i+1}|^2 \le \left(\frac{\delta(1+\beta)}{\beta_0}\right)^2\left\{2\mathbb{E}|\hat{X}_T^{i}|^2 + \mathbb{E}\sum_{t=0}^{T-1}|\hat{\Theta}_t^{i}|^2\right\}.
$$

Then

$$
\mathbb{E}|\hat{X}_T^{i}|^2 = \mathbb{E}\left|\sum_{t=0}^{T-1}\Delta\hat{X}_t^{i}\right|^2 \le T|\Delta\hat{X}_t^{i}|^2 \le \beta_1\mathbb{E}\sum_{t=0}^{T-1}(|\hat{\Theta}_t^{i}|^2 + |\hat{\Theta}_t^{i-1}|^2),
$$

where $\beta_1 > 0$ and it only relies on c and T. Thus, there exists $\beta_2 > 0$ relying on c, β, and T such that

$$
\mathbb{E}\sum_{t=0}^{T-1}|\hat{\Theta}_t^{i+1}|^2 \le \beta_2\delta^2\left\{\mathbb{E}\sum_{t=0}^{T-1}|\hat{\Theta}_t^{i}|^2 + \mathbb{E}\sum_{t=0}^{T-1}|\hat{\Theta}_t^{i-1}|^2\right\}.
$$

Furthermore, there exists $\bar{\delta} \in (0,1)$ relying on c, β, and T such that, for $0 < \delta \le \bar{\delta}$,

$$
\mathbb{E}\sum_{t=0}^{T-1}|\hat{\Theta}_t^{i+1}|^2 \le \frac{1}{4}\mathbb{E}\sum_{t=0}^{T-1}|\hat{\Theta}_t^{i}|^2 + \frac{1}{8}\mathbb{E}\sum_{t=0}^{T-1}|\hat{\Theta}_t^{i-1}|^2, \quad \forall i \ge 1.
$$

By [30, Lemma 4.1], we see that $\{\Theta_t^i\}_{t=0}^{T-1}$ is a Cauchy sequence in $\mathcal{K}^2[0,T]$. Denote its limit by $\Theta = (X,Y,Z)$. Taking the limit in (3.4), we can derive that, when $0 < \delta \le \bar{\delta}$, $\Theta = (X,Y,Z)$ solves (3.3) for $\mu = \mu_0 + \delta$. The proof is completed. □

For simplicity, for $\rho = b,\sigma,f,g$ and $a = x,y,z,\tilde{x},\tilde{y},\tilde{z},u$, we use the following abbreviations:

$$\begin{cases} \bar{\Gamma}_t = (\bar{X}_t,\bar{Y}_t,\bar{Z}_t,\mathbb{E}\bar{X}_t,\mathbb{E}\bar{Y}_t,\mathbb{E}\bar{Z}_t), \qquad \Gamma_t^\varepsilon = (X_t^\varepsilon,Y_t^\varepsilon,Z_t^\varepsilon,\mathbb{E}X_t^\varepsilon,\mathbb{E}Y_t^\varepsilon,\mathbb{E}Z_t^\varepsilon), \\ \bar{\rho}(t) = \rho(t,\bar{\Gamma}_t,\bar{u}_t), \qquad \rho^\varepsilon(t) = \rho(t,\Gamma_t^\varepsilon,u_t^\varepsilon), \\ \tilde{\rho}^\varepsilon(t) = \rho(t,\bar{\Gamma}_t,u_t^\varepsilon), \qquad \rho_a(t) = \rho_a(t,\bar{\Gamma}_t,\bar{u}_t). \end{cases}$$

Let (k,m,n,N) be a solution of the following variational equations:

$$\begin{cases} \Delta k_t = \bar{b}_x(t)k_t + \bar{b}_y(t)m_t + \bar{b}_z(t)n_t + \bar{b}_{\tilde{x}}(t)\mathbb{E}k_t + \bar{b}_{\tilde{y}}(t)\mathbb{E}m_t + \bar{b}_{\tilde{z}}(t)\mathbb{E}n_t \\ \qquad + \alpha(t)\bar{b}_u(t)\varepsilon(v_t - \bar{u}_t) \\ \qquad + \{\bar{\sigma}_x(t)k_t + \bar{\sigma}_y(t)m_t + \bar{\sigma}_z(t)n_t + \alpha(t)\bar{\sigma}_u(t)\varepsilon(v_t - \bar{u}_t)\}\Delta W_t \\ \qquad + \{\bar{\sigma}_{\tilde{x}}(t)\mathbb{E}k_t + \bar{\sigma}_{\tilde{y}}(t)\mathbb{E}m_t + \bar{\sigma}_{\tilde{z}}(t)\mathbb{E}n_t\}\Delta W_t, \\ \Delta m_t = -\bar{f}_x(t+1)k_{t+1} - \bar{f}_y(t+1)m_{t+1} - \bar{f}_z(t+1)n_{t+1} \\ \qquad - \alpha(t+1)\bar{f}_u(t+1)\varepsilon(v_{t+1} - \bar{u}_{t+1}) \\ \qquad - \bar{f}_{\tilde{x}}(t+1)\mathbb{E}k_{t+1} - \bar{f}_{\tilde{y}}(t+1)\mathbb{E}m_{t+1} - \bar{f}_{\tilde{z}}(t+1)\mathbb{E}n_{t+1} + n_t\Delta W_t + \Delta\bar{N}_t, \\ k_0 = 0, \qquad m_T = \{l_x(\bar{X}_T,\mathbb{E}\bar{X}_T) + \mathbb{E}l_{\tilde{x}}(\bar{X}_T,\mathbb{E}\bar{X}_T)\}k_T. \end{cases} \tag{3.5}$$

We proceed to introducing the following adjoint equations:

$$\begin{cases} \Delta\eta_t = -\bar{b}_x'(t+1)\eta_{t+1} - \bar{\sigma}_x'(t+1)\zeta_{t+1} + \bar{f}_x'(t+1)\xi_{t+1} + \bar{g}_x(t+1) + \zeta_t\Delta W_t \\ \qquad + \Delta\bar{V}_t + \mathbb{E}\{-\bar{b}_{\tilde{x}}'(t+1)\eta_{t+1} - \bar{\sigma}_{\tilde{x}}'(t+1)\zeta_{t+1} + \bar{f}_{\tilde{x}}'(t+1)\xi_{t+1} + \bar{g}_{\tilde{x}}(t+1)\}, \\ \Delta\xi_t = \{\bar{f}_z'(t)\xi_t - \bar{b}_z'(t)\eta_t - \bar{\sigma}_z'(t)\zeta_t + \bar{g}_z(t) \\ \qquad + \mathbb{E}(\bar{f}_{\tilde{z}}'(t)\xi_t - \bar{b}_{\tilde{z}}'(t)\eta_t - \bar{\sigma}_{\tilde{z}}'(t)\zeta_t + \bar{g}_{\tilde{z}}(t))\}\Delta W_t \\ \qquad + \bar{f}_y'(t)\xi_t - \bar{b}_y'(t)\eta_t - \bar{\sigma}_y'(t)\zeta_t + \bar{g}_y(t) \\ \qquad + \mathbb{E}\{\bar{f}_{\tilde{y}}'(t)\xi_t - \bar{b}_{\tilde{y}}'(t)\eta_t - \bar{\sigma}_{\tilde{y}}'(t)\zeta_t + \bar{g}_{\tilde{y}}(t)\}, \\ \eta_T = -h_x(\bar{X}_T,\mathbb{E}\bar{X}_T) - \mathbb{E}h_{\tilde{x}}(\bar{X}_T,\mathbb{E}\bar{X}_T) - l_x(\bar{X}_T,\mathbb{E}\bar{X}_T)\xi_T - \mathbb{E}\{l_{\tilde{x}}(\bar{X}_T,\mathbb{E}\bar{X}_T)\xi_T\}, \\ \xi_0 = 0. \end{cases} \tag{3.6}$$

Here, W, N, V are square integrable martingale processes and N, V are strongly orthogonal to W. Set

$$\begin{cases} \hat{X}_t = X_t^\varepsilon - \bar{X}_t, \qquad \hat{Y}_t = Y_t^\varepsilon - \bar{Y}_t, \qquad \hat{Z}_t = Z_t^\varepsilon - \bar{Z}_t, \qquad \hat{M}_t = M_t^\varepsilon - \bar{M}_t, \\ \tilde{X}_t = \hat{X}_t - k_t, \qquad \tilde{Y}_t = \hat{Y}_t - m_t, \qquad \tilde{Z}_t = \hat{Z}_t - n_t, \qquad \tilde{M}_t = \hat{M}_t - N_t, \end{cases}$$

then we get

$$\begin{cases} \Delta\hat{X}_t = b^\varepsilon(t) - \bar{b}(t) + (\sigma^\varepsilon(t) - \bar{\sigma}(t))\Delta W_t, \\ \Delta\hat{Y}_t = -f^\varepsilon(t+1) + \bar{f}(t+1) + \hat{Z}_t\Delta W_t + \Delta\hat{M}_t, \\ \hat{X}_0 = 0, \qquad \hat{Y}_T = 0. \end{cases} \tag{3.7}$$

For MF-FBS∆E (1.1), we give the following estimates.

Lemma 3.3 *Assume that* (A2)–(A5) *hold*, *we get*

$$\mathbb{E}\left\{\sum_{t=0}^{T}|\hat{X}_t|^2+\sum_{t=0}^{T}|\hat{Y}_t|^2+\sum_{t=0}^{T-1}|\hat{Z}_t|^2\right\}\leq c\varepsilon^2\mathbb{E}|v_\theta-\bar{u}_\theta|^2.$$

Proof According to (3.7), we have

$$\begin{aligned}
0&=\mathbb{E}\langle\hat{X}_T,\hat{Y}_T\rangle-\mathbb{E}\langle\hat{X}_0,\hat{Y}_0\rangle=\mathbb{E}\sum_{t=0}^{T-1}\Delta\langle\hat{X}_t,\hat{Y}_t\rangle\\
&=\mathbb{E}\sum_{t=0}^{T}\left\{\left\langle\hat{X}_t,\tilde{f}^\varepsilon(t)-f^\varepsilon(t)\right\rangle+\left\langle\hat{Y}_t,b^\varepsilon(t)-\tilde{b}^\varepsilon(t)\right\rangle+\left\langle\hat{Z}_t,\sigma^\varepsilon(t)-\tilde{\sigma}^\varepsilon(t)\right\rangle\right.\\
&\quad\left.+\left\langle\hat{X}_t,-\tilde{f}^\varepsilon(t)+\bar{f}(t)\right\rangle+\left\langle\hat{Y}_t,\tilde{b}^\varepsilon(t)-\bar{b}(t)\right\rangle+\left\langle\hat{Z}_t,\tilde{\sigma}^\varepsilon(t)-\bar{\sigma}(t)\right\rangle\right\}\\
&=\mathbb{E}\sum_{t=1}^{T-1}\left\langle F\left(t,\Gamma_t^\varepsilon,u_t^\varepsilon\right)-F\left(t,\bar{\Gamma}_t,u_t^\varepsilon\right),\hat{\Theta}_t\right\rangle\\
&\quad+\mathbb{E}\sum_{t=0}^{T}\left\{\left\langle\hat{X}_t,\bar{f}(t)-\tilde{f}^\varepsilon(t)\right\rangle+\left\langle\hat{Y}_t,\tilde{b}^\varepsilon(t)-\bar{b}(t)\right\rangle+\left\langle\hat{Z}_t,\tilde{\sigma}^\varepsilon(t)-\bar{\sigma}(t)\right\rangle\right\}\\
&\quad+\mathbb{E}\left\{\left\langle\hat{X}_T,-f^\varepsilon(T)+\tilde{f}^\varepsilon(T)\right\rangle+\left\langle\hat{Y}_0,b^\varepsilon(0)-\tilde{b}^\varepsilon(0)\right\rangle+\left\langle\hat{Z}_0,\sigma^\varepsilon(0)-\tilde{\sigma}^\varepsilon(0)\right\rangle\right\}.
\end{aligned}$$

Combining the above equation with the monotonic conditions, we obtain

$$\begin{aligned}
&\mathbb{E}\sum_{t=0}^{T}\left\{\left\langle\hat{X}_t,-\tilde{f}^\varepsilon(t)+\bar{f}(t)\right\rangle+\left\langle\hat{Y}_t,\tilde{b}^\varepsilon(t)-\bar{b}(t)\right\rangle+\left\langle\hat{Z}_t,\tilde{\sigma}^\varepsilon(t)-\bar{\sigma}(t)\right\rangle\right\}\\
&\quad=-\mathbb{E}\sum_{t=1}^{T-1}\left\langle F\left(t,\Gamma_t^\varepsilon,u_t^\varepsilon\right)-F\left(t,\bar{\Gamma}_t,u_t^\varepsilon\right),\hat{\Theta}_t\right\rangle\\
&\qquad-\mathbb{E}\left\{\left\langle\hat{X}_T,-f^\varepsilon(T)+\tilde{f}^\varepsilon(T)\right\rangle+\left\langle\hat{Y}_0,b^\varepsilon(0)-\tilde{b}^\varepsilon(0)\right\rangle+\left\langle\hat{Z}_0,\sigma^\varepsilon(0)-\tilde{\sigma}^\varepsilon(0)\right\rangle\right\}\\
&\quad\geq\beta\mathbb{E}\left\{\sum_{t=0}^{T}|\hat{X}_t|^2+\sum_{t=0}^{T}|\hat{Y}_t|^2+\sum_{t=0}^{T-1}|\hat{Z}_t|^2\right\}.
\end{aligned}\tag{3.8}$$

On the other hand, there is a constant $c_1>0$ such that

$$\begin{aligned}
\mathbb{E}\sum_{t=0}^{T}\left\langle\hat{X}_t,-\tilde{f}^\varepsilon(t)+\bar{f}(t)\right\rangle&\leq\mathbb{E}\sum_{t=0}^{T}\left(\frac{\beta}{2}|\hat{X}_t|^2+\frac{1}{2\beta}\left|\bar{f}(t)-\tilde{f}^\varepsilon(t)\right|^2\right)\\
&\leq\mathbb{E}\sum_{t=0}^{T}\left(\frac{\beta}{2}|\hat{X}_t|^2+\frac{c_1}{2\beta}\varepsilon^2|v_\theta-\bar{u}_\theta|^2\right).
\end{aligned}$$

Similarly, we can deduce

$$\begin{aligned}&\mathbb{E}\sum_{t=0}^{T}\bigl\{\langle\hat{X}_t,-\tilde{f}^\varepsilon(t)+\bar{f}(t)\rangle+\langle\hat{Y}_t,\tilde{b}^\varepsilon(t)-\bar{b}(t)\rangle+\langle\hat{Z}_t,\tilde{\sigma}^\varepsilon(t)-\bar{\sigma}(t)\rangle\bigr\}\\&\quad\le\frac{\beta}{2}\mathbb{E}\left\{\sum_{t=0}^{T}|\hat{X}_t|^2+\sum_{t=0}^{T}|\hat{Y}_t|^2+\sum_{t=0}^{T-1}|\hat{Z}_t|^2\right\}+\frac{3c_1}{2\beta}\varepsilon^2\mathbb{E}|v_\theta-\bar{u}_\theta|^2.\end{aligned}\tag{3.9}$$

Using (3.8)–(3.9), we finally get

$$\mathbb{E}\left\{\sum_{t=0}^{T}|\hat{X}_t|^2+\sum_{t=0}^{T}|\hat{Y}_t|^2+\sum_{t=0}^{T-1}|\hat{Z}_t|^2\right\}\le c\varepsilon^2\mathbb{E}|v_\theta-\bar{u}_\theta|^2.$$

The proof is completed. □

Remark 3.1 Under (A2)–(A5), we have the following results.

If $t=1,2,\dots,T-1$,

$$\begin{pmatrix}-\bar{f}_x(t)&-\bar{f}_y(t)&-\bar{f}_z(t)\\\bar{b}_x(t)&\bar{b}_y(t)&\bar{b}_z(t)\\\bar{\sigma}_x(t)&\bar{\sigma}_y(t)&\bar{\sigma}_z(t)\end{pmatrix}\le-\beta I_{3n},\qquad\begin{pmatrix}-\bar{f}_{\tilde{x}}(t)&-\bar{f}_{\tilde{y}}(t)&-\bar{f}_{\tilde{z}}(t)\\\bar{b}_{\tilde{x}}(t)&\bar{b}_{\tilde{y}}(t)&\bar{b}_{\tilde{z}}(t)\\\bar{\sigma}_{\tilde{x}}(t)&\bar{\sigma}_{\tilde{y}}(t)&\bar{\sigma}_{\tilde{z}}(t)\end{pmatrix}\le-\beta I_{3n}.$$

If $t=0$,

$$\begin{pmatrix}\bar{b}_y(0)&\bar{b}_z(0)\\\bar{\sigma}_y(0)&\bar{\sigma}_z(0)\end{pmatrix}\le-\beta I_{2n},\qquad\begin{pmatrix}\bar{b}_{\tilde{y}}(0)&\bar{b}_{\tilde{z}}(0)\\\bar{\sigma}_{\tilde{y}}(0)&\bar{\sigma}_{\tilde{z}}(0)\end{pmatrix}\le-\beta I_{2n}.$$

If $t=T$,

$$-\bar{f}_x(T)\le-\beta I_n,\qquad-\bar{f}_{\tilde{x}}(T)\le-\beta I_n.$$

Consequently, the coefficients of (3.5) satisfy the monotonic conditions and there exists a unique solution (k,m,n,N) to (3.5). Following the proof of Lemma 3.3, it is easy to check

$$\mathbb{E}\left\{\sum_{t=0}^{T}|k_t|^2+\sum_{t=0}^{T}|m_t|^2+\sum_{t=0}^{T-1}|n_t|^2\right\}\le c\varepsilon^2\mathbb{E}|v_\theta-\bar{u}_\theta|^2.$$

Lemma 3.4 *Assume that* (A2)–(A5) *hold, we get*

$$\mathbb{E}\left\{\sum_{t=0}^{T}|\tilde{X}_t|^2+\sum_{t=0}^{T}|\tilde{Y}_t|^2+\sum_{t=0}^{T-1}|\tilde{Z}_t|^2\right\}=o(\varepsilon^2).$$

Proof Observe that

$$\begin{aligned}\Lambda^\varepsilon(t)-\bar{\Lambda}(t)&=\tilde{\Lambda}_x(t)\hat{X}_t+\tilde{\Lambda}_y(t)\hat{Y}_t+\tilde{\Lambda}_z(t)\hat{Z}_t+\tilde{\Lambda}_{\tilde{x}}(t)\mathbb{E}\hat{X}_t\\&\quad+\tilde{\Lambda}_{\tilde{y}}(t)\mathbb{E}\hat{Y}_t+\tilde{\Lambda}_{\tilde{z}}(t)\mathbb{E}\hat{Z}_t+\alpha(t)\tilde{\Lambda}_u(t)\varepsilon(v_t-\bar{u}_t),\end{aligned}$$

where

$$\tilde{\Lambda}_a(t) = \int_0^1 \Lambda_a\big(t, \bar{\Gamma}_t + \lambda \hat{\Gamma}_t, \bar{u}_t + \lambda\big(u_t^\varepsilon - \bar{u}_t\big)\big)\, d\lambda,$$

with $a = x, y, z, \tilde{x}, \tilde{y}, \tilde{z}, u$, then we have

$$\begin{cases} \Delta \tilde{X}_t = \bar{b}_x(t)\tilde{X}_t + \bar{b}_y(t)\tilde{Y}_t + \bar{b}_z(t)\tilde{Z}_t + \kappa_1(t) + \bar{b}_{\tilde{x}}(t)\mathbb{E}\tilde{X}_t \\ \qquad + \bar{b}_{\tilde{y}}(t)\mathbb{E}\tilde{Y}_t + \bar{b}_{\tilde{z}}(t)\mathbb{E}\tilde{Z}_t + \tilde{\kappa}_1(t) \\ \qquad + \{\bar{\sigma}_{\tilde{x}}(t)\mathbb{E}\tilde{X}_t + \bar{\sigma}_{\tilde{y}}(t)\mathbb{E}\tilde{Y}_t + \bar{\sigma}_{\tilde{z}}(t)\mathbb{E}\tilde{Z}_t + \tilde{\kappa}_2(t)\}\Delta W_t \\ \qquad + \{\bar{\sigma}_x(t)\tilde{X}_t + \bar{\sigma}_y(t)\tilde{Y}_t + \bar{\sigma}_z(t)\tilde{Z}_t + \kappa_2(t)\}\Delta W_t, \\ \Delta \tilde{Y}_t = -\bar{f}_x(t+1)\tilde{X}_{t+1} - \bar{f}_y(t+1)\tilde{Y}_{t+1} - \bar{f}_z(t+1)\tilde{Z}_{t+1} - \kappa_3(t+1) + \tilde{Z}_t\Delta W_t \\ \qquad + \Delta\tilde{M}_t - \bar{f}_{\tilde{x}}(t+1)\mathbb{E}\tilde{X}_{t+1} - \bar{f}_{\tilde{y}}(t+1)\mathbb{E}\tilde{Y}_{t+1} - \bar{f}_{\tilde{z}}(t+1)\mathbb{E}\tilde{Z}_{t+1} - \tilde{\kappa}_3(t+1), \\ \tilde{X}_0 = 0, \qquad \tilde{Y}_T = 0, \end{cases} \tag{3.10}$$

where

$$\begin{cases} \kappa_1(t) = \{\tilde{b}_x(t) - \bar{b}_x(t)\}\hat{X}_t + \{\tilde{b}_y(t) - \bar{b}_y(t)\}\hat{Y}_t + \{\tilde{b}_z(t) - \bar{b}_z(t)\}\hat{Z}_t \\ \qquad + \alpha(t)\{\tilde{b}_u(t) - \bar{b}_u(t)\}\varepsilon(v_t - \bar{u}_t), \\ \tilde{\kappa}_1(t) = \{\tilde{b}_{\tilde{x}}(t) - \bar{b}_{\tilde{x}}(t)\}\mathbb{E}\hat{X}_t + \{\tilde{b}_{\tilde{y}}(t) - \bar{b}_{\tilde{y}}(t)\}\mathbb{E}\hat{Y}_t + \{\tilde{b}_{\tilde{z}}(t) - \bar{b}_{\tilde{z}}(t)\}\mathbb{E}\hat{Z}_t, \\ \kappa_2(t) = \{\tilde{\sigma}_x(t) - \bar{\sigma}_x(t)\}\hat{X}_t + \{\tilde{\sigma}_y(t) - \bar{\sigma}_y(t)\}\hat{Y}_t + \{\tilde{\sigma}_z(t) - \bar{\sigma}_z(t)\}\hat{Z}_t \\ \qquad + \alpha(t)\{\tilde{\sigma}_u(t) - \bar{\sigma}_u(t)\}\varepsilon(v_t - \bar{u}_t), \\ \tilde{\kappa}_2(t) = \{\tilde{\sigma}_{\tilde{x}}(t) - \bar{\sigma}_{\tilde{x}}(t)\}\mathbb{E}\hat{X}_t + \{\tilde{\sigma}_{\tilde{y}}(t) - \bar{\sigma}_{\tilde{y}}(t)\}\mathbb{E}\hat{Y}_t + \{\tilde{\sigma}_{\tilde{z}}(t) - \bar{\sigma}_{\tilde{z}}(t)\}\mathbb{E}\hat{Z}_t, \\ \kappa_3(t) = -\{\tilde{f}_x(t) - \bar{f}_x(t)\}\hat{X}_t - \{\tilde{f}_y(t) - \bar{f}_y(t)\}\hat{Y}_t - \{\tilde{f}_z(t) - \bar{f}_z(t)\}\hat{Z}_t \\ \qquad - \alpha(t)\{\tilde{f}_u(t) - \bar{f}_u(t)\}\varepsilon(v_t - \bar{u}_t), \\ \tilde{\kappa}_3(t) = -\{\tilde{f}_{\tilde{x}}(t) - \bar{f}_{\tilde{x}}(t)\}\mathbb{E}\hat{X}_t - \{\tilde{f}_{\tilde{y}}(t) - \bar{f}_{\tilde{y}}(t)\}\mathbb{E}\hat{Y}_t - \{\tilde{f}_{\tilde{z}}(t) - \bar{f}_{\tilde{z}}(t)\}\mathbb{E}\hat{Z}_t. \end{cases}$$

From (3.10), it yields that

$$\begin{aligned} 0 &= \mathbb{E}\langle \tilde{X}_T, \tilde{Y}_T\rangle - \mathbb{E}\langle \tilde{X}_0, \tilde{Y}_0\rangle = \mathbb{E}\sum_{t=0}^{T-1} \Delta\langle \tilde{X}_t, \tilde{Y}_t\rangle \\ &= \mathbb{E}\sum_{t=0}^{T} \big\{\big\langle \tilde{X}_t, -\bar{f}_\Gamma(t)\tilde{\Gamma}_t\big\rangle + \big\langle \tilde{Y}_t, \bar{b}_\Gamma(t)\tilde{\Gamma}_t\big\rangle + \big\langle \tilde{Z}_t, \bar{\sigma}_\Gamma(t)\tilde{\Gamma}_t\big\rangle \\ &\quad - \big\langle \tilde{X}_t, \kappa_3(t) + \tilde{\kappa}_3(t)\big\rangle + \big\langle \tilde{Y}_t, \kappa_1(t) + \tilde{\kappa}_1(t)\big\rangle + \big\langle \tilde{Z}_t, \kappa_2(t) + \tilde{\kappa}_2(t)\big\rangle\big\}, \end{aligned}$$

where

$$\begin{aligned} &\tilde{\Gamma}_t = \big(\tilde{X}_t', \tilde{Y}_t', \tilde{Z}_t', \mathbb{E}\tilde{X}_t', \mathbb{E}\tilde{Y}_t', \mathbb{E}\tilde{Z}_t'\big)', \\ &\bar{\rho}_\Gamma(t) = \big(\bar{\rho}_x(t), \bar{\rho}_y(t), \bar{\rho}_z(t), \bar{\rho}_{\tilde{x}}(t), \bar{\rho}_{\tilde{y}}(t), \bar{\rho}_{\tilde{z}}(t)\big). \end{aligned}$$

Noticing Remark 3.1, we derive

$$\begin{aligned}
&\mathbb{E}\sum_{t=0}^{T}\bigl\{\langle\tilde{X}_t,-\kappa_3(t)-\tilde{\kappa}_3(t)\rangle+\langle\tilde{Y}_t,\kappa_1(t)+\tilde{\kappa}_1(t)\rangle+\langle\tilde{Z}_t,\kappa_2(t)+\tilde{\kappa}_2(t)\rangle\bigr\}\\
&\quad\geq\beta\mathbb{E}\left\{\sum_{t=0}^{T}|\tilde{X}_t|^2+\sum_{t=0}^{T}|\tilde{Y}_t|^2+\sum_{t=0}^{T-1}|\tilde{Z}_t|^2\right\}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\mathbb{E}\langle\tilde{X}_t,-\kappa_3(t)-\tilde{\kappa}_3(t)\rangle&=\mathbb{E}\langle\tilde{X}_t,[\tilde{f}_x(t)-\bar{f}_x(t)]\hat{X}_t\rangle+\mathbb{E}\langle\tilde{X}_t,[\tilde{f}_y(t)-\bar{f}_y(t)]\hat{Y}_t\rangle\\
&\quad+\mathbb{E}\langle\tilde{X}_t,[\tilde{f}_z(t)-\bar{f}_z(t)]\hat{Z}_t\rangle+\mathbb{E}\langle\tilde{X}_t,[\tilde{f}_{\tilde{z}}(t)-\bar{f}_{\tilde{z}}(t)]\mathbb{E}\hat{Z}_t\rangle\\
&\quad+\mathbb{E}\langle\tilde{X}_t,[\tilde{f}_{\tilde{x}}(t)-\bar{f}_{\tilde{x}}(t)]\mathbb{E}\hat{X}_t\rangle+\mathbb{E}\langle\tilde{X}_t,[\tilde{f}_{\tilde{y}}(t)-\bar{f}_{\tilde{y}}(t)]\mathbb{E}\hat{Y}_t\rangle\\
&\quad+\mathbb{E}\langle\tilde{X}_t,\alpha(t)[\tilde{f}_u(t)-\bar{f}_u(t)]\varepsilon(v_t-\bar{u}_t)\rangle.
\end{aligned}$$

Let $\varepsilon=\frac{1}{\beta}$, by virtue of $ab\leq\frac{a^2}{8\varepsilon}+2\varepsilon b^2$, we have

$$\begin{aligned}
&\mathbb{E}\langle\tilde{X}_t,-\kappa_3(t)-\tilde{\kappa}_3(t)\rangle\\
&\quad\leq\frac{7\beta}{8}\mathbb{E}|\tilde{X}_t|^2+\frac{2}{\beta}\mathbb{E}\bigl\{\|\tilde{f}_x(t)-\bar{f}_x(t)\|^2|\hat{X}_t|^2+\|\tilde{f}_y(t)-\bar{f}_y(t)\|^2|\hat{Y}_t|^2+\|\tilde{f}_z(t)-\bar{f}_z(t)\|^2|\hat{Z}_t|^2\\
&\quad+\varepsilon^2\|\tilde{f}_u(t)-\bar{f}_u(t)\|^2|v_t-\bar{u}_t|^2+\|\tilde{f}_{\tilde{x}}(t)-\bar{f}_{\tilde{x}}(t)\|^2|\mathbb{E}\hat{X}_t|^2\\
&\quad+\|\tilde{f}_{\tilde{y}}(t)-\bar{f}_{\tilde{y}}(t)\|^2|\mathbb{E}\hat{Y}_t|^2+\|\tilde{f}_{\tilde{z}}(t)-\bar{f}_{\tilde{z}}(t)\|^2|\mathbb{E}\hat{Z}_t|^2\bigr\}.
\end{aligned}$$

Combining $\lim_{\varepsilon\to 0}\|\tilde{f}_\Gamma(t)-\bar{f}_\Gamma(t)\|=0$ with Lemma 3.3, we obtain

$$\mathbb{E}\langle\tilde{X}_t,-\kappa_3(t)-\tilde{\kappa}_3(t)\rangle\leq\frac{7\beta}{8}\mathbb{E}|\tilde{X}_t|^2+o(\varepsilon^2).$$

In a similar way, we have

$$\begin{aligned}
&\mathbb{E}\langle\tilde{Y}_t,\kappa_1(t)+\tilde{\kappa}_1(t)\rangle\leq\frac{7\beta}{8}\mathbb{E}|\tilde{Y}_t|^2+o(\varepsilon^2),\\
&\mathbb{E}\langle\tilde{Z}_t,\kappa_2(t)+\tilde{\kappa}_2(t)\rangle\leq\frac{7\beta}{8}\mathbb{E}|\tilde{Z}_t|^2+o(\varepsilon^2).
\end{aligned}$$

Thus,

$$\mathbb{E}\left\{\sum_{t=0}^{T}|\tilde{X}_t|^2+\sum_{t=0}^{T}|\tilde{Y}_t|^2+\sum_{t=0}^{T-1}|\tilde{Z}_t|^2\right\}\leq o(\varepsilon^2).$$

The proof is completed. □

Remark 3.2 From Lemma 3.4, we see that k_t is the first-order variation of X_t, and m_t is the first-order variation of Y_t. It is easy to derive

$$\begin{aligned} J(u_t^\varepsilon) - J(\bar{u}_t) = \mathbb{E}\sum_{t=0}^{T-1}\bigl\{ &\langle \bar{g}_x(t) + \mathbb{E}\bar{g}_{\tilde{x}}(t), k_t\rangle + \langle \bar{g}_y(t) + \mathbb{E}\bar{g}_{\tilde{y}}(t), m_t\rangle + \langle \bar{g}_z(t) + \mathbb{E}\bar{g}_{\tilde{z}}(t), n_t\rangle \\ &+ \alpha(t)\varepsilon\langle \bar{g}_u(t), v_t - \bar{u}_t\rangle\bigr\} + \mathbb{E}\langle h_x(\bar{X}_T, \mathbb{E}\bar{X}_T) + \mathbb{E}h_{\tilde{x}}(\bar{X}_T, \mathbb{E}\bar{X}_T), k_T\rangle \\ &+ o(\varepsilon). \end{aligned} \tag{3.11}$$

4 Stochastic maximum principle

In this section, we are devoted to establishing the stochastic maximum principle for fully coupled MF-FBSΔE (1.1). Define the following Hamiltonian function:

$$\begin{aligned} \mathscr{H}(t,x,y,z,\tilde{x},\tilde{y},\tilde{z},\eta,\zeta,\xi,u) = {}& \langle \eta, b(t,x,y,z,\tilde{x},\tilde{y},\tilde{z},u)\rangle + \langle \zeta, \sigma(t,x,y,z,\tilde{x},\tilde{y},\tilde{z},u)\rangle \\ &- \langle \xi, f(t,x,y,z,\tilde{x},\tilde{y},\tilde{z},u)\rangle - g(t,x,y,z,\tilde{x},\tilde{y},\tilde{z},u). \end{aligned}$$

Theorem 4.1 (Stochastic maximum principle) *Assume that* (A2)–(A5) *hold. Let $\bar{u}_t$ be the optimal control and $(\bar{X}_t, \bar{Y}_t, \bar{Z}_t)$ be the corresponding optimal trajectory of Problem $\mathcal{A}$, then for $v_t \in U_t$ and $\alpha(t) \in \mathcal{I}(t, v_t)$, one has*

$$\alpha(t)\langle \mathscr{H}_u(t, \bar{X}_t, \bar{Y}_t, \bar{Z}_t, \mathbb{E}\bar{X}_t, \mathbb{E}\bar{Y}_t, \mathbb{E}\bar{Z}_t, \eta_t, \zeta_t, \xi_t, \bar{u}_t), v_t - \bar{u}_t\rangle \le 0, \quad \mathbb{P}\text{-}a.s. \tag{4.1}$$

Proof By (3.5)–(3.6), for $t \in \{0, 1, \dots, T-1\}$, we obtain

$$\begin{aligned} \Delta\langle k_t, \eta_t\rangle &= \langle k_{t+1}, \Delta\eta_t\rangle + \langle \Delta k_t, \eta_t\rangle \\ &= \langle k_{t+1}, -\bar{b}_x'(t+1)\eta_{t+1} - \bar{\sigma}_x'(t+1)\zeta_{t+1} + \bar{f}_x'(t+1)\xi_{t+1} + \bar{g}_x(t+1)\rangle \\ &\quad + \langle k_{t+1}, \mathbb{E}[-\bar{b}_{\tilde{x}}'(t+1)\eta_{t+1} - \bar{\sigma}_{\tilde{x}}'(t+1)\zeta_{t+1} + \bar{f}_{\tilde{x}}'(t+1)\xi_{t+1} + \bar{g}_{\tilde{x}}(t+1)]\rangle \\ &\quad + \langle \bar{b}_x(t)k_t + \bar{b}_y(t)m_t + \bar{b}_z(t)n_t + \bar{b}_{\tilde{x}}(t)\mathbb{E}k_t + \bar{b}_{\tilde{y}}(t)\mathbb{E}m_t \\ &\quad + \bar{b}_{\tilde{z}}(t)\mathbb{E}n_t + \alpha(t)\varepsilon\bar{b}_u(t)(v_t - \bar{u}_t), \eta_t\rangle \\ &\quad + \langle [\bar{\sigma}_x(t)k_t + \bar{\sigma}_y(t)m_t + \bar{\sigma}_z(t)n_t + \alpha(t)\varepsilon\bar{\sigma}_u(t)(v_t - \bar{u}_t)]\Delta W_t, \zeta_t\Delta W_t\rangle \\ &\quad + \langle [\bar{\sigma}_{\tilde{x}}(t)\mathbb{E}k_t + \bar{\sigma}_{\tilde{y}}(t)\mathbb{E}m_t + \bar{\sigma}_{\tilde{z}}(t)\mathbb{E}n_t]\Delta W_t, \zeta_t\Delta W_t\rangle + \Phi_t, \end{aligned}$$

where

$$\begin{aligned} \Phi_t = {}& \langle k_t + \bar{b}_x(t)k_t + \bar{b}_y(t)m_t + \bar{b}_z(t)n_t + \bar{b}_{\tilde{x}}(t)\mathbb{E}k_t + \bar{b}_{\tilde{y}}(t)\mathbb{E}m_t \\ &+ \bar{b}_{\tilde{z}}(t)\mathbb{E}n_t + \alpha(t)\varepsilon\bar{b}_u(t)(v_t - \bar{u}_t), \zeta_t\Delta W_t\rangle \\ &+ \langle [\bar{\sigma}_x(t)k_t + \bar{\sigma}_y(t)m_t + \bar{\sigma}_z(t)n_t + \bar{\sigma}_{\tilde{x}}(t)\mathbb{E}k_t + \bar{\sigma}_{\tilde{y}}(t)\mathbb{E}m_t \\ &+ \bar{\sigma}_{\tilde{z}}(t)\mathbb{E}n_t + \alpha(t)\varepsilon\bar{\sigma}_u(t)(v_t - \bar{u}_t)]\Delta W_t, \eta_t\rangle \\ &+ \langle k_t + \bar{b}_x(t)k_t + \bar{b}_y(t)m_t + \bar{b}_z(t)n_t + \bar{b}_{\tilde{x}}(t)\mathbb{E}k_t + \bar{b}_{\tilde{y}}(t)\mathbb{E}m_t \\ &+ \bar{b}_{\tilde{z}}(t)\mathbb{E}n_t + \alpha(t)\varepsilon\bar{b}_u(t)(v_t - \bar{u}_t), \Delta\bar{V}_t\rangle \end{aligned}$$

$$
\begin{aligned}
&+\big\langle\big[\bar\sigma_x(t)k_t+\bar\sigma_y(t)m_t+\bar\sigma_z(t)n_t+\bar\sigma_{\tilde x}(t)\mathbb{E}k_t+\bar\sigma_{\tilde y}(t)\mathbb{E}m_t+\bar\sigma_{\tilde z}(t)\mathbb{E}n_t\\
&+\alpha(t)\varepsilon\bar\sigma_u(t)(v_t-\bar u_t)\big]\Delta W_t,\Delta\bar V_t\big\rangle.
\end{aligned}
$$

Since W, V are martingale processes and V is strongly orthogonal to W, then $\mathbb{E}[\Phi_t]=0$. Similarly,

$$
\begin{aligned}
\Delta\langle m_t,\xi_t\rangle &= \langle m_t,\Delta\xi_t\rangle+\langle\Delta m_t,\xi_{t+1}\rangle\\
&=-\big\langle\bar f_{\tilde x}(t+1)\mathbb{E}k_{t+1}+\bar f_{\tilde y}(t+1)\mathbb{E}m_{t+1}+\bar f_{\tilde z}(t+1)\mathbb{E}n_{t+1},\xi_{t+1}\big\rangle\\
&\quad-\big\langle\bar f_x(t+1)k_{t+1}+\bar f_y(t+1)m_{t+1}+\bar f_z(t+1)n_{t+1}\\
&\quad+\alpha(t+1)\bar f_u(t+1)\varepsilon(v_{t+1}-\bar u_{t+1}),\xi_{t+1}\big\rangle\\
&\quad+\big\langle n_t\Delta W_t,\bar f_z'(t)\xi_t-\bar b_z'(t)\eta_t-\bar\sigma_z'(t)\zeta_t+\bar g_z(t)\\
&\quad+\mathbb{E}\big[\bar f_{\tilde z}'(t)\xi_t-\bar b_{\tilde z}'(t)\eta_t-\bar\sigma_{\tilde z}'(t)\zeta_t+\bar g_{\tilde z}(t)\big]\Delta W_t\big\rangle\\
&\quad+\big\langle m_t,\bar f_y'(t)\xi_t-\bar b_y'(t)\eta_t-\bar\sigma_y'(t)\zeta_t+\bar g_y(t)\\
&\quad+\mathbb{E}\big[\bar f_{\tilde y}'(t)\xi_t-\bar b_{\tilde y}'(t)\eta_t-\bar\sigma_{\tilde y}'(t)\zeta_t+\bar g_{\tilde y}(t)\big]\big\rangle+\Upsilon_t,
\end{aligned}
$$

where

$$
\begin{aligned}
\Upsilon_t&=\big\langle n_t\Delta W_t,\xi_t+\bar f_y'(t)\xi_t-\bar b_y'(t)\eta_t-\bar\sigma_y(t)\zeta_t+\bar g_y(t)\\
&\quad+\mathbb{E}\big[\bar f_{\tilde y}'(t)\xi_t-\bar b_{\tilde y}'(t)\eta_t-\bar\sigma_{\tilde y}(t)\zeta_t+\bar g_{\tilde y}(t)\big]\big\rangle\\
&\quad+\big\langle m_t,\big\{\bar f_z'(t)\xi_t-\bar b_z'(t)\eta_t-\bar\sigma_z'(t)\zeta_t+\bar g_z(t)\\
&\quad+\mathbb{E}\big[\bar f_{\tilde z}'(t)\xi_t-\bar b_{\tilde z}'(t)\eta_t-\bar\sigma_{\tilde z}'(t)\zeta_t+\bar g_{\tilde z}(t)\big]\big\}\Delta W_t\big\rangle\\
&\quad+\big\langle\xi_t+\bar f_y'(t)\xi_t-\bar b_y'(t)\eta_t-\bar\sigma_y'(t)\zeta_t+\bar g_y(t)\\
&\quad+\mathbb{E}\big[\bar f_{\tilde y}'(t)\xi_t-\bar b_{\tilde y}'(t)\eta_t-\bar\sigma_{\tilde y}'(t)\zeta_t+\bar g_{\tilde y}(t)\big],\Delta\bar N_t\big\rangle\\
&\quad+\big\langle\big\{\bar f_z'(t)\xi_t-\bar b_z'(t)\eta_t-\bar\sigma_z'(t)\zeta_t+\bar g_z(t)\\
&\quad+\mathbb{E}\big[\bar f_{\tilde z}'(t)\xi_t-\bar b_{\tilde z}'(t)\eta_t-\bar\sigma_{\tilde z}(t)\zeta_t+\bar g_{\tilde z}(t)\big]\big\}\Delta W_t,\Delta\bar N_t\big\rangle.
\end{aligned}
$$

We further derive

$$
\begin{aligned}
&\mathbb{E}\Delta\big\{\langle k_t,\eta_t\rangle+\langle m_t,\xi_t\rangle\big\}\\
&\quad=\mathbb{E}\big\{\big\langle k_{t+1},-\bar b_x'(t+1)\eta_{t+1}-\mathbb{E}\big[\bar b_{\tilde x}'(t+1)\eta_{t+1}\big]\big\rangle+\big\langle\bar b_x(t)k_t+\bar b_{\tilde x}(t)\mathbb{E}k_t,\eta_t\big\rangle\\
&\qquad-\big\langle k_{t+1},\bar\sigma_x'(t+1)\zeta_{t+1}+\mathbb{E}\big[\bar\sigma_{\tilde x}'(t+1)\zeta_{t+1}\big]\big\rangle+\big\langle\bar\sigma_x(t)k_t+\bar\sigma_{\tilde x}(t)\mathbb{E}k_t,\zeta_t\big\rangle\\
&\qquad-\big\langle\bar f_y(t+1)m_{t+1}+\bar f_{\tilde y}(t+1)\mathbb{E}m_{t+1},\xi_{t+1}\big\rangle+\big\langle m_t,\bar f_y'(t)\xi_t+\mathbb{E}\big[\bar f_{\tilde y}'(t)\xi_t\big]\big\rangle\\
&\qquad-\big\langle\bar f_z(t+1)n_{t+1}+\bar f_{\tilde z}(t+1)\mathbb{E}n_{t+1},\xi_{t+1}\big\rangle+\big\langle n_t,\bar f_z'(t)\xi_t+\mathbb{E}\big[\bar f_{\tilde z}'(t)\xi_t\big]\big\rangle\\
&\qquad+\varepsilon\big\langle\alpha(t)\bar b_u(t)(v_t-\bar u_t),\eta_t\big\rangle+\varepsilon\big\langle\alpha(t)\bar\sigma_u(t)(v_t-\bar u_t),\zeta_t\big\rangle+\big\langle\bar g_x(t+1)+\mathbb{E}\bar g_{\tilde x}(t+1),k_{t+1}\big\rangle\\
&\qquad+\big\langle\bar g_z(t)+\mathbb{E}\bar g_{\tilde z}(t),n_t\big\rangle+\big\langle\bar g_y(t)+\mathbb{E}\bar g_{\tilde y}(t),m_t\big\rangle-\varepsilon\big\langle\alpha(t+1)\bar f_u(t+1)(v_{t+1}-\bar u_{t+1}),\xi_{t+1}\big\rangle\big\}.
\end{aligned}
$$

Accordingly,

$$
\begin{aligned}
&-\mathbb{E}\big\langle h_x(\bar{X}_T,\mathbb{E}\bar{X}_T)+\mathbb{E}h_{\tilde{x}}(\bar{X}_T,\mathbb{E}\bar{X}_T),k_T\big\rangle\\
&\quad=\mathbb{E}\big\{\langle k_T,\eta_T\rangle+\langle m_T,\xi_T\rangle-\langle k_0,\eta_0\rangle-\langle m_0,\xi_0\rangle\big\}=\mathbb{E}\sum_{t=0}^{T-1}\Delta\big\{\langle k_t,\eta_t\rangle+\langle m_t,\xi_t\rangle\big\}\\
&\quad=\mathbb{E}\sum_{t=0}^{T-1}\big\{\big\langle\bar{g}_x(t)+\mathbb{E}\bar{g}_{\tilde{x}}(t),k_t\big\rangle+\big\langle\bar{g}_y(t)+\mathbb{E}\bar{g}_{\tilde{y}}(t),m_t\big\rangle+\big\langle\bar{g}_z(t)+\mathbb{E}\bar{g}_{\tilde{z}}(t),n_t\big\rangle\big\}\\
&\qquad+\mathbb{E}\sum_{t=0}^{T}\varepsilon\alpha(t)\big\langle\bar{b}_u'(t)\eta_t+\bar{\sigma}_u'(t)\zeta_t-\bar{f}_u'(t)\xi_t,v_t-\bar{u}_t\big\rangle+\mathbb{E}\big\{\big\langle\bar{b}_x(0)k_0+\bar{b}_{\tilde{x}}(0)\mathbb{E}k_0,\eta_0\big\rangle\\
&\qquad+\big\langle k_0,\bar{\sigma}_x'(0)\mathbb{E}\zeta_0+\bar{\sigma}_{\tilde{x}}'(0)\mathbb{E}\zeta_0\big\rangle+\big\langle\bar{f}_y(0)m_0+\bar{f}_{\tilde{y}}(0)\mathbb{E}m_0,\xi_0\big\rangle+\big\langle\bar{f}_z(0)n_0+\bar{f}_{\tilde{z}}(0)\mathbb{E}n_0,\xi_0\big\rangle\big\}.
\end{aligned}
$$

Since $k_0=0$, $\xi_0=0$, then the above equation leads to

$$
\begin{aligned}
&\mathbb{E}\sum_{t=0}^{T-1}\big\{\big\langle\bar{g}_x(t)+\mathbb{E}\bar{g}_{\tilde{x}}(t),k_t\big\rangle+\big\langle\bar{g}_y(t)+\mathbb{E}\bar{g}_{\tilde{y}}(t),m_t\big\rangle+\big\langle\bar{g}_z(t)+\mathbb{E}\bar{g}_{\tilde{z}}(t),n_t\big\rangle\big\}\\
&\qquad+\mathbb{E}\big\langle h_x(\bar{X}_T,\mathbb{E}\bar{X}_T)+\mathbb{E}h_{\tilde{x}}(\bar{X}_T,\mathbb{E}\bar{X}_T),k_T\big\rangle\\
&\quad=-\varepsilon\alpha\mathbb{E}\big\langle\bar{b}_u'(\theta)\eta_\theta+\bar{\sigma}_u'(\theta)\zeta_\theta-\bar{f}_u'(\theta)\xi_\theta,v_\theta-\bar{u}_\theta\big\rangle.
\end{aligned}
$$

Combining $\lim_{\varepsilon\to0}\frac{1}{\varepsilon}[J(u_t^\varepsilon)-J(\bar{u}_t)]\ge0$ with (3.11), we have

$$
\alpha\mathbb{E}\big\langle\bar{b}_u'(\theta)\eta_\theta+\bar{\sigma}_u'(\theta)\zeta_\theta-\bar{f}_u'(\theta)\xi_\theta-\bar{g}_u(\theta),v_\theta-\bar{u}_\theta\big\rangle\le0,
$$

then (4.1) holds owing to the arbitrariness of θ. □

Remark 4.1 Theorem 4.1 establishes a more general and constructive stochastic maximum principle under weakened convexity assumption. To be specific, if the set U_t is not convex, then the discrete-time stochastic maximum principles in [15, 21] are invalid. In this sense, our work generalizes and strengthens the discrete-time stochastic maximum principle of the existing works.

Corollary 4.1 *Assume that* (A2)–(A4) *hold. Let $\bar{u}_t$ be the optimal control and $(\bar{X}_t,\bar{Y}_t,\bar{Z}_t)$ be the corresponding optimal trajectory of Problem $\mathcal{A}$. Moreover, assume that the set U_t is γ-convex, $t\in\mathbb{T}$, then for $v_t\in U_t$, one has*

$$
\big\langle\mathscr{H}_u(t,\bar{X}_t,\bar{Y}_t,\bar{Z}_t,\mathbb{E}\bar{X}_t,\mathbb{E}\bar{Y}_t,\mathbb{E}\bar{Z}_t,\eta_t,\zeta_t,\xi_t,\bar{u}_t),v_t-\bar{u}_t\big\rangle\le0,\quad\mathbb{P}\text{-}a.s.
$$

Corollary 4.2 *Assume that* (A2)–(A4) *hold. Let $\bar{u}_t$ be the optimal control and $(\bar{X}_t,\bar{Y}_t,\bar{Z}_t)$ be the corresponding optimal trajectory of Problem $\mathcal{A}$. Moreover, assume that $U_t=\operatorname{ri}U_t$, $t\in\mathbb{T}$, then for $v_t\in U_t$, one has*

$$
\big\langle\mathscr{H}_u(t,\bar{X}_t,\bar{Y}_t,\bar{Z}_t,\mathbb{E}\bar{X}_t,\mathbb{E}\bar{Y}_t,\mathbb{E}\bar{Z}_t,\eta_t,\zeta_t,\xi_t,\bar{u}_t),v_t-\bar{u}_t\big\rangle=0,\quad\mathbb{P}\text{-}a.s.
$$

In what follows, we discuss assumptions, under which the necessary condition (4.1) turns into a sufficient one.

Theorem 4.2 (Sufficient conditions for optimality) *Under* (A2)–(A5), *assume that* $h(\cdot,\cdot)$ *is convex and* $\mathscr{H}(t,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\eta_t,\zeta_t,\xi_t,\cdot)$ *is convex. Then* $\bar{u}_t$ *is an optimal control of Problem* $\mathcal{A}$ *if* (4.1) *holds.*

Proof Let u_t be an arbitrary admissible control and $(X_t,\mathbb{E}X_t,Y_t,\mathbb{E}Y_t,Z_t,\mathbb{E}Z_t)$ be the corresponding trajectory. Set $\acute{X}_t = X_t - \bar{X}_t$ and $\acute{Y}_t = Y_t - \bar{Y}_t$. Since $\acute{X}_0 = \acute{Y}_T = \xi_0 = 0$, it is derived that

$$\begin{aligned}
-\mathbb{E}\langle\eta_T,\acute{X}_T\rangle &= \mathbb{E}\bigl\{\langle\eta_T,\acute{X}_T\rangle + \langle\xi_T,\acute{Y}_T\rangle - \langle\eta_0,\acute{X}_0\rangle - \langle\xi_0,\acute{Y}_0\rangle\bigr\} \\
&= \mathbb{E}\sum_{t=0}^{T-1}\bigl\{\Delta\langle\eta_t,\acute{X}_t\rangle + \Delta\langle\xi_t,\acute{Y}_t\rangle\bigr\} \\
&= \mathbb{E}\sum_{t=0}^{T-1}\Bigl\{\bigl\langle\xi_t,-f(t)+\bar{f}(t)\bigr\rangle + \bigl\langle\eta_t,b(t)-\bar{b}(t)\bigr\rangle + \bigl\langle\zeta_t,\sigma(t)-\bar{\sigma}(t)\bigr\rangle \\
&\quad - \bigl\langle\acute{X}_t,\bar{\mathscr{H}}_x(t)+\mathbb{E}\bar{\mathscr{H}}_{\tilde{x}}(t)\bigr\rangle - \bigl\langle\acute{Y}_t,\bar{\mathscr{H}}_y(t)+\mathbb{E}\bar{\mathscr{H}}_{\tilde{y}}(t)\bigr\rangle - \bigl\langle\acute{Z}_t,\bar{\mathscr{H}}_z(t)+\mathbb{E}\bar{\mathscr{H}}_{\tilde{z}}(t)\bigr\rangle\Bigr\}.
\end{aligned}$$

Denote by $(\partial_{(x,y,z,\tilde{x},\tilde{y},\tilde{z},u)}\bar{\mathscr{H}})(t)$, etc., the Clarke generalized gradients of $\mathscr{H}$ evaluated at $(\bar{X}_t,\bar{Y}_t,\bar{Z}_t,\mathbb{E}\bar{X}_t,\mathbb{E}\bar{Y}_t,\mathbb{E}\bar{Z}_t,\bar{u}_t)$. Together with the stochastic maximum principle (4.1), it follows that

$$0 \in (\partial_u\bar{\mathscr{H}})(t), \quad \text{a.e. } t.$$

By [31, Lemma 2.3], $((\partial_a\bar{\mathscr{H}})(t),0) \in (\partial_{a,u}\bar{\mathscr{H}})(t)$, a.e. t. Besides, from [31, Lemma 2.2(4)], we get

$$\begin{aligned}
\mathscr{H}(t) - \bar{\mathscr{H}}(t) &\le (\partial_x\bar{\mathscr{H}})(t)\acute{X}_t + (\partial_{\tilde{x}}\bar{\mathscr{H}})(t)\mathbb{E}\acute{X}_t + (\partial_y\bar{\mathscr{H}})(t)\acute{Y}_t \\
&\quad + (\partial_{\tilde{y}}\bar{\mathscr{H}})(t)\mathbb{E}\acute{Y}_t + (\partial_z\bar{\mathscr{H}})(t)\acute{Z}_t + (\partial_{\tilde{z}}\bar{\mathscr{H}})(t)\mathbb{E}\acute{Z}_t.
\end{aligned}$$

Therefore, along with the Hamiltonian function, it yields that

$$\begin{aligned}
-\mathbb{E}\langle\eta_T,\acute{X}_T\rangle &\le \mathbb{E}\sum_{t=0}^{T-1}\Bigl\{\bigl\langle\xi_t,\bar{f}(t)-f(t)\bigr\rangle + \bigl\langle\eta_t,b(t)-\bar{b}(t)\bigr\rangle + \bigl\langle\zeta_t,\sigma(t)-\bar{\sigma}(t)\bigr\rangle + \bar{\mathscr{H}}(t) - \mathscr{H}(t)\Bigr\} \\
&= \mathbb{E}\sum_{t=0}^{T-1}\bigl\{g(t,\Gamma_t,\bar{u}_t,v_t) - g(t,\bar{\Gamma}_t,\bar{u}_t,\bar{v}_t)\bigr\}.
\end{aligned}$$

On the other hand, using the convexity assumption on $h(\cdot,\cdot)$, we claim that

$$\mathbb{E}\bigl\{h(X_T,\mathbb{E}X_T) - h(\bar{X}_T,\mathbb{E}\bar{X}_T)\bigr\} \ge \mathbb{E}\left\{\left(\frac{\partial\bar{h}}{\partial x}\right)(t)\acute{X}_T + \left(\frac{\partial\bar{h}}{\partial\tilde{x}}\right)(t)\mathbb{E}\acute{X}_T\right\} = \mathbb{E}\langle\eta_T,\acute{X}_T\rangle.$$

Thus,

$$\mathbb{E}\bigl\{h(X_T,\mathbb{E}X_T) - h(\bar{X}_T,\mathbb{E}\bar{X}_T)\bigr\} + \mathbb{E}\sum_{t=0}^{T-1}\bigl\{g(t,\Gamma_t,u_t,v_t) - g(t,\bar{\Gamma}_t,\bar{u}_t,\bar{v}_t)\bigr\} \ge 0.$$

The proof is completed. □

5 A Stackelberg game of MF-BSΔE

As an application, in this section, we consider a Stackelberg game of MF-BSΔE:

$$\begin{cases} \Delta Y_t = -f(t+1, Y_{t+1}, Z_{t+1}, \mathbb{E}Y_{t+1}, \mathbb{E}Z_{t+1}, u_{t+1}, v_{t+1}) + Z_t \Delta W_t + \Delta M_t, \\ Y_T = \kappa, \quad u_t, v_t \in U_t \subseteq \mathbb{R}^m. \end{cases} \tag{5.1}$$

The cost functionals for the follower and the leader are given, respectively, as follows:

$$J_1(u_t, v_t; \kappa) = \mathbb{E}\left\{\sum_{t=1}^{T} g_1(t, Y_t, Z_t, \mathbb{E}Y_t, \mathbb{E}Z_t, u_t, v_t) + h_1(Y_0, \mathbb{E}Y_0)\right\},$$

$$J_2(u_t, v_t; \kappa) = \mathbb{E}\left\{\sum_{t=1}^{T} g_2(t, Y_t, Z_t, \mathbb{E}Y_t, \mathbb{E}Z_t, u_t, v_t) + h_2(Y_0, \mathbb{E}Y_0)\right\}.$$

Here, u_t, v_t denote the control processes of the follower and leader, respectively. The admissible control sets are given by

$$\mathscr{U}_1[0,T] = \{u_t, t \in \mathbb{T} \mid u_t \in \mathcal{L}^2(\mathfrak{F}_t; \mathbb{R}^m), u_t \in U_t\},$$

$$\mathscr{U}_2[0,T] = \{v_t, t \in \mathbb{T} \mid v_t \in \mathcal{L}^2(\mathfrak{F}_t; \mathbb{R}^m), v_t \in U_t\}.$$

In this section, we make the following assumptions. Set $\Gamma = (y, z, \tilde{y}, \tilde{z})$, $i = 1, 2$.

(H1) (a) $\forall \Gamma$, $u \in \mathscr{U}_1[0,T]$, $v \in \mathscr{U}_2[0,T]$, $f(\cdot, \Gamma, u, v)$, $g_i(\cdot, \Gamma, u, v)$ are $\mathfrak{F}_t$-adapted processes.

(b) $g_i(t, 0, 0, 0) \in \mathcal{L}^2(\mathfrak{F}_t; \mathbb{R})$, $f(t, 0, 0, 0) \in \mathcal{L}^2(\mathfrak{F}_t; \mathbb{R}^n)$, $h_i(y, \tilde{y}) \in \mathcal{L}^2(\mathfrak{F}_0; \mathbb{R})$.

(c) $\forall t \in \mathbb{T}$, $f(t, \cdot, \cdot, \cdot)$, $g_i(t, \cdot, \cdot, \cdot)$ are uniformly Lipschitz continuous and differentiable on Γ, u, v; $h_i(\cdot, \cdot)$ are continuously differentiable on y, $\tilde{y}$ and all derivatives are uniformly bounded.

(d) The function f is independent of z, $\tilde{z}$ at $t = T$.

(H2) The functions $f(t, \cdot, \cdot, \cdot)$, $g_i(t, \cdot, \cdot, \cdot)$ are twice continuously differentiable on Γ, u, v; $h_i(\cdot, \cdot)$ are twice continuously differentiable on y, $\tilde{y}$, and all derivatives are uniformly bounded.

(H3) The set U_t ($t \in \mathbb{T}$) is $\pm\gamma$-convex.

Besides, throughout this section, we formally denote $f(0, \Gamma, u, v) = g_i(0, \Gamma, u, v) \equiv 0$, $i = 1, 2$. The optimal control problem to be solved can be stated in the following definition.

Definition 5.1 The pair $(\bar{u}, \bar{v}) \in \mathscr{U}_1[0,T] \times \mathscr{U}_2[0,T]$ is called an optimal solution to the Stackelberg game of MF-BSΔE if it satisfies the following statements:

(a) Given $\kappa \in \mathcal{L}^2(\mathfrak{F}; \mathbb{R}^n)$, there is a mapping $l : \mathscr{U}_2[0,T] \times \mathcal{L}^2(\mathfrak{F}; \mathbb{R}^n) \to \mathscr{U}_1[0,T]$ such that

$$J_1\big(l(v_t, \xi), v_t; \kappa\big) = \min_{u_t \in \mathscr{U}_1[0,T]} J_1(u_t, v_t; \kappa), \quad \forall v_t \in \mathscr{U}_2[0,T].$$

(b) There exists unique $\bar{v}_t \in \mathscr{U}_2[0,T]$ such that

$$J_2\big(l(\bar{v}_t, \kappa), \bar{v}_t; \kappa\big) = \min_{v_t \in \mathscr{U}_2[0,T]} J_2\big(l(\bar{v}_t, \kappa), v_t; \kappa\big).$$

(c) The optimal strategy of the follower is $\bar{u}_t = l(\bar{v}_t, \kappa)$.

5.1 Optimization for the follower

In view of the hierarchy property of the leader-follower game, the follower's optimization problem is firstly considered in this subsection. Denote by $\bar{u}_t$ and $(\bar{Y}_t, \bar{Z}_t)$, respectively, the optimal control and optimal trajectory. To begin with, we consider the admissible control $u_t^\varepsilon = \bar{u}_t + \alpha(t)\varepsilon(u_t - \bar{u}_t)$, where $u_t \in U_t$, $\alpha(t) \in \mathcal{I}(t, u_t)$. We construct a needle variation

$$\alpha(t)\varepsilon(u_t - \bar{u}_t) = \begin{cases} \alpha\varepsilon(u_\theta - \bar{u}_\theta), & t = \theta, \\ 0, & t \in \mathbb{T} \setminus \{\theta\}, \end{cases}$$

where $\alpha \in \mathcal{I}(\theta, u_\theta)$, $(\theta, u_\theta) \in T \times U_\theta$, and $\varepsilon \in (0, \gamma_1^*]$. Let $(Y_t^\varepsilon, Z_t^\varepsilon)$ be the state trajectory corresponding to the control u_t^ε. Set $\Gamma_t = (Y_t, \mathbb{E}Y_t, Z_t, \mathbb{E}Z_t)$. We introduce the following variational equation:

$$\begin{cases} \Delta m_t = -f_y(t+1, \bar{\Gamma}_{t+1}, \bar{u}_{t+1}, v_{t+1}) m_{t+1} - f_z(t+1, \bar{\Gamma}_{t+1}, \bar{u}_{t+1}, v_{t+1}) n_{t+1} \\ \qquad - f_{\tilde{y}}(t+1, \bar{\Gamma}_{t+1}, \bar{u}_{t+1}, v_{t+1}) \mathbb{E}m_{t+1} - f_{\tilde{z}}(t+1, \bar{\Gamma}_{t+1}, \bar{u}_{t+1}, v_{t+1}) \mathbb{E}n_{t+1} \\ \qquad - \alpha(t+1) f_u(t+1, \bar{\Gamma}_{t+1}, \bar{u}_{t+1}, v_{t+1}) \varepsilon(u_{t+1} - \bar{u}_{t+1}) + n_t \Delta W_t + \Delta \bar{N}_t, \\ m_T = 0, \end{cases}$$

and the adjoint equation

$$\begin{cases} \Delta X_t = f_y'(t, \bar{\Gamma}_t, \bar{u}_t, v_t) X_t + g_{1y}(t, \bar{\Gamma}_t, \bar{u}_t, v_t) + \mathbb{E}\{f_{\tilde{y}}'(t, \bar{\Gamma}_t, \bar{u}_t, v_t) X_t + g_{1\tilde{y}}(t, \bar{\Gamma}_t, \bar{u}_t, v_t)\} \\ \qquad + \{f_z'(t, \bar{\Gamma}_t, \bar{u}_t, v_t) X_t + g_{1z}(t, \bar{\Gamma}_t, \bar{u}_t, v_t) \\ \qquad + \mathbb{E}[f_{\tilde{z}}'(t, \bar{\Gamma}_t, \bar{u}_t, v_t) X_t + g_{1\tilde{z}}(t, \bar{\Gamma}_t, \bar{u}_t, v_t)]\} \Delta W_t, \\ X_0 = h_{1y}(\bar{Y}_0, \mathbb{E}\bar{Y}_0) + \mathbb{E}h_{1\tilde{y}}(\bar{Y}_0, \mathbb{E}\bar{Y}_0). \end{cases}$$

Define the Hamiltonian function

$$\mathscr{H}_1(t, y, z, \tilde{y}, \tilde{z}, u, v, x) = -\langle x, f(t, y, z, \tilde{y}, \tilde{z}, u, v)\rangle - g_1(t, y, z, \tilde{y}, \tilde{z}, u, v).$$

Using Theorems 4.1–4.2, we can immediately obtain the following statements.

Theorem 5.1 (Stochastic maximum principle) *Let* (H1), (H3) *hold and* $\kappa \in \mathcal{L}^2(\mathfrak{F}; \mathbb{R}^n)$. *Given the leader's strategy* $v_t \in \mathscr{U}_2[0, T]$, *assume that* $(\bar{Y}_t, \bar{Z}_t)$ *is the optimal trajectory and* $\bar{u}_t$ *is the optimal control of the follower, then for any* $u_t \in U_t$, *one has*

$$\alpha(t)\langle \mathscr{H}_{1u}(t, \bar{Y}_t, \bar{Z}_t, \mathbb{E}\bar{Y}_t, \mathbb{E}\bar{Z}_t, \bar{u}_t, v_t, X_t), u_t - \bar{u}_t\rangle \leq 0, \quad \mathbb{P}\text{-a.s.} \tag{5.2}$$

Theorem 5.2 (Sufficient conditions for optimality) *Let* (H1), (H3) *hold and* $\kappa \in \mathcal{L}^2(\mathfrak{F}; \mathbb{R}^n)$. *Given the leader's strategy* $v_t \in \mathscr{U}_2[0, T]$, *assume that* $h_1(\cdot, \cdot)$ *is convex and* $\mathscr{H}_1(t, \cdot, \cdot, \cdot, \cdot, \cdot, v_t, X_t)$ *is concave and Lipschitz continuous. Then* $\bar{u}_t$ *is an optimal control of the follower's problem if it satisfies* (5.2).

5.2 Optimization for the leader

Notice that the follower's optimal response $\bar{u}_t$ can be determined by the leader, the state equation of the leader turns to be a MF-FBSΔE:

$$\begin{cases} \Delta X_t = f_y'(t,\bar{\Gamma}_t,\bar{u}_t,v_t)X_t + g_{1y}(t,\bar{\Gamma}_t,\bar{u}_t,v_t) \\ \qquad + \mathbb{E}\{f_{\tilde{y}}'(t,\bar{\Gamma}_t,\bar{u}_t,v_t)X_t + g_{1\tilde{y}}(t,\bar{\Gamma}_t,\bar{u}_t,v_t)\} \\ \qquad + \{f_z'(t,\bar{\Gamma}_t,\bar{u}_t,v_t)X_t + g_{1z}(t,\bar{\Gamma}_t,\bar{u}_t,v_t) \\ \qquad + \mathbb{E}[f_{\tilde{z}}'(t,\bar{\Gamma}_t,\bar{u}_t,v_t)X_t + g_{1\tilde{z}}(t,\bar{\Gamma}_t,\bar{u}_t,v_t)]\}\Delta W_t, \\ \Delta\bar{Y}_t = -f(t+1,\bar{Y}_{t+1},\bar{Z}_{t+1},\mathbb{E}\bar{Y}_{t+1},\mathbb{E}\bar{Z}_{t+1},\bar{u}_{t+1},v_{t+1}) + \bar{Z}_t\Delta W_t + \Delta\bar{M}_t, \\ X_0 = h_{1y}(\bar{Y}_0,\mathbb{E}\bar{Y}_0) + \mathbb{E}h_{1\tilde{y}}(\bar{Y}_0,\mathbb{E}\bar{Y}_0), \bar{Y}_T = \kappa. \end{cases} \tag{5.3}$$

In this subsection, for $a = y,\tilde{y},z,\tilde{z},v,x,\tilde{x}$, we use the following abbreviations:

$$\begin{aligned} &\bar{b}(t) = f_y'(t,\bar{\Gamma}_t,\bar{u}_t,\bar{v}_t)\bar{X}_t + g_{1y}(t,\bar{\Gamma}_t,\bar{u}_t,\bar{v}_t) + \mathbb{E}\bigl\{f_{\tilde{y}}'(t,\bar{\Gamma}_t,\bar{u}_t,\bar{v}_t)\bar{X}_t + g_{1\tilde{y}}(t,\bar{\Gamma}_t,\bar{u}_t,\bar{v}_t)\bigr\}, \\ &\bar{\sigma}(t) = f_z'(t,\bar{\Gamma}_t,\bar{u}_t,\bar{v}_t)\bar{X}_t + g_{1z}(t,\bar{\Gamma}_t,\bar{u}_t,\bar{v}_t) + \mathbb{E}\bigl\{f_{\tilde{z}}'(t,\bar{\Gamma}_t,\bar{u}_t,\bar{v}_t)\bar{X}_t + g_{1\tilde{z}}(t,\bar{\Gamma}_t,\bar{u}_t,\bar{v}_t)\bigr\}, \\ &\bar{\mathscr{H}}_{2a}(t) = \mathscr{H}_{2a}(t,\bar{\Gamma}_t,\bar{u}_t,\bar{v}_t,\bar{X}_t,\mathbb{E}\bar{X}_t,p_t,\zeta_t,q_t), \\ &\bar{f}_a(t) = f_a(t,\bar{\Gamma}_t,\bar{u}_t,\bar{v}_t), \qquad \bar{g}_{2a}(t) = g_{2a}(t,\bar{\Gamma}_t,\bar{u}_t,\bar{v}_t). \end{aligned}$$

Likewise, we consider the admissible control

$$v_t^\varepsilon = \begin{cases} \bar{v}_\theta + \alpha\varepsilon(v_\theta - \bar{v}_\theta), & t = \theta, \\ 0, & t \in \mathbb{T}\setminus\{\theta\}, \end{cases}$$

where $\alpha \in \mathcal{I}(\theta, v_\theta)$, $(\theta, v_\theta) \in \mathbb{T}\times U_\theta$, and $\varepsilon \in (0,\gamma_2^*]$. Let $(X_t^\varepsilon, Y_t^\varepsilon, Z_t^\varepsilon)$ be the state trajectory corresponding to the admissible control v_t^ε. Let (k,η,ρ,N) be a solution of the following variational equations:

$$\begin{cases} \Delta k_t = \bar{b}_x(t)k_t + \bar{b}_y(t)\eta_t + \bar{b}_z(t)\rho_t + \bar{b}_{\tilde{x}}(t)\mathbb{E}k_t + \bar{b}_{\tilde{y}}(t)\mathbb{E}\eta_t + \bar{b}_{\tilde{z}}(t)\mathbb{E}\rho_t + \alpha(t)\bar{b}_v(t)\varepsilon(v_t - \bar{v}_t) \\ \qquad + [\bar{\sigma}_x(t)k_t + \bar{\sigma}_y(t)\eta_t + \bar{\sigma}_z(t)\rho_t + \alpha(t)\bar{\sigma}_v(t)\varepsilon(v_t - \bar{v}_t)]\Delta W_t \\ \qquad + [\bar{\sigma}_{\tilde{x}}(t)\mathbb{E}k_t + \bar{\sigma}_{\tilde{y}}(t)\mathbb{E}\eta_t + \bar{\sigma}_{\tilde{z}}(t)\mathbb{E}\rho_t]\Delta W_t, \\ \Delta\eta_t = -\bar{f}_y(t+1)\eta_{t+1} - \bar{f}_z(t+1)\rho_{t+1} - \alpha(t+1)\bar{f}_v(t+1)\varepsilon(v_{t+1} - \bar{v}_{t+1}) \\ \qquad - \bar{f}_{\tilde{y}}(t+1)\mathbb{E}\eta_{t+1} - \bar{f}_{\tilde{z}}(t+1)\mathbb{E}\rho_{t+1} + \rho_t\Delta W_t + \Delta N_t, \\ k_0 = \{h_{1yy}(\bar{Y}_0,\mathbb{E}\bar{Y}_0) + h_{1y\tilde{y}}(\bar{Y}_0,\mathbb{E}\bar{Y}_0) + \mathbb{E}h_{1y\tilde{y}}(\bar{Y}_0,\mathbb{E}\bar{Y}_0) + \mathbb{E}h_{1\tilde{y}\tilde{y}}(\bar{Y}_0,\mathbb{E}\bar{Y}_0)\}\eta_0, \qquad \eta_T = 0. \end{cases}$$

We define the Hamiltonian

$$\begin{aligned} \mathscr{H}_2(t,y,z,\tilde{y},\tilde{z},u,v,x,\tilde{x},p,\zeta,q) &= \bigl\langle p, b(t,y,z,\tilde{y},\tilde{z},u,v,x,\tilde{x})\bigr\rangle + \bigl\langle \zeta, \sigma(t,y,z,\tilde{y},\tilde{z},u,v,x,\tilde{x})\bigr\rangle \\ &\quad - \bigl\langle q, f(t,y,z,\tilde{y},\tilde{z},u,v)\bigr\rangle - g_2(t,y,z,\tilde{y},\tilde{z},u,v). \end{aligned}$$

Similarly, we proceed to introducing the following adjoint equations associated with MF-FBSΔE (5.3):

$$\begin{cases} \Delta p_t = -\bar{\mathscr{H}}_{2x}(t) - \mathbb{E}\bar{\mathscr{H}}_{2\tilde{x}}(t) + \zeta_t \Delta W_t + \Delta V_t, \qquad p_T = 0, \\ \Delta q_t = -\bar{\mathscr{H}}_{2y}(t) - \mathbb{E}\bar{\mathscr{H}}_{2\tilde{y}}(t) - \{\bar{\mathscr{H}}_{2z}(t) + \mathbb{E}\bar{\mathscr{H}}_{2\tilde{z}}(t)\}\Delta W_t, \\ q_0 = h_{2y}(\bar{Y}_0, \mathbb{E}\bar{Y}_0) + \mathbb{E}h_{2\tilde{y}}(\bar{Y}_0, \mathbb{E}\bar{Y}_0) - \{h_{1yy}(\bar{Y}_0, \mathbb{E}\bar{Y}_0)p_0 + \mathbb{E}h_{1\tilde{y}y}(\bar{Y}_0, \mathbb{E}\bar{Y}_0)p_0\} \\ \qquad - \mathbb{E}\{h_{1y\tilde{y}}(\bar{Y}_0, \mathbb{E}\bar{Y}_0)p_0 + \mathbb{E}h_{1\tilde{y}\tilde{y}}(\bar{Y}_0, \mathbb{E}\bar{Y}_0)p_0\}. \end{cases}$$

Here, W, N, V are square integrable martingale processes and N, V are strongly orthogonal to W.

The following conclusions are straightforward with the aid of Theorems 4.1–4.2.

Theorem 5.3 (Stochastic maximum principle) *Assume that* (H1)–(H3) *hold. Let* $\bar{v}_t$ *and* $(\bar{X}_t, \bar{Y}_t, \bar{Z}_t)$ *be the optimal control, optimal trajectory, respectively. Then, for* $v_t \in U_t$, *one has*

$$\alpha(t)\langle \mathscr{H}_{2v}(t, \bar{Y}_t, \bar{Z}_t, \mathbb{E}\bar{Y}_t, \mathbb{E}\bar{Z}_t, \bar{u}_t, \bar{v}_t, \bar{X}_t, \mathbb{E}\bar{X}_t, p_t, \zeta_t, q_t), v_t - \bar{v}_t\rangle \leq 0, \quad \mathbb{P}\text{-}a.s. \tag{5.4}$$

Theorem 5.4 (Sufficient conditions for optimality) *Under* (H1)–(H3), *assume that* $h_2(\cdot,\cdot)$ *is convex and* $\mathscr{H}_2(t,\cdot,\cdot,\cdot,\cdot,\bar{u}_t,\cdot,\cdot,\cdot,p_t,\zeta_t,q_t)$ *is convex, then* $\bar{v}_t$ *is an optimal control of the leader's problem if* (5.4) *holds.*

6 Perspectives and open problems

In this section, we give a brief exposition on the prospects that are open to the researchers. The following topics shall be explored in our future works.

Firstly, we see that the effectiveness of optimality conditions obtained in this paper substantially relies on the structure of the set U_t. So it is pregnant to discuss more general and essential convexity assumptions for the discrete-time forward–backward stochastic system.

Secondly, there are many more partially observable cases which are more constructive and inevitable for applications and are technologically demanding in their filtering procedure.

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