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Separation and stability of solutions to nonlinear systems involving Caputo–Fabrizio derivatives

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Abstract

This work mainly investigates the separation and stability of solutions to nonlinear systems involving Caputo–Fabrizio fractional derivatives. An inequality ensuring the positivity of the fractional derivative at a given point is derived, by which the sufficient conditions for the separation of solutions are obtained. The comparison principle and the inequality for the fractional derivatives of convex functions are obtained, by which the approach of the convex Lyapunov functions is extended effectively to establish the criteria for the stability of solutions in the context of Caputo–Fabrizio fractional derivatives. Applications of the main results are illustrated by using examples.

Keywords: Caputo–Fabrizio derivatives; Nonlinear systems; Stability; Separation of solutions; Convex Lyapunov functions

1 Introduction

The fractional derivative, in which the order of the derivative is permitted to be noninteger, originated in 1695. Although it is as old as the classical calculus, the world of analysis seems to be always dominated by the latter. It has been thought out of mainstream science for over 300 years, and it is only in the last 40 years that it has come back to life.

It is worth stressing that the recovery of vitality is not just another way of presenting old stories, since the fractional derivatives and fractional differential equations have been applied to numerous fields such as viscoelasticity, signal processing, biology, engineering, neuroscience and materials science in the last four decades, and have become a well-accepted instrument in the description of complex systems [1–5].

Many important results have been obtained for the investigation of the theory of differential equations with the Riemann–Liouville or Caputo fractional derivatives. For instance, the existence of solutions to the boundary value problems was discussed in [6–12] for some specific fractional differential or integro-differential equations or inclusions by the use of certain fixed point theorems; and new results were derived in [13] on exact controllability of a class of fractional neutral integro-differential systems. Moreover, the boundary value problems involving the ψ -Caputo fractional derivative were studied in [14].

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It is also worth pointing out that the concepts of fractional derivatives are also modified and further developed to adapt to the new requirements of theory and practice. Up to now, there exist several definitions of the fractional derivative in the literature, each being constructed to satisfy various constraints and to be consistent with physical background and experimental data, among which the most popular ones are the Riemann–Liouville and Caputo fractional derivatives. The integral kernels in the previous two kinds of fractional derivatives are singular, which describe the processes involving memory effects.

The retarded effects in many dynamical processes can also be depicted by using integrals with nonsingular kernels, among which the exponential kernels were used in [15, 16] to capture the temporal memory. Based on the previous considerations, a new fractional derivative with nonsingular kernel was proposed recently by Caputo and Fabrizio in [17, 18], and then the concept of the Atangana–Baleanu derivative with Mittag-Leffler kernel was introduced in [19] and the existence of solutions to some integral equations was discussed in [20, 21].

Moreover, the definition of the conformable fractional derivative was also suggested in [22] such that the chain rule holds for the new derivative, and some results for this type of derivative have been already achieved [23–26].

The Caputo and Fabrizio fractional derivative (*CFFD*) and the differential equations involving the *CFFD* have received growing attention. In recent years, a considerable literature has sprung up in the framework of *CFFDs*. The definition of the *CFFD* and its properties were deeply explored and further extended in [27–34], among which the definition of the extended *CFFD* was first introduced in [34]. The well-posedness for the differential systems with the *CFFD* was well investigated in the published studies [35–42]; among them, the existence of solutions for a coupled system of differential inclusions was discussed in [37] by using some fixed point theorems for the multivalued maps, the existence of solutions and of approximate solutions for some high-order fractional integro-differential equations were completely considered in [38] and [39–41], respectively, by means of fixed point theorems and by virtue of approaches involving α -contractive maps, and the results for the existence and dimension of the set of solutions were obtained in [42] for the second fractional integro-differential inclusion problem with the extended *CFFD*. And more importantly, the differential equations involving the *CFFD* have found a wide range of applications [43–49].

For the above-mentioned equations involving the *CFFD*, relatively speaking, there are very few results in the published studies for the separation of solutions and for the stability in the Lyapunov sense, and in the context of Caputo fractional derivatives, the former has been considered for the case where the state space is one-dimensional [2, 50–52]; while the latter has been completely discussed in [53–59] by using the Lyapunov second's method. It is noted that the direct calculation of the fractional derivative of the Lyapunov functions is far from easy, and the method of the convex Lyapunov functions was recently introduced in [60] and then developed in [61, 62], which effectively settles the difficulty.

It is also worth remarking that the Hyers–Ulam stability was discussed in [63–65] for the specific integro-differential equations and advection–reaction diffusion system with the Atangana–Baleanu derivative.

Inspired by results and techniques in the aforementioned papers, in this work we will discuss the qualitative properties and stability of solutions to the initial value problem

(IVP) with the extend *CFFD*. Concretely speaking, the first goal is to investigate the separation of trajectories of solutions to the IVP involving *CFFDs* as follows:

$$\begin{cases} \mathcal{D}_a^{(\alpha)} \omega(t) = F(t, \omega(t)), & a \leq t < \infty, \\ \omega(a) = \omega_a, \end{cases} \quad (1)$$

where $\mathcal{D}_a^{(\alpha)}$ denotes the *CFFD* of order α (defined below); ω and F are vector-valued functions on $[a, \infty)$ and \mathbb{R}^{d+1} , respectively, and $F(a, \omega_a) = 0$. The approach employed here depends on the inequality to be established in this work, which is different from any other methods used to discuss the same problems with the Caputo fractional derivative in the existing studies [2, 50–52]. Here the separation of solutions means that the trajectories of solutions with different initial values do not intersect each other.

Based on the previous discussion, under the local Lipschitz condition, we then explore the stability of the zero solution to the system

$$\begin{cases} \mathcal{D}_a^{(\alpha)} \omega(t) = p(t)f(\omega(t)), & a \leq t < \infty, \\ \omega(a) = \omega_a, \end{cases} \quad (2)$$

where f is a vector-valued function on \mathbb{R}^d , and $p(t)$ is a scalar function on $[a, \infty)$ with $p(a) = 0$.

It is also worth remarking that, in the framework of *CFFDs* and under the local Lipschitz condition, there are no global existence results for the solutions in the published studies due to the restriction of the construction of the integral equations equivalent to the fractional ones under consideration, which causes the difficulty to the discussion of the stability.

To arrive at our second goal, using the comparison principle and the inequality involving the *CFFD* of convex functions to be established in this work, we follow the idea from [61, 66] by extending the vector fields of fractional nonlinear systems and obtain the criteria for the stability of solutions under the local Lipschitz condition. Moreover, we have also to point out that the discussion of stability of autonomous systems ($p(t) \equiv 1$ in (2)) is trivial (see Remark 4.1 in Sect. 4).

The paper is divided into six parts. The Caputo–Fabrizio fractional derivative and integral are introduced in Sect. 2. We first establish a comparison principle for the *CFFDs* in Sect. 3, and then obtain an inequality involving *CFFDs* by which the result for the separation of trajectories of solutions to the IVP (1) for the case of one dimension is derived. We then establish an inequality involving the *CFFDs* of convex functions in Sect. 4, by which, combined with the comparison principle, we discuss the stability of solutions to the IVP (2). The validity of the main result is illustrated in Sect. 5 by using two examples, and finally, the conclusions are discussed in Sect. 6.

2 Preliminaries

We introduce the Caputo–Fabrizio fractional derivative and integral in this section, and show their basic properties to be used in what follows.

Given a positive integer d , the d -dimensional Euclidean space is denoted by \mathbb{R}^d , equipped with the classical norm $|\cdot|$. Let J_a be a left-closed interval with the left endpoint a and Ω be a connected open subset of \mathbb{R}^d , and denote by $C(J_a, \Omega)$ the space of all

continuous functions from J_a into Ω . The gradient vector of a function V on \mathbb{R}^d is denoted by ∇V .

Moreover, unless otherwise specified, it is always assumed that the exponent α is in the interval $(0, 1)$ throughout this work.

Given an absolutely continuous function ω in the interval $[a, b]$, the Caputo–Fabrizio fractional derivative of order α is defined [17, 18] by

$${}^{\text{CF}}D_t^\alpha \omega(t) = \frac{1}{1-\alpha} \int_a^t e_\alpha(t-s) \omega'(s) \, ds. \quad (3)$$

Here the function $e_\alpha(\cdot)$ is defined by

$$e_\alpha(t) = \exp\left(-\frac{\alpha t}{1-\alpha}\right). \quad (4)$$

Using the results obtained by integration by parts on the right-hand side of (3), we next give the weak version of the CFFD, denoted by $\mathcal{D}_a^{(\alpha)}$, without requirement for the smoothness of functions.

Definition 2.1 ([34]) Letting ω be in $C(J_a, \mathbb{R})$, the Caputo–Fabrizio fractional derivative of order α is defined by

$$\mathcal{D}_a^{(\alpha)} \omega(t) = \frac{\omega(t) - \omega(a)}{1-\alpha} e_\alpha(t-a) + \frac{\alpha}{(1-\alpha)^2} \int_a^t (\omega(t) - \omega(s)) e_\alpha(t-s) \, ds.$$

Remark 2.1

$$\mathcal{D}_a^{(\alpha)} \omega(t) = \frac{\omega(t) - \omega(a)}{1-\alpha} - \frac{\alpha}{(1-\alpha)^2} \int_a^t (\omega(s) - \omega(a)) e_\alpha(t-s) \, ds,$$

and $\mathcal{D}_a^{(\alpha)} \omega(a) = 0$.

Definition 2.2 Letting ω be in $C(J_a, \mathbb{R})$, the Caputo–Fabrizio fractional integral of order α is defined by

$$\mathcal{I}_a^{(\alpha)} \omega(t) = (1-\alpha)(\omega(t) - \omega(a)) + \alpha \int_a^t (\omega(s) - \omega(a)) \, ds.$$

Remark 2.2 Here our definition of the fractional integral is slightly different from those in [27, 33], by which a harmonic connection between $\mathcal{D}_a^{(\alpha)} \mathcal{I}_a^{(\alpha)}$ and $\mathcal{I}_a^{(\alpha)} \mathcal{D}_a^{(\alpha)}$ is established.

Lemma 2.1 If ω belongs to $C(J_a, \mathbb{R})$, then $\mathcal{D}_a^{(\alpha)} \mathcal{I}_a^{(\alpha)} \omega(t) = \omega(t) - \omega(a)$ and $\mathcal{I}_a^{(\alpha)} \mathcal{D}_a^{(\alpha)} \omega(t) = \omega(t) - \omega(a)$.

Proof Letting $\beta = \frac{\alpha}{1-\alpha}$ and using Remark 2.1 and Definition 2.2, a direct calculation yields

$$\begin{aligned}
 \mathcal{D}_a^{(\alpha)} \mathcal{I}_a^{(\alpha)} \omega(t) &= (1-\alpha) \mathcal{D}_a^{(\alpha)} (\omega(t) - \omega(a)) + \alpha \mathcal{D}_a^{(\alpha)} \left(\int_a^t (\omega(s) - \omega(a)) \, ds \right) \\
 &= \omega(t) - \omega(a) - \beta \int_a^t (\omega(s) - \omega(a)) e_\alpha(t-s) \, ds \\
 &\quad + \beta \int_a^t (\omega(s) - \omega(a)) \, ds - \beta^2 \int_a^t \int_a^s (\omega(\tau) - \omega(a)) e_\alpha(t-s) \, d\tau \, ds \\
 &= \omega(t) - \omega(a) - \beta \int_a^t (\omega(s) - \omega(a)) e_\alpha(t-s) \, ds \\
 &\quad + \beta \int_a^t (\omega(s) - \omega(a)) \, ds - \beta^2 \int_a^t \int_\tau^t (\omega(\tau) - \omega(a)) e_\alpha(t-s) \, ds \, d\tau \\
 &= \omega(t) - \omega(a) - \beta \int_a^t (\omega(s) - \omega(a)) e_\alpha(t-s) \, ds \\
 &\quad + \beta \int_a^t (\omega(s) - \omega(a)) \, ds - \beta \int_a^t (\omega(\tau) - \omega(a)) (1 - e_\alpha(t-\tau)) \, d\tau \\
 &= \omega(t) - \omega(a),
 \end{aligned}$$

and

$$\begin{aligned}
 \mathcal{I}_a^{(\alpha)} \mathcal{D}_a^{(\alpha)} \omega(t) &= (1-\alpha) \mathcal{D}_a^{(\alpha)} \omega(t) + \alpha \int_a^t \mathcal{D}_a^{(\alpha)} \omega(s) \, ds \\
 &= \omega(t) - \omega(a) - \beta \int_a^t (\omega(s) - \omega(a)) e_\alpha(t-s) \, ds \\
 &\quad + \beta \int_a^t (\omega(s) - \omega(a)) \, ds - \beta^2 \int_a^t \int_a^s (\omega(\tau) - \omega(a)) e_\alpha(s-\tau) \, d\tau \, ds \\
 &= \omega(t) - \omega(a) - \beta \int_a^t (\omega(s) - \omega(a)) e_\alpha(t-s) \, ds + \beta \int_a^t (\omega(s) - \omega(a)) \, ds \\
 &\quad - \beta^2 \int_a^t \int_\tau^t (\omega(\tau) - \omega(a)) e_\alpha(s-\tau) \, ds \, d\tau \\
 &= \omega(t) - \omega(a) - \beta \int_a^t (\omega(s) - \omega(a)) e_\alpha(t-s) \, ds \\
 &\quad + \beta \int_a^t (\omega(s) - \omega(a)) \, ds - \beta \int_a^t (\omega(\tau) - \omega(a)) (1 - e_\alpha(t-\tau)) \, d\tau \\
 &= \omega(t) - \omega(a).
 \end{aligned}$$

The assertions are thus verified. \square

For a vector-valued function $\omega(t) = (\omega_1(t), \omega_2(t), \dots, \omega_d(t))^T$, the corresponding Caputo–Fabrizio fractional integral and derivative of order α are defined respectively by

$$\mathcal{I}_a^{(\alpha)} \omega(t) = (\mathcal{I}_a^{(\alpha)} \omega_1(t), \mathcal{I}_a^{(\alpha)} \omega_2(t), \dots, \mathcal{I}_a^{(\alpha)} \omega_d(t))^T$$

and

$$\mathcal{D}_a^{(\alpha)} \omega(t) = (\mathcal{D}_a^{(\alpha)} \omega_1(t), \mathcal{D}_a^{(\alpha)} \omega_2(t), \dots, \mathcal{D}_a^{(\alpha)} \omega_d(t))^T.$$

The inequality $\omega(t) \geq 0$ on J_a means that $\omega_i(t) \geq 0$ for each t in J_a and for $i = 1, 2, \dots, d$, and $\omega(t) > 0$ is defined similarly.

3 Comparison principles and separation of trajectories

The following assertion can be directly verified by using Definition 2.2.

Lemma 3.1 *Let functions ω and v belong to $C(J_a, \mathbb{R}^d)$. If $\omega(t) \leq v(t)$ on J_a with $\omega(a) = v(a)$, then $\mathcal{I}_a^{(\alpha)} \omega(t) \leq \mathcal{I}_a^{(\alpha)} v(t)$ on J_a .*

Using Lemma 3.1, we next derive a comparison principle, which will play a key role in the discussion of the stability.

Lemma 3.2 *Let ω and v belong to $C(J_a, \mathbb{R}^d)$ with $\omega(a) \leq v(a)$. If $\mathcal{D}_a^{(\alpha)} \omega(t) \leq \mathcal{D}_a^{(\alpha)} v(t)$ on J_a , then $\omega(t) \leq v(t)$ on J_a .*

Proof The continuities of ω and v ensure that both $\mathcal{D}_a^{(\alpha)} \omega(t)$ and $\mathcal{D}_a^{(\alpha)} v(t)$ lie in $C(J_a, \mathbb{R}^d)$. Noting $\mathcal{D}_a^{(\alpha)} \omega(a) = 0 = \mathcal{D}_a^{(\alpha)} v(a)$ and using the inequality for CFFD in the conditions, we infer by Lemma 3.1 that

$$\mathcal{I}_a^{(\alpha)} \mathcal{D}_a^{(\alpha)} \omega(t) \leq \mathcal{I}_a^{(\alpha)} \mathcal{D}_a^{(\alpha)} v(t),$$

and hence by Lemma 2.1 that

$$\omega(t) - \omega(a) \leq v(t) - v(a),$$

which is equivalent to

$$\omega(t) \leq v(t) - (v(a) - \omega(a)),$$

which due to the condition $\omega(a) \leq v(a)$ yields the desired result. \square

Corollary 3.1 *Let ω belong to $C(J_a, \mathbb{R}^d)$. If $\mathcal{D}_a^{(\alpha)} \omega(t) \leq 0$ on J_a , then $\omega(t) \leq \omega(a)$ on J_a .*

An important inequality is next established to be used to discuss the separation of trajectories of solutions to IVPs.

Lemma 3.3 *Let a function χ belong to $C(J_a, \mathbb{R}^d)$. If $\chi(t) < 0$ on the subinterval $[a, t_1)$ of J_a with $\chi(t_1) = 0$, then $\mathcal{D}_a^{(\alpha)} \chi(t_1) > 0$.*

Proof From Definition 2.1 and the condition $\chi(t_1) = 0$, it follows that

$$\begin{aligned} \mathcal{D}_a^{(\alpha)} \chi(t_1) &= \frac{[\chi(t_1) - \chi(a)]e_\alpha(t_1 - a)}{1 - \alpha} + \int_a^{t_1} \frac{\alpha(\chi(t_1) - \chi(s))e_\alpha(t_1 - s)}{(1 - \alpha)^2} ds \\ &= -\frac{\chi(a)}{1 - \alpha} e_\alpha(t_1 - a) - \frac{\alpha}{(1 - \alpha)^2} \int_a^{t_1} \chi(s) e_\alpha(t_1 - s) ds. \end{aligned}$$

This, together with the assumption $\chi(t) < 0$ on $[a, t_1)$, verifies the desired assertion. \square

Remark 3.1 The preceding inequality is strict, which plays a central role in the investigation on the separation of trajectories.

Before starting with the discussion of the separation of trajectories, we here have to consider the existence of solutions.

Definition 3.1 A function ω from the interval J_a into \mathbb{R}^d is called a solution to the IVP (1) if it belongs to $C(J_a, \mathbb{R}^d)$ and solves the first equation in (1) on J_a with $\omega(a) = \omega_a$.

Lemma 3.4 Let $F : J_a \times \mathbb{R}^d \mapsto \mathbb{R}^d$ be a continuous function with $F(a, \omega_a) = 0$. Then a function ω in $C(J_a, \mathbb{R}^d)$ is a solution to the problem (1) if and only if it is a solution in $C(J_a, \mathbb{R}^d)$ to the integral equation

$$\omega(t) = \omega_a + (1 - \alpha)[F(t, \omega(t)) - F(a, \omega(a))] + \alpha \int_a^t [F(s, \omega(s)) - F(a, \omega(a))] ds. \quad (5)$$

Lemma 3.4 is easily verified by using Definitions 2.1 and 2.2 and Lemma 2.1.

Remark 3.2 If the fractional derivative in problem (1) is replaced by that in (3), then the equivalence in Lemma 3.4 does not hold generally any more.

Theorem 3.1 Let $F : [a, \infty) \times \mathbb{R}^d \mapsto \mathbb{R}^d$ be a continuous function with $F(a, \omega_a) = 0$. If F is Lipschitz on \mathbb{R}^d with respect to the state variables with the Lipschitz constant L and if $2L(1 - \alpha) < 1$, then there exists a unique solution to problem (1) defined on $[a, \infty)$.

Proof The inequality $2L(1 - \alpha) < 1$ ensures that we can choose a positive number $\mu > \alpha L[1 - 2(1 - \alpha)L]^{-1}$. Let $T > a$ and for each ω in $C([a, T], \mathbb{R}^d)$ define

$$\|\omega\|_\mu = \sup\{\exp(-\mu t)|\omega(t)| : t \in [a, T]\}.$$

And then $C([a, T], \mathbb{R}^d)$ becomes a Banach space equipped with the norm $\|\cdot\|_\mu$.

Define the operator $\mathcal{A} : C([a, T], \mathbb{R}^d) \mapsto C([a, T], \mathbb{R}^d)$ as follows:

$$\mathcal{A}\omega(t) = \omega_a + (1 - \alpha)[F(t, \omega(t)) - F(a, \omega(a))] + \alpha \int_a^t [F(s, \omega(s)) - F(a, \omega(a))] ds.$$

And then, using the Lipschitz condition, we obtain the following estimate:

$$\begin{aligned} |\mathcal{A}\omega(t) - \mathcal{A}v(t)| &\leq (1 - \alpha)L(|\omega(t) - v(t)| + |\omega(a) - v(a)|) \\ &\quad + \alpha L \int_a^t \exp(\mu s) ds \|\omega - v\|_\mu \\ &= (1 - \alpha)L(|\omega(t) - v(t)| + |\omega(a) - v(a)|) \\ &\quad + \alpha L \mu^{-1}(\exp(\mu t) - 1) \|\omega - v\|_\mu. \end{aligned}$$

Multiplying both sides of the above inequality by $\exp(-\mu t)$ yields

$$\|\mathcal{A}\omega - \mathcal{A}v\|_\mu \leq (2(1 - \alpha) + \alpha \mu^{-1})L \|\omega - v\|_\mu,$$

which implies that \mathcal{A} is a contraction because of $L(2(1 - \alpha) + \alpha\mu^{-1}) < 1$. It follows from the Banach fixed point theorem that the operator has a unique fixed point in $C([a, T], \mathbb{R}^d)$, and Lemma 3.4 ensures that this fixed point is the unique solution to problem (1). Finally, the solution is clearly defined on \mathbb{R}^+ due to arbitrariness of choice of T . The proof is complete. \square

By using Lemma 3.3 and Theorem 3.1, we now present a result for the separation of trajectories of the solutions to IVP (1) for the case where the dimension $d = 1$.

Theorem 3.2 (Separation of trajectories) *Let the dimension $d = 1$. If F satisfies the conditions in Theorem 3.1 except that $F(a, \omega_a) = 0$ is replaced by $F(a, \omega) = 0$ for any ω in \mathbb{R} , then the trajectories of any two solutions with different initial values at $t = a$ to the equation in (1) do not intersect on \mathbb{R}^+ .*

Proof Let ω_{1a} and ω_{2a} be different initial values with $\omega_{1a} < \omega_{2a}$, then, according to Theorem 3.1, the corresponding solutions to the equation in (1), denoted by $\omega_1(t)$ and $\omega_2(t)$, respectively, are defined on \mathbb{R}^+ .

Now suppose that the assertion of the theorem is false, and then, the inequality $\omega_{1a} < \omega_{2a}$ and the continuity of the solutions ensure that there exists a positive number t_1 such that, for each t in the interval $[a, t_1)$,

$$\omega_1(t) < \omega_2(t),$$

for which $\omega_1(t_1) = \omega_2(t_1)$; letting $\chi(t) = \omega_1(t) - \omega_2(t)$, we thus derive by Lemma 3.3 the inequality $\mathcal{D}_a^{(\alpha)} \chi(t_1) > 0$.

On the other hand, the relation $\omega_1(t_1) = \omega_2(t_1)$ implies

$$\mathcal{D}_a^{(\alpha)} \omega_1(t_1) = F(t_1, \omega_1(t_1)) = F(t_1, \omega_2(t_1)) = \mathcal{D}_a^{(\alpha)} \omega_2(t_1).$$

This yields $\mathcal{D}_a^{(\alpha)} \chi(t_1) = 0$, which contradicts the preceding inequality $\mathcal{D}_a^{(\alpha)} \chi(t_1) > 0$. Thus, the proof is complete. \square

Remark 3.3 Just as pointed out in Remark 3.1, the strict inequality in Lemma 3.3 leads here to a concise proof of the separation of trajectories, and the approach here employed is different from any other methods used to discuss the same problems with the Caputo fractional derivative in the existing studies [2, 50–52].

4 Convex Lyapunov functions and stability

We explore in this section the stability of the solutions to the IVP (2).

Definition 4.1 The zero solution of the equation in (2) is stable if, for any $\epsilon > 0$, there is a positive number δ such that if $|\omega_a| < \delta$, then, for the solution $\omega(t, \omega_a)$ of (2), we have $|\omega(t, \omega_a)| < \epsilon$ for all $t \geq a$.

Remark 4.1 The discussion for the stability of autonomous systems is trivial.

To see this, let $p(t) \equiv 1$ in (2), and then the system (2) becomes autonomous. Let ω be in $C(J_a, \mathbb{R}^d)$, then $\mathcal{D}_a^{(\alpha)} \omega(t)$ belongs also to $C(J_a, \mathbb{R}^d)$ with $\mathcal{D}_t^{(\alpha)} \omega(a) = 0$, which forces

$f(\omega_a) = 0$, and thus $\omega(t) \equiv \omega_a$ is the unique solution to (2). Accordingly, in order to discuss the stability of system (2) with $p(t) \equiv 1$, it must hold that $f(\omega) \equiv 0$ whenever $|\omega| < \delta$ for some $\delta > 0$, which obviously is trivial.

Definition 4.2 ([67]) A continuous function $\kappa : \mathbb{R}^+ \mapsto \mathbb{R}^+$ is said to belong to class \mathcal{K} if it is strictly increasing and $\kappa(0) = 0$. For such a function, we shall usually write $\kappa \in \mathcal{K}$.

Lemma 4.1 (Kirschbraun's theorem, [68]) *If $S \subset \mathbb{R}^m$ and $f : S \mapsto \mathbb{R}^n$ is Lipschitz, then f has a Lipschitz extension $f_E : \mathbb{R}^m \mapsto \mathbb{R}^n$ with $\text{Lip}(f_E) = \text{Lip}(f)$.*

Given a positive number r , define the subsets $\mathcal{S}(r)$ of \mathbb{R}^d by

$$\mathcal{S}(r) = \{\omega : |\omega| < r, \omega \in \mathbb{R}^d\}.$$

We now are in a position to present an important inequality involving the CFFD of composition function in this work.

Lemma 4.2 *Let $V : \mathcal{S}(r) \mapsto \mathbb{R}$ be a continuously differentiable and convex function. Then for any function ω in $C(J_a, \mathcal{S}(r))$, the inequality*

$$\mathcal{D}_a^{(\alpha)} V(\omega(t)) \leq \nabla V(\omega(t)) \cdot \mathcal{D}_a^{(\alpha)} \omega(t),$$

holds on J_a . Here and in what follows, the dot \cdot denotes an inner product on \mathbb{R}^d .

Proof The continuity of V and ω ensures that both $\mathcal{D}_a^{(\alpha)} V(\omega(t))$ and $\mathcal{D}_a^{(\alpha)} \omega(t)$ exist on J_a . Let $\psi(t, \tau) = V(\omega(t)) - V(\omega(\tau))$, and then the differentiability and convexity of V imply that [69]

$$\psi(t, \tau) \leq \nabla V(\omega(t)) \cdot (\omega(t) - \omega(\tau)).$$

Using Definition 2.1, by the inequality just derived, we deduce the following estimate:

$$\begin{aligned} \mathcal{D}_a^{(\alpha)} V(\omega(t)) &= \frac{\psi(t, a)e_\alpha(t-a)}{1-\alpha} + \int_a^t \frac{\alpha \psi(t, s)e_\alpha(t-s) ds}{(1-\alpha)^2} \\ &\leq \nabla V(\omega(t)) \cdot \left(\frac{(\omega(t) - \omega(a))e_\alpha(t-a)}{1-\alpha} + \int_a^t \frac{\alpha(\omega(t) - \omega(s))e_\alpha(t-s)}{(1-\alpha)^2} ds \right) \\ &= \nabla V(\omega(t)) \cdot \mathcal{D}_a^{(\alpha)} \omega(t), \end{aligned}$$

which verifies the desired inequality. \square

In order to discuss the stability of solutions to problem (2), let $\mathcal{S}(r)$ be defined as in Sect. 4 and we further impose several assumptions on the functions in (2):

- (A1) $p(\cdot)$ is a nonnegative and continuous real-valued function bounded on \mathbb{R}^+ and $p(a) = 0$.
- (A2) $f(\cdot) : \mathcal{S}(r) \mapsto \mathbb{R}^d$ is a continuous function with $f(0) = 0$ and satisfies on $\mathcal{S}(r)$ the Lipschitz condition with $2L(1-\alpha) \sup p(t) < 1$.

We now present a result for the stability of solutions by using convex Lyapunov functions.

Theorem 4.1 *Let conditions (A1) and (A2) be satisfied. If there exists a real-valued function $V(\omega)$ which is continuously differentiable and convex on $S(r)$ with $V(0) = 0$ such that*

$$\kappa(|\omega|) \leq V(\omega)$$

for some class- \mathcal{K} function κ and

$$\nabla V(\omega) \cdot f(\omega) \leq 0$$

on $S(r)$, then the zero solution of problem (2) is stable.

Proof Given a positive number r , choose an arbitrary positive number ε in the interval $(0, r)$. And then it follows from the continuity of $V(\omega)$ and $V(0) = 0$ that there exists a positive number $\delta < \varepsilon$ such that for any ω_a in $S(\delta)$,

$$V(\omega_a) < \kappa(\varepsilon). \quad (6)$$

Using condition (A2), we infer by Lemma 4.1 that f has a Lipschitz extension $f_E: \mathbb{R}^d \mapsto \mathbb{R}^d$ with $\text{Lip}(f_E) = \text{Lip}(f)$ and $f_E = f$ on $S(r)$, and using the function f_E just obtained and the initial data ω_a previously chosen, construct an IVP as follows:

$$\begin{cases} \mathcal{D}_t^{(\alpha)} \omega(t) = p(t)f_E(\omega(t)), & a \leq t < \infty, \\ \omega(a) = \omega_a, \end{cases} \quad (7)$$

and then it follows from Theorem 3.1 that there is a unique solution to the IVP (7), denoted by $\bar{\omega}(t, \omega_a)$ and defined on \mathbb{R}^+ , and we further claim that it is also the unique solution to the IVP (2) defined on \mathbb{R}^+ . To this end, we first verify that the inequality

$$|\bar{\omega}(t, \omega_a)| < \varepsilon \quad (8)$$

holds on \mathbb{R}^+ .

Now suppose that, contrary to our assertion, the inequality is false, and then, from the condition $|\bar{\omega}(a, \omega_a)| = |\omega_a| < \delta < \varepsilon$ and the continuity of the solution $\bar{\omega}(t, \omega_a)$, it follows that there exists a positive number $t_1 > 0$ such that for each t in the interval $[a, t_1)$,

$$|\bar{\omega}(t, \omega_a)| < \varepsilon \quad (9)$$

with $|\bar{\omega}(t_1, \omega_a)| = \varepsilon$, and observing $\varepsilon < r$, it follows from the inequality in (9) and the condition $\kappa(|\omega|) \leq V(\omega)$ in $S(r)$ that

$$\kappa(|\bar{\omega}(t, \omega_a)|) \leq V(\bar{\omega}(t, \omega_a)). \quad (10)$$

Also, by the inequality in (9) and the second inequality in the conditions, we deduce that for each t in the interval $[a, t_1)$,

$$\nabla V(\bar{\omega}(t, \omega_a)) \cdot f_E(\bar{\omega}(t, \omega_a)) = \nabla V(\bar{\omega}(t, \omega_a)) \cdot f(\bar{\omega}(t, \omega_a)) \leq 0.$$

Due to the differentiability and convexity of the function V , Lemma 4.2 and the preceding inequality imply that for each t in the interval $[a, t_1)$,

$$\mathcal{D}_a^{(\alpha)} V(\bar{\omega}(t, \omega_a)) \leq \nabla V(\bar{\omega}(t, \omega_a)) \cdot \mathcal{D}_a^{(\alpha)} \bar{\omega}(t, \omega_a) = p(t) \nabla V(\bar{\omega}(t, \omega_a)) \cdot f_E(\bar{\omega}(t, \omega_a)) \leq 0,$$

which by Corollary 3.1 yields

$$V(\bar{\omega}(t, \omega_a)) \leq V(\omega_a). \quad (11)$$

Combined with the inequalities in (8)–(11), the chain of inequalities are derived as follows:

$$\kappa(|\bar{\omega}(t, \omega_a)|) \leq V(\bar{\omega}(t, \omega_a)) \leq V(\omega_a) < \kappa(\varepsilon),$$

and for $t = t_1$, in particular,

$$\kappa(\varepsilon) \leq V(\varepsilon) \leq V(\omega_a) < \kappa(\varepsilon),$$

and this obvious contradiction verifies the previous assertion that the inequality in (8) holds on \mathbb{R}^+ .

The inequality in (8) obviously implies that for each t in \mathbb{R}^+ , $\bar{\omega}(t, \omega_a)$ lies in $\mathcal{S}(r)$ from which we get

$$f_E(\bar{\omega}(t, \omega_a)) = f(\bar{\omega}(t, \omega_a)),$$

and thus

$$\mathcal{D}_a^{(\alpha)} \bar{\omega}(t, \omega_a) = p(t) f_E(\bar{\omega}(t, \omega_a)) = p(t) f(\bar{\omega}(t, \omega_a)).$$

Therefore, the function $\bar{\omega}(t, \omega_a)$ is also a unique solution to the IVP (1) defined on \mathbb{R}^+ .

Again, using the inequality in (8), the stability is consequently verified. \square

Remark 4.2 ([67]) The condition that $\kappa(|\omega|) \leq V(\omega)$ on $\mathcal{S}(r)$ for some κ in \mathcal{K} with $V(0) = 0$ is equivalent to $V(\omega) > 0$ for ω in $\bar{\mathcal{S}}(r)$ with $\omega \neq 0$ and $V(0) = 0$. And $V(\omega)$ is said to be positive definite on $\mathcal{S}(r)$.

5 Illustrative examples

(I) Consider the following system of equations:

$$\begin{cases} \mathcal{D}_0^{(\alpha)} \omega_1(t) = (1 - \exp(-t))\omega_2, \\ \mathcal{D}_0^{(\alpha)} \omega_2(t) = -(1 - \exp(-t))\frac{\omega_1}{1+\omega_1}. \end{cases}$$

If $\alpha > \frac{7}{8}$, then the zero solution is stable.

Indeed, let $\omega = (\omega_1, \omega_2)^\top$, $\mathcal{S}(\frac{1}{4}) = \{(\omega_1, \omega_2) | \omega_1^2 + \omega_2^2 < \frac{1}{4}\}$, $f(\omega) = (\omega_2, -\frac{\omega_1}{1+\omega_1})^\top$, and $p(t) = 1 - \exp(-t)$. Then $f(\omega)$ is Lipschitz in $\mathcal{S}(\frac{1}{4})$ with the Lipschitz constant $L = 4$, and it is obvious that $2L(1 - \alpha) \sup p(t) = 8(1 - \alpha) < 1$. Let $V(\omega) = \frac{1}{2}\omega_2^2 + \omega_1 - \ln(1 + \omega_1)$. Then, the

Lyapunov function V is clearly positive definite on $\mathcal{S}(\frac{1}{4})$, and, moreover, it is also convex on $\mathcal{S}(\frac{1}{4})$ since its Hessian matrix

$$\begin{bmatrix} \frac{1}{(1+\omega_1)^2} & 0 \\ 0 & 1 \end{bmatrix}$$

is positive definite. Furthermore, a routine computation implies that

$$\nabla V(\omega) \cdot f(\omega) = 0.$$

Thus all conditions in Theorem 4.1 are satisfied and therefore the desired assertion follows.

(II) Consider the scalar equation as follows:

$$\mathcal{D}_0^{(\alpha)} \omega(t) = -(1 - \cos^2 t) \omega^3(t).$$

The zero solution of the above equation is stable for each $\alpha \in (0, 1)$.

To see this, set $p(t) = 1 - \cos^2 t$, $f(\omega) = -\omega^3$. Obviously, function f does not satisfy the global Lipschitz condition on \mathbb{R} . It satisfies the Lipschitz condition in $\mathcal{S}(\frac{1}{4}) = (-\frac{1}{4}, \frac{1}{4})$ with $L = \frac{3}{16}$, and thus it is easy to check $2L(1 - \alpha) \sup p(t) = \frac{3}{8}(1 - \alpha) < 1$ for each $\alpha \in (0, 1)$. Now, let $V(\omega) = \omega^2$. The function V is obviously positive definite on $\mathcal{S}(\frac{1}{4})$ and satisfies the inequality $\nabla V(\omega) \cdot f(\omega) = -2\omega^4 \leq 0$. The corresponding conditions in Theorem 4.1 are thus verified and consequently, the zero solution is stable for each $\alpha \in (0, 1)$.

6 Conclusions

The *CFFD* was introduced in 2015. Up to now, many results for the theory of differential equations with the *CFFD* have been obtained, and these equations have also found a large spectrum of applications. But very little has been published on the subject of the separation and Lyapunov stability of solutions to nonlinear systems involving *CFFD*.

In this work, using the techniques of inequalities we derived a sufficient condition for the separation of solutions to the specific nonlinear equation, and by establishing the inequality for the convex functions, we successfully generalized the method of convex Lyapunov functions to the context of the *CFFD* and set up the criteria for the stability in the Lyapunov sense for the nonlinear system with the *CFFD*.

We believe that the inequalities established in this work are also significant on their own. It is expected that the inequality in Lemma 3.3 might be used to further explore the comparison principles, and that in Lemma 4.2 might be utilized to investigate maximum principles for some partial differential equations involving the *CFFD*.

It is easy to check the validity of the results obtained in this work, and, moreover, it remains to further explore whether or not the results and approaches here can be extended to the systems involving the Mittag-Leffler kernel.

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Availability of data and materials

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Authors' contributions

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