

RESEARCH

Open Access



# Fractional dynamical analysis of measles spread model under vaccination corresponding to nonsingular fractional order derivative

Ghazala Nazir<sup>1</sup>, Kamal Shah<sup>1</sup>, Hussam Alrabaiah<sup>2,3\*</sup>, Hammad Khalil<sup>4</sup> and Rahmat Ali Khan<sup>1</sup>

\*Correspondence:

[hussam89@yahoo.com](mailto:hussam89@yahoo.com)

<sup>2</sup>College of Engineering, Al Ain University, Al Ain, UAE

<sup>3</sup>Mathematics Department, Tafila Technical University, Tafila, Jordan

Full list of author information is available at the end of the article

## Abstract

In this paper, our main purpose is to present an analytical solution for measles spread model with three doses of vaccination using Caputo–Fabrizio fractional derivative (CFFD). The presented solution is based on Laplace transform with Adomian decomposition method (LADM), which is an effective technique to obtain a solution for such type of problems. Our solution involves nonlinear differential equations of fractional order (FODEs) with non-singular kernel. Also, we provide analysis to verify the existence of a solution to the considered model using fixed point theory. Numerical results are presented to verify the model building analysis, which proved to be efficient in solving such problems.

**Keywords:** Fractional differential equation; Caputo–Fabrizio fractional derivative; Laplace Adomian decomposition method (LADM); Measles spread model

## 1 Introduction

Measles virus is a paramyxovirus, genus morbillivirus. It transmits through person to person with 90% secondary attack rates among susceptible persons. The virus initially infects immune cells in lungs and spreads in the whole body. As it travels in blood, it infects the capillaries in the skin, which results in red rashes on the skin. It is one of the primary causes of serious and fatal complications including pneumonia, diarrhoea and encephalitis, blindness, deafness, and impaired vision. To control this disease, health institutions recommend children to get measles vaccine. There are three types of vaccines that are commonly used: MMR (measles, mumps, and rubella), MR (measles and rubella), and MMRV (measles, mumps, rubella, and varicella). All those vaccines consist of two doses. According to Center Disease Control and Prevention (CDC), the vaccines are 95% effective against the disease [1–6]. Some mathematical models of the measles spread have been introduced by several authors [5, 6]. The measles spread model with three vaccines is given

© The Author(s) 2020. This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

by

$$\begin{cases} \frac{du}{dt} = \Lambda(1 - q) - \gamma v - \mu u - \mu_1 u - \beta u(w + x) - \phi_1 u, \\ \frac{dv}{dt} = \Lambda(q) + \mu_1 u + \phi_1 u - \gamma v - \mu v - \omega v, \\ \frac{dw}{dt} = \beta u(w + x) - \mu w - \alpha w - \mu_2 w - \phi_2 w, \\ \frac{dx}{dt} = \alpha w - \delta x - \mu x - \mu_3 x - \phi_2 x, \\ \frac{dy}{dt} = \mu_2 w + \omega v + \mu_3 x - \mu y + \phi_3 x + \phi_2 w, \end{cases} \tag{1}$$

where  $u$  represents susceptible,  $v$  represents vaccinated,  $w$  represents exposed,  $x$  represents infected, and  $y$  represents recovered individuals.  $\phi_1$  is the first vaccine,  $\phi_2$  is the second, and  $\phi_3$  is the third vaccine.  $\Lambda$  is the recruitment rate,  $\beta$  is the transmission rate,  $\mu$  is the natural death rate,  $\gamma$  is the vaccine wane rate,  $q$  is the proportion of new born,  $\alpha$  is the progression rate from exposed to infected individuals,  $\omega$  is the rate at which vaccine works well,  $\delta$  is the death rate induced by disease,  $\mu_1$  is the vaccination rate,  $\mu_2$  is the measles therapy rate of exposed individuals, and  $\mu_3$  is the recovery rate.

It has been evident that fractional calculus is increasingly used by for mathematical modeling of different real-life problems. Normally, integer order derivative does not accurately explore the dynamics as compared with fractional derivative. On the other hand, a variety of concepts have been introduced by different researchers to describe the aforesaid derivatives. The most famous definitions have been given by Riemann and Liouville, Caputo, Hadamard, etc. (see [7–10]).

Caputo and Fabrizio in 2015 introduced a new concept about fractional order derivatives based on non-singular kernel [11, 12]. Some remarkable observations have been recorded, Caputo–Fabrizio fractional integral is the fractional average of the function itself and its fractional integral in Riemann–Liouville sense. Recently, existence results on the human liver model have been investigated by using the definition of the Caputo–Fabrizio derivative. Also, mathematical models of tumor immune-surveillance using non-singular derivative have been discussed. Many different mathematical models for medical diseases (such as models of dengue fever, diabetes, and tuberculosis) have been studied with the help of Caputo–Fabrizio derivative [13–20]. In recent time, some significant work has been produced. The respective work has been carried out to investigate mainly the existence of a solution, or series-type solutions, by using various tools of applied analysis, for details, see [13, 21–23]. In an attempt to contribute to the existing literature on mathematical tools to handle similar problems, we propose the use of LADM, which is used to find analytical approximate solution to many nonlinear problems. This technique has been very well explored for ordinary as well as for fractional differential equations [15, 24–28]. Recently, a biological model of dengue fever has been investigated for qualitative theory by using fixed point theory and for analytical study via LADM, see [29]. Therefore, we utilize LADM for the series type solution of measles spread model (1) under CFFD to obtain a satisfactory results. We consider model (1) and take the CFFD of the model of order  $\sigma$

such that  $\sigma \in (0, 1]$  as given below:

$$\begin{cases} {}^{\text{CF}}D_t^\sigma u = \Lambda(1 - q) - \gamma v - \mu u - \mu_1 u - \beta u(w + x) - \phi_1 u, \\ {}^{\text{CF}}D_t^\sigma v = \Lambda(q) + \mu_1 u + \phi_1 u - \gamma v - \mu v - \omega v, \\ {}^{\text{CF}}D_t^\sigma w = \beta u(w + x) - \mu w - \alpha w - \mu_2 w - \phi_2 w, \\ {}^{\text{CF}}D_t^\sigma x = \alpha w - \delta x - \mu x - \mu_3 x - \phi_3 x, \\ {}^{\text{CF}}D_t^\sigma y = \mu_2 w + \omega v + \mu_3 x - \mu y + \phi_3 x + \phi_2 w \end{cases} \tag{2}$$

subject to the conditions

$$u(0) \geq 0, \quad v(0) \geq 0, \quad w(0) \geq 0, \quad x(0) \geq 0, \quad y(0) \geq 0.$$

We obtain the solution in the form of series for the considered problem. Also, we display the results against different values of fractional order  $\sigma \in (0, 1]$  that are numerically solved. Also, we provide results about the existence and uniqueness of solution for the concerned model. Here, we remark that some qualitative work, which addresses existence of solution, computation of series solutions, etc., has been framed in the last few years (see [30–33]).

The structure of the rest of the paper is as follows. In Sect. 2, we recall some definitions and basic ideas of the Caputo–Fabrizio derivative. In Sect. 3, we present the results of existence and uniqueness of the concerned model. In Sect. 4, we find the analytical solution of the model by using the Laplace Adomian decomposition method for the fractional order. In Sect. 5, we present and discuss the numerical results. In Sect. 6, we discuss the conclusions and some future directions.

## 2 Preliminaries

**Definition 2.1** ([14]) Let  $\varphi \in \mathcal{H}^1(0, a)$  be a space such that  $a > 0, \sigma \in (0, 1)$ , then the CFFD is recalled as follows:

$${}^{\text{CF}}D_t^\sigma (\varphi(t)) = \frac{\mathcal{M}(\sigma)}{1 - \sigma} \int_0^t \varphi'(t) \exp\left[-\sigma \frac{t - \zeta}{1 - \sigma}\right] d\zeta,$$

$\mathcal{M}(\sigma)$  is the normalization function with  $\mathcal{M}(1) = \mathcal{M}(0) = 1$ . If the function fails to exist in  $\mathcal{H}^1(0, a)$ , then the derivative can be redefined as

$${}^{\text{CF}}D_t^\sigma (\varphi(t)) = \frac{\sigma \mathcal{M}(\sigma)}{1 - \sigma} \int_0^t (\varphi(t) - \varphi(\zeta)) \exp\left[-\sigma \frac{t - \zeta}{1 - \sigma}\right] d\zeta.$$

**Definition 2.2** ([14]) Let  $\sigma \in ]0, 1[$ , then the integral of fractional order  $\sigma$  of function  $\varphi$  is defined by

$${}^{\text{CF}}I_t^\sigma [\varphi(t)] = \frac{(1 - \sigma)}{\mathcal{M}(\sigma)} \varphi(t) + \frac{\sigma}{\mathcal{M}(\sigma)} \int_0^t \varphi(\zeta) d\zeta, \quad t \geq 0.$$

**Definition 2.3** The Laplace transform of CFFD  ${}^{\text{CF}}D_t^\sigma x(t)$  is given as

$$\mathcal{L} [{}^{\text{CF}}D_t^\sigma x(t)] = \frac{s \mathcal{L}[x(t)] - x(0)}{s + \sigma(1 - s)}, \quad s \geq 0, \sigma \in (0, 1].$$

### 3 Existence and uniqueness results for measles disease model of fractional order

In this section, we investigate existence results for model (2) using the fixed point theorem due to Banach. In this regard, applying the Caputo–Fabrizio fractional integral operator on both sides of (2), we obtain the following:

$$\begin{aligned}
 u(t) - u(0) &= {}^{CF}I_t^\sigma [\Lambda(1 - q) - \gamma v - \mu u - \mu_1 u - \beta u(w + x) - \phi_1 u], \\
 v(t) - v(0) &= {}^{CF}I_t^\sigma [\Lambda(q) + \mu_1 u + \phi_1 u - \gamma v - \mu v - \omega v], \\
 w(t) - w(0) &= {}^{CF}I_t^\sigma [\beta u(w + x) - \mu w - \alpha w - \mu_2 w - \phi_2 w], \\
 x(t) - x(0) &= {}^{CF}I_t^\sigma [\alpha w - \delta x - \mu x - \mu_3 x - \phi_3 x], \\
 y(t) - y(0) &= {}^{CF}I_t^\sigma [\mu_2 w + \omega v + \mu_3 x - \mu y + \phi_3 x + \phi_2 w].
 \end{aligned}
 \tag{3}$$

Further, we define the functions as follows:

$$\begin{aligned}
 k_1(t, u) &= \Lambda(1 - q) - \gamma v - \mu u - \mu_1 u - \beta u(w + x) - \phi_1 u, \\
 k_2(t, v) &= \Lambda(q) + \mu_1 u + \phi_1 u - \gamma v - \mu v - \omega v, \\
 k_3(t, w) &= \beta u(w + x) - \mu w - \alpha w - \mu_2 w - \phi_2 w, \\
 k_4(t, x) &= \alpha w - \delta x - \mu x - \mu_3 x - \phi_3 x, \\
 k_5(t, y) &= \mu_2 w + \omega v + \mu_3 x - \mu y + \phi_3 x + \phi_2 w, \\
 \Phi(\sigma) &= \frac{(1 - \sigma)}{M(\sigma)}, \quad \Theta(\sigma) = \frac{(\sigma)}{M(\sigma)}.
 \end{aligned}
 \tag{4}$$

$$\tag{5}$$

We note here that the  $u, v, w, x, y$  are nonnegative bounded functions such that  $\|u(t)\| \leq \rho_1, \|v(t)\| \leq \rho_2, \|w(t)\| \leq \rho_3, \|x(t)\| \leq \rho_4, \|y(t)\| \leq \rho_5$ , where  $\rho_1, \rho_2, \rho_3, \rho_4, \rho_5$  are some positive constants. Now, we define the following variables:

$$\begin{aligned}
 \eta_1 &= \mu + \mu_1 + \beta\rho_3 + \beta\rho_4 + \phi_1, & \eta_2 &= \gamma + \mu + \omega, \\
 \eta_3 &= \beta\rho_1 + \mu + \alpha + \mu_2 + \phi_2, & \eta_4 &= \mu + \phi_3 + \mu + \delta, & \eta_5 &= \mu.
 \end{aligned}
 \tag{6}$$

Applying the Caputo–Fabrizio fractional integral on (3), we get the following:

$$\begin{aligned}
 u(t) &= u(0) + \Phi(\sigma)k_1(t, u) + \Theta(\sigma) \int_0^t k_1(\xi, u) d\xi, \\
 v(t) &= v(0) + \Phi(\sigma)k_2(t, v) + \Theta(\sigma) \int_0^t k_2(\xi, v) d\xi, \\
 w(t) &= w(0) + \Phi(\sigma)k_3(t, w) + \Theta(\sigma) \int_0^t k_3(\xi, w) d\xi, \\
 x(t) &= x(0) + \Phi(\sigma)k_4(t, x) + \Theta(\sigma) \int_0^t k_4(\xi, x) d\xi, \\
 y(t) &= y(0) + \Phi(\sigma)k_5(t, y) + \Theta(\sigma) \int_0^t k_5(\xi, y) d\xi.
 \end{aligned}
 \tag{7}$$

**Theorem 3.1** *Under the condition  $M = \max\{\eta_1, \eta_2, \eta_3, \eta_4, \eta_5\} < 1$ , the functions  $k_1, k_2, k_3, k_4, k_5$  satisfy Lipschitz conditions.*

*Proof* We consider the function  $k_1$  and let  $u$  and  $u_1$  be the two functions

$$\begin{aligned} \|k_1(t, u) - k_1(t, u_1)\| &= \|-\mu(u - u_1) - \mu_1(u - u_1) \\ &\quad - \beta w(u - u_1) - \beta x(u - u_1) - \phi_1(u - u_1)\| \\ &\leq \|\mu(u - u_1)\| + \|\mu_1(u - u_1)\| \\ &\quad + \|\beta w(u - u_1)\| + \|\beta x(u - u_1)\| + \|\phi_1(u - u_1)\| \\ &\leq (\mu + \mu_1 + \beta \|w\| + \beta \|x\| + \phi_1) \|u - u_1\| \\ &\leq (\mu + \mu_1 + \beta \|\rho_3\| + \beta \|\rho_4\| + \phi_1) \|u - u_1\| \\ &\leq \eta_1 \|u - u_1\|. \end{aligned} \tag{8}$$

Similar results for  $k_2, k_3, k_4, k_5$  can be obtained using  $\{v, v_1\}, \{w, w_1\}, \{x, x_1\}, \{y, y_1\}$ :

$$\begin{aligned} \|k_2(t, v) - k_2(t, v_1)\| &\leq \eta_2 \|v - v_1\|, \\ \|k_3(t, w) - k_3(t, w_1)\| &\leq \eta_3 \|w - w_1\|, \\ \|k_4(t, x) - k_4(t, x_1)\| &\leq \eta_4 \|x - x_1\|, \\ \|k_5(t, y) - k_5(t, y_1)\| &\leq \eta_5 \|y - y_1\|. \end{aligned} \quad \square$$

Using (3), we introduce the following recursive formulas:

$$\begin{aligned} u_n(t) &= \Phi(\sigma)k_1(t, u_{n-1}) + \Theta(\sigma) \int_0^t k_1(\xi, u_{n-1}) d\xi, \\ v_n(t) &= \Phi(\sigma)k_2(t, v_{n-1}) + \Theta(\sigma) \int_0^t k_2(\xi, v_{n-1}) d\xi, \\ w_n(t) &= \Phi(\sigma)k_3(t, w_{n-1}) + \Theta(\sigma) \int_0^t k_3(\xi, w_{n-1}) d\xi, \\ x_n(t) &= \Phi(\sigma)k_4(t, x_{n-1}) + \Theta(\sigma) \int_0^t k_4(\xi, x_{n-1}) d\xi, \\ y_n(t) &= \Phi(\sigma)k_5(t, y_{n-1}) + \Theta(\sigma) \int_0^t k_5(\xi, y_{n-1}) d\xi. \end{aligned} \tag{9}$$

The initial components of the above recursive formulas are determined by the given initial conditions as follows:

$$\begin{aligned} u_0(t) &= u(0), & v_0(t) &= v(0), & w_0(t) &= w(0), \\ x_0(t) &= x(0), & y_0(t) &= y(0), \end{aligned} \tag{10}$$

$$\begin{aligned} \theta_n(t) &= u_n(t) - u_{n-1}(t) = \Phi(\sigma)k_1(t, u_{n-1}) + \Theta(\sigma) \int_0^t k_1(\xi, u_{n-1}) d\xi, \\ \psi_n(t) &= v_n(t) - v_{n-1}(t) = \Phi(\sigma)k_2(t, v_{n-1}) + \Theta(\sigma) \int_0^t k_2(\xi, v_{n-1}) d\xi, \end{aligned}$$

$$\vartheta_n(t) = w_n(t) - w_{n-1}(t) = \Phi(\sigma)k_3(t, w_{n-1}) + \Theta(\sigma) \int_0^t k_3(\xi, w_{n-1}) d\xi, \tag{11}$$

$$\chi_n(t) = x_n(t) - x_{n-1}(t) = \Phi(\sigma)k_4(t, x_{n-1}) + \Theta(\sigma) \int_0^t k_4(\xi, x_{n-1}) d\xi,$$

$$\zeta_n(t) = y_n(t) - y_{n-1}(z) = \Phi(\sigma)k_5(t, y_{n-1}) + \Theta(\sigma) \int_0^t k_5(\xi, y_{n-1}) d\xi,$$

$$u_n(t) = \sum_{i=1}^n \theta_n(t), \quad v_n(t) = \sum_{i=1}^n \psi_n(t), \quad w_n(t) = \sum_{i=1}^n \vartheta_n(t), \tag{12}$$

$$x_n(t) = \sum_{i=1}^n \chi_n(t), \quad y_n(t) = \sum_{i=1}^n \zeta_n(t).$$

Next, we formulate the recursive inequalities for the differences

$$\begin{aligned} \|\theta_n(t)\| &= \|u_n(t) - u_{n-1}(t)\| \\ &= \left\| \Phi(\sigma)(k_1(t, u_{n-1}) - k_1(t, u_{n-2})) \right. \\ &\quad \left. + \Theta(\sigma) \int_0^t (k_1(\xi, u_{n-1}) - k_1(\xi, u_{n-2})) d\xi \right\| \\ &\leq \Phi(\sigma) \|k_1(t, u_{n-1}) - k_1(t, u_{n-2})\| \\ &\quad + \Theta(\sigma) \int_0^t \|k_1(\xi, u_{n-1}) - k_1(\xi, u_{n-2})\| d\xi, \end{aligned} \tag{13}$$

$$\|\theta_n(t)\| \leq \Phi(\sigma)\eta_1 \|\theta_{n-1}(t)\| + \eta_1 \Theta(\sigma) \int_0^t \|\theta_{n-1}(\xi)\| d\xi.$$

In a similar manner, we get

$$\begin{aligned} \|\psi_n(t)\| &\leq \Phi(\sigma)\eta_2 \|\psi_{n-1}(t)\| + \eta_2 \Theta(\sigma) \int_0^t \|\psi_{n-1}(\xi)\| d\xi, \\ \|\vartheta_n(t)\| &\leq \Phi(\sigma)\eta_3 \|\vartheta_{n-1}(t)\| + \eta_3 \Theta(\sigma) \int_0^t \|\vartheta_{n-1}(\xi)\| d\xi, \end{aligned} \tag{14}$$

$$\begin{aligned} \|\chi_n(t)\| &\leq \Phi(\sigma)\eta_4 \|\chi_{n-1}(t)\| + \eta_4 \Theta(\sigma) \int_0^t \|\chi_{n-1}(\xi)\| d\xi, \\ \|\zeta_n(t)\| &\leq \Phi(\sigma)\eta_5 \|\zeta_{n-1}(t)\| + \eta_5 \Theta(\sigma) \int_0^t \|\zeta_{n-1}(\xi)\| d\xi. \end{aligned} \tag{15}$$

For existence of a solution, we give the following result.

**Theorem 3.2** *If there exists a time  $t_0 > 0$  such that the following inequality holds:*

$$\Phi(\sigma)\eta_i + \Theta(\sigma)\eta_i t_0 < 1 \quad \text{for } i = 1, 2, \dots, 5, \tag{16}$$

*then the considered fractional order measles spread model with vaccination has a solution.*

*Proof* Since functions  $u(t), v(t), w(t), x(t), y(t)$  are assumed to be bounded and each function satisfies a Lipschitz condition, the following relation can be obtained using equations

(13), (14) recursively:

$$\begin{aligned}
 \|\theta_n(t)\| &\leq \|u(0)\| [\Phi(\sigma)\eta_1 + \eta_1\Theta(\sigma)t]^n, \\
 \|\psi_n(t)\| &\leq \|v(0)\| [\Phi(\sigma)\eta_2 + \eta_2\Theta(\sigma)t]^n, \\
 \|\vartheta_n(t)\| &\leq \|w(0)\| [\Phi(\sigma)\eta_3 + \eta_3\Theta(\sigma)t]^n, \\
 \|\chi_n(t)\| &\leq \|x(0)\| [\Phi(\sigma)\eta_4 + \eta_4\Theta(\sigma)t]^n, \\
 \|\zeta_n(t)\| &\leq \|y(0)\| [\Phi(\sigma)\eta_5 + \eta_5\Theta(\sigma)t]^n.
 \end{aligned}
 \tag{17}$$

Equation (16) shows the existence and smoothness of the functions defined in (12). To complete the proof, we prove that the function  $u_n(t), v_n(t), w_n(t), x_n(t), y_n(t)$  converges to a system of solutions of (3). We define  $B_n(t), C_n(t), D_n(t), E_n(t)$ , and  $F_n(t)$  as the remainder terms after  $n$  iterations, i.e.,

$$\begin{aligned}
 u(t) - u(0) &= u_n(t) - B_n(t), \\
 v(t) - v(0) &= v_n(t) - C_n(t), \\
 w(t) - w(0) &= w_n(t) - D_n(t), \\
 x(t) - x(0) &= x_n(t) - E_n(t), \\
 y(t) - y(0) &= y_n(t) - F_n(t).
 \end{aligned}
 \tag{18}$$

Then, using the Lipschitz condition for the  $k_1$ , we have

$$\begin{aligned}
 \|B_n(t)\| &= \left\| \Phi(\sigma)(k_1(t, u) - k_1(t, u_{n-1})) + \Theta(\sigma) \int_0^t (k_1(\xi, u) - k_1(\xi, u_{n-1})) d\xi \right\| \\
 &\leq \Phi(\sigma) \|k_1(t, u) - k_1(t, u_{n-1})\| + \Theta(\sigma) \int_0^t \|k_1(\xi, u) - k_1(\xi, u_{n-1})\| d\xi \\
 &\leq \Phi(\sigma)\eta_1 \|u - u_{n-1}\| + \eta_1\Theta(\sigma) \|u - u_{n-1}\| t,
 \end{aligned}$$

applying the process recursively, we get

$$\|B_n(t)\| \leq [\Phi(\sigma)\eta_1 + \eta_1\Theta(\sigma)t]^{n+1} \rho_1.
 \tag{19}$$

Then at  $t_0$  we obtain

$$\|B_n(t)\| \leq [\Phi(\sigma)\eta_1 + \eta_1\Theta(\sigma)t_0]^{n+1} \rho_1.
 \tag{20}$$

Taking the limit on equation (20) as  $n \rightarrow \infty$  and then using condition (16), we obtain  $\|B_n(t)\| \rightarrow 0$ . Using the same process as described above, we have the following relations:

$$\begin{aligned}
 \|C_n(t)\| &\leq [\Phi(\sigma)\eta_2 + \eta_2\Theta(\sigma)t_0]^{n+1} \rho_2, \\
 \|D_n(t)\| &\leq [\Phi(\sigma)\eta_3 + \eta_3\Theta(\sigma)t_0]^{n+1} \rho_3, \\
 \|E_n(t)\| &\leq [\Phi(\sigma)\eta_4 + \eta_4\Theta(\sigma)t_0]^{n+1} \rho_4, \\
 \|F_n(t)\| &\leq [\Phi(\sigma)\eta_5 + \eta_5\Theta(\sigma)t_0]^{n+1} \rho_5.
 \end{aligned}
 \tag{21}$$

Similarly, taking the limit on (21) as  $n \rightarrow \infty$  and then using condition (16), we have  $\|C_n(t)\| \rightarrow 0$ ,  $\|B_n(t)\| \rightarrow 0$ ,  $\|E_n(t)\| \rightarrow 0$ ,  $\|F_n(t)\| \rightarrow 0$ . Therefore, the existence of the system of solutions is proved. We now give conditions for the system of solutions to be unique.  $\square$

**Theorem 3.3** *System along with initial conditions has a unique system of solutions if the following conditions hold:*

$$(1 - \Phi(\sigma)\eta_i - \eta_i\Theta(\sigma)t > 0 \quad \text{for } i = 1, 2, \dots, 5. \tag{22}$$

*Proof* Assume that  $\{u_1(t), v_1(t), w_1(t), x_1(t), y_1(t)\}$  is another set of solutions of model (3) in addition to the solution set  $\{u(t), v(t), w(t), x(t), y(t)\}$  proved to exist in the above theorems.

$$\begin{aligned} u(t) - u_1(t) &= \Phi(\sigma)(k_1(t, u) - k_1(t, u_{n-1})) + \Theta(\sigma) \int_0^t (k_1(\xi, u) - k_1(\xi, u_{n-1})) d\xi, \\ \|u(t) - u_1(t)\| &= \left\| \Phi(\sigma)(k_1(t, u) - k_1(t, u_{n-1})) \right. \\ &\quad \left. + \Theta(\sigma) \int_0^t (k_1(\xi, u) - k_1(\xi, u_{n-1})) d\xi \right\| \\ &\leq \Phi(\sigma) \|k_1(t, u) - k_1(t, u_{n-1})\| \\ &\quad + \Theta(\sigma) \int_0^t \|k_1(\xi, u) - k_1(\xi, u_{n-1})\| d\xi \\ &\leq \Phi(\sigma)\eta_1 \|u - u_{n-1}\| + \eta_1\Theta(\sigma) \|u - u_{n-1}\|t. \end{aligned} \tag{23}$$

Rearranging equations (23), we get

$$\|u(t) - u_1(t)\| (1 - \Phi(\sigma)\eta_1 - \eta_1\Theta(\sigma)t) \leq 0, \tag{24}$$

$$\|u(t) - u_1(t)\| = 0, \tag{25}$$

and therefore  $u(t) = u_1(t)$ . Repeating the similar procedure to each of the following pairs  $(v(t), v_1(t))$ ,  $(w(t), w_1(t))$ ,  $(x(t), x_1(t))$ , and  $(y(t), y_1(t))$  with inequality (22) for  $i = 2, 3, 4, 5$ , respectively, we obtain

$$v(t) = v_1(t), \quad w(t) = w_1(t), \quad x(t) = x_1(t), \quad y(t) = y_1(t). \tag{26}$$

Thus, the uniqueness of the system of solutions of the fractional order system is proved. In summary, the existence of the solutions of the model described in system (3) can be obtained by requiring that  $M = \max\{\eta_1, \eta_2, \eta_3, \eta_4, \eta_5\} < 1$ , where  $\eta_1, \eta_2, \eta_3, \eta_4, \eta_5$  are the Lipschitz constants of the functions  $k_1, k_2, k_3, k_4, k_5$ , respectively. Moreover, the uniqueness of the solutions of the considered system can be established using the inequalities in equations (22).  $\square$

#### 4 Analytical solution of measles spread model (2) by Laplace Adomian decomposition method

In this section, we compute a series solution for the suggested problem. To achieve this goal, taking Laplace transform of (11), we have

$$\begin{cases} \mathcal{L}[u(t)] = \frac{u(0)}{s} + \frac{s+\sigma(1-s)}{s} \mathcal{L}[\Lambda(1-q) - \gamma v - \mu u - \mu_1 u - \beta u(w+x) - \phi_1 u], \\ \mathcal{L}[v(t)] = \frac{v(0)}{s} + \frac{s+\sigma(1-s)}{s} \mathcal{L}[\Lambda(q) + \mu_1 u + \phi_1 u - \gamma v - \mu v - \omega v], \\ \mathcal{L}[w(t)] = \frac{w(0)}{s} + \frac{s+\sigma(1-s)}{s} \mathcal{L}[\beta u(w+x) - \mu w - \alpha w - \mu_2 w - \phi_2 w], \\ \mathcal{L}[x(t)] = \frac{x(0)}{s} + \frac{s+\sigma(1-s)}{s} \mathcal{L}[\alpha w - \delta x - \mu x - \mu_3 x - \phi_3 x], \\ \mathcal{L}[y(t)] = \frac{y(0)}{s} + \frac{s+\sigma(1-s)}{s} \mathcal{L}[\mu_2 w + \omega v + \mu_3 x - \mu y + \phi_3 x + \phi_2 w]. \end{cases} \tag{27}$$

Now, assume the solution in the series form as follows:

$$\begin{aligned} u(t) &= \sum_{p=0}^{\infty} u_p(t), & v(t) &= \sum_{p=0}^{\infty} v_p(t), & w(t) &= \sum_{p=0}^{\infty} w_p(t), \\ x(t) &= \sum_{p=0}^{\infty} x_p(t), & y(t) &= \sum_{p=0}^{\infty} y_p(t). \end{aligned} \tag{28}$$

Furthermore, decompose the nonlinear terms  $u(t)w(t)$ ,  $u(t)x(t)$ , etc. in terms of Adomian polynomials as follows:

$$u(t)w(t) = \sum_{p=0}^{\infty} A_p(u, w), \quad u(t)x(t) = \sum_{p=0}^{\infty} B_p(u, x), \tag{29}$$

where the Adomian polynomial  $A_p(u, w)$  can be defined as

$$A_p(u, w) = \frac{1}{p!} \frac{d^p}{d\lambda^p} \left[ \sum_{i=0}^p \lambda^i u_i(t) \sum_{i=0}^p \lambda^i w_i(t) \right] \Bigg|_{\lambda=0}.$$

In a similar way, the other polynomials  $B_p$  can be defined.

Hence, in view of (28) and (29), system (27) becomes

$$\begin{cases} \mathcal{L}[\sum_{p=0}^{\infty} u_p(t)] = \frac{u(0)}{s} + \frac{s+\sigma(1-s)}{s} \mathcal{L}[\Lambda(1-q) - \gamma \sum_{p=0}^{\infty} v_p - (\mu + \mu_1 + \phi_1) \sum_{p=0}^{\infty} u_p \\ \quad - \beta \sum_{p=0}^{\infty} A_p(u, w) - \beta \sum_{p=0}^{\infty} B_p(u, x)], \\ \mathcal{L}[\sum_{p=0}^{\infty} v_p(t)] = \frac{v(0)}{s} + \frac{s+\sigma(1-s)}{s} \mathcal{L}[\Lambda(q) + (\mu_1 + \phi_1) \sum_{p=0}^{\infty} u_p - (\gamma + \mu + \omega) \sum_{p=0}^{\infty} v_p], \\ \mathcal{L}[\sum_{p=0}^{\infty} w_p(t)] = \frac{w(0)}{s} + \frac{s+\sigma(1-s)}{s} \mathcal{L}[\beta \sum_{p=0}^{\infty} A_p(u, w) + \beta \sum_{p=0}^{\infty} B_p(u, x) \\ \quad - (\mu + \alpha + \mu_2 + \phi_2) \sum_{p=0}^{\infty} w_p], \\ \mathcal{L}[\sum_{p=0}^{\infty} x_p(t)] = \frac{x(0)}{s} + \frac{s+\sigma(1-s)}{s} \mathcal{L}[\alpha \sum_{p=0}^{\infty} w_p - (\delta + \mu_3 + \mu + \phi_3) \sum_{p=0}^{\infty} x_p], \\ \mathcal{L}[\sum_{p=0}^{\infty} Y_p(t)] = \frac{y(0)}{s} + \frac{s+\sigma(1-s)}{s} \mathcal{L}[\omega \sum_{p=0}^{\infty} v_p \\ \quad + (\mu_2 + \phi_2) \sum_{p=0}^{\infty} w_p + (\phi_3 + \mu_3) \sum_{p=0}^{\infty} x_p - \mu \sum_{p=0}^{\infty} y_p]. \end{cases} \tag{30}$$

Now equating terms on both sides of (30), we have

$$\left\{ \begin{aligned}
 \mathcal{L}[u_0(t)] &= \frac{u_0}{s}, & \mathcal{L}[v_0(t)] &= \frac{v_0}{s}, & \mathcal{L}[w_0(t)] &= \frac{w_0}{s}, \\
 \mathcal{L}[x_0(t)] &= \frac{x_0}{s}, & \mathcal{L}[y_0(t)] &= \frac{y_0}{s}, \\
 \mathcal{L}[u_1(t)] &= \frac{s+\sigma(1-s)}{s} \mathcal{L}[\Lambda(1-q) - \gamma v_0 - (\mu + \mu_1 + \phi_1)u_0 - \beta A_0(u, w) - \beta B_0(u, x)], \\
 \mathcal{L}[v_1(t)] &= \frac{s+\sigma(1-s)}{s} \mathcal{L}[\Lambda(q) + (\mu_1 + \phi_1)u_0 - (\gamma + \mu + \omega)v_0], \\
 \mathcal{L}[w_1(t)] &= \frac{s+\sigma(1-s)}{s} \mathcal{L}[\beta A_0(u, w) + \beta B_0(u, x) - (\mu + \alpha + \mu_2 + \phi_2)w_0], \\
 \mathcal{L}[x_1(t)] &= \frac{s+\sigma(1-s)}{s} \mathcal{L}[\alpha w_0 - (\delta + \mu_3 + \mu + \phi_3)x_0], \\
 \mathcal{L}[y_1(t)] &= \frac{s+\sigma(1-s)}{s} \mathcal{L}[\omega v_0 + (\mu_2 + \phi_2)w_0 + (\mu_3 + \phi_3)x_0 - \mu y_0], \\
 \mathcal{L}[u_2(t)] &= \frac{s+\sigma(1-s)}{s} \mathcal{L}[\Lambda(1-q) - \gamma v_1 - (\mu + \mu_1 + \phi_1)u_1 - \beta A_1(u, w) - \beta B_1(u, x)], \\
 \mathcal{L}[v_2(t)] &= \frac{s+\sigma(1-s)}{s} \mathcal{L}[\Lambda(q) + (\mu_1 + \phi_1)u_1 - (\gamma + \mu + \omega)v_1], \\
 \mathcal{L}[w_2(t)] &= \frac{s+\sigma(1-s)}{s} \mathcal{L}[\beta A - 1(u, w) + \beta B_1(u, x) - (\mu + \alpha + \mu_2 + \phi_2)w_1], \\
 \mathcal{L}[x_2(t)] &= \frac{s+\sigma(1-s)}{s} \mathcal{L}[\alpha w_1 - (\delta + \mu_3 + \mu + \phi_3)x_1], \\
 \mathcal{L}[y_2(t)] &= \frac{s+\sigma(1-s)}{s} \mathcal{L}[\omega v_1 + (\mu_2 + \phi_2)w_1 + (\mu_3 + \phi_3)x_1 - \mu y_1], \\
 &\vdots \\
 \mathcal{L}[u_{p+1}(t)] &= \frac{s+\sigma(1-s)}{s} \mathcal{L}[\Lambda(1-q) - \gamma v_p - (\mu + \mu_1 + \phi_1)u_p - \beta A_p(u, w) - \beta B_p(u, x)], \\
 \mathcal{L}[v_{p+1}(t)] &= \frac{s+\sigma(1-s)}{s} \mathcal{L}[\Lambda(q) + (\mu_1 + \phi_1)u_p - (\gamma + \mu + \omega)v_p], \\
 \mathcal{L}[w_{p+1}(t)] &= \frac{s+\sigma(1-s)}{s} \mathcal{L}[\beta A_p(u, w) + \beta B_p(u, x) - (\mu + \alpha + \mu_2 + \phi_2)w_p], \\
 \mathcal{L}[x_{p+1}(t)] &= \frac{s+\sigma(1-s)}{s} \mathcal{L}[\alpha w_p - (\delta + \mu_3 + \mu + \phi_3)x_p], \\
 \mathcal{L}[y_{p+1}(t)] &= \frac{s+\sigma(1-s)}{s} \mathcal{L}[\omega v_p + (\mu_2 + \phi_2)w_p + \omega v_p + (\mu_3 \phi_3)x_p - \mu y_p], \quad p \geq 0.
 \end{aligned} \right. \tag{31}$$

Evaluating the Laplace transform in (31), we get

$$\left\{ \begin{aligned}
 u_0(t) &= u_0, & v_0(t) &= v_0, & w_0(t) &= w_0, & x_0(t) &= x_0, & y_0(t) &= z_0, \\
 u_1(t) &= [\Lambda(1-q) - \gamma v_0 - (\mu + \mu_1 + \phi_1 + \beta w_0 + \beta x_0)u_0](1 + \sigma(t-1)), \\
 v_1(t) &= 1 + \sigma(z-1)[\Lambda(q) + (\mu_1 + \phi_1)u_0 - (\gamma + \mu + \omega)], \\
 w_1(t) &= [(\beta u_0 - (\mu + \alpha + \mu_2 + \phi_2))w_0 + \beta u_0 x_0](1 + \sigma(t-1)), \\
 x_1(t) &= [\alpha w_0 - (\delta + \mu_3 + \mu + \phi_3)x_0](1 + \sigma(t-1)), \\
 y_1(t) &= [\omega v_0 + (\mu_2 + \phi_2)w_0 + (\mu_3 + \phi_3)x_0 - \mu y_0](1 + \sigma(t-1)), \\
 u_2(t) &= (1 + \sigma(t-1))\Lambda(1-q) - [\gamma(\Lambda(q) + (\mu_1 + \phi_2)u_0 - (\gamma + \mu + \omega)v_0) \\
 &\quad + (\mu + \mu_1 - \beta w_0 - \beta x_0 + \phi_1)(\Lambda(q) - \gamma v_0 - (\mu_1 - \beta w_0 - \beta x_0 + \phi))u_0 \\
 &\quad - \beta u_0(\beta u_0 - (\mu + \alpha + \mu_2 + \phi_2))w_0 + \beta u_0 x_0 + \alpha w_0 - (\delta + \mu_3 + \mu + \phi_3)x_0] \\
 &\quad (1 + \sigma^2(t-1)), \\
 v_2(t) &= (1 + \sigma(t-1))\Lambda(q) + [(\mu_1 + \phi_2)(\Lambda(1-q) - (\mu + \mu_1 + \beta w_0 + \beta x_0 + \phi_1)u_0 \\
 &\quad - (\gamma + \mu + \omega)(\Lambda(q) + (\mu_1 + \phi_2)u_0 - (\gamma + \mu + \omega)v_0)](1 + \sigma^2(t-1)), \\
 w_2(t) &= (1 + \sigma^2(t-1))[(\beta u_0 - (\mu + \alpha + \mu_2 + \phi_2))((\beta u_0 - (\mu + \alpha + \mu_2 + \phi_2))w_0 + \beta u_0 x_0) \\
 &\quad + \beta u_0(\beta u_0 - (\mu + \alpha + \mu_2 + \phi_2))w_0 + \beta u_0 x_0 + \alpha w_0 - (\delta + \mu_3 + \mu + \phi_3)x_0], \\
 x_2(t) &= (1 + \sigma^2(t-1))[\alpha((\beta u_0 - (\mu + \alpha + \mu_2 + \phi_3))w_0 + \beta u_0 x_0) \\
 &\quad + (\delta + \mu + \mu_3 + \phi_3)(\alpha w_0 + (\delta + \mu + \mu_3 + \phi_3)x_0)], \\
 y_2(t) &= 1 + \sigma^2(t-1)[(\mu_2 + \phi_2)(\beta u_0 - (\mu + \alpha + \mu_2 + \phi_2))w_0 + \beta u_0 x_0 \\
 &\quad + \omega(\Lambda(q) + (\mu + \phi)u_0 - (\gamma + \mu + \omega)v_0) + (\mu_3 + \phi_3)(\alpha w_0 - (\delta + \mu + \mu_3 + \phi_3)x_0) \\
 &\quad - \mu(\omega v_0 + (\mu_2 + \phi_2)w_0 + (\mu_3 + \phi_3)v_0 - \mu y_0)]
 \end{aligned} \right.$$

and so on. Therefore, we get the required solution as given by

$$\begin{cases} u(t) = u_0 + u_1(t) + u_2(t) + u_3(t) + \dots, \\ v(t) = v_0 + v_1(t) + v_2(t) + v_3(t) + \dots, \\ w(t) = w_0 + w_1(t) + w_2(t) + w_3(t) + \dots, \\ x(t) = x_0 + x_1(t) + x_2(t) + x_3(t) + \dots, \\ y(t) = y_0 + y_1(t) + y_2(t) + y_3(t) + \dots. \end{cases}$$

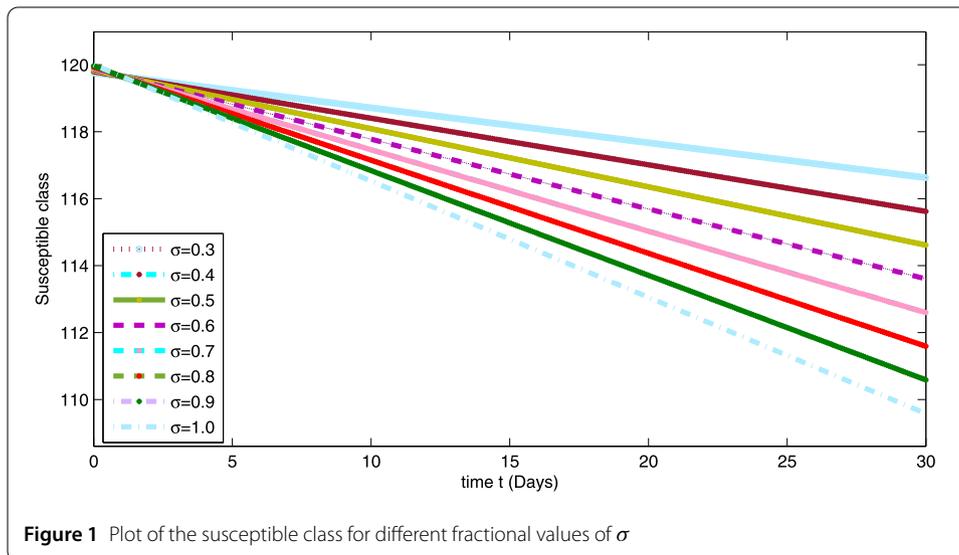
### 5 Results and discussion

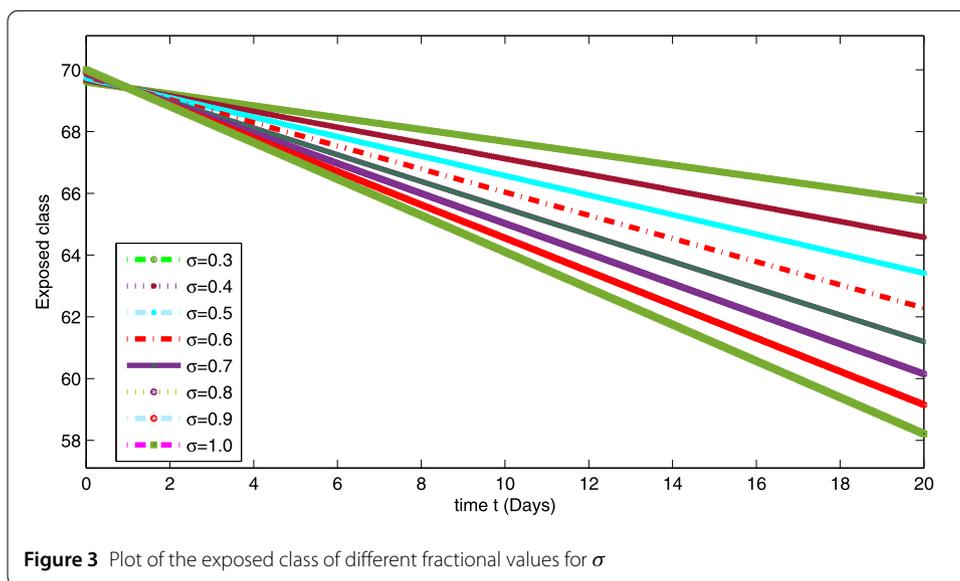
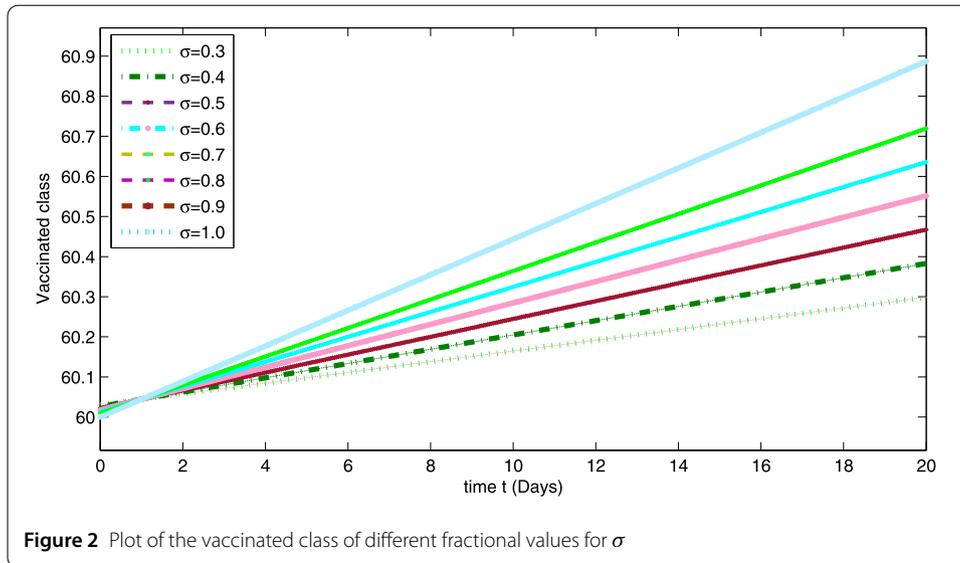
This part of the manuscript is devoted to providing numerical results and some discussions about approximate solution of the considered problem. For this purpose, we use the aforesaid technique for numerical simulation to compute the model with three parameters of vaccination. Here, we take some suitable values for the parameters as  $u_0 = 120$ ,  $v_0 = 60$ ,  $w_0 = 70$ ,  $x_0 = 50$ ,  $y_0 = 10$ ,  $\Lambda = 0.00001$ ,  $\beta = 0.000009$ ,  $\gamma = 0.00009$ ;  $\alpha = 0.000125$ ;  $\mu = 0.000875$ ,  $\mu_1 = 0.0002$ ,  $\mu_2 = 0.01$ ,  $\mu_3 = 0.0014286$ ,  $\omega = 0.00009$ ,  $q = 0.500000$ ,  $\eta = 0.030000$ ,  $\phi_1 = 0.007$ ,  $\phi_2 = 0.00009$ , and  $\phi_3 = 0.0003$ . In view of these values, we get the series solutions after three terms as follows:

$$\begin{cases} u(t) = 120 - 0.3480(1 + \sigma(t - 1)) + 6.1656(1 + \sigma^2(t - 1)), \\ v(t) = 60 + 0.0447(1 + \sigma(t - 1)) - 3.6035(1 + \sigma^2(t - 1)), \\ w(t) = 70 - 0.6467(1 + \sigma(t - 1)) + 0.0053(1 + \sigma^2(t - 1)), \\ x(t) = 50 - 1.6214(1 + \sigma(t - 1)) + 0.0528(1 + \sigma^2(t - 1)) \\ y(t) = 10 + 0.7894(1 + \sigma(t - 1)) - 0.0100(1 + \sigma^2(t - 1)). \end{cases} \tag{32}$$

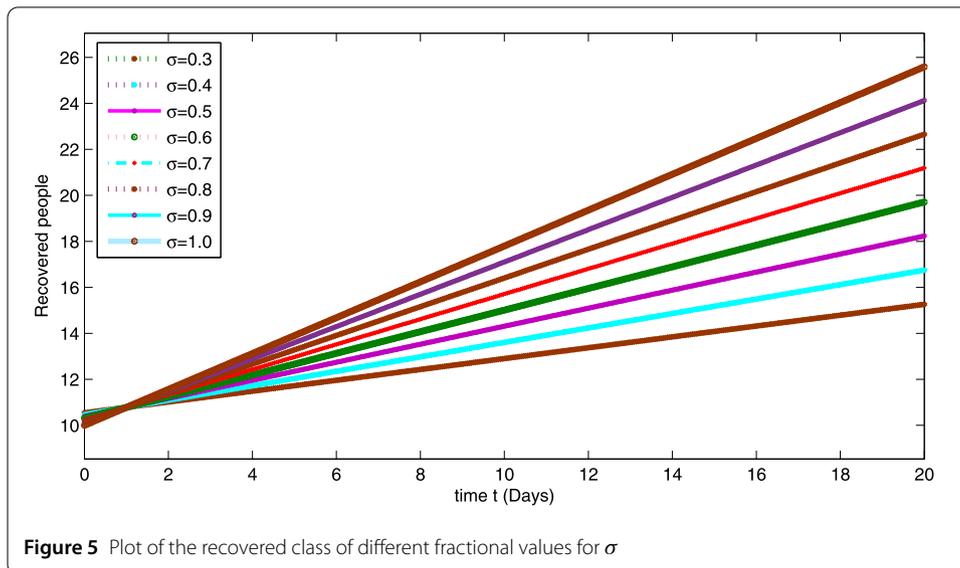
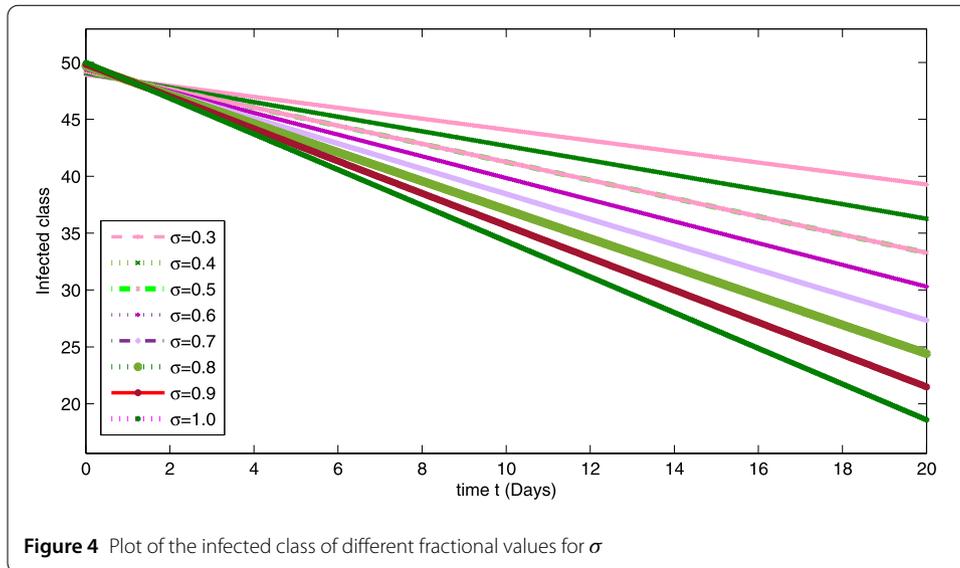
Next, we plot the solutions after three terms as given in (32) in the following Figs. 1, 2, 3, 4, 5 corresponding to different fractional order.

From Figs. 1, 2, 3, 4, 5, we plot the different classes of the model for different fractional values of  $\sigma$ . We plot the series solutions given in (32) corresponding to different fractional





order in Figs. 1, 2, 3, 4, 5 using Matlab. Figures 1, 2, 3, 4, 5 show the plots for the changes of susceptible, vaccinated, exposed, infected, and recovered for different fractional order  $\sigma$ . As can be seen, when increasing the value of  $\sigma$ , the solutions tend to the integer order solution. Further on the passage of time with proper vaccination, the density of susceptible class decreases with respect to time in days. The decay process varies at different fractional orders, while the vaccinated population increases at a given time in days. Figures 3, 4, 5 show that the exposed and infected population decreases with time as well as the recovered population is increasing with passage of time. This growth is different at different fractional order. It can be observed that at different values of  $\sigma$ , the different trajectories are shown in Figs. 1, 2, 3, 4, 5. At small fractional order, the growth or decay process is slightly faster compared to greater fractional order. From this observation, we conclude that for short memory, fractional order model is better than integer order.



### 6 Conclusion

We have developed LADM for mathematical modeling of measles spread with optimal control strategy involving CFFD. Also, some results about existence and uniqueness of solution have been developed. To the best of our knowledge, the aforesaid techniques are very rarely used to handle the analytical solutions of FODEs involving non-singular derivative of Caputo–Fabrizio type. Furthermore, the numerical results have been displayed via graphs, which indicate that the established technique can be used to handle semi-analytical solution of those FODEs involving CFFD. In the future, the mentioned method can be utilized to investigate more nonlinear problems of FODEs involving CFFD.

### Acknowledgements

The authors would like to thank the reviewers for their helpful comments and suggestions to improve the manuscript. Further the first two and last authors are thankful to HED under HEREF-46 and HEC under NRPDU:10039 of Pakistan for supporting this research work.

### Funding

The paper has no funding resources.

### Availability of data and materials

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

### Competing interests

The authors declare that there is no conflict of interest.

### Authors' contributions

GN, KS, HA, HK, and RK worked together in the derivation of the mathematical formulae and results. The authors read and approved the final manuscript.

### Author details

<sup>1</sup>Department of Mathematics, University of Malakand, Dir(L), Pakistan. <sup>2</sup>College of Engineering, Al Ain University, Al Ain, UAE. <sup>3</sup>Mathematics Department, Tafila Technical University, Tafila, Jordan. <sup>4</sup>Department of Mathematics, University of Education Lahore, Attock, Pakistan.

### Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 13 February 2020 Accepted: 8 April 2020 Published online: 22 April 2020

### References

1. Prey, R.T., Hasley, N.A.: The clinical significance of measles: a review. *J. Infect. Dis.* **189**(1), 4–16 (2004)
2. Prawato, B.P.: Stability and simulation of measles transmission model with and without vaccination. *Far East J. Math. Sci.* **102**(2), 271–281 (2017)
3. Oghere, E.O., Ako, I.I.: A mathematical model for measles disease. *Far East J. Math. Sci.* **54**(1), 47–63 (2011)
4. Diekmann, O.O., Heesterbeek, J.A., Roberts, M.G.: The construction of next generation matrices for compartmental epidemic model. *J. R. Soc. Interface* **7**(47), 873–885 (2009)
5. Pang, L., Ruan, S., Liu, S., Zaho, Z., Zhang, X.: Transmission dynamics and optimal control of measles epidemics. *Appl. Math. Comput.* **256**, 131–147 (2015)
6. Momah, A.A., Ibrahim, M.O., Uwanta, J.J., Manga, S.B.: Modelling effects of vaccination on the transmission dynamics of measles. *Int. J. Pure Appl. Math.* **88**(7), 381–390 (2013)
7. Kilbas, A.A., Srivastava, H., Trujillo, J.: *Theory and Application of Fractional Differential Equations*. North Holland Mathematics Studies, vol. 204. Elsevier, Amsterdam (2006)
8. Podlubny, I.: *Fractional Differential Equations*. Mathematics in Science and Engineering. Academic Press, New York (1999)
9. Caputo, M., Fabrizio, M.: Applications of new time and spatial fractional derivatives with exponential kernels. *Prog. Fract. Differ. Appl.* **2**, 1–11 (2016)
10. El-Saka, H.A.A.: The fractional-order SIS epidemic model with variable population size. *J. Egypt. Math. Soc.* **22**, 50–54 (2014)
11. Caputo, M., Fabrizio, M.: A new definition of fractional derivative with out singular kernel. *Prog. Fract. Differ. Appl.* **1**, 73–85 (2015)
12. Wang, Z., Yang, D., Ma, T., Sum, N.: Sun stability analysis for nonlinear fractional-order systems based on comparison principle. *Nonlinear Dyn.* **75**(1–2), 387–402 (2014)
13. Baleanu, D., Jajarmi, A., Mohammadi, H., Rezapour, S.: A new study on the mathematical modelling of human liver with Caputo–Fabrizio fractional derivative. *Chaos Solitons Fractals* **134**, Article ID 109705 (2020)
14. Jajarmi, A., Baleanu, D., Sajjadi, S.S., Asad, J.H.: A new feature of the fractional Euler–Lagrange equations for a coupled oscillator using a nonsingular operator approach. *Front. Phys.* **7**, Article ID 196 (2019)
15. Shaikh, A., Tassaddiq, A., Nisar, K.S., Baleanu, D.: Analysis of differential equations involving Caputo–Fabrizio fractional operator and its applications to reaction–diffusion equations. *Adv. Differ. Equ.* **2019**, Article ID 178 (2019)
16. Jajarmi, A., Arshad, S., Baleanu, D.: A new fractional modelling and control strategy for the outbreak of dengue fever. *Phys. A, Stat. Mech. Appl.* **535**, Article ID 122524 (2019)
17. Jajarmi, A., Ghanbari, B., Baleanu, D.: A new and efficient numerical method for the fractional modelling and optimal control of diabetes and tuberculosis co-existence. *Chaos* **29**(9), Article ID 093111 (2019)
18. Baleanu, D., Asad, J.H., Jajarmi, A.: New aspects of motion of a particle in a circular cavity. *Proc. Rom. Acad., Ser. A: Math. Phys. Tech. Sci. Inf. Sci.* **19**(2), 361–367 (2018)
19. Baleanu, D., Asad, J.H., Jajarmi, A.: The fractional model of spring pendulum: a feature within different kernels. *Proc. Rom. Acad., Ser. A: Math. Phys. Tech. Sci. Inf. Sci.* **19**(3), 447–454 (2018)
20. Yildiz, T.A., Jajarmi, A., Yildiz, B., Baleanu, D.: New aspects of time fractional optimal control problems within operators with nonsingular kernel. *Discrete Contin. Dyn. Syst., Ser. S* **13**(3), 407–428 (2020)
21. Baleanu, D., Mohammadi, H., Rezapour, S.: Analysis of the model of HIV-1 infection of CD4<sup>+</sup> T-cell with a new approach of fractional derivative. *Adv. Differ. Equ.* **2020**, Article ID 71 (2020)
22. Alizadeh, S., Baleanu, D., Rezapour, S.: Analyzing transient response of the parallel RCL circuit by using the Caputo–Fabrizio fractional derivative. *Adv. Differ. Equ.* **2020**, Article ID 55 (2020)
23. Baleanu, D., Mousalou, A., Rezapour, S.: On the existence of solutions for some infinite coefficient-symmetric Caputo–Fabrizio fractional integro-differential equations. *Bound. Value Probl.* **2017**, Article ID 145 (2017)
24. Haq, F., Shah, K., Rahman, G., Muhammad, S.: Numerical solution of fractional order smoking model via Laplace Adomian decomposition method. *Alex. Eng. J.* **57**(2), 1061–1069 (2018)

25. Ali, A., Shah, K., Khan, R.A.: Numerical treatment for traveling wave solutions of fractional Whitham–Broer–Kaup equations. *Alex. Eng. J.* **57**(3), 1991–1998 (2018)
26. Kiyemez, O.: An algorithm for solving initial value problems using Laplace Adomian decomposition. *Appl. Math. Sci.* **3**(30), 1453–1459 (2009)
27. Kuri, A.S.: A Laplace decomposition algorithm applied to a class of nonlinear differential equations. *J. Appl. Math.* **1**(4), 141–155 (2001)
28. Baleanu, D., Jajarmi, A., Sajjadi, S.S., Mozyrska, D.: A new fractional model and optimal control of a tumor-immune surveillance with non-singular derivative operator. *Chaos* **29**(8), Article ID 083127 (2019)
29. Shah, K., et al.: On a nonlinear fractional order model of dengue fever disease under Caputo–Fabrizio derivative. *Alex. Eng. J.* (2020). <https://doi.org/10.1016/j.aej.2020.02.022>
30. Aydogan, S.M., Baleanu, D., Mousalou, A., Rezapour, S.: On approximate solutions for two higher-order Caputo–Fabrizio fractional integro-differential equations. *Adv. Differ. Equ.* **2017**, Article ID 221 (2017)
31. Baleanu, D., Mousalou, A., Rezapour, S.: The extended fractional Caputo–Fabrizio derivative of order  $0 \leq \sigma < 1$  on  $C_{\mathbb{R}}[0, 1]$  and the existence of solutions for two higher-order series-type differential equations. *Adv. Differ. Equ.* **2018**, Article ID 255 (2018)
32. Aydogan, M.S., Baleanu, D., Mousalou, A., Rezapour, S.: On high order fractional integro-differential equations including the Caputo–Fabrizio derivative. *Bound. Value Probl.* **2018**, Article ID 90 (2018)
33. Baleanu, D., Rezapour, S., Saberpour, Z.: On fractional integro-differential inclusions via the extended fractional Caputo–Fabrizio derivation. *Bound. Value Probl.* **2019**, Article ID 79 (2019)

Submit your manuscript to a SpringerOpen<sup>®</sup> journal and benefit from:

- Convenient online submission
- Rigorous peer review
- Open access: articles freely available online
- High visibility within the field
- Retaining the copyright to your article

---

Submit your next manuscript at ► [springeropen.com](https://www.springeropen.com)

---