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Fractional calculus operators with Appell function kernels applied to Srivastava polynomials and extended Mittag-Leffler function

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Abstract
This article aims to establish certain image formulas associated with the fractional calculus operators with Appell function in the kernel and Caputo-type fractional differential operators involving Srivastava polynomials and extended Mittag-Leffler function. The main outcomes are presented in terms of the extended Wright function. In addition, along with the noted outcomes, the implications are also highlighted.

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1 Introduction and preliminaries

The Mittag-Leffler function is the most significant in the theory of special functions in literature; solutions are available to number of problems formulated in terms of fractional integral, differential, and difference equations. So it has become a subject of interest for many authors in the field of fractional calculus (FC) and its applications (see details [4, 9, 18, 22, 25, 27, 28, 31, 35, 41–43]). Nowadays, there are several applications of the Mittag-Leffler function such as random walks, Levy flights, telegraph equation, kinetic equation, transport, and complex system. Some of the applications are in the field of applied sciences, like rheology, fluid flow, electric networks, diffusive transport, statistical distribution theory. Other significant applications are available in the area of fractals kinetics [3, 5, 6], medical sciences [11, 17, 21], fractal calculus and applications [1, 8, 24], and in the Haar wavelet and analytical approach [14, 19, 20, 36].

The Swedish mathematician Mittag-Leffler (M-L) [26] introduced the following function, known after his name as Mittag-Leffler function:

$$E_{\theta}(x)=\sum_{n=0}^{\infty}\frac{x^{n}}{\Gamma(\theta n+1)}\quad\bigl(x\in\mathbb{C};\Re(\theta)>0\bigr).\tag{1}$$

Throughout the paper, $\mathbb{C}$, $\mathbb{R}^+$, $\mathbb{Z}_0^-$, and $\mathbb{N}$ are the usual notations for the sets of complex numbers, positive real numbers, nonpositive integers, and positive integers, respectively. Further, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Mathematician Wiman [44] generalized the M-L function (1) as follows:

$$E_{\theta,\kappa}(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(\theta n + \kappa)} \quad \bigl(t, \kappa \in \mathbb{C}; \Re(\theta) > 0\bigr). \tag{2}$$

Many authors have conferred and studied the above function in diverse areas of research (see, e.g., [10, 12, 13, 32, 33]).

Prabhakar [30] introduced the generalized Mittag-Leffler function as follows:

$$E_{\theta,\kappa}^{\gamma}(x) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\theta n + \kappa)} \frac{x^n}{n!} \quad \bigl(x, \kappa, \gamma \in \mathbb{C}; \Re(\theta) > 0\bigr), \tag{3}$$

where $(\gamma)_\tau$, $\gamma, \tau \in \mathbb{C}$; the Pochhammer symbol is defined in terms of the well-known gamma function by

$$(\gamma)_\tau := \frac{\Gamma(\gamma + \tau)}{\Gamma(\gamma)} = \begin{cases} 1 & (\tau = 0; \gamma \in \mathbb{C} \setminus \{0\}), \\ \gamma(\gamma+1)\cdots(\gamma+n-1) & (\tau = n \in \mathbb{N}; \gamma \in \mathbb{C}). \end{cases} \tag{4}$$

We recall the definition of extended M-L function, which was introduced and investigated by Özarslan and Yilmaz [29] in the manner

$$\begin{gathered} E_{\theta,\kappa}^{\gamma;c}(x;p) = \sum_{n=0}^{\infty} \frac{\mathcal{B}_p(\gamma + n, c - \gamma)(c)_n}{\mathbf{B}(\gamma, c - \gamma)\Gamma(\theta n + \kappa)} \frac{x^n}{n!} \\ \bigl(x, \kappa \in \mathbb{C}; p \geq 0; \Re(c) > \Re(\gamma) > 0, \Re(\theta) > 0\bigr), \end{gathered} \tag{5}$$

where extended beta function $\mathcal{B}_p(x,y)$ is defined as (see [7])

$$\mathcal{B}_p(x,y) = \int_0^1 u^{x-1}(1-u)^{y-1} e^{-\frac{p}{u(1-u)}}\, du \quad \bigl(\min\bigl\{\Re(p), \Re(x), \Re(y)\bigr\} > 0\bigr), \tag{6}$$

where $\mathcal{B}_0(j,k) = \mathbf{B}(j,k)$, the well-known beta function (see, e.g., [39, Sect. 1.1])

$$\mathbf{B}(j,k) = \begin{cases} \int_0^1 u^{j-1}(1-u)^{k-1}\, du & (\min\{\Re(j), \Re(k)\} > 0), \\ \frac{\Gamma(j)\Gamma(k)}{\Gamma(j+k)} & (j, k \in \mathbb{C} \setminus \mathbb{Z}_0^-). \end{cases} \tag{7}$$

Sharma and Devi [37] introduced and investigated the following extended Wright generalized hypergeometric function:

$$\begin{gathered} {}_{r+1}\Psi_{s+1}\left[\begin{matrix} (a_i, A_i)_{1,r}, (\gamma, 1); \\ (b_j, B_j)_{1,s}, (c, 1); \end{matrix} \; x; p\right] = \frac{1}{\Gamma(c-\gamma)} \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{r} \Gamma(a_i + kA_i)}{\prod_{j=1}^{s} \Gamma(b_j + kB_j)} \frac{B_p(\gamma + k, c - \gamma)x^k}{k!} \\ \bigl(\Re(c) > \Re(\gamma) > 0, \Re(p) > 0; r, s \in \mathbb{N}_0; a_i, b_j \in \mathbb{C}, A_i, B_j \in \mathbb{R}^+, i = 1, \dots, r, j = 1, \dots, s\bigr), \end{gathered} \tag{8}$$

where the empty product is perceived as unity, and it is presumed that the summation is convergent.

The Srivastava family of polynomials [38, p.1. Equation (1)] is defined as follows:

$$S_w^u(x) = \sum_{s=0}^{[w/u]} \frac{(-w)_{us}}{s!} A_{w,s} x^s, \quad w = 0, 1, 2, \ldots, u \in \mathbb{Z}^+, A_{w,s}(w,s) \geq 0, A_{w,s}(w,s) \in \mathbb{C}. \tag{9}$$

We recall the definitions of fractional integral operators associated with the Appell function F_3 (see, e.g., [40, p. 53, Eq. (6)],) in the kernel [23, 34], which are defined for τ, τ', ε, ε', $\upsilon \in \mathbb{C}$, $\Re(\upsilon) > 0$, $x \in \mathbb{R}^+$ as follows:

$$\left(I_{0+}^{\tau,\tau',\varepsilon,\varepsilon',\upsilon} f\right)(x) = \frac{x^{-\tau}}{\Gamma(\upsilon)} \int_0^x (x-t)^{\upsilon-1} t^{-\tau'} F_3\left(\tau, \tau', \varepsilon, \varepsilon'; \upsilon; 1 - \frac{t}{x}, 1 - \frac{x}{t}\right) f(t)\,\mathrm{d}t \tag{10}$$

and

$$\left(I_{-}^{\tau,\tau',\varepsilon,\varepsilon',\upsilon} f\right)(x) = \frac{x^{-\tau'}}{\Gamma(\upsilon)} \int_x^\infty (t-x)^{\upsilon-1} t^{-\tau} F_3\left(\tau, \tau', \varepsilon, \varepsilon'; \upsilon; 1 - \frac{t}{x}, 1 - \frac{x}{t}\right) f(t)\,\mathrm{d}t. \tag{11}$$

Fractional integral operators (10) and (11) were introduced by Marichev [23] and further extended and studied by Saigo and Maeda [34].

Fractional differential operators corresponding to the above integrals (10) and (11) are defined as follows:

$$\left(D_{0+}^{\tau,\tau',\varepsilon,\varepsilon',\upsilon} f\right)(x) = \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{[\Re(\upsilon)]+1} \left(I_{0+}^{-\tau',-\tau,-\varepsilon'+[\Re(\upsilon)]+1,-\varepsilon,-\upsilon+[\Re(\upsilon)]+1} f\right)(x) \tag{12}$$

and

$$\left(D_{-}^{\tau,\tau',\varepsilon,\varepsilon',\upsilon} f\right)(x) = \left(-\frac{\mathrm{d}}{\mathrm{d}x}\right)^{[\Re(\upsilon)]+1} \left(I_{-}^{-\tau',-\tau,-\varepsilon',-\varepsilon+[\Re(\upsilon)]+1,-\upsilon+[\Re(\upsilon)]+1} f\right)(x). \tag{13}$$

The following lemmas [15, 34] shall be required in the sequel.

Lemma 1.1 *Let* $\tau, \tau', \varepsilon, \varepsilon', \upsilon, \varsigma \in \mathbb{C}$ *be such that* $\Re(\upsilon) > 0$ *and*

$$\Re(\tau) > \max\left\{0, \Re\left(\tau - \tau' - \varepsilon - \upsilon\right), \Re\left(\tau' - \varepsilon'\right)\right\}.$$

Then there exists the relation

$$\left(I_{0+}^{\tau,\tau',\varepsilon,\varepsilon',\upsilon} t^{\varsigma-1}\right)(x) = \frac{\Gamma(\varsigma)\Gamma(\varsigma+\upsilon-\tau-\tau'-\varepsilon)\Gamma(\varsigma+\varepsilon'-\tau')}{\Gamma(\varsigma+\varepsilon')\Gamma(\varsigma+\upsilon-\tau-\tau')\Gamma(\varsigma+\upsilon-\tau'-\varepsilon)} x^{\varsigma-\tau-\tau'+\upsilon-1}. \tag{14}$$

Lemma 1.2 *Let* τ, τ', ε, ε', υ, $\varsigma \in \mathbb{C}$ *such that* $\Re(\upsilon) > 0$ *and*

$$\Re(\varsigma) > \max\left\{\Re(\varepsilon), \Re\left(-\tau - \tau' + \upsilon\right), \Re\left(-\tau - \varepsilon' + \upsilon\right)\right\}.$$

Then

$$\left(I_{-}^{\tau,\tau',\varepsilon,\varepsilon',\upsilon} t^{-\varsigma}\right)(x) = \frac{\Gamma(-\varepsilon+\varsigma)\Gamma(\tau+\tau'-\upsilon+\varsigma)\Gamma(\tau+\varepsilon'-\upsilon+\varsigma)}{\Gamma(\varsigma)\Gamma(\tau-\varepsilon+\varsigma)\Gamma(\tau+\tau'+\varepsilon'-\upsilon+\varsigma)} x^{-\tau-\tau'+\upsilon-\varsigma}.$$

Lemma 1.3 *Let τ, τ', ε, ε', υ, $\varsigma \in \mathbb{C}$ such that*

$$\Re(\varsigma) > \max\{0, \Re(-\tau + \varepsilon'), \Re(-\tau - \tau' - \varepsilon + \upsilon)\}.$$

Then

$$\left(D_{0+}^{\tau,\tau',\varepsilon,\varepsilon',\upsilon} t^{\varsigma-1}\right)(x) = \frac{\Gamma(\varsigma)\Gamma(-\varepsilon + \tau + \varsigma)\Gamma(\tau + \tau' + \varepsilon' - \upsilon + \varsigma)}{\Gamma(-\varepsilon + \varsigma)\Gamma(\tau + \tau' - \upsilon + \varsigma)\Gamma(\tau + \varepsilon' - \upsilon + \varsigma)} x^{\tau+\tau'-\upsilon+\varsigma-1}. \tag{15}$$

Lemma 1.4 *Let τ, τ', ε, ε', υ, $\varsigma \in \mathbb{C}$ such that*

$$\Re(\varsigma) > \max\{\Re(-\varepsilon'), \Re(\tau' + \varepsilon - \upsilon), \Re(\tau + \tau' - \upsilon) + [\Re(\upsilon)] + 1\}.$$

Then

$$\left(D_{-}^{\tau,\tau',\varepsilon,\varepsilon',\upsilon} t^{-\varsigma}\right)(x) = \frac{\Gamma(\varepsilon' + \varsigma)\Gamma(-\tau - \tau' + \upsilon + \varsigma)\Gamma(-\tau' - \varepsilon + \upsilon + \varsigma)}{\Gamma(\varsigma)\Gamma(-\tau' + \varepsilon' + \varsigma)\Gamma(-\tau - \tau' - \varepsilon + \upsilon + \varsigma)} x^{\tau+\tau'-\upsilon-\varsigma}. \tag{16}$$

The left- and right-sided Saigo fractional integral formulas are defined for $x > 0$ and $\tau, \varepsilon, \upsilon \in \mathbb{C}$, $\Re(\tau) > 0$, respectively, by (see [16])

$$\left(I_{0+}^{\tau,\varepsilon,\upsilon} f\right)(x) = \frac{x^{-\tau-\varepsilon}}{\Gamma(\tau)} \int_0^x (x-t)^{\tau-1} {}_2F_1\left(\tau + \varepsilon, -\upsilon; \tau; 1 - \frac{t}{x}\right) f(t)\,dt \tag{17}$$

and

$$\left(I_{-}^{\tau,\varepsilon,\upsilon} f\right)(x) = \frac{1}{\Gamma(\tau)} \int_x^\infty (t-x)^{\tau-1} t^{-\tau-\varepsilon} {}_2F_1\left(\tau + \varepsilon, -\upsilon; \tau; 1 - \frac{x}{t}\right) f(t)\,dt, \tag{18}$$

where ${}_2F_1(\cdot)$ is the Gauss hypergeometric series.

When $\varepsilon = -\tau$, the operators in equations (17) and (18) coincide with the classical Riemann–Liouville fractional integrals of order $\tau \in \mathbb{C}$ with $x > 0$ as follows:

$$\left(I_{0+}^{\tau} f\right)(x) = \frac{1}{\Gamma(\tau)} \int_0^x \frac{1}{(x-t)^{1-\tau}} f(t)\,dt \tag{19}$$

and

$$\left(I_{-}^{\tau} f\right)(x) = \frac{1}{\Gamma(\tau)} \int_x^\infty \frac{1}{(t-x)^{1-\tau}} f(t)\,dt. \tag{20}$$

For $\varepsilon = 0$, the operators in equations (17) and (18) yield the so-called Erdélyi–Kober integrals of order $\tau \in \mathbb{C}$ with $x > 0$ as follows:

$$\left(I_{\upsilon,\tau}^{+} f\right)(x) = \frac{x^{-\tau-\upsilon}}{\Gamma(\tau)} \int_0^x \frac{1}{(x-t)^{1-\tau}} t^{\upsilon} f(t)\,dt \tag{21}$$

and

$$\left(K_{\upsilon,\tau}^{-} f\right)(x) = \frac{x^{\upsilon}}{\Gamma(\tau)} \int_x^\infty \frac{1}{(t-x)^{1-\tau}} t^{-\tau-\upsilon} f(t)\,dt. \tag{22}$$

We also need the following lemmas [16].

Lemma 1.5 *Let $\tau,\varepsilon,\upsilon\in\mathbb{C}$ be such that $\Re(\tau)>0$, $\Re(\varsigma)>\max[0,\Re(\varepsilon-\upsilon)]$. Then*

$$\left(I_{0+}^{\tau,\varepsilon,\upsilon}t^{\varsigma-1}\right)(x)=\frac{\Gamma(\varsigma)\Gamma(\varsigma+\upsilon-\varepsilon)}{\Gamma(\varsigma-\varepsilon)\Gamma(\varsigma+\tau+\upsilon)}x^{\varsigma-\varepsilon-1}. \tag{23}$$

In particular,

$$\left(I_{0+}^{\tau}t^{\varsigma-1}\right)(x)=\frac{\Gamma(\varsigma)}{\Gamma(\varsigma+\tau)}x^{\varsigma+\tau-1},\quad \Re(\tau)>0,\Re(\varsigma)>0 \tag{24}$$

and

$$\left(I_{\upsilon,\tau}^{+}t^{\varsigma-1}\right)(x)=\frac{\Gamma(\varsigma+\upsilon)}{\Gamma(\varsigma+\tau+\upsilon)}x^{\varsigma-1},\quad \Re(\tau)>0,\Re(\varsigma)>-\Re(\upsilon). \tag{25}$$

Lemma 1.6 *Let $\tau,\varepsilon,\upsilon\in\mathbb{C}$ be such that $\Re(\tau)>0$, $\Re(\varsigma)<1+\min[\Re(\varepsilon),\Re(\upsilon)]$. Then*

$$\left(I_{-}^{\tau,\varepsilon,\upsilon}t^{\varsigma-1}\right)(x)=\frac{\Gamma(\varepsilon-\varsigma+1)\Gamma(\upsilon-\varsigma+1)}{\Gamma(1-\varsigma)\Gamma(\tau+\varepsilon+\upsilon-\varsigma+1)}x^{\varsigma-\varepsilon-1}. \tag{26}$$

In particular,

$$\left(I_{-}^{\tau}t^{\varsigma-1}\right)(x)=\frac{\Gamma(1-\tau-\varsigma)}{\Gamma(1-\varsigma)}x^{\varsigma+\tau-1},\quad 0<\Re(\tau)<1-\Re(\varsigma) \tag{27}$$

and

$$\left(K_{\upsilon,\tau}^{-}t^{\varsigma-1}\right)(x)=\frac{\Gamma(\upsilon-\varsigma+1)}{\Gamma(\tau+\upsilon-\varsigma+1)}x^{\varsigma-1},\quad \Re(\varsigma)<1+\Re(\upsilon). \tag{28}$$

2 Fractional calculus of extended Mittag-Leffler function

In this section, we present certain image formulas for product of Srivastava polynomials and generalized EMLF in view of the generalized fractional integral and differential calculus and consider some corollaries as particular cases.

Theorem 2.1 *Let $\tau,\tau',\varepsilon,\varepsilon',c,\theta,\kappa,\upsilon,\chi,\varsigma\in\mathbb{C}$, $x\in\mathbb{R}^{+}$ with $\Re(\upsilon)>0$ and $\Re(\varsigma)>\max\{0,\Re(\tau+\tau'+\varepsilon-\upsilon),\Re(\tau-\varepsilon')\}$ and $p\geq 0$, then*

$$\begin{aligned}&\left(I_{0+}^{\tau,\tau',\varepsilon,\varepsilon',\upsilon}t^{\varsigma-1}S_n^m(t)E_{\theta,\kappa}^{\chi,c}(t;p)\right)(x)\\&\quad=\frac{x^{\varsigma-\tau-\tau'+\upsilon-1}}{\Gamma(\chi)}\sum_{s=0}^{[n/m]}\frac{(-n)_{m.s}}{s!}A_{n,s}x^{s}\\&\qquad\times{}_5\psi_5\left[\begin{matrix}(c,1),(\varsigma+s,1),(\varsigma+s+\upsilon-\tau-\tau'-\varepsilon,1),(\varsigma+s+\varepsilon'-\tau',1),(\chi,1);\\(\kappa,\theta),(\varsigma+s+\varepsilon',1),(\varsigma+s+\upsilon-\tau'-\varepsilon,1),(\varsigma+s+\upsilon-\tau-\tau',1),(c,1);\end{matrix}\ (x;p)\right].\end{aligned} \tag{29}$$

Proof Let I_1 denote LHS of (29). Using (5) and (9) therein, we have

$$\begin{aligned}I_1&=\left(I_{0+}^{\tau,\tau',\varepsilon,\varepsilon',\upsilon}t^{\varsigma-1}S_n^m(t)E_{\theta,\kappa}^{\chi,c}(t;p)\right)(x)\\&=\left(I_{0+}^{\tau,\tau',\varepsilon,\varepsilon',\upsilon}t^{\varsigma-1}\sum_{s=0}^{[n/m]}\frac{(-n)_{m.s}}{s!}A_{n,s}(t)^{s}\sum_{n=0}^{\infty}\frac{\mathcal{B}_p(\chi+n,c-\chi)}{\mathbf{B}(\chi,c-\chi)}\frac{(c)_n}{\Gamma(\theta n+\kappa)}\frac{t^n}{n!}\right)(x).\end{aligned}$$

Under the valid conditions mentioned with this theorem, interchanging the integration and summation order allows us to write

$$I_1 = \sum_{s=0}^{[n/m]} \frac{(-n)_{m.s}}{s!} A_{n,s} \sum_{n=0}^{\infty} \frac{\mathcal{B}_p(\chi+n, c-\chi)}{\mathbf{B}(\chi, c-\chi)} \frac{(c)_n}{\Gamma(\theta n+\kappa) n!} \bigl(I_{0+}^{\tau,\tau',\varepsilon,\varepsilon',\upsilon} t^{\varsigma+n+s-1}\bigr)(x).$$

Applying Lemma (1.1), we get

$$\begin{aligned}
I_1 &= \sum_{s=0}^{[n/m]} \frac{(-n)_{m.s}}{s!} A_{n,s} \sum_{n=0}^{\infty} \frac{\mathcal{B}_p(\chi+n, c-\chi)}{\mathbf{B}(\chi, c-\chi)} \frac{(c)_n}{\Gamma(\theta n+\kappa) n!} \\
&\quad \times \frac{\Gamma(\varsigma+s+n)\Gamma(\varsigma+s+\upsilon-\tau-\tau'-\varepsilon+n)\Gamma(\varsigma+s+\varepsilon'-\tau'+n)}{\Gamma(\varsigma+s+\varepsilon'+n)\Gamma(\varsigma+s+\upsilon-\tau-\tau'+n)\Gamma(\varsigma+s+\upsilon-\tau'-\varepsilon+n)} \\
&\quad \times x^{\varsigma+n+s+\upsilon-\tau-\tau'-1} \\
&= \sum_{s=0}^{[n/m]} \frac{(-n)_{m.s}}{s!} A_{n,s} \sum_{n=0}^{\infty} \frac{\mathcal{B}_p(\chi+n, c-\chi)}{\Gamma(\chi)\Gamma(c-\chi)} \frac{\Gamma(c+n)}{\Gamma(\theta n+\kappa) n!} \\
&\quad \times \frac{\Gamma(\varsigma+s+n)\Gamma(\varsigma+s+\upsilon-\tau-\tau'-\varepsilon+n)\Gamma(\varsigma+s+\varepsilon'-\tau'+n)}{\Gamma(\varsigma+s+\varepsilon'+n)\Gamma(\varsigma+s+\upsilon-\tau-\tau'+n)\Gamma(\varsigma+s+\upsilon-\tau'-\varepsilon+n)} \\
&\quad \times x^{\varsigma+s+n+\upsilon-\tau-\tau'-1} \\
&= \frac{x^{\varsigma+\upsilon-\tau-\tau'-1}}{\Gamma(\chi)} \sum_{s=0}^{[n/m]} \frac{(-n)_{m,s}}{s!} A_{n,s} x^s \sum_{n=0}^{\infty} \frac{\mathcal{B}_p(\chi+n, c-\chi)}{\Gamma(c-\chi)} \frac{\Gamma(c+n)}{\Gamma(\theta n+\kappa)} \\
&\quad \times \frac{\Gamma(\varsigma+s+n)\Gamma(\varsigma+s+\upsilon-\tau-\tau'-\varepsilon+n)\Gamma(\varsigma+s+\varepsilon'-\tau'+n)}{\Gamma(\varsigma+s+\varepsilon'+n)\Gamma(\varsigma+s+\upsilon-\tau-\tau'+n)\Gamma(\varsigma+s+\upsilon-\tau'-\varepsilon+n)} \frac{x^n}{n!}.
\end{aligned}$$

Making use of (8), we get the outcome needed. □

Corollary *Let* $\tau, \varepsilon, c, \theta, \kappa, \upsilon, \varsigma \in \mathbb{C}$, $x \in \mathbb{R}^+$ *with* $\Re(\upsilon) > 0$ *and* $\Re(\varsigma) > \max\{0, \Re(\varepsilon-\upsilon)\}$ *and* $p \geq 0$, *then*

$$\begin{aligned}
&\bigl(I_{0+}^{\tau,\varepsilon,\upsilon} t^{\varsigma-1} S_n^m(t) E_{\theta,\kappa}^{\chi,c}(t;p)\bigr)(x) \\
&\quad = \frac{x^{\varsigma-\varepsilon-1}}{\Gamma(\chi)} \sum_{s=0}^{[n/m]} \frac{(-n)_{m.s}}{s!} A_{n,s} x^s \\
&\qquad \times {}_4\psi_4 \left[\begin{matrix} (c,1), (\varsigma+s,1), (\varsigma+s+\upsilon-\varepsilon,1), (\chi,1); \\ (c,1), (\kappa,\theta), (\varsigma+s-\varepsilon,1), (\varsigma+s+\upsilon+\tau,1); \end{matrix} \quad (x;p) \right].
\end{aligned} \tag{30}$$

Corollary *Let* $\tau, c, \theta, \kappa, \upsilon, \varsigma \in \mathbb{C}$, $x \in \mathbb{R}^+$ *with* $\Re(\upsilon) > 0$ *and* $\Re(\varsigma) > \Re(\upsilon)$ *and* $p \geq 0$, *then*

$$\begin{aligned}
\bigl(I_{\upsilon,\tau}^{+} t^{\varsigma-1} S_n^m(t) E_{\theta,\kappa}^{\chi,c}(t;p)\bigr)(x) &= \frac{x^{\varsigma-1}}{\Gamma(\chi)} \sum_{s=0}^{[n/m]} \frac{(-n)_{m.s}}{s!} A_{n,s} x^s \\
&\quad \times {}_3\psi_3 \left[\begin{matrix} (c,1), (\varsigma+s+\upsilon,1), (\chi,1); \\ (c,1), (\kappa,\theta), (\varsigma+s+\upsilon+\tau,1); \end{matrix} \quad (x;p) \right].
\end{aligned} \tag{31}$$

Theorem 2.2 *Let $\tau,\tau',\varepsilon,\varepsilon',\theta,\kappa,\upsilon,\chi,\varsigma\in\mathbb{C}$, $x\in\mathbb{R}^+$ with $\Re(\tau)>0$ and $\Re(\varsigma)>\max\{\Re(\varepsilon),\Re(-\tau-\tau'+\upsilon),\Re(-\tau-\varepsilon'+\upsilon)\}$ and $p\geq 0$, then*

$$\begin{aligned}&\bigl(I_{-}^{\tau,\tau',\varepsilon,\varepsilon',\upsilon}t^{-\varsigma}S_n^m(t)E_{\theta,\kappa}^{\chi,c}(1/t;p)\bigr)(x)\\&=\frac{x^{-\varsigma-\tau-\tau'+\upsilon}}{\Gamma(\chi)}\sum_{s=0}^{[n/m]}\frac{(-n)_{m.s}}{s!}A_{n,s}x^s\\&\quad\times{}_5\psi_5\left[\begin{matrix}(c,1),(\varsigma-s-\varepsilon,1),(\tau+\tau'-\upsilon+\varsigma-s,1),(\tau+\varepsilon'-\upsilon+\varsigma-s,1),(\chi,1);\\(c,1),(\kappa,\theta),(\varsigma,1),(\tau-\varepsilon+\varsigma-s,1),(\tau+\tau'-\varepsilon'-\upsilon+\varsigma-s,1),(\upsilon,1);\end{matrix}\ (1/x;p)\right].\end{aligned}\tag{32}$$

Proof Let I_2 be LHS of (32), then using (5) and (9), we have

$$\begin{aligned}I_2&=\bigl(I_{-}^{\tau,\tau',\varepsilon,\varepsilon',\upsilon}t^{-\varsigma}S_n^m(t)E_{\theta,\kappa}^{\chi,c}(1/t;p)\bigr)(x)\\&=\left(I_{-}^{\tau,\tau',\varepsilon,\varepsilon',\upsilon}t^{-\varsigma}\sum_{s=0}^{[n/m]}\frac{(-n)_{m.s}}{s!}A_{n,s}(t)^s\sum_{n=0}^{\infty}\frac{\mathcal{B}_p(\chi+n,c-\chi)}{\mathbf{B}(\chi,c-\chi)}\frac{(c)_n}{\Gamma(\theta n+\kappa)}\frac{t^{-n}}{n!}\right)(x).\end{aligned}$$

Now, under the conditions of the validity of this theorem, switching the order of integration and summation, we get

$$I_2=\sum_{s=0}^{[n/m]}\frac{(-n)_{m.s}}{s!}A_{n,s}\sum_{n=0}^{\infty}\frac{\mathcal{B}_p(\chi+n,c-\chi)}{\mathbf{B}(\chi,c-\chi)}\frac{(c)_n}{\Gamma(\theta n+\kappa)n!}\bigl(I_{-}^{\tau,\tau',\varepsilon,\varepsilon',\upsilon}t^{-\varsigma+s-n}\bigr)(x).$$

Applying Lemma (1.2), we get

$$\begin{aligned}I_2&=\sum_{s=0}^{[n/m]}\frac{(-n)_{m.s}}{s!}A_{n,s}\sum_{n=0}^{\infty}\frac{\mathcal{B}_p(\chi+n,c-\chi)}{\mathbf{B}(\chi,c-\chi)}\frac{(c)_n}{\Gamma(\theta n+\kappa)n!}\\&\quad\times\frac{\Gamma(-\varepsilon+\varsigma-s+n)\Gamma(\tau+\tau'+\varsigma-s-\upsilon+n)\Gamma(\tau+\varsigma-s+\varepsilon'-\upsilon+n)}{\Gamma(\varsigma-s+n)\Gamma(\varsigma-s-\varepsilon+\tau+n)\Gamma(\varsigma-s+\tau+\tau'-\varepsilon'-\upsilon+n)}\\&\quad\times x^{-\varsigma+s-n+\upsilon-\tau-\tau'}\\&=\sum_{s=0}^{[n/m]}\frac{(-n)_{m.s}}{s!}A_{n,s}\sum_{n=0}^{\infty}\frac{\mathcal{B}_p(\chi+n,c-\chi)}{\Gamma(\chi)\Gamma(c-\chi)}\frac{\Gamma(c+n)}{\Gamma(\theta n+\kappa)n!}\\&\quad\times\frac{\Gamma(-\varepsilon+\varsigma-s+n)\Gamma(\tau+\tau'+\varsigma-s-\upsilon+n)\Gamma(\tau+\varsigma-s+\varepsilon'-\upsilon+n)}{\Gamma(\varsigma-s+n)\Gamma(\varsigma-s-\varepsilon+\tau+n)\Gamma(\varsigma-s+\tau+\tau'-\varepsilon'-\upsilon+n)}\\&\quad\times x^{-\varsigma+s-n+\upsilon-\tau-\tau'}\\&=\frac{x^{-\varsigma+\upsilon-\tau-\tau'}}{\Gamma(\chi)}\sum_{s=0}^{[n/m]}\frac{(-n)_{m.s}}{s!}A_{n,s}x^s\sum_{n=0}^{\infty}\frac{\mathcal{B}_p(\chi+n,c-\chi)}{\Gamma(c-\chi)}\frac{\Gamma(c+n)}{\Gamma(\theta n+\kappa)}\\&\quad\times\frac{\Gamma(-\varepsilon+\varsigma-s+n)\Gamma(\tau+\tau'+\varsigma-s-\upsilon+n)\Gamma(\tau+\varsigma-s+\varepsilon'-\upsilon+n)}{\Gamma(\varsigma-s+n)\Gamma(\varsigma-s-\varepsilon+\tau+n)\Gamma(\varsigma-s+\tau+\tau'-\varepsilon'-\upsilon+n)}\frac{x^{-n}}{n!}.\end{aligned}$$

Again, by employing (8), we obtain the desired result. □

Corollary *Let $\tau,\varepsilon,c,\theta,\kappa,\chi,\upsilon,\varsigma\in\mathbb{C}$, $x\in\mathbb{R}^+$ with $\Re(\upsilon)>0$ and $\Re(\varsigma)>\max\{\Re(\varepsilon),\Re(-\upsilon)\}$ and $p\geq 0$, then*

$$\begin{aligned}&\bigl(I_{-}^{\tau,\varepsilon,\upsilon}t^{-\varsigma}S_n^m(t)E_{\theta,\kappa}^{\chi,c}(1/t;p)\bigr)(x)\\&=\frac{x^{-\varsigma-\varepsilon}}{\Gamma(\chi)}\sum_{s=0}^{[n/m]}\frac{(-n)_{m.s}}{s!}A_{n,s}x^s\\&\quad\times{}_4\psi_4\left[\begin{matrix}(c,1),(\varepsilon+\varsigma-s,1),(\upsilon+\varsigma-s,1),(\chi,1);\\(c,1),(\kappa,\theta),(\varsigma-s,1),(\tau+\varepsilon+\upsilon-s+\varsigma,1);\end{matrix}\quad(1/x;p)\right].\end{aligned}\tag{33}$$

Corollary *Let $\tau,\varepsilon,c,\theta,\kappa,\chi,\upsilon,\varsigma\in\mathbb{C}$, $x\in\mathbb{R}^+$ with $\Re(\upsilon)>0$ and $\Re(\varsigma)>\Re(-\upsilon)$ and $p\geq 0$, then*

$$\begin{aligned}&\bigl(K_{\upsilon,\tau}^{-}t^{-\varsigma}S_n^m(t)E_{\theta,\kappa}^{\chi,c}(1/t;p)\bigr)(x)\\&=\frac{x^{-\varsigma}}{\Gamma(\chi)}\sum_{s=0}^{[n/m]}\frac{(-n)_{m.s}}{s!}A_{n,s}x^s\\&\quad\times{}_3\psi_3\left[\begin{matrix}(c,1),(\upsilon+\varsigma-s,1),(\chi,1);\\(c,1),(\kappa,\theta),(\tau+\upsilon-s+\varsigma,1);\end{matrix}\quad(1/x;p)\right].\end{aligned}\tag{34}$$

Theorem 2.3 *Let $\tau,\tau',\varepsilon,\varepsilon',c,\theta,\kappa,\chi,\upsilon,\varsigma\in\mathbb{C}$ and $\Re(\varsigma)>\max\{0,\Re(-\tau+\varepsilon),\Re(-\tau-\tau'-\varepsilon'+\upsilon)\}$ and $p\geq 0$. Also let $x\in\mathbb{R}^+$, then*

$$\begin{aligned}&\bigl(D_{0+}^{\tau,\tau',\varepsilon,\varepsilon',\upsilon}t^{\varsigma-1}S_n^m(t)E_{\theta,\kappa}^{\chi,c}(t;p)\bigr)(x)\\&=\frac{x^{\varsigma+\tau+\tau'-\upsilon-1}}{\Gamma(\chi)}\sum_{s=0}^{[n/m]}\frac{(-n)_{m.s}}{s!}A_{n,s}x^s\\&\quad\times{}_5\psi_5\left[\begin{matrix}(c,1),(\varsigma+s,1),(\varsigma+s+\tau-\varepsilon,1),(\tau+\tau'+\varepsilon'-\upsilon+\varsigma+s,1),(\chi,1);\\(c,1),(\kappa,\theta),(\varsigma+s-\varepsilon,1),(\tau+\tau'-\upsilon+\varsigma+s,1),(\tau+\varsigma+s-\upsilon+\varepsilon',1);\end{matrix}\quad(x;p)\right].\end{aligned}\tag{35}$$

Proof Let I_3 be LHS of (35), then using (5), we have

$$\begin{aligned}I_3&=\bigl(D_{0+}^{\tau,\tau',\varepsilon,\varepsilon',\upsilon}t^{\varsigma-1}S_n^m(t)E_{\theta,\kappa}^{\chi,c}(t;p)\bigr)(x)\\&=\left(D_{0+}^{\tau,\tau',\varepsilon,\varepsilon',\upsilon}t^{\varsigma-1}\sum_{s=0}^{[n/m]}\frac{(-n)_{m.s}}{s!}A_{n,s}(t)^s\sum_{n=0}^{\infty}\frac{\mathcal{B}_p(\chi+n,c-\chi)}{\mathbf{B}(\chi,c-\chi)}\frac{(c)_n}{\Gamma(\theta n+\kappa)}\frac{t^n}{n!}\right)(x).\end{aligned}$$

Employing the exchange in the orders of integration and summation, we obtain

$$I_3=\sum_{s=0}^{[n/m]}\frac{(-n)_{m.s}}{s!}A_{n,s}\sum_{n=0}^{\infty}\frac{\mathcal{B}_p(\chi+n,c-\chi)}{\mathbf{B}(\chi,c-\chi)}\frac{(c)_n}{\Gamma(\theta n+\kappa)n!}\bigl(D_{0+}^{\tau,\tau',\varepsilon,\varepsilon',\upsilon}t^{\varsigma+s+n-1}\bigr)(x).$$

Applying Lemma (1.3), we get

$$I_3=\sum_{s=0}^{[n/m]}\frac{(-n)_{m.s}}{s!}A_{n,s}\sum_{n=0}^{\infty}\frac{\mathcal{B}_p(\chi+n,c-\chi)}{\mathbf{B}(\chi,c-\chi)}\frac{(c)_n}{\Gamma(\theta n+\kappa)n!}$$

$$\begin{aligned}
&\times \frac{\Gamma(\varsigma+s+n)\Gamma(\varsigma+s+\tau-\varepsilon+n)\Gamma(\varsigma+s+\tau+\tau'+\varepsilon'-\upsilon+n)}{\Gamma(\varsigma+s-\varepsilon+n)\Gamma(\varsigma+s+\tau+\tau'-\upsilon+n)\Gamma(\varsigma+s+\tau+\varepsilon'-\upsilon+n)} \\
&\times x^{\varsigma+s+n-\upsilon+\tau+\tau'-1} \\
={}& \frac{x^{\varsigma-\upsilon+\tau+\tau'-1}}{\Gamma(\chi)} \sum_{s=0}^{[n/m]} \frac{(-n)_{m.s}}{s!} A_{n,s} x^s \sum_{0}^{\infty} \frac{\mathcal{B}_p(\chi+n,c-\chi)}{\Gamma(c-\chi)} \frac{\Gamma(c+n)}{\Gamma(\theta n+\kappa)} \\
&\times \frac{\Gamma(\varsigma+s+n)\Gamma(\varsigma+s+\tau-\varepsilon+n)\Gamma(\varsigma+s+\tau+\tau'+\varepsilon'-\upsilon+n)}{\Gamma(\varsigma+s-\varepsilon+n)\Gamma(\varsigma+s+\tau+\tau'-\upsilon+n)\Gamma(\varsigma+s+\tau+\varepsilon'-\upsilon+n)} \frac{x^n}{n!}.
\end{aligned}$$

Finally, on exercising (8) therein, we easily figure out the desired result. □

Theorem 2.4 *Let* $\tau,\tau',\varepsilon,\varepsilon',c,\theta,\kappa,\chi,\upsilon,\varsigma\in\mathbb{C}$ *and*

$$\Re(\varsigma) > \max\{\Re(-\varepsilon'), \Re(\tau'+\varepsilon-\upsilon), \Re(\tau+\tau'-\upsilon)+[\Re(\upsilon)]+1\}$$

and $p\geq 0$. *Also let* $x\in\mathbb{R}^+$, *then*

$$\begin{aligned}
&\left(D_{-}^{\tau,\tau',\varepsilon,\varepsilon',\upsilon} t^{-\varsigma} S_n^m(t) E_{\theta,\kappa}^{\chi,c}(1/t;p)\right)(x) \\
&= \frac{x^{\tau+\tau'-\upsilon-\varsigma}}{\Gamma(\chi)} \sum_{s=0}^{[n/m]} \frac{(-n)_{m.s}}{s!} A_{n,s} x^s \\
&\quad \times {}_5\psi_5\left[\begin{matrix}(c,1),(\varepsilon'+\varsigma-s,1),(\varsigma-s-\tau-\tau'+\upsilon,1),(\varsigma-s+\varepsilon-\tau',1),(\chi,1); \\ (c,1),(\kappa,\theta),(\varsigma-s,1),(\varsigma-s-\tau'+\varepsilon',1),(\varsigma-s-\tau-\tau'+\upsilon-\varepsilon',1);\end{matrix}\ (1/x;p)\right].
\end{aligned} \tag{36}$$

Proof Let I_4 be LHS of (36), then using (5), we have

$$\begin{aligned}
I_4 &= \left(D_{-}^{\tau,\tau',\varepsilon,\varepsilon',\upsilon} t^{-\varsigma} S_n^m(t) E_{\theta,\kappa}^{\chi,c}(1/t;p)\right)(x) \\
&= \left(D_{-}^{\tau,\tau',\varepsilon,\varepsilon',\upsilon} t^{-\varsigma} \sum_{s=0}^{[n/m]} \frac{(-n)_{m.s}}{s!} A_{n,s}(t)^s \sum_{n=0}^{\infty} \frac{\mathcal{B}_p(\chi+n,c-\chi)}{\mathbf{B}(\chi,c-\chi)} \frac{(c)_n}{\Gamma(\theta n+\kappa)} \frac{t^{-n}}{n!}\right)(x).
\end{aligned}$$

Interchanging the order of integration and summation, under the valid condition, we obtain

$$I_4 = \sum_{s=0}^{[n/m]} \frac{(-n)_{m.s}}{s!} A_{n,s} \sum_{n=0}^{\infty} \frac{\mathcal{B}_p(\chi+n,c-\chi)}{\mathbf{B}(\chi,c-\chi)} \frac{(c)_n}{\Gamma(\theta n+\kappa)n!} \left(D_{-}^{\tau,\tau',\varepsilon,\varepsilon',\upsilon} t^{-\varsigma+s-n}\right)(x).$$

Applying Lemma (1.4), we get

$$\begin{aligned}
I_4 &= \sum_{s=0}^{[n/m]} \frac{(-n)_{m.s}}{s!} A_{n,s} \sum_{n=0}^{\infty} \frac{\mathcal{B}_p(\chi+n,c-\chi)}{\mathbf{B}(\chi,c-\chi)} \frac{(c)_n}{\Gamma(\theta n+\kappa)n!} \\
&\quad \times \frac{\Gamma(\varepsilon'+\varsigma-s+n)\Gamma(\varsigma-s-\tau-\tau'+\upsilon+n)\Gamma(\varsigma-s-\tau'+\varepsilon+\upsilon+n)}{\Gamma(\varsigma-s+n)\Gamma(\varsigma-s-\tau'+\varepsilon'+n)\Gamma(\varsigma-s-\tau-\tau'-\varepsilon'+\upsilon+n)} \\
&\quad \times x^{\tau+\tau'-\varsigma-\upsilon+s-n}
\end{aligned}$$

$$= \frac{x^{\tau+\tau'-\varsigma-\upsilon}}{\Gamma(\chi)} \sum_{s=0}^{[n/m]} \frac{(-n)_{m.s}}{s!} A_{n,s} x^s \sum_{n=0}^{\infty} \frac{\mathcal{B}_p(\chi+n, c-\chi)}{\Gamma(c-\chi)} \frac{\Gamma(c+n)}{\Gamma(\theta n+\kappa)}$$
$$\times \frac{\Gamma(\varepsilon'+\varsigma-s+n)\Gamma(\varsigma-s-\tau-\tau'+\upsilon+n)\Gamma(\varsigma-s-\tau'+\varepsilon+\upsilon+n)}{\Gamma(\varsigma-s+n)\Gamma(\varsigma-s-\tau'+\varepsilon'+n)\Gamma(\varsigma-s-\tau-\tau'-\varepsilon'+\upsilon+n)} \frac{x^{-n}}{n!}.$$

Again, by using (8), we get the desired result. □

3 Caputo-type fractional differentiation of extended Mittag-Leffler function

The left- and right-sided Caputo-type fractional differential operators with Gauss hypergeometric function in the kernel are defined as follows:

$$\left({}^{c}D_{0+}^{\tau,\varepsilon,\upsilon} f\right)(x) = \left(I_{0+}^{-\tau+\Lambda,-\varepsilon,-\upsilon+\Lambda} f^{(\Lambda)}\right)(x) \tag{37}$$

and

$$\left({}^{c}D_{-}^{\tau,\varepsilon,\upsilon} f\right)(x) = (-1)^{\Lambda}\left(I_{-}^{-\tau+\Lambda,-\varepsilon+\Lambda,\tau+\upsilon} f^{(\Lambda)}\right)(x), \tag{38}$$

where $\tau,\varepsilon,\upsilon,\varsigma \in \mathbb{C}$, $\Lambda = [\Re(\tau)] + 1$, and $x \in \mathbb{R}^+$.

The left- and right-sided Caputo-type *generalized* fractional differential operators are given by

$$\left({}^{c}D_{0+}^{\tau,\tau',\varepsilon,\varepsilon',\upsilon} f\right)(x) = (-1)^{\Lambda'}\left(I_{0+}^{-\tau,-\tau',-\varepsilon'+\Lambda',-\varepsilon,-\upsilon+\Lambda',-\upsilon+\Lambda'} f^{(\Lambda')}\right)(x) \tag{39}$$

and

$$\left({}^{c}D_{-}^{\tau,\tau',\varepsilon,\varepsilon',\upsilon} f\right)(x) = (-1)^{\Lambda'}\left(I_{-}^{-\tau,-\tau',-\varepsilon,-\varepsilon'+\Lambda',-\upsilon+\Lambda',-\upsilon+\Lambda'} f^{(\Lambda')}\right)(x), \tag{40}$$

where $\tau,\tau',\varepsilon,\varepsilon',\upsilon,\varsigma \in \mathbb{C}$, $\Lambda' = [\Re(\upsilon)] + 1$, and $x \in \mathbb{R}^+$.

The following lemmas are required to demonstrate the intended outcome.

Lemma 3.1 ([15]) *Let* $\tau,\tau',\varepsilon,\varepsilon',\upsilon,\varsigma \in \mathbb{C}$ *and* $\Lambda' = [\Re(\upsilon)] + 1$ *with* $\Re(\varsigma) - \Lambda' > \max\{0, \Re(-\tau+\varepsilon), \Re(-\tau-\tau'-\varepsilon'+\upsilon)\}$ *and* $p \geq 0$. *Then*

$$\left({}^{c}D_{0+}^{\tau,\tau',\varepsilon,\varepsilon',\upsilon} t^{\varsigma-1}\right)(x)$$
$$= \frac{\Gamma(\varsigma)\Gamma(\varsigma+\tau-\varepsilon-\Lambda')\Gamma(\varsigma+\tau+\tau'+\varepsilon'-\upsilon-\Lambda')}{\Gamma(\varsigma-\varepsilon-\Lambda')\Gamma(\varsigma+\tau+\tau'-\upsilon)\Gamma(\varsigma+\tau+\varepsilon'-\upsilon-\Lambda')} x^{\varsigma-\upsilon+\tau+\tau'-1}.$$

Lemma 3.2 [15] *Let* $\tau,\tau',\varepsilon,\varepsilon',\upsilon,\varsigma \in \mathbb{C}$ *and* $\Lambda' = [\Re(\upsilon)] + 1$ *with* $\Re(\varsigma) + \Lambda' > \max\{\Re(-\varepsilon'), \Re(\tau'+\varepsilon-\upsilon), \Re(\tau+\tau'-\upsilon) + [\Re(\upsilon)] + 1\}$. *Then*

$$\left({}^{c}D_{-}^{\tau,\tau',\varepsilon,\varepsilon',\upsilon} t^{-\varsigma}\right)(x) = \frac{\Gamma(\varsigma+\varepsilon'+\Lambda')\Gamma(\varsigma-\tau-\tau'+\upsilon)\Gamma(\varsigma-\tau'-\varepsilon+\upsilon+\Lambda')}{\Gamma(\varsigma)\Gamma(\varsigma-\tau'+\varepsilon'+\Lambda')\Gamma(\varsigma-\tau-\tau'-\varepsilon+\upsilon+\Lambda')} x^{\tau+\tau'-\upsilon-\varsigma}.$$

Theorem 3.3 *Let $\tau,\tau',\varepsilon,\varepsilon',c,\theta,\kappa,\chi,\upsilon,\varsigma\in\mathbb{C}$, $x\in\mathbb{R}^+$, and $\Lambda'=[\Re(\upsilon)+1]$ $\Re(\varsigma)-\Lambda'>\max\{0,\Re(-\tau+\varepsilon'),\Re(-\tau-\tau'-\varepsilon'+\upsilon)\}$ and $p\geq 0$, then*

$$\begin{aligned}
&\left({}^{c}D_{0+}^{\tau,\tau',\varepsilon,\varepsilon',\upsilon}t^{\varsigma-1}S_n^m(t)E_{\theta,\kappa}^{\chi,c}(t;p)\right)(x)\\
&=\frac{x^{\varsigma+\tau+\tau'-\upsilon+1}}{\Gamma(\chi)}\sum_{s=0}^{[n/m]}\frac{(-n)_{m.s}}{s!}A_{n,s}x^s\\
&\quad\times{}_5\psi_5\left[\begin{matrix}(c,1),(\varsigma,1),(\varsigma+s+\tau-\varepsilon-\Lambda',1),(\tau+\tau'+\varepsilon'-\upsilon-\Lambda'+\varsigma+s,1),(\chi,1);\\(c,1),(\kappa,\theta),(\varsigma+s-\varepsilon-\Lambda',1),(\tau+\tau'-\upsilon+\varsigma+s,1),(\tau+\varsigma+s-\upsilon+\varepsilon'-\Lambda',1);\end{matrix}\ (x;p)\right].
\end{aligned}\tag{41}$$

Proof Let I_5 be LHS of (41), then using (5), we have

$$\begin{aligned}
I_5&=\left({}^{c}D_{0+}^{\tau,\tau',\varepsilon,\varepsilon',\upsilon}t^{\varsigma-1}S_n^m(t)E_{\theta,\kappa}^{\chi,c}(t;p)\right)(x)\\
&=\left({}^{c}D_{0+}^{\tau,\tau',\varepsilon,\varepsilon',\upsilon}t^{\varsigma-1}\sum_{s=0}^{[n/m]}\frac{(-n)_{m.s}}{s!}A_{n,s}(t)^s\sum_{n=0}^{\infty}\frac{\mathbf{B}_p(\chi+n,c-\chi)}{\mathbf{B}(\chi,c-\chi)}\frac{(c)_n}{\Gamma(\theta n+\kappa)}\frac{t^n}{n!}\right)(x).
\end{aligned}$$

Interchange in the order of integration and summation under the verified condition in this theorem allows us to write

$$I_5=\sum_{s=0}^{[n/m]}\frac{(-n)_{m.s}}{s!}A_{n,s}\sum_{n=0}^{\infty}\frac{\mathbf{B}_p(\chi+n,c-\chi)}{\mathbf{B}(\chi,c-\chi)}\frac{(c)_n}{\Gamma(\theta n+\kappa)n!}\left({}^{c}D_{0+}^{\tau,\tau',\varepsilon,\varepsilon',\upsilon}t^{\varsigma+s+n-1}\right)(x).$$

Application of Lemma (3.1) leads to

$$\begin{aligned}
I_5&=\sum_{s=0}^{[n/m]}\frac{(-n)_{m.s}}{s!}A_{n,s}\sum_{n=0}^{\infty}\frac{\mathbf{B}_p(\chi+n,c-\chi)}{\mathbf{B}(\chi,c-\chi)}\frac{(c)_n}{\Gamma(\theta n+\kappa)n!}\\
&\quad\times\frac{\Gamma(\varsigma+s+n)\Gamma(\varsigma+s+\tau-\varepsilon-\Lambda'+n)\Gamma(\varsigma+s+\tau+\tau'+\varepsilon'-\upsilon-\Lambda'+n)}{\Gamma(\varsigma+s-\varepsilon-\Lambda'+n)\Gamma(\varsigma+s+\tau+\tau'-\upsilon+n)\Gamma(\varsigma+s+\tau+\varepsilon'-\upsilon-\Lambda'+n)}\\
&\quad\times x^{\varsigma+s+n-\upsilon+\tau+\tau'+1}\\
&=\frac{x^{\varsigma-\upsilon+\tau+\tau'+1}}{\Gamma(\chi)}\sum_{s=0}^{[n/m]}\frac{(-n)_{m.s}}{s!}A_{n,s}x^s\sum_{n=0}^{\infty}\frac{\mathbf{B}_p(\chi+n,c-\chi)}{\Gamma(c-\chi)}\frac{\Gamma(c+n)}{\Gamma(\theta n+\kappa)}\\
&\quad\times\frac{\Gamma(\varsigma+s+n)\Gamma(\varsigma+s+\tau-\varepsilon-\Lambda'+n)\Gamma(\varsigma+s+\tau+\tau'+\varepsilon'-\upsilon-\Lambda'+n)}{\Gamma(\varsigma+s-\varepsilon-\Lambda'+n)\Gamma(\varsigma+s+\tau+\tau'-\upsilon+n)\Gamma(\varsigma+s+\tau+\varepsilon'-\upsilon-\Lambda'+n)}\frac{x^n}{n!}.
\end{aligned}$$

In view of (8), we obtain the intended result. □

Theorem 3.4 *Let $\tau,\tau',\varepsilon,\varepsilon',c,\theta,\kappa,\chi,\upsilon,\varsigma\in\mathbb{C}$ and $\Lambda'=[\Re(\upsilon)+1]$ with*

$$\Re(\varsigma)+\Lambda'>\max\{\Re(-\varepsilon'),\Re(\tau+\tau'-\upsilon)+\Lambda'\}$$

and $p \geq 0$. *Also let* $x \in \mathbb{R}^+$, *then*

$$\begin{aligned}&\left({}^cD_-^{\tau,\tau',\varepsilon,\varepsilon',\upsilon} t^{-\varsigma} S_n^m(t) E_{\theta,\kappa}^{\chi,c}(1/t;p)\right)(x)\\&=\frac{x^{\varsigma+\tau+\tau'-\upsilon}}{\Gamma(\chi)}\sum_{s=0}^{[n/m]}\frac{(-n)_{m.s}}{s!}A_{n,s}x^s\\&\quad\times{}_5\psi_5\left[\begin{matrix}(c,1),(\varepsilon'+\varsigma+s+m,1),(-\tau-\tau'+\upsilon+\varsigma+s,1),(\varsigma+s-\tau'-\varepsilon+\upsilon+m,1),(\chi,1);\\(c,1),(\kappa,\theta),(\varsigma+s,1),(-\tau'+\varepsilon'+m+\varsigma+s,1),(-\tau-\tau'-\varepsilon+\varsigma+s+\upsilon+m,1);\end{matrix}\ (1/x;p)\right].\end{aligned}\tag{42}$$

Proof Let I_6 be LHS of (42), then using (5), we have

$$\begin{aligned}I_6&=\left({}^cD_-^{\tau,\tau',\varepsilon,\varepsilon',\upsilon} t^{-\varsigma} S_n^m(t) E_{\theta,\kappa}^{\chi,c}(1/t;p)\right)(x)\\&=\left({}^cD_-^{\tau,\tau',\varepsilon,\varepsilon',\upsilon} t^{-\varsigma}\sum_{s=0}^{[n/m]}\frac{(-n)_{m.s}}{s!}A_{n,s}(t)^s\sum_{n=0}^{\infty}\frac{\mathbf{B}_p(\chi+n,c-\chi)}{\mathbf{B}(\chi,c-\chi)}\frac{(c)_n}{\Gamma(\theta n+\kappa)}\frac{t^{-n}}{n!}\right)(x).\end{aligned}$$

Interchanging the order of integration and summation under the verified condition in this theorem, we obtain

$$I_6=\sum_{s=0}^{[n/m]}\frac{(-n)_{m.s}}{s!}A_{n,s}\sum_{n=0}^{\infty}\frac{\mathbf{B}_p(\chi+n,c-\chi)}{\mathbf{B}(\chi,c-\chi)}\frac{(c)_n}{\Gamma(\theta n+\kappa)n!}\left({}^cD_-^{\tau,\tau',\varepsilon,\varepsilon',\upsilon} t^{s-n-\varsigma}\right)(x).$$

Applying Lemma (3.2), we get

$$\begin{aligned}I_6&=\sum_{s=0}^{[n/m]}\frac{(-n)_{m.s}}{s!}A_{n,s}\sum_{n=0}^{\infty}\frac{\mathbf{B}_p(\chi+n,c-\chi)}{\mathbf{B}(\chi,c-\chi)}\frac{(c)_n}{\Gamma(\theta n+\kappa)n!}\\&\quad\times\frac{\Gamma(\varsigma-s+\varepsilon'+\Lambda'+n)\Gamma(\varsigma-s-\tau-\tau'+\upsilon+n)\Gamma(\varsigma-s-\tau'-\varepsilon+\upsilon+\Lambda'+n)}{\Gamma(\varsigma-s+n)\Gamma(\varsigma-s-\tau'+\varepsilon'+\Lambda'+n)\Gamma(\varsigma-s-\tau-\tau'-\varepsilon+\upsilon+\Lambda'+n)}\\&\quad\times x^{-\varsigma+s-n-\upsilon+\tau+\tau'}\\&=\frac{x^{-\varsigma-\upsilon+\tau+\tau'}}{\Gamma(\chi)}\sum_{s=0}^{[n/m]}\frac{(-n)_{m.s}}{s!}A_{n,s}x^s\sum_{n=0}^{\infty}\frac{\mathbf{B}_p(\chi+n,c-\chi)}{\Gamma(c-\gamma)}\frac{\Gamma(c+n)}{\Gamma(\theta n+\kappa)}\\&\quad\times\frac{\Gamma(\varsigma-s+\varepsilon'+\Lambda'+n)\Gamma(\varsigma-s-\tau-\tau'+\upsilon+n)\Gamma(\varsigma-s-\tau'-\varepsilon+\upsilon+\Lambda'+n)}{\Gamma(\varsigma-s+n)\Gamma(\varsigma-s-\tau'+\varepsilon'+\Lambda'+n)\Gamma(\varsigma-s-\tau-\tau'-\varepsilon+\upsilon+\Lambda'+n)}\\&\quad\times\frac{x^{-n}}{n!}.\end{aligned}$$

By using (8), we get the required result. □

4 Concluding remarks and discussion

In the current work, we have established the fractional calculus operators with Appell function kernels and Caputo-type fractional differential operators for the function involving the product of Srivastava's polynomials and extended Mittag-Leffler function. Various special cases of the derived results in the paper can be evaluated by taking suitable values of parameters involved. Further, one can obtain a number of image formulas involving classical orthogonal polynomials as particular cases of our results, by giving special values to the coefficient $A_{n,l}$, in the family of polynomials, which includes the polynomials viz. Laguerre, Hermite, Jacobi, Konhauser polynomials, and many others.

For example, if we put $n = 0$, then we see that the general class of polynomials $S_n^m(x)$ reduces to unity, i.e., $S_0^m(x) \to 1$, and we immediately obtain the result due to Araci et al. [2].

Also, if we set $m = 2$ and $A_{n,l} = (-1)^l$, then the general class of polynomials stated in equation (9) reduces to

$$S_n^2(x) \to x^{\frac{n}{2}} H_n\left(\frac{1}{2\sqrt{x}}\right), \tag{43}$$

where $H_n(x)$ denotes the familiar Hermite polynomials defined as

$$H_n(x) = \sum_{l=0}^{\lfloor n/m \rfloor} (-1)^l \frac{n!}{l!(n-2l)!} (2x)^{n-2l}. \tag{44}$$

Eventually, it is easy to discover a comprehensive representation of more generalized special functions that are widely used in applied sciences.

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