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A filter method for inverse nonlinear sideways heat equation

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Abstract

In this paper, we study a sideways heat equation with a nonlinear source in a bounded domain, in which the Cauchy data at $x = \mathcal{X}$ are given and the solution in $0 \leq x < \mathcal{X}$ is sought. The problem is severely ill-posed in the sense of Hadamard. Based on the fundamental solution to the sideways heat equation, we propose to solve this problem by the filter method of degree α , which generates a well-posed integral equation. Moreover, we show that its solution converges to the exact solution uniformly and strongly in $\mathcal{L}^p(\omega, \mathcal{X}; \mathcal{L}^2(\mathbb{R}))$, $\omega \in [0, \mathcal{X})$ under a priori assumptions on the exact solution. The proposed regularized method is illustrated by numerical results in the final section.

MSC: 35K05; 35K99; 47J06; 47H10

Keywords: Backward problem; Nonlinear heat equation; Ill-posed problem; Cauchy problem; Regularization method; Error estimate

1 Introduction

In this paper, we determine the surface temperature $u(x, t)$ for $0 \leq x < \mathcal{X}$ from the known temperature measurements $u(\mathcal{X}, t) = \phi(t)$ and heat-flux measurement $\frac{\partial u}{\partial x}(\mathcal{X}, t) = \psi(t)$ when $u(x, t)$ satisfies the following system:

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(u)G(x, t; u), & (x, t) \in (0, \mathcal{X}) \times \mathbb{R}, \\ u(x, t)|_{t \rightarrow \pm\infty} = 0, & (x, t) \in (0, \mathcal{X}) \times \mathbb{R}, \\ u(\mathcal{X}, t) = \phi(t), & t \in \mathbb{R}, \\ \frac{\partial u}{\partial x}(\mathcal{X}, t) = \psi(t), & t \in \mathbb{R}, \end{cases} \quad (1.1)$$

where $\phi, \psi \in \mathcal{L}^2(\mathbb{R})$ are given functions. The source terms $f(u), G(u)$ are globally Lipschitz functions satisfying (2.15a) and (2.15b), respectively.

The problem called the *inverse nonlinear sideways heat equation* (INSHE for short) is a model of a problem where one wants to determine the temperature on both sides of a thick wall, but where one side is inaccessible to measurements. In many dynamic heat transfer situations, one wishes to determine the temperature on the surface of a body, where the surface itself is inaccessible for measurements. The physical situation at the surface may

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be unsuitable for attaching a sensor, or the accuracy of a surface measurement may be seriously impaired by the presence of the sensor. Typical practical applications are the estimation of the heat flux and the temperature at the surface of the body under investigation, e.g., re-entry vehicles, calorimeter-type instrumentation, and combustion chambers [1, 2, 5, 11, 13, 15, 16, 19, 20]. In such cases, one is restricted to interior measurements, and from these one wishes to compute the surface temperature.

Cannon (1984) [4] considered the *direct problem* for the homogeneous heat equation in the quarter plane ($x \geq 0, t \geq 0$):

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, & x \geq 0, t \geq 0, \\ u(x, 0) = 0, & x \geq 0, \\ u(0, t) = \phi(t), & t \geq 0. \end{cases} \tag{1.2}$$

The functions $\phi(\cdot)$ and $u(x, \cdot)$ are to be in $\mathcal{L}^2(\mathbb{R})$ (ϕ and u vanish for $t < 0$). The author proved that, for each $\phi \in \mathcal{L}^2(\mathbb{R})$, (1.2) has a unique solution u with $u(x, \cdot) \in \mathcal{L}^2(\mathbb{R})$ for each $x \geq 0$.

Fredrik Berntsson (1999) [3] considered the sideways heat equation

$$\begin{cases} \frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}, & 0 < x < 1, t \geq 0, \\ u(x, 0) = 0, & 0 < x < 1, \\ u(1, t) = g(t), & t \geq 0 \\ \frac{\partial u}{\partial x}(1, t) = h(t), & t \geq 0. \end{cases} \tag{1.3}$$

The author used the spectral method to solve problem (1.3). Error estimates for the regularized solution were derived, and a procedure for selecting an appropriate regularization parameter was given.

In recent years, linear homogeneous problem (1.1), i.e., $f(u)G(x, t; u) = 0$, has been researched by many authors, and various numerical methods have been proposed, e.g., the *boundary element Tikhonov regularization method* (Lesnic et al. (1996) [9]), the *conjugate gradient method* (Hao (2012) [8]), the *difference regularization method* (Xiong et al. (2006a) [17]), the *“optimal filtering” method* (Seidman & Elden (1990) [14]), the *Fourier method* (Xiong et al. 2006b [18]), the *quasi-reversibility method* (Elden (1987) [6], Liu & Wei (2013) [10]), the *wavelet, wavelet-Galerkin*, and the *spectral regularization methods* (Elden et al. (2000) [7], Reginska & Elden (1997) [12]), to mention only a few.

The more important but challenging semilinear sideways heat equation with the heat source depends nonlinearly on the temperature, which occurs in many applications related to reaction-diffusion. The function $f(u)G(u)$ is known as a special type of locally Lipschitz function. For example, if we choose $f(u) := u, G(x, t; u) := \sin u$ (individually they are globally Lipschitz), then

$$\begin{aligned} |f(u)G(x, t; u) - f(v)G(x, t; v)| &= |u \sin u - v \sin v| \\ &\leq |u - v| \sin u + v |\sin u - \sin v| \\ &= |u - v| \sin u + 2v \cos \frac{u + v}{2} \sin \frac{|u - v|}{2} \\ &\leq |u - v| |u + v| \leq \max\{u, v\} |u - v|. \end{aligned} \tag{1.4}$$

Although there are some works on the nonlinear case, the literature on the case of locally Lipschitz sources $f(u)G(x, t; u)$ is quite scarce. Our results extend problem (1.3), and we propose a new filter method to establish regularized solutions of problem (1.1) in the case of the locally Lipschitz function $f(u)G(x, t; u)$.

The paper is organized as follows. In Sect. 2, the formulation of problem and regularization methods is given. In Sect. 3, a stability estimate in $\mathcal{L}^p(\omega, \mathcal{X}; \mathcal{L}^2(\mathbb{R}))$, $\omega \in [0, \mathcal{X})$ is proved under a priori condition of the exact solution and the locally Lipschitz source term. Finally, we present a numerical result to illustrate the proposed regularized method in Sect. 4.

2 Mathematical problem and mild solution of (INSHE)

For $w \in \mathcal{L}^2(\mathbb{R})$, we have the Fourier transform

$$\widehat{w}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} w(t)e^{-i\xi t} dt, \quad \xi \in \mathbb{R},$$

and the \mathcal{L}^2 norm of w is

$$\|w\|_{\mathcal{L}^2(\mathbb{R})}^2 = \|\widehat{w}\|_{\mathcal{L}^2(\mathbb{R})}^2 = \int_{-\infty}^{\infty} |\widehat{w}(\xi)|^2 d\xi. \tag{2.1}$$

Suppose that the solution of problem (1.1) is represented as a Fourier transform

$$u(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{u}(x, \xi)e^{i\xi t} d\xi \tag{2.2}$$

with

$$\widehat{u}(x, \xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x, t)e^{-i\xi t} dt. \tag{2.3}$$

Throughout this paper, we let $\mathcal{W}(x, t; u) = f(u)G(x, t; u)$, $\forall (x, t) \in (0, \mathcal{X}) \times \mathbb{R}$.

From (1.1), we have the following systems of second order ordinary equation:

$$\begin{cases} -\partial_{xx}\widehat{u}(x, \xi) + i\xi\widehat{u}(x, \xi) = \widehat{\mathcal{W}}(u)(x, \xi), \\ \widehat{u}(\mathcal{X}, \xi) = \widehat{\phi}(\xi), \\ \partial_x\widehat{u}(\mathcal{X}, \xi) = \widehat{\psi}(\xi). \end{cases} \tag{2.4}$$

We thus have after some direct calculation

$$\begin{aligned} \widehat{u}(x, \xi) &= \cosh((\mathcal{X} - x)\sqrt{i\xi})\widehat{u}(\mathcal{X}, \xi) - \frac{\sinh((\mathcal{X} - x)\sqrt{i\xi})}{\sqrt{i\xi}}\partial_x\widehat{u}(\mathcal{X}, \xi) \\ &\quad - \int_x^{\mathcal{X}} \frac{\sinh((z - x)\sqrt{i\xi})}{\sqrt{i\xi}}\widehat{\mathcal{W}}(u)(z, \xi) dz, \end{aligned} \tag{2.5}$$

where

$$\sqrt{i\xi} = \sqrt{\frac{|\xi|}{2}}(1 + \sigma i), \quad \sigma = \text{sign}(\xi), \xi \in \mathbb{R}.$$

Remark 2.1 In (2.5), for $\xi = 0$, $\frac{\sinh(y\sqrt{i\xi})}{\sqrt{i\xi}}$ is defined as y , since

$$\lim_{\xi \rightarrow 0} \frac{\sinh(y\sqrt{i\xi})}{\sqrt{i\xi}} = y.$$

Moreover, for $\xi = 0$, we have

$$\widehat{u}(x, 0) = \widehat{u}(\mathcal{X}, 0) - (\mathcal{X} - x)\widehat{\partial_x u}(\mathcal{X}, 0) - \int_x^{\mathcal{X}} (z - x)\widehat{\mathcal{W}}(u)(z, 0) dz.$$

From (2.5), the exact form of u is given by

$$\begin{aligned} u(x, t) = & \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\cosh((\mathcal{X} - x)\sqrt{i\xi})\widehat{\phi}(\xi))e^{i\xi t} d\xi \\ & - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{\sinh((\mathcal{X} - x)\sqrt{i\xi})}{\sqrt{i\xi}} \widehat{\psi}(\xi) \right) e^{i\xi t} d\xi \\ & - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\int_x^{\mathcal{X}} \frac{\sinh((z - x)\sqrt{i\xi})}{\sqrt{i\xi}} \widehat{\mathcal{W}}(u)(z, \xi) dz \right) e^{i\xi t} d\xi. \end{aligned} \tag{2.6}$$

We say that u is a mild solution of problem (1.1) if u satisfies integral (2.6). We know that the three functions

$$\cosh((\mathcal{X} - x)\sqrt{i\xi}), \quad \frac{\sinh((\mathcal{X} - x)\sqrt{i\xi})}{\sqrt{i\xi}}, \quad \frac{\sinh((z - x)\sqrt{i\xi})}{\sqrt{i\xi}} \tag{2.7}$$

are unbounded as a function of the variable ξ . Consequently, small errors in high frequency components can blow up and completely destroy the solution for $0 < x < z < \mathcal{X}$. A natural idea to stabilize the problem is to replace them by a bounded approximation. In a natural way, we can replace the terms in (2.7) by

$$\cosh^{\gamma(\delta)}((\mathcal{X} - x)\sqrt{i\xi}), \quad \frac{\sinh^{\gamma(\delta)}((\mathcal{X} - x)\sqrt{i\xi})}{\sqrt{i\xi}}, \quad \frac{\sinh^{\gamma(\delta)}((z - x)\sqrt{i\xi})}{\sqrt{i\xi}}$$

(respectively), with $\delta > 0$ is a small positive number representing the level of noise and the parameter $\gamma(\delta) > 0$ is small (regularization parameter). We introduce the first regularized solution $U_{\gamma(\delta)}^\delta$ obtained by

$$\begin{aligned} U_{\gamma(\delta)}^\delta(x, t) = & \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\cosh^{\gamma(\delta)}((\mathcal{X} - x)\sqrt{i\xi})\widehat{\phi}^\delta(\xi))e^{i\xi t} d\xi \\ & - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{\sinh^{\gamma(\delta)}((\mathcal{X} - x)\sqrt{i\xi})}{\sqrt{i\xi}} \widehat{\psi}^\delta(\xi) \right) e^{i\xi t} d\xi \\ & - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\int_x^{\mathcal{X}} \frac{\sinh^{\gamma(\delta)}((z - x)\sqrt{i\xi})}{\sqrt{i\xi}} \widehat{\mathcal{W}}(U_{\gamma(\delta)}^\delta)(z, \xi) dz \right) e^{i\xi t} d\xi. \end{aligned} \tag{2.8}$$

Here $\cosh^{\gamma(\delta)}(y\sqrt{i\xi})$, $\sinh^{\gamma(\delta)}(y\sqrt{i\xi})$ are defined for all $0 \leq y \leq \mathcal{X}$ and $\xi \in \mathbb{R}$ in the following:

$$\cosh^{\gamma(\delta)}(y\sqrt{i\xi}) = \cosh(y\sqrt{i\xi}) + \frac{1}{2}(\mathcal{F}_{\gamma(\delta)}^\alpha(\mathcal{X}, \xi) - 1)e^{\sqrt{i\xi}y}, \tag{2.9}$$

$$\sinh^{\gamma(\delta)}(\gamma\sqrt{i\xi}) = \sinh(\gamma\sqrt{i\xi}) + \frac{1}{2}(\mathcal{F}_{\gamma(\delta)}^\alpha(\mathcal{X}, \xi) - 1)e^{\sqrt{i\xi}\gamma}, \tag{2.10}$$

where

$$\mathcal{F}_{\gamma(\delta)}^\alpha(\mathcal{X}, \xi) = (1 + \gamma(\delta)e^{\sqrt{|\xi|/2}\mathcal{X}})^{-\alpha} \quad \text{for } \alpha \in \mathbb{N}^*. \tag{2.11}$$

We introduce some notations and assumptions that are needed for our analysis.

Definition 2.1 (Gevrey space) The Gevrey class of functions of order $\theta \geq 0$ defined as

$$\text{“GEVREY”} := \left\{ w \in \mathcal{L}^2(\mathbb{R}) : \int_{-\infty}^{\infty} \exp(\sqrt{2|\xi|}\theta) |\widehat{w}(\xi)|^2 d\xi \leq \infty \right\} \tag{2.12}$$

is equipped with the norm defined by

$$\|w\|_{G_\theta(\mathbb{R})} = \sqrt{\int_{-\infty}^{\infty} \exp(\sqrt{2|\xi|}\theta) |\widehat{w}(\xi)|^2 d\xi} \leq \infty. \tag{2.13}$$

Definition 2.2 For a Hilbert space \mathbb{X} , we denote by $\mathcal{L}^p(0, \mathcal{X}; \mathbb{X})$ (respectively, $\mathcal{C}([0, \mathcal{X}]; \mathbb{X})$) the Banach spaces of measurable (respectively, continuous functions) functions $w : [0, \mathcal{X}] \rightarrow \mathbb{X}$ such that

$$\begin{aligned} \|w\|_{\mathcal{L}^p(0, \mathcal{X}; \mathbb{X})} &= \left(\int_0^{\mathcal{X}} \|w(x)\|_{\mathbb{X}}^p dx \right)^{\frac{1}{p}} < \infty, \quad 1 \leq p < \infty, \\ \|w\|_{\mathcal{L}^\infty(0, \mathcal{X}; \mathbb{X})} &= \text{ess sup}_{0 \leq x \leq \mathcal{X}} \|w(x)\|_{\mathbb{X}} < \infty, \quad p = \infty \\ &\left(\text{respectively, } \|w\|_{\mathcal{C}([0, \mathcal{X}]; \mathbb{X})} = \sup_{0 \leq x \leq \mathcal{X}} \|w(x)\|_{\mathbb{X}} < \infty \right). \end{aligned}$$

We assume the following:

(H₁) The data $\phi, \psi \in \mathcal{L}^2(\mathbb{R})$ are noisy and are represented by the observation data $\phi^\delta, \psi^\delta \in \mathcal{L}^2(\mathbb{R})$ satisfying

$$\|\phi^\delta - \phi\|_{\mathcal{L}^2(\mathbb{R})} \leq \delta, \quad \|\psi^\delta - \psi\|_{\mathcal{L}^2(\mathbb{R})} \leq \delta, \tag{2.14}$$

here $\delta > 0$ is a small positive number representing the level of noise.

(H₂) The source functions $f : \mathbb{R} \rightarrow \mathbb{R}$ and $G : [0, \mathcal{X}] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ are globally Lipschitz-continuous, i.e., there exist the constants $K_f, K_G \geq 0$ such that

$$|f(u) - f(v)| \leq K_f |u - v|, \tag{2.15a}$$

$$|G(x, t; u) - G(x, t; v)| \leq K_G |u - v| \tag{2.15b}$$

for all $(x, t) \in [0, \mathcal{X}] \times \mathbb{R}$, $u, v \in \mathbb{R}$.

(H₃) There exist the constants $B_f, B_G \geq 0$ such that

$$\|f(u)\|_{\mathcal{L}^\infty(0, \mathcal{X}; \mathcal{L}^2(\mathbb{R}))} \leq B_f, \tag{2.16a}$$

$$\|G(x, \cdot; u)\|_{\mathcal{L}^\infty(0, \mathcal{X}; \mathcal{L}^2(\mathbb{R}))} \leq B_G \tag{2.16b}$$

for all $u \in \mathcal{C}([0, \mathcal{X}]; \mathcal{L}^2(\mathbb{R}))$.

3 Error estimate in $\mathcal{L}^p(\omega, \mathcal{X}; \mathcal{L}^2(\mathbb{R})), 0 \leq \omega < \mathcal{X}$

First, we have the following lemmas which will be useful.

Lemma 3.1 For $\xi \in \mathbb{R}, \alpha \in \mathbb{N}^*$, we have the following inequalities:

$$(a) \quad |\mathcal{F}_{\gamma(\delta)}^\alpha(\mathcal{X}, \xi)e^{\sqrt{i\xi}x}| \leq |\gamma(\delta)|^{-\frac{\alpha x}{\mathcal{X}}} \quad \text{for all } 0 \leq x \leq \mathcal{X}, \tag{3.1a}$$

$$(b) \quad |(\mathcal{F}_{\gamma(\delta)}^\alpha(\mathcal{X}, \xi) - 1)e^{-\sqrt{i\xi}x}| \leq |\gamma(\delta)|^{-\frac{\alpha x}{\mathcal{X}}} \quad \text{for all } 0 \leq x \leq \mathcal{X}, \tag{3.1b}$$

$$(c) \quad |\cosh^{\gamma(\delta)}(\sqrt{i\xi}x)| \leq |\gamma(\delta)|^{-\frac{\alpha x}{\mathcal{X}}} \quad \text{for all } 0 \leq x \leq \mathcal{X}, \tag{3.1c}$$

$$(d) \quad |\sinh^{\gamma(\delta)}(\sqrt{i\xi}x)| \leq |\gamma(\delta)|^{-\frac{\alpha x}{\mathcal{X}}} \quad \text{for all } 0 \leq x \leq \mathcal{X}. \tag{3.1d}$$

Proof (a) From (2.11) and Euler’s formula, we have

$$\begin{aligned} |\mathcal{F}_{\gamma(\delta)}^\alpha(\mathcal{X}, \xi)e^{\sqrt{i\xi}x}| &= \left| \frac{e^{\sqrt{i\xi}x}}{(1 + \gamma(\delta)e^{\sqrt{i\xi/2}\mathcal{X}})^\alpha} \right| = \frac{|e^{\sqrt{i\xi}x}|}{(1 + \gamma(\delta)e^{\sqrt{i\xi/2}\mathcal{X}})^\alpha} \leq \frac{e^{-\alpha\sqrt{i\xi/2}(\mathcal{X}-x)}}{(e^{-\sqrt{i\xi/2}\mathcal{X}} + \gamma(\delta))^\alpha} \\ &= \frac{e^{-\alpha\sqrt{i\xi/2}(\mathcal{X}-x)}}{(e^{-\sqrt{i\xi/2}\mathcal{X}} + \gamma(\delta))^{-\frac{\alpha(\mathcal{X}-x)}{\mathcal{X}}}(e^{-\sqrt{i\xi/2}\mathcal{X}} + \gamma(\delta))^{\frac{\alpha x}{\mathcal{X}}}} \\ &\leq \frac{1}{(e^{-\sqrt{i\xi/2}\mathcal{X}} + \gamma(\delta))^{\frac{\alpha x}{\mathcal{X}}}} \leq |\gamma(\delta)|^{-\frac{\alpha x}{\mathcal{X}}}. \end{aligned} \tag{3.2}$$

(b) Similarly, from (2.11) and for $a < b$, we have $a^\alpha - b^\alpha \leq (a - b)^\alpha$, so

$$\begin{aligned} |(\mathcal{F}_{\gamma(\delta)}^\alpha(\mathcal{X}, \xi) - 1)e^{-\sqrt{i\xi}x}| &\leq |\gamma(\delta)|^\alpha \frac{|e^{-\sqrt{i\xi}x}|}{(e^{-\sqrt{i\xi/2}\mathcal{X}} + \gamma(\delta))^\alpha} \\ &\leq |\gamma(\delta)|^\alpha \frac{e^{-\alpha\sqrt{i\xi/2}x}}{(e^{-\sqrt{i\xi/2}\mathcal{X}} + \gamma(\delta))^{\frac{\alpha x}{\mathcal{X}}}(e^{-\sqrt{i\xi/2}\mathcal{X}} + \gamma(\delta))^{-\frac{\alpha(\mathcal{X}-x)}{\mathcal{X}}}} \\ &\leq \frac{|\gamma(\delta)|^\alpha}{(e^{-\sqrt{i\xi/2}\mathcal{X}} + \gamma(\delta))^{-\frac{\alpha(\mathcal{X}-x)}{\mathcal{X}}}} \leq |\gamma(\delta)|^{-\frac{\alpha x}{\mathcal{X}}}. \end{aligned} \tag{3.3}$$

(c) From (2.9) and (3.1a), we obtain

$$\begin{aligned} |\cosh^{\gamma(\delta)}(\sqrt{i\xi}x)| &\leq \frac{1}{2}|\mathcal{F}_{\gamma(\delta)}^\alpha(\mathcal{X}, \xi)e^{\sqrt{i\xi}x}| + \frac{1}{2}|e^{-\sqrt{i\xi}x}| \\ &\leq \frac{1}{2}|\gamma(\delta)|^{-\frac{\alpha x}{\mathcal{X}}} + \frac{1}{2} \\ &\leq |\gamma(\delta)|^{-\frac{\alpha x}{\mathcal{X}}}. \end{aligned}$$

From (2.9), an argument similar to the previous one implies (3.1d). □

Lemma 3.2 For $0 \leq x \leq \mathcal{X}, \xi \in \mathbb{R}$, we have

$$\begin{aligned} \widehat{u}(x, \xi) - \frac{\widehat{\partial_x u}(x, \xi)}{\sqrt{i\xi}} &= e^{\sqrt{i\xi}(\mathcal{X}-x)} \left(\widehat{\phi}(\xi) - \frac{\widehat{\psi}(\xi)}{\sqrt{i\xi}} \right) - \int_x^\mathcal{X} \frac{e^{\sqrt{i\xi}(z-x)}}{\sqrt{i\xi}} \widehat{\mathcal{W}}(u)(z, \xi) dz. \end{aligned} \tag{3.4}$$

Proof Differentiating (2.6) with respect to x gives

$$\begin{aligned}
 -\frac{\widehat{\partial_x u}(x, \xi)}{\sqrt{i\xi}} &= \sinh((\mathcal{X} - x)\sqrt{i\xi})\widehat{\phi}(\xi) - \frac{\cosh((\mathcal{X} - x)\sqrt{i\xi})}{\sqrt{i\xi}}\widehat{\psi}(\xi) \\
 &\quad - \int_x^{\mathcal{X}} \frac{\sinh((z - x)\sqrt{i\xi})}{\sqrt{i\xi}}\widehat{\mathcal{W}}(u)(z, \xi) \, dz,
 \end{aligned}
 \tag{3.5}$$

and adding (3.5) to (2.6) completes the proof. □

Our result is in the next theorem.

Theorem 3.1 *Let $\gamma(\delta) \in (0, 1)$ be such that*

$$\begin{cases} \lim_{\delta \rightarrow 0^+} \gamma(\delta) = 0, \\ \lim_{\delta \rightarrow 0^+} \delta |\gamma(\delta)|^{-\alpha} = 0 \quad \text{for } \alpha \in \mathbb{N}^*. \end{cases}
 \tag{3.6}$$

Then the nonlinear integral equation (2.8) has a solution $U_{\gamma(\delta)}^\delta \in \mathcal{C}([0, \mathcal{X}]; \mathcal{L}^2(\mathbb{R}))$. Assume that problem (1.1) has a solution u satisfying

$$\|u\|_{\mathcal{L}^\infty(0, \mathcal{X}; G_{\mathcal{X}}(\mathbb{R}))} + \frac{1}{\rho_0} \|\partial_x u\|_{\mathcal{L}^\infty(0, \mathcal{X}; G_{\mathcal{X}}(\mathbb{R}))} \leq P(u)
 \tag{3.7}$$

for some known constant $P(u) > 0$. Then (for all $\omega \in [0, \mathcal{X})$)

$$\begin{aligned}
 &\|U_{\gamma(\delta)}^\delta - u\|_{\mathcal{L}^p(\omega, \mathcal{X}; \mathcal{L}^2(\mathbb{R}))} \\
 &\leq C(\alpha, p, \mathcal{X}, \rho_0, \tilde{K}) (|\gamma(\delta)|^{-\alpha} \delta + P(u)) \left(\frac{|\gamma(\delta)|^{\frac{p\alpha\omega}{\mathcal{X}}} - |\gamma(\delta)|^{p\alpha}}{\log(\frac{1}{|\gamma(\delta)|})} \right)^{\frac{1}{p}},
 \end{aligned}
 \tag{3.8}$$

where the positive constant $C(\alpha, p, \mathcal{X}, \rho_0, \tilde{K})$ is a positive constant independent of ω and δ .

Remark 3.1 From (3.6) we infer that the right-hand side of (3.8) tends to zero as $\delta \rightarrow 0^+$. Let us choose a parameter regularization $\gamma(\delta) = \delta^\kappa$ ($\kappa \leq \frac{1}{\alpha}$), and then the error $\|U_{\gamma(\delta)}^\delta - u\|_{\mathcal{L}^p(\omega, \mathcal{X}; \mathcal{L}^2(\mathbb{R}))}$ is of the order $(\frac{\delta^{\frac{\kappa p \alpha \omega}{\mathcal{X}} - \delta^{\kappa p \alpha}}}{\log(\frac{1}{\delta^\kappa})})^{\frac{1}{p}}$, which goes to zero as $\delta \rightarrow 0^+$ for all $\omega \in [0, \mathcal{X})$.

Proof The proof is divided into two parts.

1st part. Integral equation (2.8) has a unique solution $U_{\gamma(\delta)}^\delta \in \mathcal{C}([0, \mathcal{X}]; \mathcal{L}^2(\mathbb{R}))$. We put

$$\begin{aligned}
 \frac{1}{d} = N &= \left\lfloor \frac{|\gamma(\delta)|^{-\alpha} B_f B_G}{\mathcal{U} - |\gamma(\delta)|^{-\alpha} (\|\phi_j^\delta\|_{\mathcal{L}^2(\mathbb{R})} + \|\psi_j^\delta\|_{\mathcal{L}^2(\mathbb{R})})} \right\rfloor + \lfloor B_G K_f + B_f K_G \rfloor, \\
 \mathcal{U} &> |\gamma(\delta)|^{-\alpha} (\|\phi_j^\delta\|_{\mathcal{L}^2(\mathbb{R})} + \|\psi_j^\delta\|_{\mathcal{L}^2(\mathbb{R})}),
 \end{aligned}
 \tag{3.9}$$

where $\lfloor y \rfloor$ is the integer part of the real number y . Let us define

$$\mathcal{X}_j = \mathcal{X} - j\bar{d}, \quad j = 0, 1, \dots, N,$$

and

$$\begin{aligned} \mathcal{M}_j &= \left\{ \vartheta \in \mathcal{C}([\mathcal{X}_{j+1}, \mathcal{X}_j]; \mathcal{L}^2(\mathbb{R})) \mid (\vartheta(\mathcal{X}_j, t), \partial_x \vartheta(\mathcal{X}_j, t)) = (\phi_j^\delta(t), \psi_j^\delta(t)), \right. \\ &\quad \left. \|\vartheta\|_\oplus = \sup_{x \in [\mathcal{X}_j, \mathcal{X}_{j+1}]} \|\vartheta(x, \cdot)\|_{\mathcal{L}^2(\mathbb{R})} \leq \bar{U}, x \in [\mathcal{X}_{j+1}, \mathcal{X}_j] \right\}, \quad j = 0, 1, \dots, N-1. \end{aligned}$$

We also define (for $\mathcal{X}_j - \bar{d} \leq x \leq z \leq \mathcal{X}_j$) $j = 0, 1, \dots, N-1$

$$\begin{aligned} \mathcal{B}_j^{\gamma(\delta)}(\vartheta)(x, t) &:= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \cosh^{\gamma(\delta)}((\mathcal{X}_j - x)\sqrt{i\xi}) \widehat{\phi}_j^\delta(\xi) e^{i\xi t} d\xi \\ &\quad - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\sinh^{\gamma(\delta)}((\mathcal{X}_j - x)\sqrt{i\xi})}{\sqrt{i\xi}} \widehat{\psi}_j^\delta(\xi) e^{i\xi t} d\xi \\ &\quad - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\int_x^{\mathcal{X}_j} \frac{\sinh^{\gamma(\delta)}((z-x)\sqrt{i\xi})}{\sqrt{i\xi}} \widehat{\mathcal{W}}(\vartheta)(z, \xi) dz \right) e^{i\xi t} d\xi. \end{aligned}$$

For $\vartheta \in \mathcal{M}_j$ and $\mathcal{X}_j - \bar{d} \leq x \leq z \leq \mathcal{X}_j$, using Lemma 3.1, we obtain

$$\begin{aligned} \|\mathcal{B}_j^{\gamma(\delta)}(\vartheta)(\cdot, t)\|_{\mathcal{L}^2(\mathbb{R})} &\leq |\gamma(\delta)|^{\frac{\alpha(x-\mathcal{X}_j)}{\mathcal{X}_j}} \|\phi_j^\delta\|_{\mathcal{L}^2(\mathbb{R})} + |\gamma(\delta)|^{\frac{\alpha(x-\mathcal{X}_j)}{\mathcal{X}_j}} \|\psi_j^\delta\|_{\mathcal{L}^2(\mathbb{R})} \\ &\quad + \int_x^{\mathcal{X}_j} |\gamma(\delta)|^{\frac{\alpha(x-z)}{\mathcal{X}_j}} \|\mathcal{W}(z, \cdot; \vartheta)\|_{\mathcal{L}^2(\mathbb{R})} dz. \end{aligned}$$

Multiplying by $|\gamma(\delta)|^{-\frac{\alpha x}{\mathcal{X}_j}}$ on both sides, we obtain

$$\begin{aligned} |\gamma(\delta)|^{-\frac{\alpha x}{\mathcal{X}_j}} \|\mathcal{B}_j^{\gamma(\delta)}(\vartheta)(\cdot, t)\|_{\mathcal{L}^2(\mathbb{R})} &\leq |\gamma(\delta)|^{-\alpha} (\|\phi_j^\delta\|_{\mathcal{L}^2(\mathbb{R})} + \|\psi_j^\delta\|_{\mathcal{L}^2(\mathbb{R})}) \\ &\quad + \int_x^{\mathcal{X}_j} |\gamma(\delta)|^{-\frac{\alpha z}{\mathcal{X}_j}} \|\mathcal{W}(z, \cdot; \vartheta)\|_{\mathcal{L}^2(\mathbb{R})} dz. \end{aligned} \tag{3.10}$$

Using (H_3) , the second term in (3.10) becomes

$$\begin{aligned} &\int_x^{\mathcal{X}_j} |\gamma(\delta)|^{-\frac{\alpha z}{\mathcal{X}_j}} \|\mathcal{W}(z, \cdot; \vartheta)\|_{\mathcal{L}^2(\mathbb{R})} dz \\ &= \int_x^{\mathcal{X}_j} |\gamma(\delta)|^{-\frac{\alpha z}{\mathcal{X}_j}} \|f(\vartheta)\|_{\mathcal{L}^2(\mathbb{R})} \|G(z, \cdot; \vartheta)\|_{\mathcal{L}^2(\mathbb{R})} dz \\ &\leq (\mathcal{X}_j - x) |\gamma(\delta)|^{-\alpha} B_f B_G. \end{aligned} \tag{3.11}$$

From (3.10), (3.11), we have for $\mathcal{X}_j - \bar{d} \leq x \leq z \leq \mathcal{X}_j$

$$\|\mathcal{B}_j^{\gamma(\delta)}(\vartheta)\|_\oplus \leq |\gamma(\delta)|^{-\alpha} (\|\phi_j^\delta\|_{\mathcal{L}^2(\mathbb{R})} + \|\psi_j^\delta\|_{\mathcal{L}^2(\mathbb{R})} + (\mathcal{X}_j - x) B_f B_G).$$

From (3.9), we have

$$\frac{1}{\bar{d}} \geq \frac{|\gamma(\delta)|^{-\alpha} B_f B_G}{\bar{U} - |\gamma(\delta)|^{-\alpha} (\|\phi_j^\delta\|_{\mathcal{L}^2(\mathbb{R})} + \|\psi_j^\delta\|_{\mathcal{L}^2(\mathbb{R})})}. \tag{3.12}$$

This implies that

$$\bar{d} < \frac{\mathcal{U} - |\gamma(\delta)|^{-\alpha} (\|\phi_j^\delta\|_{\mathcal{L}^2(\mathbb{R})} + \|\psi_j^\delta\|_{\mathcal{L}^2(\mathbb{R})})}{|\gamma(\delta)|^{-\alpha} B_f B_G}, \tag{3.13}$$

so $\|\mathcal{B}_j^{\gamma(\delta)}(\vartheta)\|_{\oplus} \leq \mathcal{U}$, i.e., $\mathcal{B}_j^{\gamma(\delta)}(\mathcal{M}_j) \subset \mathcal{M}_j$.

Now, for $\vartheta_1, \vartheta_2 \in \mathcal{M}_j$, we invoke Lemma 3.1 to deduce that, for $\mathcal{X}_j - \bar{d} \leq x \leq z \leq \mathcal{X}_j$,

$$\begin{aligned} & \|\mathcal{B}_j^{\gamma(\delta)}(\vartheta_1)(x, \cdot) - \mathcal{B}_j^{\gamma(\delta)}(\vartheta_2)(x, \cdot)\|_{\mathcal{L}^2(\mathbb{R})} \\ & \leq \int_x^{\mathcal{X}_j} |\gamma(\delta)|^{-\frac{\alpha(x-z)}{\mathcal{X}_j}} \|\mathcal{W}(z, \cdot; \vartheta_1) - \mathcal{W}(z, \cdot; \vartheta_2)\|_{\mathcal{L}^2(\mathbb{R})} dz. \end{aligned} \tag{3.14}$$

From hypotheses (H_2) and (H_3) , we infer that

$$\begin{aligned} & \|\mathcal{W}(x, \cdot; \vartheta_1) - \mathcal{W}(x, \cdot; \vartheta_2)\|_{\mathcal{L}^2(\mathbb{R})} \\ & = \|f(\vartheta_1)G(x, \cdot; \vartheta_1) - f(\vartheta_2)G(x, \cdot; \vartheta_2)\|_{\mathcal{L}^2(\mathbb{R})} \\ & \leq \|G(x, \cdot; \vartheta_1)\|_{\mathcal{L}^2(\mathbb{R})} \|f(\vartheta_1) - f(\vartheta_2)\|_{\mathcal{L}^2(\mathbb{R})} \\ & \quad + \|f(\vartheta_2)\|_{\mathcal{L}^2(\mathbb{R})} \|G(x, \cdot; \vartheta_1) - G(x, \cdot; \vartheta_2)\|_{\mathcal{L}^2(\mathbb{R})} \\ & \leq (B_G K_f + B_f K_G) \|\vartheta_1(x, \cdot) - \vartheta_2(x, \cdot)\|_{\mathcal{L}^2(\mathbb{R})}. \end{aligned} \tag{3.15}$$

We put (3.15) into (3.14) and multiply both sides of the above result by $|\gamma(\delta)|^{-\frac{\alpha x}{\mathcal{X}_j}}$ to get

$$\begin{aligned} & |\gamma(\delta)|^{-\frac{\alpha x}{\mathcal{X}_j}} \|\mathcal{B}_j^{\gamma(\delta)}(\vartheta_1)(x, \cdot) - \mathcal{B}_j^{\gamma(\delta)}(\vartheta_2)(x, \cdot)\|_{\mathcal{L}^2(\mathbb{R})} \\ & \leq \int_x^{\mathcal{X}_j} |\gamma(\delta)|^{-\frac{\alpha z}{\mathcal{X}_j}} (B_G K_f + B_f K_G) \|\vartheta_1(z, \cdot) - \vartheta_2(z, \cdot)\|_{\mathcal{L}^2(\mathbb{R})} dz \\ & \leq (\mathcal{X}_j - x)(B_G K_f + B_f K_G) \|\vartheta_1 - \vartheta_2\|_{\oplus}, \end{aligned}$$

so

$$\|\mathcal{B}_j^{\gamma(\delta)}(\vartheta_1) - \mathcal{B}_j^{\gamma(\delta)}(\vartheta_2)\|_{\oplus} \leq \bar{d}(B_G K_f + B_f K_G) \|\vartheta_1 - \vartheta_2\|_{\oplus}. \tag{3.16}$$

From (3.9), we have

$$\frac{1}{\bar{d}} \geq B_G K_f + B_f K_G,$$

which implies that

$$\bar{d} \leq \frac{1}{B_G K_f + B_f K_G}. \tag{3.17}$$

Hence we have $\|\mathcal{B}_j^{\gamma(\delta)}(\vartheta_1) - \mathcal{B}_j^{\gamma(\delta)}(\vartheta_2)\|_{\oplus} \leq \varrho \|\vartheta_1 - \vartheta_2\|_{\oplus}$, $\varrho \in (0, 1)$, so $\mathcal{B}_j^{\gamma(\delta)}$ is a contraction on \mathcal{M}_j . The Banach fixed point principle guarantees that there exists unique $\vartheta \in \mathcal{M}_j$ such

that

$$\mathcal{B}_j^{\gamma(\delta)}(\vartheta)(x, t) = \vartheta(x, t).$$

In fact, for $j = 0$, we put $\phi_0^\delta(t) = \phi^\delta(t)$ and $\psi_0^\delta(t) = \psi^\delta(t)$. Using the results just obtained, we can find $U_{\gamma(\delta),0}^\delta \in \mathcal{C}([\mathcal{X}_1, \mathcal{X}]; \mathcal{L}^2(\mathbb{R}))$ such that $\mathcal{B}_0(U_{\gamma(\delta),0}^\delta) = U_{\gamma(\delta),0}^\delta$. Assume that we can find $U_{\gamma(\delta),j-1}^\delta \in \mathcal{C}([\mathcal{X}_j, \mathcal{X}_{j-1}]; \mathcal{L}^2(\mathbb{R}))$ such that $\mathcal{B}_{j-1}(U_{\gamma(\delta),j-1}^\delta) = U_{\gamma(\delta),j-1}^\delta$. We now prove that we can extend this solution to the interval $[\mathcal{X}_{j+1}, \mathcal{X}_j]$. Indeed, put $\phi_j^\delta(t) = U_{\gamma(\delta),j}^\delta(\mathcal{X}_j, t)$ and $\psi_j^\delta(t) = \partial_x U_{\gamma(\delta),j}^\delta(\mathcal{X}_j, t)$. We can find $U_{\gamma(\delta),j}^\delta \in \mathcal{C}([\mathcal{X}_{j+1}, \mathcal{X}_j]; \mathcal{L}^2(\mathbb{R}))$ such that $\mathcal{B}_j U_{\gamma(\delta),j}^\delta = U_{\gamma(\delta),j}^\delta$. So we can extend the solution $U_{\gamma(\delta)}^\delta$ on $[\mathcal{X}_{j+1}, \mathcal{X}_j]$ by putting $U_{\gamma(\delta)}^\delta(x, t) = U_{\gamma(\delta),j}^\delta(x, t)$ for $\mathcal{X}_{j+1} \leq x \leq \mathcal{X}_j$. By induction, we can complete the proof of this step.

2nd part. Error estimate $\|U_{\gamma(\delta)}^\delta - u\|_{\mathcal{L}^p(\omega, \mathcal{X}; \mathcal{L}^2(\mathbb{R}))}$. Note

$$\begin{aligned} & \|U_{\gamma(\delta)}^\delta - u\|_{\mathcal{L}^p(\omega, \mathcal{X}; \mathcal{L}^2(\mathbb{R}))} \\ & \leq \|U_{\gamma(\delta)}^\delta - V_{\gamma(\delta)}\|_{\mathcal{L}^p(\omega, \mathcal{X}; \mathcal{L}^2(\mathbb{R}))} + \|V_{\gamma(\delta)} - u\|_{\mathcal{L}^p(\omega, \mathcal{X}; \mathcal{L}^2(\mathbb{R}))}, \end{aligned} \tag{3.18}$$

where $V_{\gamma(\delta)}(x, t)$ is defined by

$$\begin{aligned} V_{\gamma(\delta)}(x, t) = & \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\cosh^{\gamma(\delta)}((\mathcal{X} - x)\sqrt{i\xi})\widehat{\phi}(\xi))e^{i\xi t} d\xi \\ & - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{\sinh^{\gamma(\delta)}((\mathcal{X} - x)\sqrt{i\xi})}{\sqrt{i\xi}} \widehat{\psi}(\xi) \right) e^{i\xi t} d\xi \\ & - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\int_x^{\mathcal{X}} \frac{\sinh^{\gamma(\delta)}((z - x)\sqrt{i\xi})}{\sqrt{i\xi}} \widehat{\mathcal{W}}(V_{\gamma(\delta)})(z, \xi) dz \right) e^{i\xi t} d\xi. \end{aligned} \tag{3.19}$$

From (2.6) and Lemma 3.2, we have

$$\begin{aligned} \widehat{u}(x, \xi) = & \cosh^{\gamma(\delta)}((\mathcal{X} - x)\sqrt{i\xi})\widehat{\phi}(\xi) - \frac{\sinh^{\gamma(\delta)}((\mathcal{X} - x)\sqrt{i\xi})}{\sqrt{i\xi}} \widehat{\psi}(\xi) \\ & - \int_x^{\mathcal{X}} \frac{\sinh^{\gamma(\delta)}((z - x)\sqrt{i\xi})}{\sqrt{i\xi}} \widehat{\mathcal{W}}(u)(z, \xi) dz + \frac{1}{2}(1 - \mathcal{F}_{\gamma(\delta)}^\alpha(\mathcal{X}, \xi)) \\ & \times \left(e^{\sqrt{i\xi}(\mathcal{X} - x)} \left(\widehat{\phi}(\xi) + \frac{\widehat{\psi}(\xi)}{\sqrt{i\xi}} \right) - \int_x^{\mathcal{X}} \frac{e^{\sqrt{i\xi}(z - x)}}{\sqrt{i\xi}} \widehat{\mathcal{W}}(u)(z, \xi) dz \right) \\ = & \cosh^{\gamma(\delta)}((\mathcal{X} - x)\sqrt{i\xi})\widehat{\phi}(\xi) - \frac{\sinh^{\gamma(\delta)}((\mathcal{X} - x)\sqrt{i\xi})}{\sqrt{i\xi}} \widehat{\psi}(\xi) \\ & - \int_x^{\mathcal{X}} \frac{\sinh^{\gamma(\delta)}((z - x)\sqrt{i\xi})}{\sqrt{i\xi}} \widehat{\mathcal{W}}(u)(z, \xi) dz \\ & + \frac{1}{2}(1 - \mathcal{F}_{\gamma(\delta)}^\alpha(\mathcal{X}, \xi)) \left[\widehat{u}(x, \xi) - \frac{\widehat{\partial_x u}(x, \xi)}{\sqrt{i\xi}} \right]. \end{aligned} \tag{3.20}$$

Note

$$\begin{aligned} & \|V_{\gamma(\delta)}(x, \cdot) - u(x, \cdot)\|_{\mathcal{L}^2(\mathbb{R})}^2 \\ & = \|\widehat{V_{\gamma(\delta)}}(x, \cdot) - \widehat{u}(x, \cdot)\|_{\mathcal{L}^2(\mathbb{R})}^2 \end{aligned}$$

$$\begin{aligned}
 &\leq 2 \int_{-\infty}^{\infty} \int_x^{\mathcal{X}} \left| \frac{\sinh^{\gamma(\delta)}((z-x)\sqrt{i\xi})}{\sqrt{i\xi}} \right|^2 |\widehat{\mathcal{W}}(V_{\gamma(\delta)})(z, \xi) - \widehat{\mathcal{W}}(u)(z, \xi)|^2 dz d\xi \\
 &\quad + 2 \int_{-\infty}^{\infty} \left| \frac{1}{2} (\mathcal{F}_{\gamma(\delta)}^{\alpha}(\mathcal{X}, \xi) - 1) e^{-\sqrt{i\xi}x} \right|^2 \left| e^{\sqrt{i\xi}x} \widehat{u}(x, \xi) - e^{\sqrt{i\xi}x} \frac{\partial_x \widehat{u}(x, \xi)}{\sqrt{i\xi}} \right|^2 d\xi \\
 &\leq 2 \int_{-\infty}^{\infty} \left(\int_x^{\mathcal{X}} \left| \frac{\sinh^{\gamma(\delta)}((z-x)\sqrt{i\xi})}{\sqrt{i\xi}} \right|^2 dz \int_x^{\mathcal{X}} |\widehat{\mathcal{W}}(V_{\gamma(\delta)})(z, \xi) - \widehat{\mathcal{W}}(u)(z, \xi)|^2 dz \right) d\xi \\
 &\quad + 4 \int_{-\infty}^{\infty} \left| \frac{1}{2} (\mathcal{F}_{\gamma(\delta)}^{\alpha}(\mathcal{X}, \xi) - 1) e^{-\sqrt{i\xi}x} \right|^2 \left(\left| e^{\sqrt{i\xi}x} \widehat{u}(x, \xi) \right|^2 + \left| e^{\sqrt{i\xi}x} \frac{\partial_x \widehat{u}(x, \xi)}{\sqrt{i\xi}} \right|^2 \right) d\xi.
 \end{aligned}$$

From Lemma 3.1, we get

$$\begin{aligned}
 &\|V_{\gamma(\delta)}(x, \cdot) - u(x, \cdot)\|_{\mathcal{L}^2(\mathbb{R})}^2 \\
 &\leq 2 \int_{-\infty}^{\infty} \frac{\mathcal{X}}{\rho_0} |\gamma(\delta)|^{\frac{2\alpha(x-z)}{\mathcal{X}}} \int_x^{\mathcal{X}} |\widehat{\mathcal{W}}(V_{\gamma(\delta)})(z, \xi) - \widehat{\mathcal{W}}(u)(z, \xi)|^2 dz d\xi \\
 &\quad + 4 \int_{-\infty}^{\infty} |\gamma(\delta)|^{\frac{2\alpha x}{\mathcal{X}}} \left(e^{\sqrt{2|\xi|\mathcal{X}}} |\widehat{u}(x, \xi)|^2 + \frac{e^{\sqrt{2|\xi|\mathcal{X}}}}{\rho_0} |\partial_x \widehat{u}(x, \xi)|^2 \right) d\xi \\
 &\leq 2 \int_x^{\mathcal{X}} \frac{\mathcal{X}}{\rho_0} |\gamma(\delta)|^{\frac{2\alpha(x-z)}{\mathcal{X}}} \|\widehat{\mathcal{W}}(V_{\gamma(\delta)})(z, \cdot) - \widehat{\mathcal{W}}(u)(z, \cdot)\|_{\mathcal{L}^2(\mathbb{R})}^2 dz \\
 &\quad + 4 |\gamma(\delta)|^{\frac{2\alpha x}{\mathcal{X}}} \left(\|\widehat{u}(x, \cdot)\|_{G_{\mathcal{X}}(\mathbb{R})}^2 + \frac{1}{\rho_0} \|\partial_x \widehat{u}(x, \cdot)\|_{G_{\mathcal{X}}(\mathbb{R})}^2 \right).
 \end{aligned}$$

Hence, we get

$$\begin{aligned}
 &\|V_{\gamma(\delta)}(x, \cdot) - u(x, \cdot)\|_{\mathcal{L}^2(\mathbb{R})}^2 \\
 &\leq 2 \int_x^{\mathcal{X}} \frac{\mathcal{X}}{\rho_0} |\gamma(\delta)|^{\frac{2\alpha(x-z)}{\mathcal{X}}} \|\mathcal{W}(z, \cdot; V_{\gamma(\delta)}) - \mathcal{W}(z, \cdot; u)\|_{\mathcal{L}^2(\mathbb{R})}^2 dz \\
 &\quad + 4 |\gamma(\delta)|^{\frac{2\alpha x}{\mathcal{X}}} \left(\|u\|_{\mathcal{L}^\infty(0, \mathcal{X}; G_{\mathcal{X}}(\mathbb{R}))}^2 + \frac{1}{\rho_0} \|\partial_x u\|_{\mathcal{L}^\infty(0, \mathcal{X}; G_{\mathcal{X}}(\mathbb{R}))}^2 \right) \\
 &\leq 2 \int_x^{\mathcal{X}} \frac{\mathcal{X}}{\rho_0} |\gamma(\delta)|^{\frac{2\alpha(x-z)}{\mathcal{X}}} \|\mathcal{W}(z, \cdot; V_{\gamma(\delta)}) - \mathcal{W}(z, \cdot; u)\|_{\mathcal{L}^2(\mathbb{R})}^2 dz + 4 |\gamma(\delta)|^{\frac{2\alpha x}{\mathcal{X}}} P^2(u).
 \end{aligned}$$

Moreover, like in (3.15), we obtain

$$\begin{aligned}
 \|\mathcal{W}(x, \cdot; V_{\gamma(\delta)}) - \mathcal{W}(x, \cdot; u)\|_{\mathcal{L}^2(\mathbb{R})} &\leq (K_f B_G + K_G B_f) \|V_{\gamma(\delta)}(x, \cdot) - u(x, \cdot)\|_{\mathcal{L}^2(\mathbb{R})} \\
 &\leq \widetilde{K} \|V_{\gamma(\delta)}(x, \cdot) - u(x, \cdot)\|_{\mathcal{L}^2(\mathbb{R})},
 \end{aligned}$$

where $\widetilde{K} := K_f B_G + K_G B_f$.

Note

$$\begin{aligned}
 &\|V_{\gamma(\delta)}(x, \cdot) - u(x, \cdot)\|_{\mathcal{L}^2(\mathbb{R})}^2 \\
 &\leq 4 |\gamma(\delta)|^{\frac{2\alpha x}{\mathcal{X}}} P^2(u) + \frac{2\mathcal{X}}{\rho_0} \widetilde{K} \int_x^{\mathcal{X}} |\gamma(\delta)|^{\frac{2\alpha(x-z)}{\mathcal{X}}} \|V_{\gamma(\delta)}(z, \cdot) - u(z, \cdot)\|_{\mathcal{L}^2(\mathbb{R})}^2 dz.
 \end{aligned}$$

Thus, we have the following inequality:

$$\begin{aligned} & |\gamma(\delta)|^{-\frac{2\alpha x}{\mathcal{X}}} \|V_{\gamma(\delta)}(x, \cdot) - u(x, \cdot)\|_{\mathcal{L}^2(\mathbb{R})}^2 \\ & \leq 4P^2(u) + \frac{2\mathcal{X}}{\rho_0} \tilde{K} \int_x^{\mathcal{X}} |\gamma(\delta)|^{-\frac{2\alpha z}{\mathcal{X}}} \|V_{\gamma(\delta)}(z, \cdot) - u(z, \cdot)\|_{\mathcal{L}^2(\mathbb{R})}^2 dz. \end{aligned}$$

From Gronwall’s inequality, we conclude that

$$|\gamma(\delta)|^{-\frac{2\alpha x}{\mathcal{X}}} \|V_{\gamma(\delta)}(x, \cdot) - u(x, \cdot)\|_{\mathcal{L}^2(\mathbb{R})}^2 \leq 4P^2(u) \exp\left(\frac{2\mathcal{X}(\mathcal{X} - x)\tilde{K}}{\rho_0}\right).$$

Hence, we deduce that

$$\begin{aligned} & \|V_{\gamma(\delta)} - u\|_{\mathcal{L}^p(\omega, \mathcal{X}; \mathcal{L}^2(\mathbb{R}))} \\ & \leq 2\frac{\mathcal{X}}{p\alpha} P(u) \exp\left(\frac{\mathcal{X}^2 \tilde{K}}{\rho_0}\right) \left(\frac{|\gamma(\delta)|^{\frac{p\alpha\omega}{\mathcal{X}}} - |\gamma(\delta)|^{p\alpha}}{\log\left(\frac{1}{|\gamma(\delta)|}\right)}\right)^{\frac{1}{p}}. \end{aligned} \tag{3.21}$$

Next, we estimate $\|U_{\gamma(\delta)}^\delta - V_{\gamma(\delta)}\|_{\mathcal{L}^p(\omega, \mathcal{X}; \mathcal{L}^2(\mathbb{R}))}$. Using the basic inequality $(a + b + c)^2 \leq 3(a^2 + b^2 + c^2)$ and Hölder’s inequality, we obtain

$$\begin{aligned} & \|U_{\gamma(\delta)}^\delta(x, \cdot) - V_{\gamma(\delta)}(x, \cdot)\|_{\mathcal{L}^2(\mathbb{R})}^2 \\ & = \|\widehat{U_{\gamma(\delta)}^\delta}(x, \cdot) - \widehat{V_{\gamma(\delta)}}(x, \cdot)\|_{\mathcal{L}^2(\mathbb{R})}^2 \\ & \leq 3 \int_{-\infty}^\infty |\cosh^{\gamma(\delta)}((\mathcal{X} - x)\sqrt{i\xi})|^2 |\widehat{\phi}^\delta(\xi) - \widehat{\phi}(\xi)|^2 d\xi \\ & \quad + 3 \int_{-\infty}^\infty \left|\frac{\sinh^{\gamma(\delta)}((\mathcal{X} - x)\sqrt{i\xi})}{\sqrt{i\xi}}\right|^2 |\widehat{\psi}^\delta(\xi) - \widehat{\psi}(\xi)|^2 d\xi \\ & \quad + 3 \int_{-\infty}^\infty \int_x^{\mathcal{X}} \left|\frac{\sinh^{\gamma(\delta)}((z - x)\sqrt{i\xi})}{\sqrt{i\xi}}\right|^2 dz \\ & \quad \times \int_x^{\mathcal{X}} |\widehat{\mathcal{W}}(U_{\gamma(\delta)}^\delta)(z, \xi) - \widehat{\mathcal{W}}(V_{\gamma(\delta)})(z, \xi)|^2 dz d\xi. \end{aligned}$$

Similar calculations as in (3.15) yield

$$\|\mathcal{W}(x, \cdot, U_{\gamma(\delta)}^\delta) - \mathcal{W}(x, \cdot, V_{\gamma(\delta)})\|_{\mathcal{L}^2(\mathbb{R})} \leq \tilde{K} \|U_{\gamma(\delta)}^\delta(x, \cdot) - V_{\gamma(\delta)}(x, \cdot)\|_{\mathcal{L}^2(\mathbb{R})}.$$

From Lemma 3.1 and using the Lipschitzian property of \mathcal{W} , we get the following inequality:

$$\begin{aligned} & \|U_{\gamma(\delta)}^\delta(x, \cdot) - V_{\gamma(\delta)}(x, \cdot)\|_{\mathcal{L}^2(\mathbb{R})}^2 \\ & \leq 3|\gamma(\delta)|^{\frac{2\alpha(\mathcal{X}-\mathcal{X})}{\mathcal{X}}} \|\phi^\delta - \phi\|_{\mathcal{L}^2(\mathbb{R})}^2 + 3|\gamma(\delta)|^{\frac{2\alpha(\mathcal{X}-\mathcal{X})}{\mathcal{X}}} \|\psi^\delta - \psi\|_{\mathcal{L}^2(\mathbb{R})}^2 \\ & \quad + 3(\mathcal{X} - x)\tilde{K} \int_x^{\mathcal{X}} |\gamma(\delta)|^{\frac{2\alpha(x-z)}{\mathcal{X}}} \|U_{\gamma(\delta)}^\delta(z, \cdot) - V_{\gamma(\delta)}(z, \cdot)\|_{\mathcal{L}^2(\mathbb{R})}^2 dz \\ & \leq 6|\gamma(\delta)|^{\frac{2\alpha(\mathcal{X}-\mathcal{X})}{\mathcal{X}}} \delta^2 + 3\mathcal{X}\tilde{K} \int_x^{\mathcal{X}} |\gamma(\delta)|^{\frac{2\alpha(x-z)}{\mathcal{X}}} \|U_{\gamma(\delta)}^\delta(z, \cdot) - V_{\gamma(\delta)}(z, \cdot)\|_{\mathcal{L}^2(\mathbb{R})}^2 dz. \end{aligned}$$

This implies that

$$\begin{aligned}
 & |\gamma(\delta)|^{-\frac{2\alpha x}{\mathcal{X}}} \left\| U_{\gamma(\delta)}^\delta(x, \cdot) - V_{\gamma(\delta)}(x, \cdot) \right\|_{\mathcal{L}^2(\mathbb{R})}^2 \\
 & \leq 6|\gamma(\delta)|^{-2\alpha} \delta^2 + 3\mathcal{X}\tilde{K} \int_x^{\mathcal{X}} |\gamma(\delta)|^{-\frac{2\alpha z}{\mathcal{X}}} \left\| U_{\gamma(\delta)}^\delta(z, \cdot) - V_{\gamma(\delta)}(z, \cdot) \right\|_{\mathcal{L}^2(\mathbb{R})}^2 dz.
 \end{aligned} \tag{3.22}$$

Applying Grönwall’s inequality, we have

$$|\gamma(\delta)|^{-\frac{2\alpha x}{\mathcal{X}}} \left\| U_{\gamma(\delta)}^\delta(x, \cdot) - V_{\gamma(\delta)}(x, \cdot) \right\|_{\mathcal{L}^2(\mathbb{R})}^2 \leq 6|\gamma(\delta)|^{-2\alpha} \delta^2 \exp\{3\mathcal{X}(\mathcal{X} - x)\tilde{K}\}.$$

Thus

$$\begin{aligned}
 & \left\| U_{\gamma(\delta)}^\delta - V_{\gamma(\delta)} \right\|_{\mathcal{L}^p(\omega, \mathcal{X}; \mathcal{L}^2(\mathbb{R}))} \\
 & \leq \sqrt{6} \frac{\mathcal{X}}{p\alpha} |\gamma(\delta)|^{-\alpha} \delta \exp\left\{ \frac{3}{2} \mathcal{X}^2 \tilde{K} \right\} \left(\frac{|\gamma(\delta)|^{\frac{p\alpha\omega}{\mathcal{X}}} - |\gamma(\delta)|^{p\alpha}}{\log(\frac{1}{\gamma(\delta)})} \right)^{\frac{1}{p}}.
 \end{aligned} \tag{3.23}$$

Combining (3.18), (3.21), and (3.23), we obtain

$$\begin{aligned}
 & \left\| U_{\gamma(\delta)}^\delta - u \right\|_{\mathcal{L}^p(\omega, \mathcal{X}; \mathcal{L}^2(\mathbb{R}))} \\
 & \leq \left\| U_{\gamma(\delta)}^\delta - V_{\gamma(\delta)} \right\|_{\mathcal{L}^p(\omega, \mathcal{X}; \mathcal{L}^2(\mathbb{R}))} + \left\| V_{\gamma(\delta)} - u \right\|_{\mathcal{L}^p(\omega, \mathcal{X}; \mathcal{L}^2(\mathbb{R}))} \\
 & \leq C(\alpha, \mathcal{X}, \rho_0, \tilde{K}) (|\gamma(\delta)|^{-\alpha} \delta + P(u)) \left(\frac{|\gamma(\delta)|^{\frac{p\alpha\omega}{\mathcal{X}}} - |\gamma(\delta)|^{p\alpha}}{\log(\frac{1}{\gamma(\delta)})} \right)^{\frac{1}{p}},
 \end{aligned}$$

where

$$C(\alpha, p, \mathcal{X}, \rho_0, \tilde{K}) = \max \left\{ 2 \frac{\mathcal{X}}{p\alpha} \exp\left\{ \frac{\mathcal{X}^2}{\rho_0} \tilde{K} \right\}; \sqrt{6} \frac{\mathcal{X}}{p\alpha} \exp\left\{ \frac{3}{2} \mathcal{X}^2 \tilde{K} \right\} \right\}.$$

The proof of the theorem is completed. □

4 Numerical test

In this section, we show a numerical simulation for the following inverse sideways heat equation by the filter method:

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(u)G(x, t; u), & (x, t) \in (0, \mathcal{X}) \times \mathbb{R}, \\ u(x, t)|_{t \rightarrow \pm\infty} = 0, & (x, t) \in (0, \mathcal{X}) \times \mathbb{R}, \\ u(\mathcal{X}, t) = \phi(t), & t \in \mathbb{R}, \\ \frac{\partial u}{\partial x}(\mathcal{X}, t) = \psi(t), & t \in \mathbb{R}. \end{cases} \tag{4.1}$$

The tests are performed using software MATLAB R2014b (version 64-bit). Problem (4.1) is considered for $(x, t) \in (0, 1) \times (-\infty, +\infty)$ and the functions are

$$f = u, \quad G = 1 - 2t, \tag{4.2}$$

$$\phi = \exp(-t^2) \sin(1), \quad \psi = \exp(-t^2) \cos(1). \tag{4.3}$$

Based on the Fourier transform, for $\mathcal{F} \in \mathcal{L}^2(\mathbb{R})$, we have

$$\widehat{\mathcal{F}}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathcal{F}(t)e^{-i\xi t} dt, \quad \xi \in \mathbb{R}.$$

The exact solution of problem (4.1)–(4.3) is given by

$$\begin{aligned} u(x, t) = & \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\cosh((\mathcal{X} - x)\sqrt{i\xi})\widehat{\phi}(\xi))e^{i\xi t} d\xi \\ & - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{\sinh((\mathcal{X} - x)\sqrt{i\xi})}{\sqrt{i\xi}} \widehat{\psi}(\xi) \right) e^{i\xi t} d\xi \\ & - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\int_x^{\mathcal{X}} \frac{\sinh((z - x)\sqrt{i\xi})}{\sqrt{i\xi}} \widehat{\mathcal{W}}(u)(z, \xi) dz \right) e^{i\xi t} d\xi. \end{aligned} \tag{4.4}$$

The data $\phi, \psi \in \mathcal{L}^2(\mathbb{R})$ are noisy and are represented by the observation data $\phi^\delta, \psi^\delta \in \mathcal{L}^2(\mathbb{R})$ satisfying

$$\|\phi^\delta - \phi\|_{\mathcal{L}^2(\mathbb{R})} \leq \delta, \quad \|\psi^\delta - \psi\|_{\mathcal{L}^2(\mathbb{R})} \leq \delta, \tag{4.5}$$

here $\delta > 0$ is a small positive number representing the level of noise ($\delta \rightarrow 0^+$).

We recall the regularized solution $U_{\gamma(\delta)}^\delta$ obtained by

$$\begin{aligned} U_{\gamma(\delta)}^\delta(x, t) = & \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\cosh^{\gamma(\delta)}((\mathcal{X} - x)\sqrt{i\xi})\widehat{\phi}^\delta(\xi))e^{i\xi t} d\xi \\ & - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{\sinh^{\gamma(\delta)}((\mathcal{X} - x)\sqrt{i\xi})}{\sqrt{i\xi}} \widehat{\psi}^\delta(\xi) \right) e^{i\xi t} d\xi \\ & - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\int_x^{\mathcal{X}} \frac{\sinh^{\gamma(\delta)}((z - x)\sqrt{i\xi})}{\sqrt{i\xi}} \widehat{\mathcal{W}}(U_{\gamma(\delta)}^\delta)(z, \xi) dz \right) e^{i\xi t} d\xi. \end{aligned} \tag{4.6}$$

Here $\cosh^{\gamma(\delta)}(y\sqrt{i\xi}), \sinh^{\gamma(\delta)}(y\sqrt{i\xi})$ are defined for all $0 \leq y \leq \mathcal{X}$ and $\xi \in \mathbb{R}$ in the following:

$$\cosh^{\gamma(\delta)}(y\sqrt{i\xi}) = \cosh(y\sqrt{i\xi}) + \frac{1}{2}(\mathcal{F}_{\gamma(\delta)}^\alpha(\mathcal{X}, \xi) - 1)e^{\sqrt{i\xi}y}, \tag{4.7}$$

$$\sinh^{\gamma(\delta)}(y\sqrt{i\xi}) = \sinh(y\sqrt{i\xi}) + \frac{1}{2}(\mathcal{F}_{\gamma(\delta)}^\alpha(\mathcal{X}, \xi) - 1)e^{\sqrt{i\xi}y}, \tag{4.8}$$

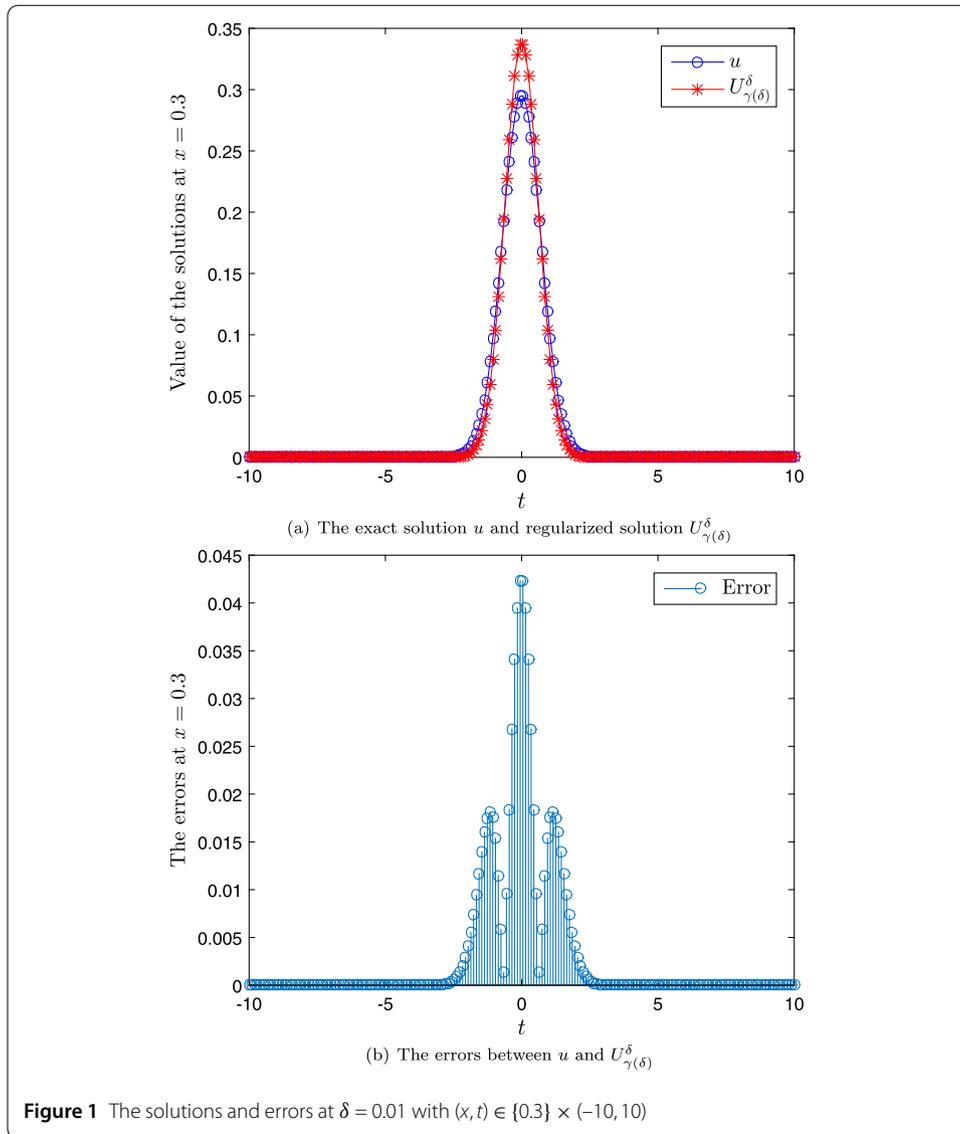
where

$$\mathcal{F}_{\gamma(\delta)}^\alpha(\mathcal{X}, \xi) = (1 + \gamma(\delta)e^{\sqrt{|\xi|/2}\mathcal{X}})^{-\alpha} \quad \text{for } \alpha \in \mathbb{N}^*. \tag{4.9}$$

Next, we consider the problem of computing the Fourier transform as follows:

Let $m, n \in \mathbb{R}, m < n$ and assume that

$$\mathcal{F}(t) = \begin{cases} \mathcal{F}(t) & \text{if } m < t < n, \\ 0 & \text{otherwise.} \end{cases}$$



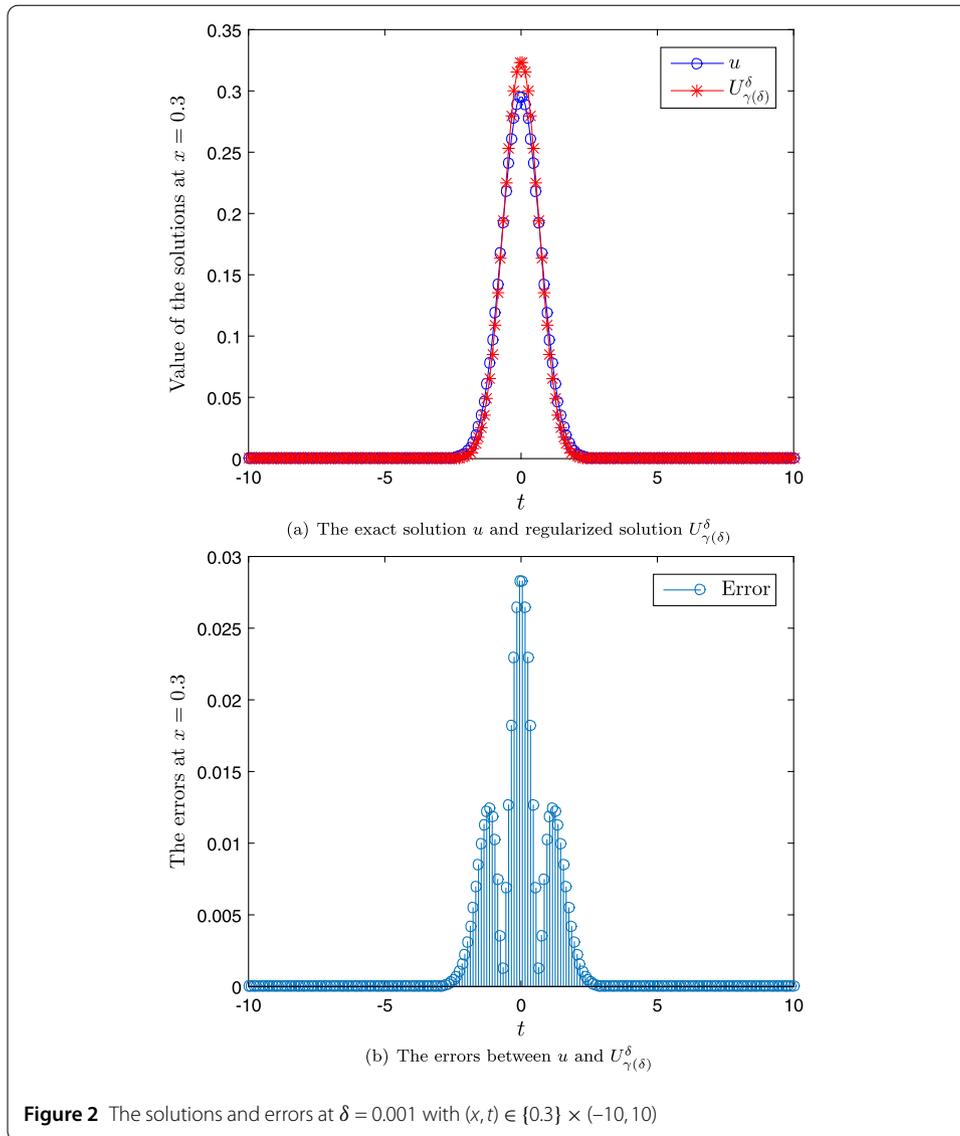
Put $h_t = \frac{n-m}{N_t}$, $t_i = ih_t + n$, $i = \overline{1, N_t}$, respectively. Noting that $\xi_k = \frac{2\pi(k - \frac{N_t}{2})}{n-m}$, $k = \overline{1, N_t}$, we obtain

$$\widehat{\mathcal{F}}(\xi_k) = \frac{1}{\sqrt{2\pi}} \int_m^n \mathcal{F}(t) e^{-i\xi_k t} dt = \frac{1}{\sqrt{2\pi}} \sum_{i=1}^{N_t} \mathcal{F}(t_i) e^{-i\xi_k t_i} h_t. \tag{4.10}$$

We set up a uniform Cartesian grid $(x, t) \in (0, 1) \times (T_1, T_2)$, which can be generated as follows:

$$x_p = p\Delta x, \quad p = \overline{0, P_x}, \quad \Delta x = \frac{1}{P_x},$$

$$t_q = q\Delta t + T_1, \quad q = \overline{0, Q_t}, \quad \Delta t = \frac{T_2 - T_1}{Q_t}.$$



To illustrate the theoretical results, we consider this example with some conditions given by

$$\begin{aligned}
 P_x &= 200, & Q_t &= N_t = 200, & m &= -100, & n &= 100, \\
 \alpha &= 2, & T_1 &= -10, & T_2 &= 10, \\
 \gamma(\delta) &= \delta^{1/3}, & \delta &\in \{10^{-2}, 10^{-3}, 10^{-4}\}.
 \end{aligned}$$

The error between the exact and regularized solutions is evaluated by

$$\text{Err} = \sqrt{\frac{1}{Q_t + 1} \sum_{q=0}^{Q_t} [U_{\gamma(\delta)}^\delta(\cdot, t_q) - u(\cdot, t_q)]^2}. \tag{4.11}$$

Figures 1(a), 2(a), 3(a) show the graphs of the exact and regularized solutions at $x \in \{0.3, 0.5, 0.8\}$ for $\delta \in \{10^{-2}, 10^{-3}, 10^{-4}\}$. The errors between u and $U_{\gamma(\delta)}^\delta$ at $x \in \{0.3, 0.5, 0.8\}$

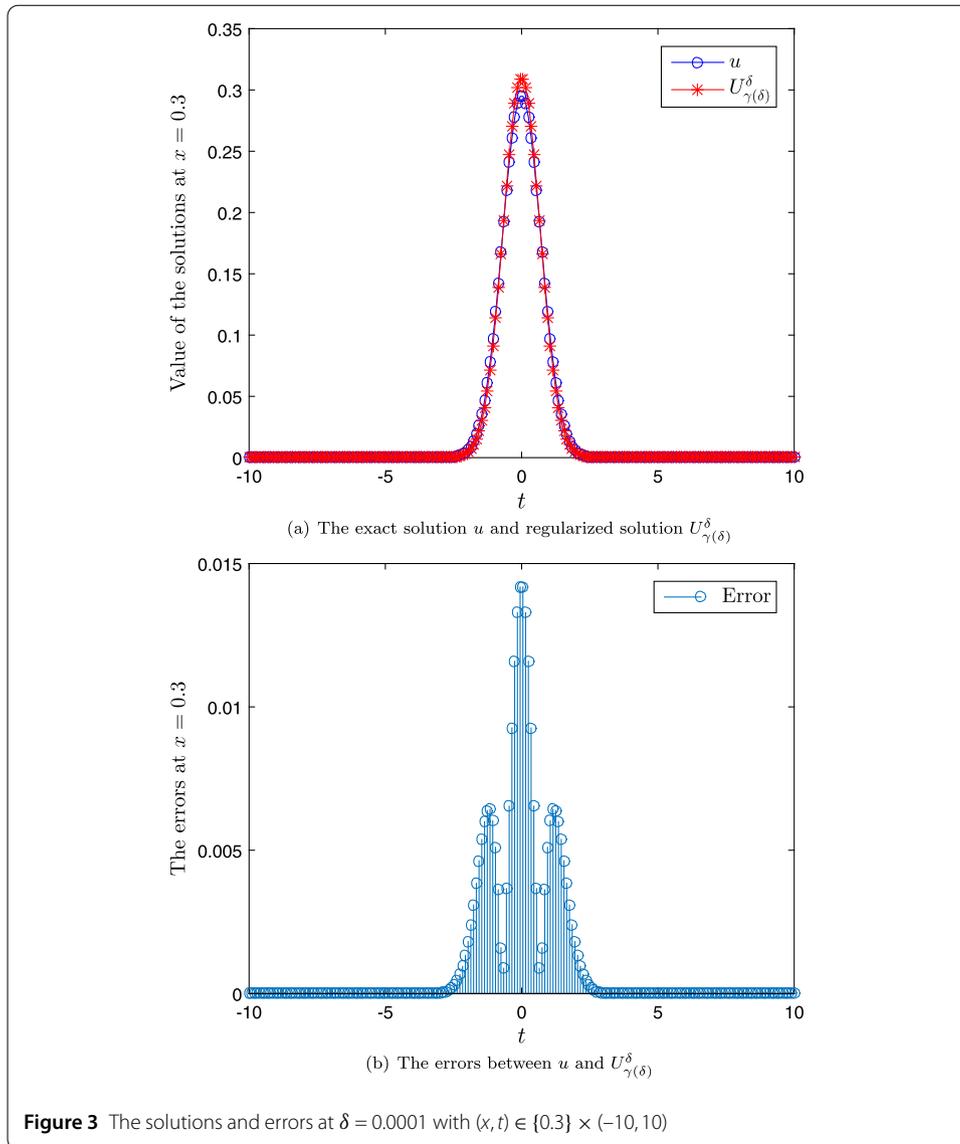


Table 1 The error between the exact and regularized solutions at $\delta \in \{0.01, 0.001, 0.0001\}$ and $x \in \{0.3, 0.5, 0.8\}$

Err(x)	$\delta = 10^{-2}$	$\delta = 10^{-3}$	$\delta = 10^{-4}$
Err(0.3)	0.002012711736159	0.001389166469924	0.000721673458996
Err(0.5)	0.003399412278741	0.002346838740302	0.001219296599176
Err(0.8)	0.005590567951672	0.003862720100876	0.002007822431434

for various amounts of noise $\delta \in \{10^{-2}, 10^{-3}, 10^{-4}\}$ are shown in Table 1. For convenience of comparison, we give the contour graphs between the exact and regularized solutions (see Figs. 1(b), 2(b), 3(b)).

From them, we observe that the errors at $\delta = 0.001$ are greater than those at $\delta = 0.0001$ and smaller than those at $\delta = 0.01$. Furthermore, with the smaller errors of input data, the results obtained are more accurate, which verifies the theoretical results.

Acknowledgements

The authors wish to express their sincere appreciation to the editor and the anonymous referees for their valuable comments and suggestions.

Funding

Not applicable.

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no financial and non-financial competing interests.

Authors' contributions

All authors contributed equally. All the authors read and approved the final manuscript.

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Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 12 August 2019 Accepted: 23 March 2020 Published online: 07 April 2020

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