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Positive periodic solutions for high-order differential equations with multiple delays in Banach spaces

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Abstract

This paper deals with the existence of positive ω-periodic solutions for nth-order ordinary differential equation with delays in Banach space E of the form

$$L_n u(t) = f\big(t, u(t-\tau_1), \ldots, u(t-\tau_m)\big), \quad t \in \mathbb{R},$$

where $L_n u(t) = u^{(n)}(t) + \sum_{i=0}^{n-1} a_i u^{(i)}(t)$ is the nth-order linear differential operator, $a_i \in \mathbb{R}$ $(i = 0, 1, \ldots, n-1)$ are constants, $f : \mathbb{R} \times E^m \to E$ is a continuous function which is ω-periodic with respect to t, and $\tau_i > 0$ $(i = 1, 2, \ldots, m)$ are constants which denote the time delays. We first prove the existence of ω-periodic solutions of the corresponding linear problem. Then the strong positivity estimation is established. Finally, two existence theorems of positive ω-periodic solutions are proved. Our discussion is based on the theory of fixed point index in cones.

MSC: 34B18; 34C25

Keywords: nth-order differential equation; Delays; Positive ω-periodic solution; Existence; Fixed point index theory

1 Introduction

In recent years, the existence of periodic solutions for differential equations has been studied by many authors. But in some practice models, only positive periodic solutions are more important. For second-order differential equations without delay, the existence of positive periodic solutions has been discussed extensively, see [1, 5, 10–12] and the references therein. In [2], Chu and Zhou considered the periodic solutions for the third-order differential equation

$$u'''(t) + \rho^3 u(t) = f\big(t, u(t)\big), \quad t \in [0, 2\pi],$$

where $\rho \in (0, \frac{1}{\sqrt{3}})$ is a constant and $f \in C([0, 2\pi] \times (0, \infty), [0, +\infty))$. By using the Krasnoselskii fixed point theorem in cones, they proved the existence of positive 2π-periodic solutions. In [4], Feng studied the third-order differential equation

$$u'''(t) + \delta u''(t) + \varrho u'(t) = f\bigl(t, u(t)\bigr), \quad t \in [0, 2\pi],$$

where δ and ϱ are positive constants. By utilizing the Guo–Krasnoselskii fixed point theorem in cones, he established some existence and multiplicity results of positive 2π-periodic solutions. For the more general case, in [7], Li proved some existence theorems of positive 2π-periodic solutions for the high-order differential equation

$$L_n u(t) = f\bigl(t, u(t)\bigr), \quad t \in \mathbb{R},$$

where $L_n u(t) = u^{(n)}(t) + \sum_{i=0}^{n-1} a_i u^{(i)}(t)$ is the nth-order linear differential operator, $a_i \in \mathbb{R}$ ($i = 0, 1, \ldots, n-1$) are constants. However, in these works, the authors did not consider the effect of the delay in the equation. Recently, Li [8] discussed the existence of positive ω-periodic solutions of the second-order differential equation with delays of the form

$$-u''(t) + a(t)u(t) = f\bigl(t, u(t-\tau_1), \ldots, u(t-\tau_m)\bigr), \quad t \in \mathbb{R},$$

where $a \in C(\mathbb{R}, (0, \infty))$ is an ω-periodic function, $f : \mathbb{R} \times [0, \infty)^m \to [0, \infty)$ is a continuous function that is ω-periodic with respect to t, and $\tau_1, \tau_2, \ldots, \tau_m$ are positive constants. The results obtained in [8] can deal with the case of second-order differential equations, but for high-order differential equations, for example,

$$L_n u(t) = \frac{1}{2}u^2(t-\tau_1) + \frac{1}{4}u^2(t-\tau_2) + \frac{1}{8}u^2(t-\tau_3), \quad t \in \mathbb{R},$$

the results of [8] are not valid.

Motivated by the papers mentioned above, we consider the existence of positive ω-periodic solutions for the nth-order nonlinear ordinary differential equations in Banach space E

$$L_n u(t) = f\bigl(t, u(t-\tau_1), \ldots, u(t-\tau_m)\bigr), \quad t \in \mathbb{R}, \tag{1.1}$$

where $f : \mathbb{R} \times E^m \to E$ is a continuous function that is ω-periodic with respect to t, and $\tau_1, \tau_2, \ldots, \tau_m$ are positive constants which denote the time delays.

The main features and crucial technique of the present paper are summarized as follows:

(i) In this paper, we discuss the effect of multiple delays in the high-order ordinary differential equation in abstract Banach spaces, which has seldom been studied before.

(ii) Since the integral operator Q is not compact in abstract Banach spaces, the fixed theorems of completely continuous mapping are not valid for this problem. In order to overcome this difficulty, we provide a measure of non-compactness condition $(R1)$ on nonlinear term f, which is much weaker than some existing results. And we prove that the operator Q is a condensing mapping, see Lemma 2.7.

(iii) By utilizing the perturbation method, we obtain the existence of positive ω-periodic solution of the linear differential equation corresponding to Eq. (1.1).

Then the strong positivity estimation of the operator T is established by using the positivity of $G_n(t,s)$ and T_n, see Lemma 2.3.

(iv) In our main results Theorem 3.1 and Theorem 3.2, we provide some order conditions on nonlinearity f to guarantee the existence of positive ω-periodic solutions of Eq. (1.1), which are much easier to verify in application.

The rest of this paper is organized as follows. In Sect. 2, we introduce some preliminaries and prove the existence of positive solutions of the corresponding linear problem. The main results of this paper are presented in Sect. 3. Some remarks are given to show the superiority of this work.

2 Preliminaries

Let $I = [0,\omega]$, $C(I,\mathbb{R})$ be the Banach space of all continuous functions furnished with the norm $\|u\|_C = \max_{t\in I} |u(t)|$. For $\forall h \in C(I,\mathbb{R})$, we first consider the linear boundary value problem (LBVP)

$$\begin{cases} L_n u(t) = 0, \quad t \in I, \\ u^{(i)}(0) = u^{(i)}(\omega), \quad i = 0,1,\dots,n-2, \\ u^{(n-1)}(0) - u^{(n-1)}(\omega) = 1. \end{cases} \tag{2.1}$$

Denote by $P_n(\lambda)$ the characteristic polynomial of L_n:

$$P_n(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0.$$

Let $\mathcal{N}(P_n(\lambda)) = \{\lambda \in \mathbb{C} : P_n(\lambda) = 0\}$, where $\mathbb{C}$ denotes the complex plane. By Lemma 3 of [7], we get the following lemma.

Lemma 2.1 *Let $P_n(\lambda)$, the characteristic polynomial of L_n, satisfy*

(P) $\mathcal{N}(P_n(\lambda)) \subset \{z \in \mathbb{C} : |\operatorname{Im} z| < \frac{\pi}{\omega}\}$.

Then, if $a_0 > 0$, LBVP(2.1) has a unique solution $r_n(t) > 0$ for any $t \in I$.

Let E be a Banach space whose positive cone K is normal. Denote by $C_\omega(\mathbb{R},E)$ the Banach space of all E-valued ω-periodic continuous functions on $\mathbb{R}$ endowed with the norm $\|u\|_C = \max_{t\in I} \|u(t)\|$. Let $K_C = C_\omega(\mathbb{R},K)$ be the normal cone of $C_\omega(\mathbb{R},E)$.

Definition 2.1 A function $u \in C^n_\omega(\mathbb{R},E)$ is called a positive ω-periodic solution of Eq. (1.1) if $u(t) > 0$ for any $t \in \mathbb{R}$ and $u(t)$ satisfies Eq. (1.1).

Lemma 2.2 *Let assumption (P) hold and $a_0 > 0$. Then, for $\forall h \in C_\omega(\mathbb{R},E)$, the linear equation*

$$L_n u(t) = h(t), \quad t \in \mathbb{R} \tag{2.2}$$

has a unique ω-periodic solution given by

$$u(t) = \int_{t-\omega}^{t} G_n(t,s)h(s)\,ds := (T_n h)(t), \quad t \in \mathbb{R}, \tag{2.3}$$

where

$$G_n(t,s)=\begin{cases} r_n(t-s), & 0\le s\le t\le\omega,\\ r_n(\omega+t-s), & 0\le t<s\le\omega,\end{cases}$$

where $r_n(t)\in C^\infty(I,\mathbb{R})$ *is the unique solution of LBVP*(2.1), *and* $T_n:C_\omega(\mathbb{R},E)\to C_\omega(\mathbb{R},E)$ *is a positive and bounded linear operator, whose norm satisfies* $\|T_n\|=\frac{1}{a_0}$.

Proof From Lemma 2.1, if assumption (P) holds and $a_0>0$, LBVP(2.1) has a unique solution $r_n(t)>0$ for any $t\in I$. By Lemma 1 of [7], the linear periodic boundary value problem(LPBVP)

$$\begin{cases} L_nu(t)=h(t), & t\in I,\\ u^{(i)}(0)=u^{(i)}(\omega)\end{cases}\tag{2.4}$$

has a unique solution $u\in C^n(I,E)$, which is given by the expression

$$u(t)=\int_0^\omega G_n(t,s)h(s)\,ds.$$

Since the ω-extension of the solution of LPBVP(2.4) is the ω-periodic solution of linear equation (2.2). Inversely, the ω-periodic solution of linear equation (2.2) restricted to $[0,\omega]$ is the solution of LPBVP(2.4). Hence, for $\forall h\in C_\omega(\mathbb{R},E)$, linear equation (2.2) has a unique ω-periodic solution, which is given by (2.3).

Clearly, $T_n:C_\omega(\mathbb{R},E)\to C^n_\omega(\mathbb{R},E)\hookrightarrow C_\omega(\mathbb{R},E)$ is a positive operator. It remains to prove that T_n is bounded and $\|T_n\|=\frac{1}{a_0}$. On the one hand, for $\forall h\in C_\omega(\mathbb{R},E)$, the inequality

$$\bigl\|T_nh(t)\bigr\|=\left\|\int_{t-\omega}^t G_n(t,s)h(s)\,ds\right\|\le\frac{1}{a_0}\|h\|_C$$

implies $\|T_n\|\le\frac{1}{a_0}$. This means that T_n is bounded.

On the other hand, let $h_0(t)\equiv1$ for all $t\in\mathbb{R}$. Then $h_0\in C_\omega(\mathbb{R},E)$ and $\|h_0\|_C=1$. So,

$$\begin{aligned}\bigl\|T_nh_0(t)\bigr\| &= \left\|\int_{t-\omega}^t G_n(t,s)h_0(s)\,ds\right\|\\ &= \left\|\int_0^\omega G_n(t,s)\,ds\right\|\|h_0\|_C\\ &= \frac{1}{a_0}\|h_0\|_C.\end{aligned}$$

Therefore, $\|T_n\|=\frac{1}{a_0}$. □

In order to prove the existence of positive ω-periodic solutions of Eq. (1.1), for $\forall h\in C_\omega(\mathbb{R},E)$, we consider the linear differential equation with delay of the form

$$L_nu(t)+\rho u(t-\tau)=h(t),\quad t\in\mathbb{R},\tag{2.5}$$

where $\rho\ge0$ is a constant.

If $r_n(t) > 0$ for $t \in I$, let $m_n = \min_{t\in I} r_n(t)$ and $M_n = \max_{t\in I} r_n(t)$. Then $0 < m_n \le r_n(t) \le M_n$. By Lemmas 2.1 and 2.2, we obtain the following lemma.

Lemma 2.3 *Assume that* (P) *holds and* $0 \le \rho < \gamma a_0$, *where* $\gamma = \frac{m_n}{M_n}$. *Then, for* $\forall h \in C_\omega(\mathbb{R}, E)$, *linear delayed differential equation* (2.5) *has a unique* ω*-periodic solution* $u := Th \in C_\omega(\mathbb{R}, E)$, *and* $T : C_\omega(\mathbb{R}, E) \to C_\omega(\mathbb{R}, E)$ *is a linear bounded operator satisfying* $\|T\| \le \frac{1}{a_0-\rho}$. *If* $h \in C_\omega(\mathbb{R}, K)$, *then* $T : C_\omega(\mathbb{R}, K) \to C_\omega(\mathbb{R}, K)$ *is a positive linear bounded operator satisfying the strong positivity estimate*

$$(Th)(t) \ge \gamma (Th)(s), \quad \forall t, s \in \mathbb{R}.$$

Proof By Lemma 2.2, the ω-periodic solution of Eq. (2.5) is expressed by

$$u(t) = \int_{t-\omega}^{t} G_n(t,s)\big[h(s) - \rho u(s-\tau)\big]\,ds, \quad t \in \mathbb{R}. \tag{2.6}$$

Define an operator B by

$$Bu(t) = \rho u(t-\tau), \quad t \in \mathbb{R}. \tag{2.7}$$

Then $B : C_\omega(\mathbb{R}, E) \to C_\omega(\mathbb{R}, E)$ is a linear bounded operator with $\|B\| \le \rho$. Hence, by (2.6) and (2.7), we have

$$(I + T_nB)u(t) = T_nh(t), \quad t \in \mathbb{R}. \tag{2.8}$$

Since $\|T_nB\| \le \|T_n\|\|B\| \le \frac{\rho}{a_0} < 1$, by the perturbation theorem, $(I + T_nB)^{-1}$ exists and

$$(I + T_nB)^{-1} = \sum_{i=0}^{\infty} (-1)^i (T_nB)^i = \sum_{i=0}^{\infty} (T_nB)^{2i}(I - T_nB). \tag{2.9}$$

By direct calculation, we get

$$\big\|(I + T_nB)^{-1}\big\| \le \frac{a_0}{a_0 - \rho}. \tag{2.10}$$

Hence, by (2.8), we have

$$u(t) = (I + T_nB)^{-1}T_nh(t) := Th(t), \quad t \in \mathbb{R}, \tag{2.11}$$

which is an ω-periodic solution of (2.5). By (2.10) and (2.11), we have

$$\begin{aligned} \big\|Th(t)\big\| &= \big\|(I + T_nB)^{-1}T_nh(t)\big\| \\ &\le \big\|(I + T_nB)^{-1}\big\|\|T_n\| \cdot \|h\|_C \\ &\le \frac{1}{a_0 - \rho}\|h\|_C. \end{aligned}$$

Therefore,

$$\|T\| \le \frac{1}{a_0 - \rho}.$$

Furthermore, for $\forall h \in C_\omega(\mathbb{R},K)$, we prove that $T : C_\omega(\mathbb{R},K) \to C_\omega(\mathbb{R},K)$ is a positive operator and $(Th)(t) \ge \gamma(Th)(s)$ for any $t,s \in \mathbb{R}$. By (2.9) and (2.11), we have

$$Th(t) = \sum_{i=0}^{\infty}(T_nB)^{2i}(I - T_nB)T_nh(t), \quad t \in \mathbb{R}.$$

Since

$$T_nh(t) = \int_{t-\omega}^{t} G_n(t,s)h(s)\,ds \ge m_n \int_0^\omega h(s)\,ds$$

and

$$T_nh(t) \le M_n \int_0^\omega h(s)\,ds$$

and

$$(T_nB)T_nh(t) \le \frac{\rho M_n}{a_0}\int_0^\omega h(s)\,ds,$$

we get

$$\begin{aligned}(I - T_nB)T_nh(t) &= T_nh(t) - (T_nB)T_nh(t)\\ &\ge m_n \int_0^\omega h(s)\,ds - \frac{\rho M_n}{a_0}\int_0^\omega h(s)\,ds\\ &= \left(m_n - \frac{\rho M_n}{a_0}\right)\int_0^\omega h(s)\,ds.\end{aligned}$$

Since $h \in C_\omega(\mathbb{R},K)$, $h(t) \not\equiv 0$ for all $t \in \mathbb{R}$. There exist $[c,d] \subset I$ and $\varepsilon > 0$ such that

$$h(t) > \varepsilon, \quad t \in [c,d],$$

from which we get $\int_0^\omega h(s)\,ds \ge \int_c^d h(s)\,ds > \varepsilon(d-c) > 0$. Due to $\rho < \gamma a_0$, we get

$$\begin{aligned}(I - T_nB)T_nh(t) &\ge \left(m_n - \frac{\rho M_n}{a_0}\right)\int_0^\omega h(s)\,ds\\ &\ge \frac{a_0 m_n - \rho M_n}{a_0}\varepsilon(d-c)\\ &> 0.\end{aligned}$$

Therefore, $T : C_\omega(\mathbb{R},K) \to C_\omega(\mathbb{R},K)$ is a positive operator. Moreover, for any $h \in C_\omega(\mathbb{R},K)$, by (2.11), we have

$$(I + T_nB)(Th)(t) = T_nh(t) = \int_{t-\omega}^{t} G_n(t,s)h(s)\,ds, \quad t \in \mathbb{R}.$$

So, we have

$$(I + T_nB)(Th)(t) \ge m_n \int_0^\omega h(s)\,ds \tag{2.12}$$

and

$$\int_0^\omega h(s)\,ds \geq \frac{1}{M_n}(I + T_nB)(Th)(t). \tag{2.13}$$

Consequently, for any $t, s \in \mathbb{R}$, we have

$$(I + T_nB)(Th)(t) \geq \gamma(I + T_nB)(Th)(s). \tag{2.14}$$

Hence, $(Th)(t) \geq \gamma(Th)(s)$ for any $t, s \in \mathbb{R}$. □

Let E be a separable Banach space. Denote by $\beta_E(\cdot)$ and $\beta_C(\cdot)$ the Hausdorff measure of non-compactness(MNC) of the bounded set in E and $C_\omega(\mathbb{R}, E)$, respectively. Let $D \subset C_\omega(\mathbb{R}, E)$ be bounded, set $D(t) = \{u(t) : u \in D\} \subset E$ for $t \in \mathbb{R}$. Then $\beta_E(D(t)) \leq \beta_C(D)$. The following lemmas for the MNC are cited from [3, 6, 13].

Lemma 2.4 *Let $D \subset C(I, E)$ be a bounded and equicontinuous subset. Then $\beta_E(D(t))$ is continuous on I and*

$$\beta_C(D) = \max_{t \in I} \beta_E\bigl(D(t)\bigr) = \beta_E\bigl(D(I)\bigr),$$

where $D(I) := \{u(t) : u \in D, t \in I\}$.

Lemma 2.5 *Let $D \subset E$ be bounded. Then there exists a countable subset $D_0 \subset D$ such that*

$$\beta_E(D) \leq 2\beta_E(D_0).$$

Lemma 2.6 *Let $D = \{u_n\} \subset C(I, E)$ be a bounded and countable subset. Then $\beta_E(D(t))$ is Lebesgue integrable on I and*

$$\beta_E\left(\left\{\int_I u_n(t)\,dt\right\}\right) \leq 2\int_I \beta_E\bigl(D(t)\bigr)\,dt.$$

Now, we consider the existence of positive ω-periodic solutions for the high-order differential equation with delays of the form (1.1). By Lemma 2.3, we define an operator $Q : C_\omega(\mathbb{R}, E) \to C_\omega(\mathbb{R}, E)$ by

$$\begin{aligned}(Qu)(t) = {}& (I + T_nB)^{-1} \\ & \times \int_{t-\omega}^t G_n(t,s)\bigl[f\bigl(s, u(s-\tau_1), \ldots, u(s-\tau_m)\bigr) + \rho u(s-\tau_1)\bigr]\,ds, \quad t \in \mathbb{R}.\end{aligned} \tag{2.15}$$

By the continuity of f, the operator $Q : C_\omega(\mathbb{R}, E) \longrightarrow C_\omega(\mathbb{R}, E)$ is continuous. The positive ω-periodic solution of the high-order differential equation (1.1) is equivalent to the positive fixed point of Q. It is noted that the integral operator Q is not compact in an abstract Banach space. In order to employ the topological degree theory of condensing mapping, it demands that the nonlinear term f satisfies some MNC conditions. Thus, we make the following assumption.

($R1$) For $\forall r > 0, f \in C(\mathbb{R} \times K_r^m, E)$ is bounded and

$$\beta_E\big(f(t, D_1, D_2, \ldots, D_m) + \rho D_1\big) \le \sum_{i=1}^{m} M_i \beta_E(D_i), \quad t \in \mathbb{R},$$

where $K_r = \{u \in K : \|u\| \le r\}$, $D_i \subset K_r$ $(i = 1, 2, \ldots, m)$ are arbitrarily countable subsets, M_i $(i = 1, 2, \ldots, m)$ are positive constants satisfying

$$\sum_{i=1}^{m} M_i < \frac{a_0 - \rho}{4}. \tag{2.16}$$

Lemma 2.7 *Suppose that condition* ($R1$) *holds. Then* $Q : C_\omega(\mathbb{R}, K_r) \longrightarrow C_\omega(\mathbb{R}, K)$ *is a condensing mapping.*

Proof For any $r > 0$, let $K_{r,C} = \{u \in C_\omega(\mathbb{R}, K) : u(t) \in K_r, t \in \mathbb{R}\}$. Since $f(\mathbb{R} \times K_r^m, E)$ is bounded, there exists a constant $\overline{M} > 0$ such that

$$\big\|f(t, x_1, \ldots, x_m)\big\| \le \overline{M} \tag{2.17}$$

for any $t \in \mathbb{R}$ and $x_i \in K_r$, $i = 1, 2, \ldots, m$. Hence, for any $u \in K_{r,C}$, by (2.15), we have

$$\begin{aligned}
\big\|(Qu)(t)\big\| &= \left\|(I + T_n B)^{-1} \int_{t-\omega}^{t} G_n(t,s)\big[f\big(s, u(s-\tau_1), \ldots, u(s-\tau_m)\big) + \rho u(s-\tau_1)\big]\, ds\right\| \\
&\le \frac{a_0}{a_0 - \rho} \int_{t-\omega}^{t} G_n(t,s)\big\|f\big(s, u(s-\tau_1), \ldots, u(s-\tau_m)\big) + \rho u(s-\tau_1)\big\|\, ds \\
&\le \frac{a_0}{a_0 - \rho} \int_{0}^{\omega} G_n(t,s)[\overline{M} + \rho r]\, ds \\
&= \frac{\overline{M} + \rho r}{a_0 - \rho}.
\end{aligned}$$

Then $Q(K_{r,C})$ is bounded. Clearly, $Q(K_{r,C})$ is equicontinuous. Hence, by Lemmas 2.4 and 2.5, there exists a countable subset $D_\ell = \{u_\ell\}_{\ell=1}^{\infty} \subset K_{r,C}$ such that

$$\beta_C\big(Q(K_{r,C})\big) \le 2\beta_C\big(Q(D_\ell)\big) = 2 \max_{t \in J} \beta_E\big(Q(D_\ell)(t)\big). \tag{2.18}$$

By assumption ($R1$) and Lemma 2.6, we have

$$\begin{aligned}
&\beta_E\big(Q(D_\ell)(t)\big) \\
&\quad = \beta_E\left(\left\{(I + T_n B)^{-1} \int_{t-\omega}^{t} G_n(t,s)\big[f\big(s, u_\ell(s-\tau_1), \ldots, u_\ell(s-\tau_m)\big) + \rho u_\ell(s-\tau_1)\big]\, ds\right\}\right) \\
&\quad \le \frac{2a_0}{a_0 - \rho} \int_{t-\omega}^{t} G_n(t,s)\beta_E\big(\big\{f\big(s, u_\ell(s-\tau_1), \ldots, u_\ell(s-\tau_m)\big) + \rho u_\ell(s-\tau_1)\big\}\big)\, ds \\
&\quad \le \frac{2a_0}{a_0 - \rho} \sum_{i=1}^{m} M_i \int_{t-\omega}^{t} G_n(t,s)\beta_E\big(\big\{u_\ell(s-\tau_i)\big\}\big)\, ds
\end{aligned}$$

$$\leq \frac{2a_0}{a_0-\rho}\sum_{i=1}^{m} M_i \int_0^{\omega} G_n(t,s)\,ds\beta_C(D_\ell)$$
$$\leq \frac{2\sum_{i=1}^{m} M_i}{a_0-\rho}\beta_C(K_{r,C}).$$

Consequently, we have

$$\beta_C\bigl(Q(K_{r,C})\bigr) \leq 2\max_{t\in J}\beta_E\bigl(Q(D_\ell)(t)\bigr) \leq \frac{4\sum_{i=1}^{m} M_i}{a_0-\rho}\beta_C(K_{r,C}). \tag{2.19}$$

Hence, $Q: C_\omega(\mathbb{R},K_r) \longrightarrow C_\omega(\mathbb{R},K)$ is a condensing mapping due to (2.16). □

Remark 1 In Lemma 2.7, if the nonlinearity f satisfies linear growth condition, for example, f satisfies the following condition:

($R2$) There exist constants $\overline{C}_i > 0$ ($i = 1,2,\ldots,m$) and $b > 0$ such that

$$f(t,x_1,\ldots,x_m) \leq \sum_{i=1}^{m} \overline{C}_i x_i + b$$

for any $t \in \mathbb{R}$ and $x_i \in K$, $i = 1,2,\ldots,m$, then (2.17) holds for $x_i \in K_r$, $i = 1,2,\ldots,m$, with $\overline{M} = \sum_{i=1}^{m} \overline{C}_i r + b$.

Define a cone $\varXi$ in $C_\omega(\mathbb{R},K)$ by

$$\varXi = \bigl\{u \in C_\omega(\mathbb{R},K) : u(t) \geq \gamma u(s), \forall t,s \in \mathbb{R}\bigr\}, \tag{2.20}$$

where $\gamma = \frac{m_n}{M_n}$. Then we can obtain the following lemma.

Lemma 2.8 $Q(C_\omega(\mathbb{R},K)) \subset \varXi$.

Proof For any $t,s \in \mathbb{R}$ and $u \in C_\omega(\mathbb{R},K)$, by (2.15), we have

$$\begin{aligned}(I+T_nB)(Qu)(t) &= \int_{t-\omega}^{t} G_n(t,\theta)\bigl[f\bigl(\theta,u(\theta-\tau_1),\ldots,u(\theta-\tau_m)\bigr) + \rho u(\theta-\tau_1)\bigr]\,d\theta \\ &\geq m_n \int_0^{\omega} f\bigl(\theta,u(\theta-\tau_1),\ldots,u(\theta-\tau_m)\bigr) + \rho u(\theta-\tau_1)\,d\theta\end{aligned}$$

and

$$\begin{aligned}(I+T_nB)(Qu)(s) &= \int_{s-\omega}^{s} G_n(s,\theta)\bigl[f\bigl(\theta,u(\theta-\tau_1),\ldots,u(\theta-\tau_m)\bigr) + \rho u(\theta-\tau_1)\bigr]\,d\theta \\ &\leq M_n \int_0^{\omega} f\bigl(\theta,u(\theta-\tau_1),\ldots,u(\theta-\tau_m)\bigr) + \rho u(\theta-\tau_1)\,d\theta.\end{aligned}$$

It follows from the above two inequalities that

$$(I+T_nB)(Qu)(t) \geq \gamma (I+T_nB)(Qu)(s), \quad \forall t,s \in \mathbb{R}.$$

Hence $Q(C_\omega(\mathbb{R},K)) \subset \varXi$. □

Let E be a Banach space and $D \subset E$ be a closed convex cone in E. Assume that Ω is a bounded open subset of E with boundary $\partial\Omega$ and $D \cap \Omega \neq \emptyset$, and $Q : D \cap \bar{\Omega} \to D$ is a condensing mapping. If $Qu \neq u$ for any $u \in D \cap \partial\Omega$, the fixed point index $i(Q, D \cap \Omega, D)$ is well defined. If $i(Q, D \cap \Omega, D) \neq 0$, then Q has a fixed point in $D \cap \Omega$. In the proof of the main results, the following two lemmas are useful.

Lemma 2.9 ([9]) *Let Ω be a bounded open subset of E with $\theta \in \Omega$ and $Q : D \cap \bar{\Omega} \to D$ be a condensing mapping. If*

$$\lambda Qu \neq u, \quad \forall u \in D \cap \partial\Omega, 0 < \lambda \leq 1,$$

then $i(Q, D \cap \Omega, D) = 1$.

Lemma 2.10 ([9]) *Let Ω be a bounded open subset of E and $Q : D \cap \bar{\Omega} \to D$ be a condensing mapping. If there exists $e \in D \setminus \{\theta\}$ such that*

$$u - Qu \neq \mu e, \quad \forall u \in D \cap \partial\Omega, \mu \geq 0,$$

then $i(Q, D \cap \Omega, D) = 0$.

3 Existence of positive ω-periodic solutions

Let E be a separable Banach space and $K \subset E$ be a positive cone of E. For any positive constants R and r, let

$$\Omega_R = \big\{u \in C_\omega(\mathbb{R}, K) : \|u\|_C < R\big\}, \qquad \Omega_r = \big\{u \in C_\omega(\mathbb{R}, K) : \|u\|_C < r\big\}. \tag{3.1}$$

Then $\partial\Omega_R = \{u \in C_\omega(\mathbb{R}, K) : \|u\|_C = R\}$ and $\partial\Omega_r = \{u \in C_\omega(\mathbb{R}, K) : \|u\|_C = r\}$. Define an operator $Q : C_\omega(\mathbb{R}, K) \to C_\omega(\mathbb{R}, K)$ by (2.15). Then, by Lemmas 2.7 and 2.8, $Q : C_\omega(\mathbb{R}, K) \to C_\omega(\mathbb{R}, K)$ is a condensing mapping when assumption $(R1)$ holds. We will prove that the operator Q has at least one fixed point in $\Omega_{r,R} := \Omega_R \setminus \overline{\Omega_r}$, which is the positive ω-periodic solution of Eq. (1.1).

Theorem 3.1 *Suppose that (P) holds and $0 \leq \rho < \gamma a_0$. Let $f \in C(\mathbb{R} \times K^m, K)$ satisfy assumption $(R1)$. Then Eq. (1.1) has at least one positive ω-periodic solution if the following conditions hold:*

$(H1)$ *There exist positive constants $c_1, \ldots, c_m$ satisfying $\sum_{i=1}^m c_i < \gamma^2 a_0$ and $\delta > 0$ such that*

$$f(t, x_1, \ldots, x_m) \leq \sum_{i=1}^m c_i x_i$$

for any $t \in \mathbb{R}$ and $x_i \in K$ with $\|x_i\| < \delta$, $i = 1, 2, \ldots, m$.

$(H2)$ *There exist positive constants $d_1, \ldots, d_m$ satisfying $\sum_{i=1}^m d_i > a_0$ and $h_0 \in C_\omega(\mathbb{R}, K)$ such that*

$$f(t, x_1, \ldots, x_m) \geq \sum_{i=1}^m d_i x_i - h_0(t)$$

for any $t \in \mathbb{R}$ and $x_i \in \Xi$, $i = 1, 2, \ldots, m$.

Proof Let Ξ be the closed convex cone of $C_\omega(\mathbb{R},K)$ defined by (2.20). Define an operator $Q : C_\omega(\mathbb{R},K) \to C_\omega(\mathbb{R},K)$ by (2.15). We show that Q has a fixed point in $\Xi \cap \Omega_{r,R}$ for $r > 0$ small enough and $R > 0$ sufficiently large.

Let $r \in (0,\delta)$, where δ is the positive constant in assumption $(H1)$. We prove that Q satisfies the conditions of Lemma 2.9 in $\Xi \cap \Omega_r$, namely,

$$\lambda Qu \neq u, \quad \forall u \in \Xi \cap \partial\Omega_r, 0 < \lambda \leq 1.$$

In fact, if there exist $u_0 \in \Xi \cap \partial\Omega_r$ and $0 < \lambda_0 \leq 1$ such that

$$\lambda_0 Qu_0 = u_0,$$

then by the definition of Q and Lemma 2.3, u_0 satisfies the delayed differential equation

$$L_n u_0(t) + \rho u_0(t-\tau_1) = \lambda_0 f\big(t, u_0(t-\tau_1), \dots, u_0(t-\tau_m)\big) + \lambda_0 \rho u_0(t-\tau_1), \quad t \in \mathbb{R},$$

i.e.,

$$L_n u_0(t) \leq f\big(t, u_0(t-\tau_1), \dots, u_0(t-\tau_m)\big), \quad t \in \mathbb{R}. \tag{3.2}$$

Since $u_0 \in \partial\Omega_r$, by the definition of Ω_r, we have

$$0 \leq \big\|u_0(t-\tau_i)\big\| \leq \|u_0\|_C = r < \delta, \quad i = 1,2,\dots,m, t \in \mathbb{R}.$$

It follows from $(H1)$ that

$$f\big(t, u_0(t-\tau_1), \dots, u_0(t-\tau_m)\big) \leq \sum_{i=1}^{m} c_i u_0(t-\tau_i), \quad t \in \mathbb{R}.$$

Hence, by (3.2), we have

$$L_n u_0(t) \leq \sum_{i=1}^{m} c_i u_0(t-\tau_i), \quad t \in \mathbb{R}.$$

Integrating both sides of this inequality from 0 to ω and using the periodicity of u_0, we have

$$a_0 \int_0^\omega u_0(\nu)\,d\nu \leq \sum_{i=1}^{m} c_i \int_0^\omega u_0(\nu-\tau_i)\,d\nu = \sum_{i=1}^{m} c_i \int_0^\omega u_0(\theta)\,d\theta.$$

By the definition of cone Ξ, we have

$$\int_0^\omega u_0(\nu)\,d\nu \geq \int_0^\omega \gamma u_0(s)\,d\nu \geq \gamma\omega u_0(s), \quad \forall \nu, s \in \mathbb{R} \tag{3.3}$$

and

$$\int_0^\omega u_0(\nu)\,d\nu \leq \frac{1}{\gamma}\omega u_0(t), \quad \forall \nu, t \in \mathbb{R}. \tag{3.4}$$

By the arbitrariness of $t, s \in \mathbb{R}$ in (3.3) and (3.4), choosing $t = s$, we get

$$a_0\gamma\omega u_0(s) \le a_0 \int_0^\omega u_0(\nu)\,d\nu \le \sum_{i=1}^m c_i \int_0^\omega u_0(\nu)\,d\nu \le \frac{1}{\gamma}\omega u_0(s)\sum_{i=1}^m c_i.$$

Consequently,

$$\left(a_0\gamma\omega - \frac{1}{\gamma}\omega\sum_{i=1}^m c_i\right)u_0(s) \le 0.$$

Since $\sum_{i=1}^m c_i \le a_0\gamma^2$, it follows that $u_0(s) \le 0$ for $s \in \mathbb{R}$, which is a contradiction to $u_0 \in \partial\Omega_r$. Hence, for any $u \in \varXi \cap \partial\Omega_r$ and $0 < \lambda \le 1$, we have

$$\lambda Qu \ne u.$$

By Lemma 2.9, we have

$$i(Q, \varXi \cap \Omega_r, \varXi) = 1. \tag{3.5}$$

Let $e \in C(\mathbb{R}, K)$ with $e(t) \equiv 1$ for any $t \in \mathbb{R}$. Then $e \in \varXi \setminus \{\theta\}$. We show that Q satisfies the conditions of Lemma 2.10 in $\varXi \cap \partial\Omega_R$, that is,

$$u - Qu \ne \mu e, \quad \forall u \in \varXi \cap \partial\Omega_R, \mu \ge 0 \tag{3.6}$$

for $R > 0$ sufficiently large. In fact, if there exist $u_1 \in \varXi \cap \partial\Omega_R$ and $\mu_1 \ge 0$ such that

$$u_1 - \mu_1 e = Qu_1.$$

Then, by the definition of Q and Lemma 2.3, u_1 satisfies the delayed differential equation

$$L_n u_1(t) - \mu_1 a_0 = f\bigl(t, u_1(t-\tau_1), \ldots, u_1(t-\tau_m)\bigr), \quad t \in \mathbb{R}.$$

Since $u_1 \in \varXi \cap \partial\Omega_R$, by condition $(H2)$, we have

$$L_n u_1(t) \ge \sum_{i=1}^m d_i u_1(t-\tau_i) - h_0(t).$$

Integrating both sides of this inequality from 0 to ω and using the periodicity of u_1, we have

$$\begin{aligned} a_0 \int_0^\omega u_1(\nu)\,d\nu &\ge \sum_{i=1}^m d_i \int_0^\omega u_1(\nu - \tau_i)\,d\nu - \int_0^\omega h_0(\nu)\,d\nu \\ &= \sum_{i=1}^m d_i \int_0^\omega u_1(\nu)\,d\nu - \int_0^\omega h_0(\nu)\,d\nu. \end{aligned}$$

This implies

$$\left(\sum_{i=1}^{m} d_i - a_0\right)\int_0^{\omega} u_1(\nu)\,d\nu \le \int_0^{\omega} h_0(\nu)\,d\nu \le \omega\|h_0\|_C. \tag{3.7}$$

Since $u_1 \in \Xi \cap \partial\Omega_R$, by the definition of Ξ, we have

$$u_1(\nu) \ge \gamma u_1(s), \quad \forall \nu, s \in \mathbb{R}.$$

Hence, by (3.7) and $\sum_{i=1}^{m} d_i > a_0$, we have

$$\left(\sum_{i=1}^{m} d_i - a_0\right)\omega\gamma u_1(s) \le \omega\|h_0\|_C.$$

So,

$$u_1(s) \le \frac{\|h_0\|_C}{\gamma(\sum_{i=1}^{m} d_i - a_0)} \triangleq R^*.$$

Let $R > \max\{R^*, \delta\}$. Then (3.6) is satisfied. By Lemma 2.10, we have

$$i(Q, \Xi \cap \Omega_R, \Xi) = 0. \tag{3.8}$$

Combining (3.5) with (3.8), we have

$$i(Q, \Xi \cap \Omega_{r,R}, \Xi) = i(Q, \Xi \cap \Omega_R, \Xi) - i(Q, \Xi \cap \Omega_r, \Xi) = -1 \ne 0.$$

Hence, Q has at least one fixed point in $\Xi \cap \Omega_{r,R}$, which is the positive ω-periodic solution of Eq. (1.1). □

Remark 2 If we choose

$$f\big(t, u(t-\tau_1), \dots, u(t-\tau_m)\big) = \sum_{i=1}^{m} \frac{1}{i} u^2(t-\tau_i), \quad \forall u \in K_C,$$

we can prove that $(H1)$ and $(H2)$ hold. Hence, conditions $(H1)$ and $(H2)$ allow $f(t, x_1, \dots, x_m)$ to be superlinear growth on $x_1, \dots, x_m$.

Theorem 3.2 *Suppose that assumption* (P) *holds and* $0 \le \rho < \gamma a_0$. *Let* $f \in C(\mathbb{R} \times K^m, K)$ *satisfy* $(R1)$. *Then Eq.* (1.1) *has at least one positive* ω*-periodic solution if the following conditions hold*:

$(H3)$ *There exist positive constants* $d_1, d_2, \dots, d_m$ *satisfying* $\sum_{i=1}^{m} d_i > a_0$ *and* $\delta > 0$ *such that*

$$f(t, x_1, \dots, x_m) \ge \sum_{i=1}^{m} d_i x_i$$

for any $t \in \mathbb{R}$ *and* $x_i \in K$ *with* $\|x_i\| < \delta$, $i = 1, 2, \dots, m$.

$(H4)$ *There exist positive constants* $c_1, c_2, \dots, c_m$ *satisfying* $\sum_{i=1}^{m} c_i < a_0$ *and* $h_1 \in C_\omega(\mathbb{R}, K)$ *such that*

$$f(t, x_1, \dots, x_m) \leq \sum_{i=1}^{m} c_i x_i + h_1(t)$$

for any $t \in \mathbb{R}$ *and* $x_i \in \Xi$, $i = 1, 2, \dots, m$.

Proof For any $0 < r < R < +\infty$, choose Ξ, Ω_r, Ω_R, and $\Omega_{r,R}$ as in the proof of Theorem 3.1. Define an operator Q by (2.15), then by $(R1)$, $Q : C_\omega(\mathbb{R}, K) \longrightarrow C_\omega(\mathbb{R}, K)$ is a condensing mapping. We will show that the operator Q has at least one fixed point in $\Xi \cap \Omega_{r,R}$.

Let $r \in (0, \delta)$. On the one hand, we prove that Q satisfies the conditions of Lemma 2.10 in $\Xi \cap \partial\Omega_r$. Choose $e(t) \equiv 1$ for any $t \in \mathbb{R}$, then $e \in \Xi \setminus \{\theta\}$. For any $u \in \Xi \cap \partial\Omega_r$ and $\mu \geq 0$, we will show that

$$u - \mu e \neq Qu.$$

In fact, if there exist $u_0 \in \Xi \cap \partial\Omega_r$ and $\mu_0 \geq 0$ such that

$$u_0 - \mu_0 e = Qu_0,$$

then u_0 satisfies the delayed differential equation

$$L_n u_0(t) + \rho u_0(t - \tau_1) - \mu_0 a_0 = f\bigl(t, u_0(t - \tau_1), \dots, u_0(t - \tau_m)\bigr) + \rho u_0(t - \tau_1), \quad t \in \mathbb{R},$$

namely,

$$L_n u_0(t) - \mu_0 a_0 = f\bigl(t, u_0(t - \tau_1), \dots, u_0(t - \tau_m)\bigr), \quad t \in \mathbb{R}.$$

Since $u_0 \in \Xi \cap \partial\Omega_r$, we have

$$0 \leq \bigl\| u_0(t - \tau_i) \bigr\| \leq \|u_0\|_C = r < \delta.$$

So, by condition $(H3)$, we have

$$f\bigl(t, u_0(t - \tau_1), \dots, u_0(t - \tau_m)\bigr) \geq \sum_{i=1}^{m} d_i u_0(t - \tau_i)$$

for any $t \in \mathbb{R}$. Hence

$$L_n u_0(t) - \mu_0 a_0 \geq \sum_{i=1}^{m} d_i u_0(t - \tau_i).$$

Integrating both sides of this inequality from 0 to ω, we have

$$a_0 \int_0^\omega u_0(\nu)\, d\nu - \mu_0 a_0 \omega \geq \sum_{i=1}^{m} d_i \int_0^\omega u_0(\nu - \tau_i)\, d\nu = \sum_{i=1}^{m} d_i \int_0^\omega u_0(\nu)\, d\nu.$$

So, we have

$$\left(\sum_{i=1}^{m} d_i - a_0\right)\int_0^{\omega} u_0(\nu)\,d\nu \le -\mu_0 a_0 \omega.$$

Furthermore, we get

$$\left(\sum_{i=1}^{m} d_i - a_0\right)\gamma\omega u_0(s) \le -\mu_0 a_0 \omega.$$

In view of $\sum_{i=1}^{m} d_i > a_0$, $\gamma > 0$, and $\omega > 0$, we obtain that $u_0(s) \le -\mu_0 a_0 \omega < 0$ for any $s \in \mathbb{R}$, which contradicts $u_0 \in \partial\Omega_r$. Hence, all the conditions of Lemma 2.10 hold. By Lemma 2.10, we have

$$i(Q, \Xi \cap \partial\Omega_r, \Xi) = 1. \tag{3.9}$$

On the other hand, we show that the conditions of Lemma 2.9 are satisfied when R is large enough. That is, for any $u \in \Xi \cap \partial\Omega_R$ and $0 < \lambda \le 1$ such that

$$u \ne \lambda Qu.$$

In fact, if there exist $u_1 \in \Xi \cap \partial\Omega_R$ and $0 < \lambda_1 \le 1$ satisfying

$$u_1 = \lambda_1 Q u_1,$$

then by the definition of Q, we have

$$L_n u_1(t) + \rho u_1(t-\tau_1) = \lambda_1 f\bigl(t, u_1(t-\tau_1), \ldots, u_1(t-\tau_m)\bigr) + \lambda_1 \rho u_1(t-\tau_1), \quad t \in \mathbb{R}.$$

Hence, we get

$$L_n u_1(t) \le f\bigl(t, u_1(t-\tau_1), \ldots, u_1(t-\tau_m)\bigr), \quad t \in \mathbb{R}.$$

By virtue of $u_1 \in \Xi \cap \partial\Omega_R$ and $(H4)$, we have

$$f\bigl(t, u_1(t-\tau_1), \ldots, u_1(t-\tau_m)\bigr) \le \sum_{i=1}^{m} c_i u_1(t-\tau_i) + h_1(t), \quad t \in \mathbb{R}.$$

Thus,

$$L_n u_1(t) \le \sum_{i=1}^{m} c_i u_1(t-\tau_i) + h_1(t), \quad t \in \mathbb{R}.$$

Integrating both sides of this inequality from 0 to ω, we have

$$a_0 \int_0^{\omega} u_1(\nu)\,d\nu \le \sum_{i=1}^{m} c_i \int_0^{\omega} u_1(\nu)\,d\nu + \int_0^{\omega} h_1(\nu)\,d\nu,$$

namely,

$$\left(a_0 - \sum_{i=1}^{m} c_i\right) \int_0^{\omega} u_1(\nu)\,d\nu \le \int_0^{\omega} h_1(\nu)\,d\nu \le \omega \|h_1\|_C.$$

Since $u_1 \in \Xi$, it follows that

$$u_1(\nu) \ge \gamma u_1(s), \quad \forall \nu, s \in \mathbb{R}.$$

Hence, from the above inequality, we have

$$\left(a_0 - \sum_{i=1}^{m} c_i\right) \gamma \omega u_1(s) \le \omega \|h_1\|_C.$$

Consequently, we have

$$\|u_1\|_C \le \frac{\|h_1\|_C}{(a_0 - \sum_{i=1}^{m} c_i)\gamma} \triangleq \overline{R}.$$

Let $R > \max\{\overline{R}, r\}$. Then all the conditions of Lemma 2.9 are satisfied. By Lemma 2.9, we have

$$i(Q, \Xi \cap \partial\Omega_R, \Xi) = 1. \tag{3.10}$$

Combining (3.9) with (3.10) and by utilizing the additivity of fixed point index, we have

$$i(Q, \Xi \cap \Omega_{r,R}, \Xi) = i(Q, \Xi \cap \Omega_R, \Xi) - i(Q, \Xi \cap \Omega_r, \Xi) = 1 \neq 0.$$

Hence, Q has at least one fixed point in $\Xi \cap \Omega_{r,R}$, which is the positive ω-periodic solution of Eq. (1.1). □

Remark 3 If we choose

$$f\big(t, u(t-\tau_1), \ldots, u(t-\tau_m)\big) = \sum_{i=1}^{m} \frac{1}{i} u^{\frac{1}{2}}(t-\tau_i), \quad \forall u \in K_C,$$

we can prove that $(H3)$ and $(H4)$ hold. Hence, conditions $(H3)$ and $(H4)$ allow $f(t, x_1, \ldots, x_m)$ to be sublinear growth on $x_1, \ldots, x_m$.

3.1 Conclusion

In the present work, we establish some sufficient conditions on nonlinear term f to guarantee the existence of positive ω-periodic solutions of Eq. (1.1) in abstract Banach spaces. By using perturbation methods, we first prove the existence of positive ω-periodic solutions of the linear problem corresponding to Eq. (1.1). Then the strong positivity estimation of the operator T is established. The existence of positive ω-periodic solutions of Eq. (1.1) is proved by utilizing fixed point index in cones.

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