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Exponential spline for the numerical solutions of linear Fredholm integro-differential equations

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Abstract

In this paper, we introduce a new scheme based on the exponential spline function for solving linear second-order Fredholm integro-differential equations. Our approach consists of reducing the problem to a set of linear equations. We prove the convergence analysis of the method applied to the solution of integro-differential equations. The method is described and illustrated with numerical examples. The results reveal that the method is accurate and easy to apply. Moreover, results are compared with the method in (J. Comput. Appl. Math. 290:633–640, 2015).

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1 Introduction

Integro-differential equations have gained a lot of interest in multitude of uses, specifically in sciences related to nature and engineering. Special usages of the integro-differential equations are visible in the mathematical modeling on spatio-temporal development of epidemics [44]. Generally, it is impossible to get an analytical answer for such equations. Because of that, various numerical methods have been devoted to finding the approximate solutions to such equations. The numerical solution of this type of integro-differential equations is discussed by a large number of authors. A few of these solutions are as follows: approximate solution that is obtained by using spline functions [1], Jacobi-spectral method for integro-delay differential equations with weakly singular kernels [25], polynomial spline functions that have free boundary condition for solving the first-order integro-differential equations whose order of derivative is one [34], quartic trigonometric B-spline algorithm for numerical solution of the regularized long wave equation [15], an effective application of differential quadrature method based on modified cubic B-splines to numerical solutions of the KdV equation [3], and the exponential cubic b-spline collocation method for the Kuramoto–Sivashinsky equation in [18].

Recently, many authors have investigated the numerical methods for integral equations. These methods include a cubic spline approximation in C^2 to the solution of the Volterra integral equation of the second kind [33], quintic B-spline method [30], Bernstein operational matrix of derivative [4], hybrid of block pulse functions and normalized Bernstein polynomials [5], iterative method [49], sinc-collocation method [48], bivariate splines on nonuniform partitions [36], Jacobi operational matrices for solving delay or advanced integro-differential equations [40], the tau approximation for the Volterra–Hammerstein integral equations [21], b-spline collocation and cubature formulas [12] and [37], wavelet method [6], Walsh function method [35], Chebyshev finite difference method [13], differential transform method [7], Legendre polynomial method [39], an approximating solution, based on Lagrange interpolation and spline functions, to treat functional integral equations of Fredholm type and Volterra type [20], CAS wavelets method [22], an efficient matrix method based on Bell polynomials for solving nonlinear Fredholm-Volterra integral equations [32], collocation methods [10], Taylor polynomial methods [46], and Bernoulli matrix method [9]. Xuhao Li and Patricia J.Y. Wong in [26–29] have successfully applied non-polynomial spline to fractional diffusion problems. Besides, non-polynomial splines have also been applied to solve a system of second-order boundary value problems in the mid-knots of the mesh [14]. In addition, Sezer 's method is discussed by Sezer et al. for approximating different types of integral and differential equations, especially Fredholm integro-differential equation [2]. Some papers have also developed numerical methods based on B-spline collocation method, for example, the extended B-spline collocation method for numerical solutions of Fisher equation in [17], numerical solutions of the Gardner equation by extended form of the cubic b-splines in [24], and in [23] generation of the trigonometric cubic b-spline collocation solutions for the Kuramoto–Sivashinsky (KS) equation.

For second-order impulsive integro-differential equations, periodic boundary value problems are discussed in [47]. Moreover, for second-order impulsive integro-differential equations, a class of three-point boundary value problems in Banach space have been developed in [19]. Yüzbaşi et al. in [50–56] used the non-polynomial functions to solve differential equations that have been based on non-polynomial functions set $\{1, e^{-t}, e^{-2t}, \ldots\}$.

In this paper, based on the non-polynomial spline basis and quasilinearization method to solve the nonlinear Volterra integral equation [31], we want to use the non-polynomial spline functions to develop a numerical method for the solution of the Fredholm integro-differential equation

$$\begin{cases} u''(x) + p(x)u'(x) + q(x)u(x) = f(x) + \int_a^b k(t,x)u(t)\,dt, \\ u(a) = \alpha, \qquad u(b) = \beta, \qquad t \in I := [0,1], \end{cases} \tag{1}$$

where $p(x)$, $q(x)$, $k(t,x)$ are known functions and are considered sufficiently smooth, and also $u(x)$ is an unknown function to be determined. In [45] the existence of solutions has been discussed. In this paper, the basic ideas are developed to establish an algorithm that can be easily implemented and applied to second-order linear Volterra integro-differential equations. The aim of present work is to explore exponential spline interpolation with multiple parameters and devise a method to determine these parameters and also produce the minimum error. The main advantage of our algorithm is that it can be used directly without using assumption or transformation formulae.

The next sections of this paper are organized as follows. In Sect. 2, non-polynomial spline method to solve second-order boundary value problems of Fredholm integro-differential equation is described. In Sect. 3, the convergence of the method is explained.

The efficiency of the method by solving some examples and comparison of the numerical solutions with some other existing methods in [11] is shown in Sect. 4. Finally, a short conclusion is given.

2 Exponential spline

Proof of the existence and uniqueness of the non-polynomial interpolation function is presented in [42] and [43] (Sect. 2.3); in addition, the error analysis in non-polynomial interpolation function is proved in [41]. In [41] interpolation function has been presented as the form

$$Q_n(x)=\sum_{i=0}^{n} c_i y_i(x), \tag{2}$$

where $\{y_0(x), y_1(x), \dots, y_n(x)\}$ are continuous functions which are real-valued and linearly independent on $[a, b]$; moreover $c_0, c_1, \dots, c_n$ are coefficients which are determined by the interpolation conditions. The following form can be considered as a special item of (2):

$$Q_n(x) \in \operatorname{span}\{e^{0\lambda x}, e^{\lambda x}, e^{2\lambda x}, \dots, e^{n\lambda x}\}.$$

Let Ω be a partition of the interval $[a, b]$, defined by the knots x_i, such that $\Omega : a = x_0 < x_1 < \cdots < x_n = b$, with step size $h = \frac{b-a}{n}$. We denote the exponential spline function that interpolates the values $u_0, u_1, \dots, u_n$ of the function of $u(x)$ by $S_i(x, \lambda)$ as follows:

$$S_\Omega(x,\lambda) = a_i e^{\lambda(x-x_i)} + b_i e^{2\lambda(x-x_i)} + c_i e^{3\lambda(x-x_i)} + d_i e^{4\lambda(x-x_i)}. \tag{3}$$

The coefficients introduced in equations (3) are real and λ is an arbitrary parameter. To derive expression for the coefficients in equations of (3) in terms of u_i, u_{i+1}, M_i and M_{i+1}, we first denote

$$\begin{cases} \text{(a)} \begin{cases} S_\Omega(x_i,\lambda) = u_i, & S_\Omega(x_{i+1},\lambda) = u_{i+1}, \\ S''_\Omega(x_i,\lambda) = M_i, & S''_\Omega(x_{i+1},\lambda) = M_{i+1}, \end{cases} \\ \text{(b)} \begin{cases} S_\Omega(x_i,\lambda) = u_i, & S_\Omega(x_{i+1},\lambda) = u_{i+1}, \\ S'_\Omega(x_i,\lambda) = m_i, & S'_\Omega(x_{i+1},\lambda) = m_{i+1}. \end{cases} \end{cases} \tag{4}$$

By using algebraic manipulation of (3) and (4)(a), we obtain the following relations:

$$\begin{aligned}
\bar{a}_i &= e^{-\theta}\big(-e^{3\theta}\big(-5+7e^{\theta}\big)M_i + \big(7-5e^{\theta}\big)M_{i+1} \\
&\quad + 4\lambda^2\big(e^{3\theta}\big(-20+7e^{\theta}\big)u_i + \big(-7+20e^{2\theta}\big)u_{i+1}\big)\big), \\
\bar{b}_i &= e^{-2\theta}\big(e^{3\theta}\big(-8+7e^{\theta}+7e^{2\theta}\big)M_i + \big(-7-7e^{\theta}+8e^{2\theta}\big)M_{i+1} \\
&\quad - \lambda^2\big(e^{3\theta}\big(-128+7e^{\theta}+7e^{2\theta}\big)u_i + \big(-7-7e^{\theta}+128e^{2\theta}\big)u_{i+1}\big)\big), \\
\bar{c}_i &= e^{-2\theta}\big(\big(e^{2\theta}+e^{3\theta}-4e^{4\theta}\big)M_i - \big(-4+e^{\theta}+e^{2\theta}\big)M_{i+1} \\
&\quad + 4\lambda^2\big(e^{2\theta}\big(-4-4e^{\theta}+e^{2\theta}\big)u_i + \big(-1+4e^{\theta}+4e^{2\theta}\big)u_{i+1}\big)\big), \\
\bar{d}_i &= e^{-2\theta}\big(e^{2\theta}\big(-3+5e^{\theta}\big)M_i + \big(-3+5e^{\theta}\big)M_{i+1} \\
&\quad - \lambda^2\big(e^{2\theta}\big(-27+5e^{\theta}\big)u_i + \big(-5+27e^{\theta}\big)u_{i+1}\big)\big),
\end{aligned}$$

$$\vartheta = 3\bigl(-1+e^{\theta}\bigr)\bigl(7-18e^{\theta}+7e^{2\theta}\bigr)\lambda^2,$$
$$a_i = \frac{\bar{a}_i}{\vartheta}, \qquad a_i = \frac{\bar{b}_i}{\vartheta}, \qquad c_i = \frac{3\bar{c}_i}{\vartheta}, \qquad d_i = \frac{\bar{d}_i}{\vartheta}.$$

To develop the consistency relations between the values of spline and its derivatives at knots, consider the following relation:

$$\alpha M_{i-1} + 2\beta M_i + \alpha M_{i+1} = \frac{1}{h^2}(u_{i+1} - 2u_i + u_{i-1}), \tag{5}$$

where

$$\alpha = \frac{1}{(h\lambda)^2}\left(\frac{\lambda h}{\sinh(\lambda h)} - 1\right), \qquad \beta = \frac{1}{(h\lambda)^2}\left(1 - \frac{\coth(\lambda h)}{\lambda}\right),$$

and $\theta = h\lambda$.

Pay attention that exponential spline functions relation (5) will be identical with ordinary spline functions as $\theta \to 0$, which $(\alpha,\beta) \to (\frac{1}{6},\frac{1}{3})$. Moreover, assuming $\alpha = \frac{1}{12}$, $\beta = \frac{5}{12}$, we get the following relation:

$$M_{i-1} + 10M_i + M_{i+1} = \frac{12}{h^2}(u_{i+1} - 2u_i + u_{i-1}), \quad i = 1,2,\ldots,n-1. \tag{6}$$

By expanding (5) in Taylor series about x_i, we obtain the following local truncation error:

$$\begin{aligned} T_i &= (2\alpha + 2\beta - 1)h^2 u_i'' + \frac{1}{12}(12\alpha - 1)h^4 u_i^{(4)} + \frac{1}{360}(30\alpha - 1)h^6 u_i^{(6)} \\ &\quad + \frac{(56\alpha - 1)h^8 u^{(8)}}{20160} + O\bigl(h^{10}\bigr). \end{aligned} \tag{7}$$

Similarly, by using (3) and (4)(b), we get

$$\alpha m_{i-1} + 2\beta m_i + \alpha m_{i+1} = \frac{\alpha + \beta}{h}(u_{i+1} - u_{i-1}), \quad i = 1,2,\ldots,n-1, \tag{8}$$

and also by expanding (8) in Taylor series about x_i, we obtain the following local truncation error:

$$T_i = \left(\frac{2\alpha - \beta}{3(\alpha + \beta)}\right)h^3 u_i^{(3)} + \left(\frac{4\alpha - \beta}{60(\alpha + \beta)}\right)h^5 u_i^{(5)} + \left(\frac{6\alpha - \beta}{2520(\alpha + \beta)}\right)h^7 u_i^{(7)} + O\bigl(h^9\bigr).$$

In the matrix notation, equations (5) and (8) have the following forms:

$$WM = \bar{R}U, \qquad Zm = SU,$$

where W, R, Z, and S are coefficient matrices in (5) and (8). We approximate $m_0 = \frac{-3u_0+4u_1-u_2}{2h}$ and $m_n = \frac{3u_{n-2}-4u_{n-1}-u_n}{2h}$, and also M_i for $i = 0, n$, by using second-order approximation. From (3) and (4) we have

$$\begin{aligned} S_i(x) &= \bigl(e^{\lambda(x-x_i)}\bigl(5e^{2\theta}M_i - 7e^{3\theta}M_i - 5M_{i+1} + 7e^{-\theta}M_{i+1} - 80e^{2\theta}\lambda^2 u_i + 28e^{3\theta}\lambda^2 u_i \\ &\quad + 80\lambda^2 u_{i+1} - 28e^{-\theta}\lambda^2 u_{i+1}\bigr)\bigr) \end{aligned}$$

$$
\begin{aligned}
&/\bigl(3\bigl(-1+e^{\theta}\bigr)\bigl(7-18e^{\theta}+7e^{2\theta}\bigr)\lambda^2\bigr)\\
&+\bigl(e^{4\lambda(x-x_i)}\bigl(-3M_i+5e^{\theta}M_i-5e^{-2\theta}M_{i+1}+3e^{-\theta}M_{i+1}+27\lambda^2u_i-5e^{\theta}\lambda^2u_i\\
&+5e^{-2\theta}\lambda^2u_{i+1}-27e^{-\theta}\lambda^2u_{i+1}\bigr)\bigr)\\
&/\bigl(3\bigl(-1+e^{\theta}\bigr)\bigl(7-18e^{\theta}+7e^{2\theta}\bigr)\lambda^2\bigr)\\
&+\bigl(e^{2\lambda(x-x_i)}\bigl(-8e^{\theta}M_i+7e^{2\theta}M_i+7e^{3\theta}M_i+8M_{i+1}-7e^{-2\theta}M_{i+1}-7e^{-\theta}M_{i+1}\\
&+128\lambda^2e^{\theta}u_i-7e^{2\theta}\lambda^2u_i\bigr)\bigr)\\
&/\bigl(3(-1+e^{\theta})(7-18e^{\theta}+7e^{2\theta})\lambda^2\bigr)\\
&+\frac{e^{2\lambda(x-x_i)}(-7\lambda^2e^{3\theta}u_i-128\lambda^2u_{i+1}+7e^{-2\theta}\lambda^2u_{i+1}+7e^{-\theta}\lambda^2u_{i+1})}{3(-1+e^{\theta})(7-18e^{\theta}+7e^{2\theta})\lambda^2}\\
&+\bigl(e^{3\lambda(x-x_i)}\bigl(M_i+e^{\theta}M_i-4e^{2\theta}M_i-M_{i+1}+4e^{-2\theta}M_{i+1}-e^{-\theta}M_{i+1}\\
&-16\lambda^2u_i-16e^{\theta}\lambda^2u_i\bigr)\bigr)\\
&/\bigl((-1+e^{\theta})(7-18e^{\theta}+7e^{2\theta})\lambda^2\bigr)\\
&+\frac{e^{3\lambda(x-x_i)}(+4\lambda^2e^{2\theta}u_i+16\lambda^2u_{i+1}+4e^{-2\theta}\lambda^2u_{i+1}+16e^{-\theta}\lambda^2u_{i+1})}{(-1+e^{\theta})(7-18e^{\theta}+7e^{2\theta})\lambda^2}+O\bigl(h^4\bigr). \qquad (9)
\end{aligned}
$$

To discretize the integro-differential equation of (1) by using equation (4), we obtain

$$
\begin{aligned}
&u''_i+p_iu'_i+q_iu_i\\
&\quad=f_i+\int_a^b k(t,x)u(t)\,dt\\
&\quad=f_i+\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}}k(t,x_i)u(t)\,dt,\\
&\quad\approx f_i+\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}}k(t,x_i)S_j(t)\,dt,\\
&\quad=f_i+\sum_{j=0}^{n-1}\frac{e^{2\theta}((5-7e^{\theta})M_j+\lambda^2(-80+28e^{\theta})u_j)}{3(-1+e^{\theta})(7-18e^{\theta}+7e^{2\theta})\lambda^2}\int_{t_j}^{t_{j+1}}k(t,x_i)e^{\lambda(t-t_j)}\,dt\\
&\qquad+\sum_{j=0}^{n-1}\frac{(-5+7e^{-\theta})M_{j+1}+\lambda^2(80-28e^{-\theta})u_{j+1}}{3(-1+e^{\theta})(7-18e^{\theta}+7e^{2\theta})\lambda^2}\int_{t_j}^{t_{j+1}}k(t,x_i)e^{\lambda(t-t_j)}\,dt\\
&\qquad+\sum_{j=0}^{n-1}\frac{e^{\theta}((-8+7e^{\theta}+7e^{2\theta})M_j+\lambda^2(128-7e^{\theta}-7e^{2\theta})u_j)}{3(-1+e^{\theta})(7-18e^{\theta}+7e^{2\theta})\lambda^2}\int_{t_j}^{t_{j+1}}k(t,x_i)e^{2\lambda(t-t_j)}\,dt\\
&\qquad+\sum_{j=0}^{n-1}\frac{(8-7e^{-\theta}-7e^{-2\theta})M_{j+1}+\lambda^2(-128+7e^{-\theta}+7e^{-2\theta})u_{j+1}}{3(-1+e^{\theta})(7-18e^{\theta}+7e^{2\theta})\lambda^2}\int_{t_j}^{t_{j+1}}k(t,x_i)e^{2\lambda(t-t_j)}\,dt\\
&\qquad+\sum_{j=0}^{n-1}\frac{(1+e^{\theta}-4e^{2\theta})M_j+\lambda^2(-16-16e^{\theta}+4e^{2\theta})u_j}{(-1+e^{\theta})(7-18e^{\theta}+7e^{2\theta})\lambda^2}\int_{t_j}^{t_{j+1}}k(t,x_i)e^{3\lambda(t-t_j)}\,dt\\
&\qquad+\sum_{j=0}^{n-1}\frac{(-1-e^{-\theta}+4e^{-2\theta})M_{j+1}+\lambda^2(+16+16e^{-\theta}-4e^{-2\theta})u_{j+1}}{(-1+e^{\theta})(7-18e^{\theta}+7e^{2\theta})\lambda^2}
\end{aligned}
$$

$$\begin{aligned}
&\times \int_{t_j}^{t_{j+1}} k(t,x_i)e^{3\lambda(t-t_j)}\,dt \\
&+ \sum_{j=0}^{n-1} \frac{(-3+5e^{\theta})M_j + \lambda^2(27-5e^{\theta})u_j}{3(-1+e^{\theta})(7-18e^{\theta}+7e^{2\theta})\lambda^2} \int_{t_j}^{t_{j+1}} k(t,x_i)e^{4\lambda(t-t_j)}\,dt \\
&+ \sum_{j=0}^{n-1} \frac{e^{-\theta}((3-5e^{-\theta})M_{j+1} + \lambda^2(-27+5e^{-\theta})u_{j+1})}{3(-1+e^{\theta})(7-18e^{\theta}+7e^{2\theta})\lambda^2} \int_{t_j}^{t_{j+1}} k(t,x_i)e^{4\lambda(t-t_j)}\,dt.
\end{aligned}$$

We let

$$\begin{cases}
a(i,j) = \int_{t_j}^{t_{j+1}} k(t,x_i)e^{\lambda(t-t_j)}\,dt, & b(i,j+1) = \int_{t_j}^{t_{j+1}} k(t,x_i)e^{\lambda(t-t_j)}\,dt, \\
c(i,j) = \int_{t_j}^{t_{j+1}} k(t,x_i)e^{2\lambda(t-t_j)}\,dt, & d(i,j+1) = \int_{t_j}^{t_{j+1}} k(t,x_i)e^{2\lambda(t-t_j)}\,dt, \\
e(i,j) = \int_{t_j}^{t_{j+1}} k(t,x_i)e^{3\lambda(t-t_j)}\,dt, & r(i,j+1) = \int_{t_j}^{t_{j+1}} k(t,x_i)e^{3\lambda(t-t_j)}\,dt, \\
g(i,j) = \int_{t_j}^{t_{j+1}} k(t,x_i)e^{4\lambda(t-t_j)}\,dt, & h(i,j+1) = \int_{t_j}^{t_{j+1}} k(t,x_i)e^{4\lambda(t-t_j)}\,dt,
\end{cases}$$

and introduce the following relations:

$$\begin{cases}
a(i,n) = 0, \quad b(i,0) = 0, \quad c(i,n) = 0, \quad d(i,0) = 0, \\
e(i,n) = 0, \quad r(i,0) = 0, \quad g(i,n) = 0, \quad h(i,0) = 0.
\end{cases}$$

We can write the defined notations in the form of the matrix as follows: $A = (a_{i,j})$, $B = (b_{i,j})$, $C = (c_{i,j})$, $D = (d_{i,j})$, $\bar{E} = (e_{i,j})$, $R = (r_{i,j})$, $G = (g_{i,j})$, $H = (h_{i,j})$, $Q = (q_{i,j})$, $P = (p_{i,j})$ also if suppose $M \approx \hat{M} = (\widehat{M}_0, \widehat{M}_1, \dots, \widehat{M}_{n-1}, \widehat{M}_n)^T$, $U \approx \hat{U} = (\widehat{u}_0, \widehat{u}_1, \dots, \widehat{u}_{n-1}, \widehat{u}_n)^T$, $m \approx \hat{m} = (\widehat{m}_0, \widehat{m}_1, \dots, \widehat{m}_{n-1}, \widehat{m}_n)^T$, and $F = (f_0, f_1, \dots, f_{n-1}, f_n)^T$. After substitution, we get

$$\begin{aligned}
&\hat{M} + P\hat{m} + Q\hat{U} \\
&= F + \frac{e^{2\theta}(5-7e^{\theta})}{3(-1+e^{\theta})(7-18e^{\theta}+7e^{2\theta})\lambda^2} A\hat{M} + \frac{e^{2\theta}\lambda^2(-80+28e^{\theta})}{3(-1+e^{\theta})(7-18e^{\theta}+7e^{2\theta})\lambda^2} A\hat{U} \\
&\quad + \frac{(-5+7e^{-\theta})}{3(-1+e^{\theta})(7-18e^{\theta}+7e^{2\theta})\lambda^2} B\hat{M} + \frac{\lambda^2(80-28e^{-\theta})}{3(-1+e^{\theta})(7-18e^{\theta}+7e^{2\theta})\lambda^2} B\hat{U} \\
&\quad + \frac{e^{\theta}(-8+7e^{\theta}+7e^{2\theta})}{3(-1+e^{\theta})(7-18e^{\theta}+7e^{2\theta})\lambda^2} C\hat{M} + \frac{e^{\theta}\lambda^2(128-7e^{\theta}-7e^{2\theta})}{3(-1+e^{\theta})(7-18e^{\theta}+7e^{2\theta})\lambda^2} C\hat{U} \\
&\quad + \frac{e^{\theta}(-8+7e^{\theta}+7e^{2\theta})}{3(-1+e^{\theta})(7-18e^{\theta}+7e^{2\theta})\lambda^2} D\hat{M} + \frac{e^{\theta}\lambda^2(128-7e^{\theta}-7e^{2\theta})}{3(-1+e^{\theta})(7-18e^{\theta}+7e^{2\theta})\lambda^2} D\hat{U} \\
&\quad + \frac{8-7e^{-\theta}-7e^{-2\theta}}{3(-1+e^{\theta})(7-18e^{\theta}+7e^{2\theta})\lambda^2} \bar{E}\hat{M} + \frac{\lambda^2(-128+7e^{-\theta}+7e^{-2\theta})}{3(-1+e^{\theta})(7-18e^{\theta}+7e^{2\theta})\lambda^2} \bar{E}\hat{U} \\
&\quad + \frac{-1-e^{-\theta}+4e^{-2\theta}}{(-1+e^{\theta})(7-18e^{\theta}+7e^{2\theta})\lambda^2} R\hat{M} + \frac{\lambda^2(16+16e^{-\theta}-4e^{-2\theta})}{(-1+e^{\theta})(7-18e^{\theta}+7e^{2\theta})\lambda^2} R\hat{U} \\
&\quad + \frac{-3+5e^{\theta}}{3(-1+e^{\theta})(7-18e^{\theta}+7e^{2\theta})\lambda^2} G\hat{M} + \frac{\lambda^2(27-5e^{\theta})}{3(-1+e^{\theta})(7-18e^{\theta}+7e^{2\theta})\lambda^2} G\hat{U} \\
&\quad + \frac{e^{-\theta}(3-5e^{-\theta})}{3(-1+e^{\theta})(7-18e^{\theta}+7e^{2\theta})\lambda^2} H\hat{M} \\
&\quad + \frac{e^{-\theta}\lambda^2(-27+5e^{-\theta})}{3(-1+e^{\theta})(7-18e^{\theta}+7e^{2\theta})\lambda^2} H\hat{U}.
\end{aligned} \tag{10}$$

By solving the above system, an approximation solution of equation (1) will be gotten. Now, the u_i function can be approximated by using the exponential spline $\widehat{S}_i$, where

$$\begin{aligned}\widehat{S}_i(x) = {} & \bigl(e^{\lambda(x-x_i)}\bigl(5e^{2\theta}\widehat{M}_i - 7e^{3\theta}\widehat{M}_i - 5\widehat{M}_{i+1} + 7e^{-\theta}\widehat{M}_{i+1} - 80e^{2\theta}\lambda^2\widehat{u}_i + 28e^{3\theta}\lambda^2\widehat{u}_i \\ & + 80\lambda^2\widehat{u}_{i+1} - 28e^{-\theta}\lambda^2\widehat{u}_{i+1}\bigr)\bigr) \\ & /\bigl(3\bigl(-1+e^{\theta}\bigr)\bigl(7-18e^{\theta}+7e^{2\theta}\bigr)\lambda^2\bigr) \\ & + \bigl(e^{4\lambda(x-x_i)}\bigl(-3\widehat{M}_i + 5e^{\theta}\widehat{M}_i - 5e^{-2\theta}\widehat{M}_{i+1} + 3e^{-\theta}\widehat{M}_{i+1} + 27\lambda^2\widehat{u}_i \\ & - 5e^{\theta}\lambda^2\widehat{u}_i + 5e^{-2\theta}\lambda^2\widehat{u}_{i+1} - 27e^{-\theta}\lambda^2\widehat{u}_{i+1}\bigr)\bigr) \\ & /\bigl(3\bigl(-1+e^{\theta}\bigr)\bigl(7-18e^{\theta}+7e^{2\theta}\bigr)\lambda^2\bigr) \\ & + \bigl(e^{2\lambda(x-x_i)}\bigl(-8e^{\theta}\widehat{M}_i + 7e^{2\theta}\widehat{M}_i + 7e^{3\theta}\widehat{M}_i + 8\widehat{M}_{i+1} - 7e^{-2\theta}\widehat{M}_{i+1} - 7e^{-\theta}\widehat{M}_{i+1} \\ & + 128\lambda^2e^{\theta}\widehat{u}_i - 7e^{2\theta}\lambda^2\widehat{u}_i\bigr)\bigr) \\ & /\bigl(3\bigl(-1+e^{\theta}\bigr)\bigl(7-18e^{\theta}+7e^{2\theta}\bigr)\lambda^2\bigr) \\ & + \frac{e^{2\lambda(x-x_i)}(-7\lambda^2e^{3\theta}\widehat{u}_i - 128\lambda^2\widehat{u}_{i+1} + 7e^{-2\theta}\lambda^2\widehat{u}_{i+1} + 7e^{-\theta}\lambda^2\widehat{u}_{i+1})}{3(-1+e^{\theta})(7-18e^{\theta}+7e^{2\theta})\lambda^2} \\ & + \bigl(e^{3\lambda(x-x_i)}\bigl(\widehat{M}_i + e^{\theta}\widehat{M}_i - 4e^{2\theta}\widehat{M}_i - \widehat{M}_{i+1} + 4e^{-2\theta}\widehat{M}_{i+1} - e^{-\theta}\widehat{M}_{i+1} \\ & - 16\lambda^2\widehat{u}_i - 16e^{\theta}\lambda^2\widehat{u}_i\bigr)\bigr) \\ & /\bigl(\bigl(-1+e^{\theta}\bigr)\bigl(7-18e^{\theta}+7e^{2\theta}\bigr)\lambda^2\bigr) \\ & + \frac{e^{3\lambda(x-x_i)}(+4\lambda^2e^{2\theta}\widehat{u}_i + 16\lambda^2\widehat{u}_{i+1} + 4e^{-2\theta}\lambda^2\widehat{u}_{i+1} + 16e^{-\theta}\lambda^2\widehat{u}_{i+1})}{(-1+e^{\theta})(7-18e^{\theta}+7e^{2\theta})\lambda^2} + O\bigl(h^4\bigr). \qquad (11)\end{aligned}$$

In consequence, for all $i = 0(1)n-1$ and $x \in (x_i, x_{i+1})$, we get

$$\bigl|S_i(x) - \widehat{S}_i(x)\bigr| \equiv \kappa_0 h^4, \qquad (12)$$

and similarly we get

$$\bigl|S_i''(x) - \widehat{S}_i''(x)\bigr| \equiv \kappa_1 h^2. \qquad (13)$$

See [38].

3 Convergence of the method

In this section the convergence of the method is proved. To do this, we consider equation (10) in a matrix form as follows:

$$\begin{aligned}& \hat{M} + P\hat{m} + Q\hat{U} \\ & \quad = F + \eta e^{2\theta}\bigl(5-7e^{\theta}\bigr)A\hat{M} + \eta e^{2\theta}\lambda^2\bigl(-80+28e^{\theta}\bigr)A\hat{U} \\ & \qquad + \eta\bigl(-5+7e^{-\theta}\bigr)B\hat{M} + \eta\lambda^2\bigl(80-28e^{-\theta}\bigr)B\hat{U} \\ & \qquad + \eta e^{\theta}\bigl(-8+7e^{\theta}+7e^{2\theta}\bigr)C\hat{M} + \eta e^{\theta}\lambda^2\bigl(128-7e^{\theta}-7e^{2\theta}\bigr)C\hat{U} \\ & \qquad + \eta e^{\theta}\bigl(-8+7e^{\theta}+7e^{2\theta}\bigr)D\hat{M} + \eta e^{\theta}\lambda^2\bigl(128-7e^{\theta}-7e^{2\theta}\bigr)D\hat{U}\end{aligned}$$

$$
\begin{aligned}
&+ \eta\left(8 - 7e^{-\theta} - 7e^{-2\theta}\right)\bar{E}\hat{M} + \eta\lambda^2\left(-128 + 7e^{-\theta} + 7e^{-2\theta}\right)\bar{E}\hat{U} \\
&+ \eta\left(-1 - e^{-\theta} + 4e^{-2\theta}\right)R\hat{M} + \eta\lambda^2\left(16 + 16e^{-\theta} - 4e^{-2\theta}\right)R\hat{U} \\
&+ \eta\left(-3 + 5e^{\theta}\right)G\hat{M} + \eta\lambda^2\left(27 - 5e^{\theta}\right)G\hat{U} \\
&+ \eta e^{-\theta}\left(3 - 5e^{-\theta}\right)H\hat{M} + \eta e^{-\theta}\lambda^2\left(-27 + 5e^{-\theta}\right)H\hat{U},
\end{aligned} \tag{14}
$$

where

$$
\eta = \frac{1}{3(-1 + e^{\theta})(7 - 18e^{\theta} + 7e^{2\theta})\lambda^2}.
$$

Using (14), we get the following expression:

$$
\begin{aligned}
&\widehat{M} + P\widehat{m} + Q\widehat{U} \\
&\quad = F + \eta\left(e^{2\theta}\left(5 - 7e^{\theta}\right)A + \left(-5 + 7e^{-\theta}\right)B + e^{\theta}\left(-8 + 7e^{\theta} + 7e^{2\theta}\right)C \right.\\
&\qquad \left. + \eta e^{\theta}\left(-8 + 7e^{\theta} + 7e^{2\theta}\right)D\right)\widehat{M} \\
&\qquad + \eta\left(\left(8 - 7e^{-\theta} - 7e^{-2\theta}\right)\bar{E} + \left(-1 - e^{-\theta} + 4e^{-2\theta}\right)R + \left(-3 + 5e^{\theta}\right)G \right.\\
&\qquad \left. + e^{-\theta}\left(3 - 5e^{-\theta}\right)H\right)\widehat{M} \\
&\qquad + \eta\lambda^2\left(e^{2\theta}\left(-80 + 28e^{\theta}\right)A + \left(80 - 28e^{-\theta}\right)B + e^{\theta}\left(128 - 7e^{\theta} - 7e^{2\theta}\right)C \right.\\
&\qquad \left. + e^{\theta}\left(128 - 7e^{\theta} - 7e^{2\theta}\right)D\right)\widehat{U} \\
&\qquad + \eta\lambda^2\left(\left(-128 + 7e^{-\theta} + 7e^{-2\theta}\right)\bar{E} + \left(16 + 16e^{-\theta} - 4e^{-2\theta}\right)R + \left(27 - 5e^{\theta}\right)G \right.\\
&\qquad \left. + e^{-\theta}\left(-27 + 5e^{-\theta}\right)H\right)\widehat{U} \\
&\quad \Rightarrow \quad W^{-1}\bar{R}\widehat{U} + PZ^{-1}S\widehat{U} + Q\widehat{U} = F + H_1 W^{-1}\bar{R}\widehat{U} + H_2\widehat{U},
\end{aligned} \tag{15}
$$

where

$$
\begin{aligned}
H_1 &= \eta e^{2\theta}\left(5 - 7e^{\theta}\right)A + \eta\left(-5 + 7e^{-\theta}\right)B + \eta e^{\theta}\left(-8 + 7e^{\theta} + 7e^{2\theta}\right)C \\
&\quad + \eta e^{\theta}\left(-8 + 7e^{\theta} + 7e^{2\theta}\right)D + \eta\left(8 - 7e^{-\theta} - 7e^{-2\theta}\right)\bar{E} \\
&\quad + \eta\left(-1 - e^{-\theta} + \eta 4e^{-2\theta}\right)R + \eta\left(-3 + 5e^{\theta}\right)G + \eta e^{-\theta}\left(3 - 5e^{-\theta}\right)H, \\
H_2 &= \eta\lambda^2\left(e^{2\theta}\left(-80 + 28e^{\theta}\right)A + \eta\left(80 - 28e^{-\theta}\right)B \right.\\
&\quad \left. + \eta e^{\theta}\left(128 - 7e^{\theta} - 7e^{2\theta}\right)C + \eta e^{\theta}\left(128 - 7e^{\theta} - 7e^{2\theta}\right)D\right) \\
&\quad + \eta\lambda^2\left(\left(-128 + 7e^{-\theta} + 7e^{-2\theta}\right)\bar{E} + \eta\left(16 + 16e^{-\theta} - 4e^{-2\theta}\right)R \right.\\
&\quad \left. + \eta\left(27 - 5e^{\theta}\right)G + \eta e^{-\theta}\left(-27 + 5e^{-\theta}\right)H\right).
\end{aligned}
$$

So the exact solution can be written as follows:

$$
Q\left[I - \left(-Q^{-1}W^{-1}\bar{R} - Q^{-1}PZ^{-1}S + Q^{-1}H_1 W^{-1}\bar{R} + Q^{-1}H_2\right)\right]\overline{U} = F + T, \tag{16}
$$

where $\overline{U} = [u(x_0), u(x_1), \ldots, u(x_n)]^T$ is the (n+1)-dimensional column vector of the exact solution, the vector of local truncation error is displayed as $T = [t_0, t_2, \ldots, t_n]^T$. According to (15) and (16), we have

$$Q\bigl[I-\bigl(-Q^{-1}W^{-1}\bar{R}-Q^{-1}PZ^{-1}S+Q^{-1}H_1W^{-1}\bar{R}+Q^{-1}H_2\bigr)\bigr]E=T, \tag{17}$$

where $E=(e_j)$ indicates the column vector of e_i, $i=0,1,2,\dots,n$, which is $(n+1)$-dimensional. Since $A_{n\times n}$ is a diagonally-dominant matrix, then $|A_{n\times n}|\neq 0$. We need the following lemma for analysis of convergence.

Lemma 1 *Let N be an $n\times n$ matrix with $\|N\|_\infty<1$. So, the matrix $(I-N)$ is invertible. Moreover, $\|(I-N)^{-1}\|_\infty\leq\frac{1}{1-\|N\|_\infty}$.*

Lemma 2 *The matrices W and Z are invertible.*

Proof For $\alpha=\frac{1}{6}$, $\beta=\frac{1}{3}$ and $\alpha=\frac{1}{12}$, $\beta=\frac{5}{12}$, the matrices W and Z are diagonally-dominant matrices, then are invertible. By using the inversion of general tridiagonal matrices [16] and [8], it is easy to prove that $\|W^{-1}\|_\infty\leq 1$ for $\alpha=\frac{1}{12}$, $\beta=\frac{5}{12}$ and $\|Z^{-1}\|_\infty\leq 1$ for $\alpha=\frac{1}{6}$, $\beta=\frac{1}{3}$. We need to show that the inverse of $Q[I-(-Q^{-1}W^{-1}\bar{R}-Q^{-1}PZ^{-1}S+Q^{-1}H_1W^{-1}\bar{R}+Q^{-1}H_2)]$ exists. Now, if Q is a diagonal matrix with the inverse Q^{-1}, we can derive the following lemma. We obtain $\|Q^{-1}\|_\infty\leq\frac{1}{\max|q_{ii}|}=\xi$. □

Lemma 3 *The matrix $[I-(-Q^{-1}W^{-1}\bar{R}-Q^{-1}PZ^{-1}S+Q^{-1}H_1W^{-1}\bar{R}+Q^{-1}H_2)]$ is nonsingular, provided*

$$\xi\bigl(\eta_2h^2+\eta_1\eta_3h^2+\eta_2\eta_5\|k\|_\infty(b-a)h^4+\|k\|_\infty(b-a)\eta_6\bigr)<1.$$

Proof Obviously, for $i=0,1,\dots,n$, it can be verified as follows:

$$\begin{cases}\|A\|_\infty=\|B\|_\infty\leq\|k\|_\infty(b-a)(\frac{e^\theta-1}{\theta}),\\ \|C\|_\infty=\|D\|_\infty\leq\|k\|_\infty(b-a)(\frac{e^{2\theta}-1}{2\theta}),\\ \|E\|_\infty=\|R\|_\infty\leq\|k\|_\infty(b-a)(\frac{e^{3\theta}-1}{3\theta}),\\ \|G\|_\infty=\|H\|_\infty\leq\|k\|_\infty(b-a)(\frac{e^{4\theta}-1}{4\theta}),\\ \|P\|_\infty=\operatorname{Max}|p(x_i)|\leq\eta_3,\\ \|Q\|_\infty=\operatorname{Max}|q(x_i)|\leq\eta_4,\\ \|S\|_\infty\leq\eta_1h^2,\|\bar{R}\|_\infty\leq\eta_2h^2,\\ \|H_1\|_\infty\leq\|k\|_\infty(b-a)h^2\eta_5,\\ \|H_2\|_\infty\leq\|k\|_\infty(b-a)\eta_6,\end{cases} \tag{18}$$

where

$$\begin{aligned}\eta_5={}&\left|\frac{1}{3(e^\theta-1)(-18e^\theta+7e^{2\theta}+7)\theta^2}\right|\biggl(\left|\frac{e^{2\theta}(5-7e^\theta)(e^\theta-1)}{\theta}\right|+\left|\frac{(7e^{-\theta}-5)(e^\theta-1)}{\theta}\right|\\&+\left|\frac{e^\theta(7e^\theta+7e^{2\theta}-8)(e^{2\theta}-1)}{2\theta}\right|+\left|\frac{e^\theta(7e^\theta+7e^{2\theta}-8)(e^{2\theta}-1)}{2\theta}\right|\\&+\left|\frac{(-7e^{-\theta}-7e^{-2\theta}+8)(e^{3\theta}-1)}{3\theta}\right|+\left|\frac{(-e^{-\theta}+4e^{-2\theta}-1)(e^{3\theta}-1)}{3\theta}\right|\\&+\left|\frac{(5e^\theta-3)(e^{4\theta}-1)}{4\theta}\right|+\left|\frac{e^{-\theta}(3-5e^{-\theta})(e^{4\theta}-1)}{4\theta}\right|\biggr),\end{aligned}$$

$$\begin{aligned}
\eta_6 = {} & \left|\frac{1}{3(e^{\theta}-1)(-18e^{\theta}+7e^{2\theta}+7)}\right|\left(\left|\frac{e^{2\theta}(28e^{\theta}-80)(e^{\theta}-1)}{\theta}\right| + \left|\frac{(80-28e^{-\theta})(e^{\theta}-1)}{\theta}\right|\right. \\
& + \left|\frac{e^{\theta}(-7e^{\theta}-7e^{2\theta}+128)(e^{2\theta}-1)}{2\theta}\right| + \left|\frac{e^{\theta}(-7e^{\theta}-7e^{2\theta}+128)(e^{2\theta}-1)}{2\theta}\right| \\
& + \left|\frac{(7e^{-\theta}+7e^{-2\theta}-128)(e^{3\theta}-1)}{3\theta}\right| + \left|\frac{(16e^{-\theta}-4e^{-2\theta}+16)(e^{3\theta}-1)}{3\theta}\right| \\
& \left. + \left|\frac{(27-5e^{\theta})(e^{4\theta}-1)}{4\theta}\right| + \left|\frac{e^{-\theta}(5e^{-\theta}-27)(e^{4\theta}-1)}{4\theta}\right|\right).
\end{aligned}$$

By using Lemma 1, we get

$$\begin{aligned}
& \|Q^{-1}\|\left(\|W^{-1}\|\|\bar{R}\| + \|P\|\|Z^{-1}\|\|S\| + \|H_1\|\|W^{-1}\|\|\bar{R}\| + \|H_2\|\right) < 1, \\
& \xi\left(\eta_2 h^2 + \eta_1\eta_3 h^2 + \eta_2\eta_5\|k\|_{\infty}(b-a)h^4 + \|k\|_{\infty}(b-a)\eta_6\right) < 1.
\end{aligned}$$

□

Theorem 1 *Assume $f(x) \in C^4(I)$, $k(t,x) \in C^4(I \times I)$ in a way that*

$$\xi\left(\eta_2 h^2 + \eta_1\eta_3 h^2 + \eta_2\eta_5\|k\|_{\infty}(b-a)h^4 + \|k\|_{\infty}(b-a)\eta_6\right) < 1.$$

Therefore consider a unique approximating solution and the obtained error $E := U - \hat{S}$ satisfies

$$\|E\| \equiv O(h^2),$$

where $\Omega := [a,b]$; moreover, α, η_l for $l = 1,2,3,4,5,6$ are constants.

Proof By using equation (17) and Lemma 1, we get

$$\|E\| \leq \frac{\|Q^{-1}\|\|T\|}{1 - \|Q^{-1}\|(\|W^{-1}\|\|\bar{R}\| + \|P\|\|Z^{-1}\|\|S\| + \|H_1\|\|W^{-1}\|\|\bar{R}\| + \|H_2\|)}. \tag{19}$$

By substituting $\|T\| \leq \frac{h^6}{240}\phi_4$ and (18) in (19), we get

$$\|E\| \equiv O(h^2).$$

Therefore, we have

$$\|U - \hat{S}\|_{\infty} \leq \zeta_2 h^2. \tag{20}$$

Therefore, applying (12) and (20) leads to

$$\|U - \hat{S}\|_{\infty} \leq \|U - S\|_{\infty} + \|S - \hat{S}\|_{\infty} \leq \zeta_2 h^2 + \zeta_0 h^4 \equiv O(h^2).$$

Then it may follow $\|E\| \to 0$ if $h \to 0$. So, for ($\alpha = \frac{1}{12}, \beta = \frac{5}{12}$) and ($\alpha = \frac{1}{6}, \beta = \frac{1}{3}$), we established the convergence of second-order method because we approximated m_0, m_n, M_0, and M_n by second-order methods. Therefore, α and β do not affect the second order of convergence. □

4 Numerical results

Here, we apply our method for $\alpha = \frac{1}{12}$, $\beta = \frac{5}{12}$ on some examples of the second-order boundary value problems of Fredholm integro-differential equation. First, the absolute error is calculated and then compared with the well-known methods in [11]. Note that numerical results are derived by *MAPLE 14*.

Example 1 As the first example, consider the following boundary value problem:

$$u''(x) + xu'(x) + \pi^2 u(x) = f(x) + \int_0^1 k(t,x)u(t)\,dx, \quad x \in [0,1],$$

subject to boundary conditions

$$u(0) = u(1) = 0,$$

where $k(t,x) = x + t$, $f(x) = \pi x \cos(\pi x) - \frac{2x+1}{\pi}$, and $u(x) = \sin(\pi x)$ is exact solution. The absolute errors are presented in Tables 1 and 2. The convergence ratio (C.R.) is gotten as follows:

$$\text{C.R.} = \log_2 \frac{E(h)}{E(\frac{h}{2})}, \tag{21}$$

where the maximum absolute error is shown by $E(h)$.

Example 2 Consider the following boundary value problem [11]:

$$u''(x) = f(x) + \int_0^1 k(t,x)u(t)\,dx, \quad x \in [0,1],$$

with boundary conditions

$$u(0) = u(1) = 0,$$

Table 1 $E(h)$ of Example 1

n	Our method	C.R.	Method in [11]	C.R
16	1.2194e-3		1.6002e-2	
32	3.7902e-4	1.6858	4.0613e-3	1.98
64	1.0683e-4	1.8271	1.0192e-3	1.99
128	2.8367e-5	1.9131	2.5504e-4	1.99

Table 2 Comparison of solutions obtained by the presented method with the exact solution of Example 1 while $n = 128$

x	Exact solution	Approximating solution	Obtained errors
0.125	0.382683	0.382694	1.11e-5
0.250	0.707107	0.707027	2.07e-5
0.375	0.923879	0.923906	2.67e-5
0.500	1.000000	1.000028	2.83e-5
0.625	0.923880	0.923904	2.54e-5
0.750	0.707107	0.707125	1.88e-5
0.875	0.382683	0.382693	0.98e-5

where $k(t,x) = e^{xt}$, $f(x) = 2 + \frac{x+2+e^x(x-2)}{x^3}$, and $u(x) = x(x-1)$ is the exact solution. The absolute errors are presented in Tables 3 and 4.

Example 3 Consider the following boundary value problem:

$$u''(x) - u'(x) = f(x) + \int_0^1 k(t,x)u(t)\,dx, \quad x \in [0,1],$$

subject to boundary conditions

$$u(0) = u(1) = 0,$$

where $k(t,x) = xt$, $f(x) = \frac{1+41x^3+31x^5-46x^4+8x-25x^2-13x^6+2x^7}{(x^2-x+1)^2} + \frac{121}{120}x - \frac{1}{6}\sqrt{3}\pi x$, and $u(x) = \ln(x^2 - x + 1) - \frac{(x^2-x)^2}{2}$ is the exact solution. The absolute errors are presented in Tables 5 and 6.

Table 3 $E(h)$ of Example 2

n	Our method	C.R.	Best in [11]	C.R
16	6.1104e-4		6.5606e-4	
32	1.5751e-4	1.9558	1.6398e-4	2.0003
64	4.0018e-5	1.9767	4.0991e-5	2.0001
128	1.0087e-5	1.9882	1.0248e-5	2.0000

Table 4 Comparison of solutions obtained by the presented method with the exact solution of Example 2 while $n = 128$

x	Exact solution	Approximating solution	Obtained errors
0.125	−1.09367e-1	−1.09375e-1	8.00000e-6
0.250	−1.87492e-1	−1.87500e-1	8.00000e-6
0.375	−2.34367e-1	−2.34375e-1	8.00000e-6
0.500	−2.49993e-1	−2.50000e-1	7.00000e-6
0.625	−2.34368e-1	−2.34375e-1	7.00000e-6
0.750	−1.87492e-1	−1.87500e-1	8.00000e-6
0.875	−1.09366e-1	−1.09375e-1	9.00000e-6

Table 5 $E(h)$ of Example 3

n	Our method	C.R.
16	2.0224e-3	
32	5.3502e-4	1.9184
64	1.3644e-4	1.9713
128	3.4381e-5	1.9886

Table 6 Comparison of solutions obtained by the presented method with the exact solution of Example 3 while $n = 128$

x	Exact solution	Approximating solution	Obtained errors
0.125	−1.2181e-1	−1.2184e-1	3.000e-5
0.250	−2.2522e-1	−2.2524e-1	2.000e-5
0.375	−2.9453e-1	−2.9456e-1	3.000e-5
0.500	−3.1893e-1	−3.1895e-1	2.000e-5
0.625	−2.9454e-1	−2.9455e-1	1.000e-5
0.750	−2.2522e-1	−2.2523e-1	1.000e-5
0.875	−1.2181e-1	−1.2182e-1	1.000e-5

5 Conclusion

In our knowledge, so far the exponential spline functions have not been yet applied for approximating the second-order integro-differential equations. In this study, according to the exponential method in [31], a suitable method is presented to approximate second-order integro-differential equations. The proposed algorithm is novel for second-order integro-differential equations. The second-order convergence of the proposed method has been derived and the computational outcomes have been found to be conformable with theoretical expectations. Our method shows better accuracy compared to the existing method in [11].

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