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Shannon type inequalities via time scales theory

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Abstract

The aim of present paper is to obtain Shannon type inequalities using the extended version of Jensen's inequality in time scales settings. The concept of differential entropy of a continuous random variable on time scales is introduced, and its bounds for some particular distributions are also estimated.

Keywords: Shannon entropy; Time scale calculus; Convex function

1 Introduction and preliminaries

In recent times, Shannon entropy and Zipf–Mandelbrot law have been the topics of great interest, see for example [1, 9, 11, 12]. The concept of Shannon entropy, the central source of information theory, is sometimes referred to as measure of uncertainty. Shannon entropy allows to estimate the average minimum number of bits needed to encode a string of symbols based on the alphabet size and the frequency of the symbols.

The following definition of Shannon entropy is given in [8].

Definition 1 The Shannon entropy of positive probability distribution $\mathbf{r} = (r_1, r_2, \dots, r_n)$ is defined by

$$S(\mathbf{r}) := - \sum_{i=1}^n r_i \log(r_i).$$

A fundamental inequality related to the notion of Shannon entropy is the following inequality given in [16]:

$$\sum_{i=1}^n r_i \log \frac{1}{r_i} \leq \sum_{i=1}^n r_i \log \frac{1}{f_i}, \tag{1}$$

which is valid for all $r_i, f_i > 0$ with

$$\sum_{i=1}^n r_i = \sum_{i=1}^n f_i = 1.$$

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Equality holds in (1) if and only if $r_i = f_i$ for all i . This result sometimes called the fundamental lemma of information theory has extensive applications (see [14]). In [13], Matic *et al.* gave the refinement of Shannon’s inequality in its discrete and integral forms by presenting upper estimates of the difference between its two sides. In [10], Sadia *et al.* studied some interesting results related to the bounds of the Shannon entropy by using nonincreasing (nondecreasing) sequences of real numbers.

One of the main approaches to unifying continuous and discrete mathematics is time scale calculus which was founded by German mathematician Stefan Hilger in 1988. A time scale is an arbitrary nonempty closed subset of the real numbers. For an introduction to the theory of dynamic equations on time scales, see [6]. In [7], Guseinov studied the process of Riemann and Lebesgue integration on time scales. Bohner and Guseinov [4, 5] defined the multiple Riemann and multiple Lebesgue integration on time scales and compared the Lebesgue Δ -integral with the Riemann Δ -integral. Various authors examined certain integral inequalities on time scales. In [2], Agarwal *et al.* proved the time scales version of Jensen’s inequality. In [18], Wong *et al.* proved the extended version of Jensen’s inequality on time scales. In [3], Anwar *et al.* derived a series of known inequalities, their extensions, and some new inequalities in the theory of dynamic equations on time scales by applying the theory of isotonic linear functionals. In [15], Rozarija Mikić and Josip Pečarić obtained lower and upper bounds for the difference in Jensen’s inequality and in the Edmundson–Lah–Ribarić inequality in time scales calculus which holds for the class of n -convex functions.

In the following considerations, \mathbb{T} denotes a time scale.

Definition 2 ([6]) A function $f : \mathbb{T} \rightarrow \mathbb{R}$ is called rd-continuous provided it is continuous at right-dense points of \mathbb{T} and its left-sided limits exist (finite) at left-dense points of \mathbb{T} . The set of rd-continuous functions $f : \mathbb{T} \rightarrow \mathbb{R}$ is denoted here by C_{rd} .

Definition 3 ([6]) A function $F : \mathbb{T} \rightarrow \mathbb{R}$ is called antiderivative of $f : \mathbb{T} \rightarrow \mathbb{R}$ if $F^\Delta(t) = f(t)$ for all $t \in \mathbb{T}^k$ and the delta integral

$$\int_{t_0}^t f(\tau) \Delta \tau = F(t) - F(t_0) \quad \text{for all } t, t_0 \in \mathbb{T}. \tag{2}$$

The following theorems are useful in the proof of the main results.

Theorem 1 ([6], Existence of antiderivatives) *Every rd-continuous function has an antiderivative.*

Theorem 2 ([18]) *Let $I \subset \mathbb{R}$ and assume that $r \in C_{rd}([a, b]_{\mathbb{T}}, \mathbb{R})$ with*

$$\int_a^b |r(s)| \Delta s > 0,$$

where $a, b \in \mathbb{T}$. If $g \in C(I, \mathbb{R})$ is convex and $\xi \in C_{rd}([a, b]_{\mathbb{T}}, I)$, then

$$g\left(\frac{\int_a^b |r(s)| \xi(s) \Delta s}{\int_a^b |r(s)| \Delta s}\right) \leq \frac{\int_a^b |r(s)| g(\xi(s)) \Delta s}{\int_a^b |r(s)| \Delta s}. \tag{3}$$

The inequality in (3) is strict if g is strictly convex.

2 Main results

Throughout the paper ‘log’ refers to logarithms to base \bar{b} for some fixed $\bar{b} > 1$. We initiate with the following result.

Theorem 3 *Let $r \in C_{rd}([a, b]_{\mathbb{T}}, \mathbb{R}^+)$ and assume that*

$$\int_a^b r(s) \Delta s > 0,$$

where $a, b \in \mathbb{T}$. If $\xi, \frac{1}{\xi} \in C_{rd}([a, b]_{\mathbb{T}}, \mathbb{R}^+)$ such that $\int_a^b r(s) \xi(s) \Delta s < \infty$ and $\int_a^b \frac{r(s)}{\xi(s)} \Delta s < \infty$, then we have

$$0 \leq \log \left[\frac{\int_a^b r(s) \xi(s) \Delta s}{\int_a^b r(s) \Delta s} \right] - \frac{\int_a^b r(s) \log \xi(s) \Delta s}{\int_a^b r(s) \Delta s} \tag{4}$$

$$\leq \log \left[\frac{\int_a^b r(s) \xi(s) \Delta s \int_a^b \frac{r(s)}{\xi(s)} \Delta s}{(\int_a^b r(s) \Delta s)^2} \right] \tag{5}$$

$$\leq \frac{1}{\ln \bar{b}} \left[\frac{\int_a^b r(s) \xi(s) \Delta s \int_a^b \frac{r(s)}{\xi(s)} \Delta s}{(\int_a^b r(s) \Delta s)^2} - 1 \right]. \tag{6}$$

Proof Use inequality (3) for the convex function $g(x) = -\log x, x > 0$ to get inequality (4). Replace ξ by $\frac{1}{\xi}$ in inequality (4), which implies

$$-\frac{\int_a^b r(s) \log \xi(s) \Delta s}{(\int_a^b r(s) \Delta s)} = \frac{\int_a^b r(s) \log \frac{1}{\xi(s)} \Delta s}{(\int_a^b r(s) \Delta s)} \leq \log \left(\frac{\int_a^b \frac{r(s)}{\xi(s)} \Delta s}{\int_a^b r(s) \Delta s} \right). \tag{7}$$

Now, by adding $\log \left(\frac{\int_a^b r(s) \xi(s) \Delta s}{\int_a^b r(s) \Delta s} \right)$ on both sides of (7), we get

$$\log \left(\frac{\int_a^b r(s) \xi(s) \Delta s}{\int_a^b r(s) \Delta s} \right) - \frac{\int_a^b r(s) \log \xi(s) \Delta s}{(\int_a^b r(s) \Delta s)} \leq \log \left(\frac{\int_a^b r(s) \xi(s) \Delta s}{\int_a^b r(s) \Delta s} \right) + \log \left(\frac{\int_a^b \frac{r(s)}{\xi(s)} \Delta s}{\int_a^b r(s) \Delta s} \right),$$

which is inequality (5). Inequality (6) is a straightforward outcome of the following inequality given in [13]:

$$\log x \leq \frac{1}{\ln \bar{b}} (x - 1) \quad (x > 0), \tag{8}$$

with

$$x = \frac{1}{(\int_a^b r(s) \Delta s)^2} \left(\int_a^b r(s) \xi(s) \Delta s \int_a^b \frac{r(s)}{\xi(s)} \Delta s \right). \quad \square$$

2.1 Shannon entropy

Consider X to be a continuous random variable with a nonnegative density function $r(s)$ on \mathbb{T} such that $\int_a^b r(s) \Delta s = 1$, whenever the integral exists, we have the following definition.

Definition 4 The nominal differential entropy of X on time scale is defined by

$$h_{\bar{b}}(X) := \int_a^b r(s) \log \frac{1}{r(s)} \Delta s \quad (\bar{b} > 1). \tag{9}$$

The following result is the time scale extension of integral Shannon inequality [13, Theorem 18]. Moreover, one can get results related to Shannon entropy by choosing time scale to be the set of integers with positive probability distributions in the following result.

Theorem 4 Let $a, b \in \mathbb{T}$, $a < b$ and assume that $r, f \in C_{rd}([a, b]_{\mathbb{T}}, \mathbb{R})$ are positive functions with $\int_a^b r(s) \Delta s > 0$ and $\lambda := \int_a^b f(s) \Delta s < \infty$. Suppose that for $\bar{b} > 1$ at least one of the following Δ -integrals is finite:

$$Q_r := \int_a^b r(s) \log \frac{1}{r(s)} \Delta s \quad \text{and} \quad Q_f := \int_a^b r(s) \log \frac{1}{f(s)} \Delta s.$$

If $\int_a^b \frac{r^2(s)}{f(s)} \Delta s < \infty$, then

$$\begin{aligned} 0 &\leq \log \left(\frac{\lambda}{\int_a^b r(s) \Delta s} \right) + \frac{1}{(\int_a^b r(s) \Delta s)} (Q_f - Q_r) \\ &\leq \log \left[\frac{\lambda}{(\int_a^b r(s) \Delta s)^2} \int_a^b \frac{r^2(s)}{f(s)} \Delta s \right] \\ &\leq \frac{1}{\ln \bar{b}} \left[\frac{\lambda}{(\int_a^b r(s) \Delta s)^2} \int_a^b \frac{r^2(s)}{f(s)} \Delta s - 1 \right]. \end{aligned} \tag{10}$$

Proof Apply Theorem 3 with $\xi(s) = \frac{f(s)}{r(s)}$ ($s \in \mathbb{T}$) and $\lambda = \int_a^b f(s) \Delta s = \int_a^b r(s) \xi(s) \Delta s < \infty$ to get

$$\begin{aligned} 0 &\leq \log \left(\frac{\lambda}{\int_a^b r(s) \Delta s} \right) - \frac{1}{(\int_a^b r(s) \Delta s)} \int_a^b r(s) \log \frac{f(s)}{r(s)} \Delta s \\ &\leq \log \left[\frac{\lambda}{(\int_a^b r(s) \Delta s)^2} \int_a^b \frac{r^2(s)}{f(s)} \Delta s \right] \\ &\leq \frac{1}{\ln \bar{b}} \left[\frac{\lambda}{(\int_a^b r(s) \Delta s)^2} \int_a^b \frac{r^2(s)}{f(s)} \Delta s - 1 \right]. \end{aligned}$$

Since

$$\log x < x \quad (x > 0), \tag{11}$$

therefore replacing x by $\frac{f(s)}{r(s)}$ and multiplying both sides by $r(s)$ in (11), we get

$$r(s) \log \frac{f(s)}{r(s)} < r(s) \frac{f(s)}{r(s)},$$

thus

$$J := \int_a^b r(s) \log \frac{f(s)}{r(s)} \Delta s < \infty \quad \text{as} \quad \int_a^b f(s) \Delta s < \infty.$$

Whenever Q_r is finite, then $Q_r - J = Q_f$ is also finite, further if Q_f is finite, then $Q_f + J = Q_r$ is finite as well. Therefore we may write $J = Q_r - Q_f$, and consequently the desired result is proved. \square

Corollary 1 *Let $a, b \in \mathbb{T}$, $a < b$, and $r, f \in C_{rd}([a, b]_{\mathbb{T}}, \mathbb{R}^+)$ with $\lambda := \int_a^b f(s) \Delta s < \infty$. Suppose that for $\bar{b} > 1$ at least one of the following Δ -integrals is finite:*

$$Q_r := \int_a^b r(s) \log \frac{1}{r(s)} \Delta s \quad \text{and} \quad Q_f := \int_a^b r(s) \log \frac{1}{f(s)} \Delta s.$$

If $\int_a^b \frac{r^2(s)}{f(s)} \Delta s < \infty$, then

$$\begin{aligned} 0 &\leq \log \lambda + (Q_f - Q_r) \\ &\leq \log \left[\lambda \int_a^b \frac{r^2(s)}{f(s)} \Delta s \right] \\ &\leq \frac{1}{\ln \bar{b}} \left[\lambda \int_a^b \frac{r^2(s)}{f(s)} \Delta s - 1 \right]. \end{aligned}$$

Proof Use $\int_a^b r(s) \Delta s = 1$ in Theorem 4 to get the required result. \square

Remark 1 Choose $\mathbb{T} = \mathbb{R}$ in Theorem 4 with $\int_a^b r(s) \Delta s = 1$ to get [13, Theorem 18].

In the proof of our next result, we need the following weighted Grüss type inequality on time scales established by Sarikaya *et al.* in [17].

Theorem 5 *Let $\xi, g \in C_{rd}$ and $\xi, g : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}$ be two Δ -integrable functions on $[a, b]_{\mathbb{T}}$ and $r \in C_{rd}$ be a positive function with $\int_a^b r(s) \Delta s > 0$. Then, for*

$$\alpha \leq \xi(s) \leq A, \quad \beta \leq g(s) \leq B \quad \forall s \in [a, b]_{\mathbb{T}},$$

we have

$$\left| \frac{\int_a^b r(s) \xi(s) g(s) \Delta s}{\int_a^b r(s) \Delta s} - \frac{\int_a^b r(s) \xi(s) \Delta s \int_a^b r(s) g(s) \Delta s}{(\int_a^b r(s) \Delta s)^2} \right| \leq \frac{1}{4} (A - \alpha)(B - \beta). \tag{12}$$

Lemma 1 *Suppose that the assumptions of Theorem 3 are satisfied. If*

$$0 < m \leq \xi(s) \leq M \quad \forall s \in [a, b]_{\mathbb{T}}, \tag{13}$$

then

$$0 \leq \log \left[\frac{\int_a^b r(s) \xi(s) \Delta s}{\int_a^b r(s) \Delta s} \right] - \frac{\int_a^b r(s) \log \xi(s) \Delta s}{(\int_a^b r(s) \Delta s)} \tag{14}$$

$$\leq \log \left[\frac{1}{4} \left(\sqrt{Q} + \frac{1}{\sqrt{Q}} \right)^2 \right] \tag{15}$$

$$\leq \frac{1}{\ln \bar{b}} \left[\frac{1}{4} \left(\sqrt{Q} - \frac{1}{\sqrt{Q}} \right)^2 \right], \tag{16}$$

where $m, M \in \mathbb{R}_+$ and $\varrho := \frac{M}{m}$. Further, if

$$\varrho \leq \Phi(\varepsilon) := 2\bar{b}^\varepsilon - 1 + 2\sqrt{\bar{b}^\varepsilon(\bar{b}^\varepsilon - 1)} \tag{17}$$

for $\varepsilon > 0$, then

$$0 \leq \log \left[\frac{\int_a^b r(s)\xi(s)\Delta s}{\int_a^b r(s)\Delta s} \right] - \frac{\int_a^b r(s)\log \xi(s)\Delta s}{(\int_a^b r(s)\Delta s)} \leq \varepsilon. \tag{18}$$

Proof Inequality (14) is the same as (4). From (13) one gets

$$0 < \frac{1}{M} \leq \frac{1}{\xi(s)} \leq \frac{1}{m} \quad \text{for all } s \in [a, b]_{\mathbb{T}}.$$

Set $g = \frac{1}{\xi}$ in inequality (12) to get

$$\frac{\int_a^b r(s)\xi(s)\Delta s \int_a^b r(s)\frac{1}{\xi(s)}\Delta s}{(\int_a^b r(s)\Delta s)^2} - 1 \leq \frac{1}{4}(M - m)\left(\frac{1}{m} - \frac{1}{M}\right)$$

or

$$\begin{aligned} \frac{\int_a^b r(s)\xi(s)\Delta s \int_a^b r(s)\frac{1}{\xi(s)}\Delta s}{(\int_a^b r(s)\Delta s)^2} &\leq \frac{1}{4}\left[\left(\frac{M - m}{m} - \frac{M - m}{M}\right) + 4\right] \\ &= \frac{1}{4}\left[\left(\frac{M}{m} + \frac{m}{M} - 2\right) + 4\right] \\ &= \frac{1}{4}\left[\left(\varrho + \frac{1}{\varrho} - 2\right) + 4\right] \\ &= \frac{1}{4}\left[\left(\sqrt{\varrho} - \frac{1}{\sqrt{\varrho}}\right)^2 + 4\right] \\ &= \frac{1}{4}\left(\sqrt{\varrho} + \frac{1}{\sqrt{\varrho}}\right)^2. \end{aligned}$$

Since log is strictly increasing, we have

$$\log \left[\frac{\int_a^b r(s)\xi(s)\Delta s \int_a^b r(s)\frac{1}{\xi(s)}\Delta s}{(\int_a^b r(s)\Delta s)^2} \right] \leq \log \left[\frac{1}{4}\left(\sqrt{\varrho} + \frac{1}{\sqrt{\varrho}}\right)^2 \right]. \tag{19}$$

Using inequality (19) together with (5) gives (15). However, (16) can be easily derived from the elementary inequality (8). Further, set

$$\log \left[\frac{1}{4}\left(\sqrt{\varrho} + \frac{1}{\sqrt{\varrho}}\right)^2 \right] \leq \varepsilon,$$

or

$$\left[\frac{1}{4}\left(\sqrt{\varrho} + \frac{1}{\sqrt{\varrho}}\right)^2 \right] \leq \bar{b}^\varepsilon,$$

therefore

$$\varrho^2 - 2\varrho(2\bar{b}^\varepsilon - 1) + 1 \leq 0,$$

which holds if and only if

$$2\bar{b}^\varepsilon - 1 - 2\sqrt{\bar{b}^\varepsilon(\bar{b}^\varepsilon - 1)} \leq \varrho \leq 2\bar{b}^\varepsilon - 1 + 2\sqrt{\bar{b}^\varepsilon(\bar{b}^\varepsilon - 1)}.$$

Since

$$\begin{aligned} [2\bar{b}^\varepsilon - 1 + 2\sqrt{\bar{b}^\varepsilon(\bar{b}^\varepsilon - 1)}]^{-1} &= \frac{1}{2\bar{b}^\varepsilon - 1 + 2\sqrt{\bar{b}^\varepsilon(\bar{b}^\varepsilon - 1)}} \\ &= \frac{(2\bar{b}^\varepsilon - 1) - 2\sqrt{\bar{b}^\varepsilon(\bar{b}^\varepsilon - 1)}}{(2\bar{b}^\varepsilon - 1)^2 - 4(\bar{b}^\varepsilon(\bar{b}^\varepsilon - 1))} \\ &= \frac{(2\bar{b}^\varepsilon - 1) - 2\sqrt{\bar{b}^\varepsilon(\bar{b}^\varepsilon - 1)}}{4\bar{b}^{2\varepsilon} + 1 - 4\bar{b}^\varepsilon - 4\bar{b}^{2\varepsilon} + 4\bar{b}^\varepsilon} \\ &= 2\bar{b}^\varepsilon - 1 - 2\sqrt{\bar{b}^\varepsilon(\bar{b}^\varepsilon - 1)}, \end{aligned}$$

inequality (18) follows from (14) and it holds whenever ϱ satisfies (17). □

Remark 2 Let $\mathbb{T} = \mathbb{R}$ with $\int_a^b r(s)\Delta s = 1$ in Lemma 1 to get [13, Lemma 2].

Theorem 6 Assume the conditions of Theorem 4, and let

$$0 < m \leq \frac{r(s)}{f(s)} \leq M \quad \forall s \in [a, b]_{\mathbb{T}}. \tag{20}$$

Then

$$\begin{aligned} 0 &\leq \left(\frac{\int_a^b r(s) \log \frac{1}{f(s)} \Delta s}{\int_a^b r(s) \Delta s} \right) - \left(\frac{\int_a^b r(s) \log \frac{1}{r(s)} \Delta s}{\int_a^b r(s) \Delta s} \right) + \log \left(\frac{\lambda}{\int_a^b r(s) \Delta s} \right) \\ &\leq \log \frac{(M + m)^2}{4Mm} \\ &\leq \frac{1}{4 \ln \bar{b}} \frac{(M - m)^2}{Mm}. \end{aligned}$$

Also, if $\frac{M}{m} \leq \Phi(\varepsilon) := 2\bar{b}^\varepsilon - 1 + 2\sqrt{\bar{b}^\varepsilon(\bar{b}^\varepsilon - 1)}$ for some $\varepsilon > 0$, then

$$0 \leq \left(\frac{\int_a^b r(s) \log \frac{1}{f(s)} \Delta s}{\int_a^b r(s) \Delta s} \right) - \left(\frac{\int_a^b r(s) \log \frac{1}{r(s)} \Delta s}{\int_a^b r(s) \Delta s} \right) + \log \left(\frac{\lambda}{\int_a^b r(s) \Delta s} \right) \leq \varepsilon.$$

Proof Apply Lemma 1 with $\xi(s) = \frac{f(s)}{r(s)}$ ($s \in [a, b]_{\mathbb{T}}$) and

$$0 < \frac{1}{M} \leq \xi(s) \leq \frac{1}{m} \quad \forall s \in [a, b]_{\mathbb{T}}$$

to obtain the desired results. □

Corollary 2 Consider the assumptions of Theorem 6 with $\int_a^b r(s)\Delta s = 1$, then we obtain

$$\begin{aligned} 0 &\leq \int_a^b r(s) \log \frac{1}{f(s)} \Delta s - \int_a^b r(s) \log \frac{1}{r(s)} \Delta s + \log \lambda \\ &\leq \log \frac{(M+m)^2}{4Mm} \\ &\leq \frac{1}{4 \ln b} \frac{(M-m)^2}{Mm}. \end{aligned}$$

Remark 3 Let $\mathbb{T} = \mathbb{R}$ with $\int_a^b r(s)\Delta s = 1$ in Theorem 6 to get [13, Theorem 19].

2.2 Entropy of continuous random variable

In the sequel, we denote mean and variance of a continuous random variable X by $\mu_m = \int_a^b sr(s)\Delta s$ and $v^2 = \int_a^b (s - \mu_m)^2 r(s)\Delta s$ respectively.

Theorem 7 Consider a continuous random variable X and density function $r(s)$ ($s \in \mathbb{T}$).

(a) If X has a finite mean μ_m and variance v^2 with

$$\int_a^b r^2(s) \exp\left[\frac{1}{2v^2}(s - \mu_m)^2\right] \Delta s < \infty,$$

then $h_{\bar{b}}(X)$ is finite and

$$\begin{aligned} 0 &\leq \log(v\sqrt{2\pi}e) - h_{\bar{b}}(X) + \log(\lambda) \\ &\leq \log\left\{\lambda v\sqrt{2\pi} \int_a^b r^2(s) \exp\left[\frac{1}{2v^2}(s - \mu_m)^2\right] \Delta s\right\} \\ &\leq \frac{1}{\ln b} \left\{\lambda v\sqrt{2\pi} \int_a^b r^2(s) \exp\left[\frac{1}{2v^2}(s - \mu_m)^2\right] \Delta s - 1\right\}, \end{aligned}$$

where $\lambda = \int_a^b (1/v\sqrt{2\pi}) \exp[-(s - \mu_m)^2/2v^2] \Delta s > 0$.

(b) Suppose that X has finite mean and $r(s) = 0$ for all $s < 0$. If

$$\int_0^\infty r^2(s) \exp(s/\mu_m) \Delta s < \infty,$$

then $h_{\bar{b}}(X)$ is finite and

$$\begin{aligned} 0 &\leq \log(\mu_m e) - h_{\bar{b}}(X) + \log(\lambda) \\ &\leq \log\left[\lambda \mu_m \int_0^\infty r^2(s) \exp(s/\mu_m) \Delta s\right] \\ &\leq \frac{1}{\ln b} \left[\lambda \mu_m \int_0^\infty r^2(s) \exp(s/\mu_m) \Delta s - 1\right], \end{aligned}$$

where $\lambda = \int_0^\infty (1/\mu_m) \exp(-s/\mu_m) \Delta s > 0$.

Proof

- (a) As the variance v^2 of X is finite, which implies that $\mu_m = \int_a^b sr(s)\Delta s$ and $v^2 = \int_a^b (s - \mu_m)^2 r(s)\Delta s > 0$ are well defined real numbers, we can define $f(s) = (1/v\sqrt{2\pi}) \exp[-(s - \mu_m)^2/2v^2] > 0$ ($s \in \mathbb{T}$) to get $\lambda = \int_a^b f(s)\Delta s > 0$ and

$$\begin{aligned} \int_a^b r(s) \log \frac{1}{f(s)} \Delta s &= \frac{1}{\ln b} \int_a^b r(s) \ln \left(\frac{1}{f(s)} \right) \Delta s \\ &= \frac{1}{\ln b} \int_a^b r(s) \ln [(v\sqrt{2\pi}) \exp[(s - \mu_m)^2/2v^2]] \Delta s \\ &= \frac{1}{\ln b} \int_a^b r(s) [\ln(v\sqrt{2\pi}) + \ln(\exp[(s - \mu_m)^2/2v^2])] \Delta s \\ &= \frac{1}{\ln b} \int_a^b r(s) \left[\ln(v\sqrt{2\pi}) + \frac{(s - \mu_m)^2}{2v^2} \right] \Delta s \\ &= \frac{1}{\ln b} \left[\ln(v\sqrt{2\pi}) \int_a^b r(s) \Delta s + \frac{1}{2v^2} \int_a^b r(s)(s - \mu_m)^2 \Delta s \right] \\ &= \frac{1}{\ln b} \left[\ln(v\sqrt{2\pi}) + \frac{1}{2v^2} \cdot v^2 \right] \\ &= \frac{1}{\ln b} \left[\ln(v\sqrt{2\pi}) + \frac{1}{2} \right] \\ &= \frac{1}{\ln b} [\ln(v\sqrt{2\pi}) + \ln \sqrt{e}] \\ &= \frac{1}{\ln b} \ln(v\sqrt{2\pi e}) \\ &= \log(v\sqrt{2\pi e}). \end{aligned}$$

Now apply Corollary 1 to get the stated result.

- (b) Under the given conditions, we have mean $\mu_m = \int_0^\infty sr(s)\Delta s > 0$, and we may define $f(s) = (1/\mu_m) \exp(-s/\mu_m)$ ($s \in [0, \infty)_{\mathbb{T}}$) such that $\lambda = \int_0^\infty f(s)\Delta s > 0$ and

$$\begin{aligned} \int_0^\infty r(s) \log \frac{1}{f(s)} \Delta s &= \frac{1}{\ln b} \int_a^b r(s) \ln \frac{1}{f(s)} \Delta s \\ &= \frac{1}{\ln b} \int_a^b r(s) \ln [\mu_m \exp(s/\mu_m)] \Delta s \\ &= \frac{1}{\ln b} \int_a^b r(s) [\ln \mu_m + \ln(\exp(s/\mu_m))] \Delta s \\ &= \frac{1}{\ln b} \int_0^\infty r(s) \left(\ln \mu_m + \frac{s}{\mu_m} \right) \Delta s \\ &= \frac{1}{\ln b} \left[\ln \mu_m \int_0^\infty r(s) \Delta s + \frac{1}{\mu_m} \int_0^\infty sr(s) \Delta s \right] \\ &= \frac{1}{\ln b} \left[\ln \mu_m + \frac{1}{\mu_m} \cdot \mu_m \right] \\ &= \frac{1}{\ln b} (\ln \mu_m + \ln e) \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{\ln b} \ln(\mu_m e) \\
 &= \log(\mu_m e).
 \end{aligned}$$

Again apply Corollary 1 to obtain the required result. □

Corollary 3 Assume $\mathbb{T} = \mathbb{R}$ in Theorem 7 to get [13, Theorem 21 a, b].

Remark 4 Theorem 7 shows that $h_{\bar{b}}(X) \approx \log(\lambda v \sqrt{2\pi} e)$ whenever the distribution of X is nearly equal to the Gaussian distribution with variance v^2 . If the distribution of X is close to the exponential distribution with mean μ_m , then we have $h_{\bar{b}}(X) \approx \log(\lambda \mu_m e)$.

Theorem 8

(a) Under the assumptions of Theorem 7(a), if

$$0 < \delta \leq r(s) \exp\left[\frac{1}{2v^2}(s - \mu_m)^2\right] \leq \theta \quad \forall s \in \mathbb{T},$$

then

$$\begin{aligned}
 0 &\leq \log(v\sqrt{2\pi} e) - h_{\bar{b}}(X) + \log(\lambda) \\
 &\leq \log \frac{(\theta + \delta)^2}{4\delta\theta} \\
 &\leq \frac{1}{4\ln \bar{b}} \frac{(\theta - \delta)^2}{\delta\theta},
 \end{aligned}$$

where $\delta, \theta \in \mathbb{R}_+$ and $\lambda = \int_a^b (1/v\sqrt{2\pi}) \exp[-(s - \mu_m)^2/2v^2] \Delta s > 0$.

(b) Consider the assumptions of Theorem 7(b), if

$$0 < \delta \leq r(s) \exp(s/\mu_m) \leq \theta \quad \forall s \in [0, \infty)_{\mathbb{T}},$$

then

$$\begin{aligned}
 0 &\leq \log(\mu_m e) - h_{\bar{b}}(X) + \log(\lambda) \\
 &\leq \log \frac{(\theta + \delta)^2}{4\delta\theta} \\
 &\leq \frac{1}{4\ln \bar{b}} \frac{(\theta - \delta)^2}{\delta\theta},
 \end{aligned}$$

where $\lambda = \int_0^\infty (1/\mu_m) \exp(-s/\mu_m) \Delta s > 0$.

Proof

- (a) In Corollary 2 replace m and M by $v\sqrt{2\pi}\delta$ and $v\sqrt{2\pi}\theta$ respectively and $f(s)$ as in the proof of Theorem 7(a).
- (b) In Corollary 2 replace m and M with $\mu_m\delta$ and $\mu_m\theta$ respectively and $f(s)$ as in the proof of Theorem 7(b). □

Corollary 4 Consider $\mathbb{T} = \mathbb{R}$ in Theorem 8 to get [13, Theorem 22 a, b].

The following generalization of Jensen’s inequality on time scales established by Anwar [3] et al. is needed in the proof of our next result.

Theorem 9 *Let $J \subset \mathbb{R}$ be an interval and assume that $\Psi \in C(J, \mathbb{R})$ is convex. Consider f to be Δ -integrable on D such that $f(D) \subset J$, where $D \subset ([a_1, b_1] \cap \mathbb{T}_1 \times \cdots \times [a_n, b_n] \cap \mathbb{T}_n)$ and $\mathbb{T}_1, \mathbb{T}_2, \dots, \mathbb{T}_n$ are time scales. Moreover, let $p : D \rightarrow \mathbb{R}$ be Δ -integrable such that $\int_D |p(s)| \Delta s > 0$. Then*

$$\Psi \left(\frac{\int_D |p(s)| f(s) \Delta s}{\int_D |p(s)| \Delta s} \right) \leq \frac{\int_D |p(s)| \Psi(f(s)) \Delta s}{\int_D |p(s)| \Delta s}. \tag{21}$$

The following result is a generalization of Theorem 3.

Proposition 1 *Let $\mathbb{T}_1, \mathbb{T}_2, \dots, \mathbb{T}_n$ be time scales. For $a_i, b_i \in \mathbb{T}_i$ with $a_i < b_i, 1 \leq i \leq n$, let $D \subset ([a_1, b_1] \cap \mathbb{T}_1 \times \cdots \times [a_n, b_n] \cap \mathbb{T}_n)$ be Lebesgue Δ -measurable, and let $\psi : D \rightarrow (0, \infty)$ be a positive Δ -integrable function such that $\int_D |\psi(w)| \Delta w > 0$. If $\xi, \frac{1}{\xi} : D \rightarrow (0, \infty)$ are two positive Δ -integrable functions such that*

$$\int_D |\psi(w) \xi(w)| \Delta w < \infty \quad \text{and} \quad \int_D \left| \frac{\psi(w)}{\xi(w)} \right| \Delta w < \infty,$$

then we get

$$\begin{aligned} 0 &\leq \log \left[\frac{\int_D \psi(w) \xi(w) \Delta w}{\int_D \psi(w) \Delta w} \right] - \frac{\int_D \psi(w) \log \xi(w) \Delta w}{(\int_D \psi(w) \Delta w)} \\ &\leq \log \left[\frac{\int_D \psi(w) \xi(w) \Delta w \int_D \frac{\psi(w)}{\xi(w)} \Delta w}{(\int_D \psi(w) \Delta w)^2} \right] \\ &\leq \frac{1}{\ln b} \left[\frac{\int_D \psi(w) \xi(w) \Delta w \int_D \frac{\psi(w)}{\xi(w)} \Delta w}{(\int_D \psi(w) \Delta w)^2} - 1 \right]. \end{aligned} \tag{22}$$

Proof Use inequality (21) and follow similar steps as in the proof of Theorem 3 to get the stated result. □

Corollary 5 *Assume the conditions of Proposition 1 with $\int_D \psi(w) \Delta w = 1$, then we have*

$$\begin{aligned} 0 &\leq \log \left[\int_D \psi(w) \xi(w) \Delta w \right] - \int_D \psi(w) \log \xi(w) \Delta w \\ &\leq \log \left[\int_D \psi(w) \xi(w) \Delta w \int_D \frac{\psi(w)}{\xi(w)} \Delta w \right] \\ &\leq \frac{1}{\ln b} \left[\int_D \psi(w) \xi(w) \Delta w \int_D \frac{\psi(w)}{\xi(w)} \Delta w - 1 \right]. \end{aligned}$$

Remark 5 Choose $\mathbb{T} = \mathbb{R}$ with $\int_D \psi(w) \Delta w = 1$ in Proposition 1 to get [13, Proposition 1].

Suppose that X and Z are random variables whose distributions have density functions $r(s)$ and $r(z)$ respectively, and let $r(s, z)$ be the joint density function for (X, Z) . Denote

$$D_X := \{s \in X : r(s) > 0\}, \quad D_Z := \{z \in Z : r(z) > 0\} \tag{23}$$

and

$$D := \{(s, z) \in X \times Z : r(s, z) > 0\}. \tag{24}$$

Definition 5 The differential \bar{b} -entropy of X on time scales is defined by

$$h_{\bar{b}}(X) := \int_{D_X} r(s) \log \frac{1}{r(s)} \Delta s. \tag{25}$$

By analogy, we may state the following definition.

Definition 6 The differential conditional \bar{b} -entropy of X given Z on time scales is defined by

$$h_{\bar{b}}(X|Z) := \int \int_D r(s, z) \log \frac{1}{r(s|z)} \Delta s \Delta z. \tag{26}$$

Theorem 10 Suppose that X and Z are random variables whose distributions have density functions $r(s)$ and $r(z)$ respectively, and let $r(s, z)$ be the joint density function for (X, Z) . Let

$$R' := \int \int_D r(z) \Delta s \Delta z < \infty \quad \text{and} \quad \int \int_D r(z) r^2(s|z) \Delta s \Delta z < \infty.$$

Then $h_{\bar{b}}(X|Z)$ exists and

$$\begin{aligned} 0 &\leq \log R' - h_{\bar{b}}(X|Z) \\ &\leq \log \left[R' \int \int_D r(z) r^2(s|z) \Delta s \Delta z \right] \\ &\leq \frac{1}{\ln \bar{b}} \left[R' \int \int_D r(z) r^2(s|z) \Delta s \Delta z - 1 \right]. \end{aligned}$$

Proof Apply Corollary 5 with $n = 2$ and

$$\psi'(w) = r(s, z), \quad \xi(w) = \frac{1}{r(s|z)} = \frac{r(z)}{r(s, z)} \quad \text{for } w = (s, z) \in D,$$

to get

$$\begin{aligned} 0 &\leq \log \left[\int \int_D r(s, z) \frac{r(z)}{r(s, z)} \Delta s \Delta z \right] - \int \int_D r(s, z) \log \frac{1}{r(s|z)} \Delta s \Delta z \\ &= \log R' - h_{\bar{b}}(X|Z) \\ &\leq \log \left[R' \int \int_D r(s, z) r(s|z) \Delta s \Delta z \right] \\ &= \log \left[R' \int \int_D r(z) r^2(s|z) \Delta s \Delta z \right] \\ &\leq \frac{1}{\ln \bar{b}} \left[R' \int \int_D r(z) r^2(s|z) \Delta s \Delta z - 1 \right]. \quad \square \end{aligned}$$

With the help of (23) and (24), we can define the differential mutual information between X and Z on time scales by

$$i_{\bar{b}}(X, Z) := h_{\bar{b}}(X) - h_{\bar{b}}(X|Z),$$

where $h_{\bar{b}}(X)$ and $h_{\bar{b}}(X|Z)$ are given by (25) and (26), respectively. It is straightforward to see that

$$i_{\bar{b}}(X, Z) = \int \int_D r(s, z) \log \frac{r(s, z)}{r(s)r(z)} \Delta s \Delta z.$$

Theorem 11 *Suppose that X and Z are random variables whose distributions have density functions $r(s)$ and $r(z)$ respectively, and let $r(s, z)$ be the joint density function for (X, Z) . Define*

$$S := \int \int_D r(s)r(z) \Delta s \Delta z.$$

If

$$\int \int_D \frac{r^2(s, z)}{r(s)r(z)} \Delta s \Delta z < \infty,$$

then $i_{\bar{b}}(X, Z)$ exists and

$$\begin{aligned} 0 &\leq \log S + i_{\bar{b}}(X, Z) \\ &\leq \log \left[S \int \int_D \frac{r^2(s, z)}{r(s)r(z)} \Delta s \Delta z \right] \\ &\leq \frac{1}{\ln \bar{b}} \left[S \int \int_D \frac{r^2(s, z)}{r(s)r(z)} \Delta s \Delta z - 1 \right]. \end{aligned}$$

Proof Use Corollary 5 with $n = 2$ and

$$\psi(w) = r(s, z), \quad \xi(w) = \frac{r(s)r(z)}{r(s, z)} \quad \text{for } w = (s, z) \in D,$$

to get

$$\begin{aligned} 0 &\leq \log \left[\int \int_D r(s, z) \frac{r(s)r(z)}{r(s, z)} \Delta s \Delta z \right] - \int \int_D r(s, z) \log \frac{r(s)r(z)}{r(s, z)} \Delta s \Delta z \\ &= \log \left[\int \int_D r(s, z) \frac{r(s)r(z)}{r(s, z)} \Delta s \Delta z \right] + \int \int_D r(s, z) \log \frac{r(s, z)}{r(s)r(z)} \Delta s \Delta z \\ &= \log S + i_{\bar{b}}(X, Z) \\ &\leq \log \left[S \int \int_D r(s, z) \frac{r(s, z)}{r(s)r(z)} \Delta s \Delta z \right] \\ &= \log \left[S \int \int_D \frac{r^2(s, z)}{r(s)r(z)} \Delta s \Delta z \right] \\ &\leq \frac{1}{\ln \bar{b}} \left[S \int \int_D \frac{r^2(s, z)}{r(s)r(z)} \Delta s \Delta z - 1 \right]. \quad \square \end{aligned}$$

3 Conclusion

In the paper, Shannon type inequalities on time scales have been established by using the time scales version of Jensen's inequality. Bounds are obtained for some Shannon type inequalities which have direct association to information theory. Differential entropy on time scales has been introduced and its bounds for some particular distributions have been obtained. The given results are the generalization of corresponding results established by Matić, Pearce, and Pečarić in [13], and the idea may stimulate further research in the theory of Shannon entropy, delta integrals, and generalized convex functions.

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Authors' contributions

All authors jointly worked on the results and they read and approved the final manuscript.

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