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Square-mean piecewise almost automorphic mild solutions to a class of impulsive stochastic evolution equations

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Abstract

In this paper, we introduce the concept of square-mean piecewise almost automorphic function. By using the theory of semigroups of operators and the contraction mapping principle, the existence of square-mean piecewise almost automorphic mild solutions for linear and nonlinear impulsive stochastic evolution equations is investigated. In addition, the exponential stability of square-mean piecewise almost automorphic mild solutions for nonlinear impulsive stochastic evolution equations is obtained by the generalized Gronwall–Bellman inequality. Finally, we provide an illustrative example to justify the results.

Keywords: Square-mean piecewise almost automorphic function; Theory of semigroups of operators; Contraction mapping principle; Impulsive stochastic evolution equations; Generalized Gronwall–Bellman inequality

1 Introduction

The concept of the almost automorphic function is proposed by Bochner in the paper [1], which is an important generalization of the classical almost periodic function and is related to some aspects of differential geometry. Also, the almost automorphic solutions for differential systems have been extensively investigated in [2–9]. Moreover, the square-mean almost automorphic function is defined as almost automorphic function in stochastic process, which has more extensive applications (see [10–15]). On the other hand, the theory of impulsive evolution equations has become an active area of investigation, since it fully considers the impact of instantaneous changes on the whole process, and has the characteristics of differential equations and difference equations. There are several interesting results concerning the existence and stability of solutions, especially for piecewise almost periodic type solutions and piecewise almost automorphic type solutions for impulsive evolution equations (see [16–25]). In [21], the authors introduced a PC-almost automorphic function and investigated the existence of PC-almost automorphic solution to impulsive fractional functional differential equations by α-resolvent family of bounded linear operators, Sadovskii's fixed point theorem and Schauder's fixed point theorem. In [25], by introducing the concept of equipotentially almost automorphic sequence, the definition of weighted piecewise pseudo almost automorphic function on time scale is proposed, and the existence and stability of the weighted piecewise pseudo almost automorphic mild solutions to abstract impulsive dynamic equation on time scale is investigated.

However, besides impulsive effects, stochastic effects likewise exist in real systems. Therefore, we must import the stochastic effects into the investigation of impulsive evolution systems. Recently, many authors studied the square-mean piecewise almost periodic solutions for impulsive stochastic evolution equations (see [26–30]). In [28], the authors introduced the concept of square-mean piecewise almost periodic functions for impulsive stochastic processes, and studied the existence and stability of square-mean piecewise almost periodic solutions for linear and nonlinear impulsive stochastic differential equations. Furthermore, Yan et al. [31–34] discussed the square-mean piecewise pseudo almost periodic solutions and the square-mean piecewise weighted pseudo almost periodic solutions for impulsive stochastic evolution equations. However, there are few studies on the square-mean piecewise almost automorphic solutions of impulsive stochastic evolution equations.

Based on this, in this paper, we will construct a square-mean piecewise almost automorphic function and study its composite properties. Further we use these properties to prove the existence of the square-mean piecewise almost automorphic mild solutions for two types of impulsive stochastic evolution equations. Also, the stability of the square-mean piecewise almost automorphic mild solutions for impulsive stochastic evolution equations is studied by the generalized Gronwall–Bellman inequality. In the end, we give an example to illustrate our results.

2 Preliminaries

Throughout this paper, let $(H, \|\cdot\|)$ be a real and separable Hilbert space. Let (Ω, F, P) be a complete probability space. Let $L^2(P,H)$ be a space of the H-valued random variables x such that $E\|x\|^2 = \int_\Omega \|x\|^2\,dP < \infty$, and $(L^2(P,H), \|\cdot\|_2)$ is a Hilbert space when it is equipped with the norm $\|x\|_2 = (\int_\Omega \|x\|^2\,dP)^{1/2}$.

Definition 2.1 A stochastic process $x : R \to L^2(P,H)$ is said to be stochastically bounded if there exists $M > 0$ such that $E\|x(t)\|^2 \le M$ for all $t \in R$.

Definition 2.2 A stochastic process $x : R \to L^2(P,H)$ is said to be stochastically continuous, if

$$\lim_{t\to s} E\left\|x(t) - x(s)\right\|^2 = 0$$

for all $s \in R$.

Let $T = \{\{t_i\}_{i\in Z} \mid \gamma = \inf_{i\in Z}(t_{i+1} - t_i) > 0\}$. For $\{t_i\}_{i\in Z} \in T$, let $PC(R, L^2(P,H))$ be the space consisting of all stochastically bounded piecewise continuous functions $f : R \to L^2(P,H)$ such that f is stochastically continuous in $t \neq t_i$, $i \in Z$, and $f(t_i^+)$, $f(t_i^-)$ exist, $f(t_i) = f(t_i^-)$. Let $PC(R \times L^2(P,H), L^2(P,H))$ be the space formed by all stochastically uniformly bounded piecewise continuous functions $f : R \times L^2(P,H) \to L^2(P,H)$ such that $f(\cdot, x) \in PC(R, L^2(P,H))$ and $f(t, \cdot)$ is continuous.

Definition 2.3 A function $\phi \in PC(R, L^2(P,H))$ is said to be square-mean piecewise almost automorphic if the following conditions are fulfilled:

(i) $\{t_i^j : i \in Z\}_{j\in Z}$ are equipotentially almost automorphic, that is, for any sequence $\{s_n\} \subseteq Z$, there exists a subsequence $\{\tau_n\}$ such that

$$\lim_{n\to\infty} t_k^{\tau_n} = \eta_k$$

and

$$\lim_{n\to\infty} \eta_k^{-\tau_n} = t_k$$

for each $k \in Z$.

(ii) For any sequence $\{s'_n\} \subseteq R$, there exist a subsequence $\{s_n\} \subseteq \{s'_n\}$ and $\varphi \in PC(R, L^2(P,H))$ such that

$$\lim_{n\to\infty} E\big\|\phi(t+s_n) - \varphi(t)\big\|^2 = 0$$

and

$$\lim_{n\to\infty} E\big\|\varphi(t-s_n) - \phi(t)\big\|^2 = 0$$

for all $t \in R$ and $t \neq t_i$.

Denote by $AA_T(R, L^2(P,H))$ the set of all square-mean piecewise almost automorphic functions.

Definition 2.4 A function $f \in PC(R \times L^2(P,H), L^2(P,H))$ is said to be square-mean piecewise almost automorphic in $t \in R$ uniformly in $x \in L^2(P,H)$, if for any $t \in R$ and $t \neq t_i$, $x \in L^2(P,H)$ such that $f(\cdot,x) \in AA_T(R, L^2(P,H))$. Denote by $AA_T(R \times L^2(P,H), L^2(P,H))$ the set of all such functions.

Theorem 2.1 *Let $\phi \in AA_T(R, L^2(P,H))$, then $R(\phi)$ is a relatively compact set of $L^2(P,H)$.*

Proof Let $\{\phi(x_n)\} \subseteq L^2(P,H)$. Since $\phi \in AA_T(R, L^2(P,H))$, by Definition 2.3, there exists a subsequence $\{x'_n\} \subseteq \{x_n\}$ such that $\lim_{n\to\infty} E\|\phi(x'_n) - \varphi(0)\|^2 = 0$, that is, $\{\phi(x'_n)\}$ is the convergent subsequence of $\{\phi(x_n)\}$ in $L^2(P,H)$. Therefore, $R(\phi) = \{\phi(x) : x \in R\}$ is a relatively compact set of $L^2(P,H)$. □

Theorem 2.2 *Assume $f \in AA_T(R \times L^2(R,H), L^2(P,H))$, and that f satisfies the following Lipschitz continuous condition: there exists a number $L > 0$ such that, for any $x, y \in L^2(P,H)$,*

$$E\big\|f(t,x) - f(t,y)\big\|^2 \leq LE\|x-y\|^2, \quad t \in R, t \neq t_i.$$

If $g \in AA_T(R, L^2(P,H))$, then $f(\cdot, g(\cdot)) \in AA_T(R, L^2(P,H))$.

Proof Since $g \in AA_T(R, L^2(P,H))$ and $f \in AA_T(R \times L^2(R,H), L^2(P,H))$, for any sequence $\{s_n\} \subseteq Z$, there exists a subsequence $\{s'_n\} \subseteq \{s_n\}$ such that

$$\lim_{n\to\infty} E\big\|f\big(t+s'_n, x\big) - \widetilde{f}(t,x)\big\|^2 = 0, \tag{1}$$

$$\lim_{n\to\infty} E\big\|\widetilde{f}\big(t-s_n',x\big)-f(t,x)\big\|^2=0, \tag{2}$$

$$\lim_{n\to\infty} E\big\|g\big(t+s_n'\big)-\widetilde{g}(t)\big\|^2=0, \tag{3}$$

$$\lim_{n\to\infty} E\big\|\widetilde{g}\big(t-s_n'\big)-g(t)\big\|^2=0. \tag{4}$$

Let $F(t)=f(t,g(t))$, $G(t)=\widetilde{f}(t,\widetilde{g}(t))$, therefore, we only need to prove

$$\lim_{n\to\infty} E\big\|F\big(t+s_n'\big)-G(t)\big\|^2=0$$

and

$$\lim_{n\to\infty} E\big\|G\big(t-s_n'\big)-F(t)\big\|^2=0.$$

Since $E\|f(t,x)-f(t,y)\|^2\le LE\|x-y\|^2$, we have

$$\begin{aligned}
&E\big\|F\big(t+s_n'\big)-G(t)\big\|^2\\
&\quad=E\big\|f\big(t+s_n',g\big(t+s_n'\big)\big)-\widetilde{f}\big(t,\widetilde{g}(t)\big)\big\|^2\\
&\quad=E\big\|f\big(t+s_n',g\big(t+s_n'\big)\big)-f\big(t+s_n',\widetilde{g}(t)\big)+f\big(t+s_n',\widetilde{g}(t)\big)-\widetilde{f}\big(t,\widetilde{g}(t)\big)\big\|^2\\
&\quad\le 2E\big\|f\big(t+s_n',g\big(t+s_n'\big)\big)-f\big(t+s_n',\widetilde{g}(t)\big)\big\|^2+2E\big\|f\big(t+s_n',\widetilde{g}(t)\big)-\widetilde{f}\big(t,\widetilde{g}(t)\big)\big\|^2\\
&\quad\le 2LE\big\|g\big(t+s_n'\big)-\widetilde{g}(t)\big\|^2+2E\big\|f\big(t+s_n',\widetilde{g}(t)\big)-\widetilde{f}\big(t,\widetilde{g}(t)\big)\big\|^2.
\end{aligned} \tag{5}$$

Combining (1), (3) and (5), we have

$$\lim_{n\to\infty} E\big\|F\big(t+s_n'\big)-G(t)\big\|^2=0.$$

Similarly,

$$\lim_{n\to\infty} E\big\|G\big(t-s_n'\big)-F(t)\big\|^2=0.$$

Therefore, $f(\cdot,g(\cdot))\in AA_T(R,L^2(P,H))$. □

Lemma 2.1 *If $\phi\in AA_T(R,L^2(P,H))$, then $\{\phi(t_i):i\in Z\}$ is a square-mean almost automorphic sequence.*

The proof of Lemma 2.1 is similar to the proof of Lemma 3.12 in [25], one may refer to [25] for more details.

Theorem 2.3 *Assume $\phi\in AA_T(R,L^2(P,H))$, $\{I_i(\cdot):i\in Z\}$ is a square-mean almost automorphic function sequence, that is, for any $x\in L^2(P,H)$, $\{I_i(x):i\in Z\}$ is a square-mean almost automorphic sequence, if I_i satisfies the following Lipschitz continuous condition: there exists a number $L>0$, for any $x,y\in L^2(P,H)$, $i\in Z$,*

$$E\big\|I_i(x)-I_i(y)\big\|^2\le LE\|x-y\|^2,$$

then $\{I_i(\phi(t_i)):i\in Z\}$ is a square-mean almost automorphic sequence.

Proof Since $\phi \in AA_T(R, L^2(P,H))$, by Lemma 2.1, $\{\phi(t_i) : i \in Z\}$ is a square-mean almost automorphic sequence. Let

$$I(t,x) = I_i(x) + (t-i)\big[I_{i+1}(x) - I_i(x)\big], \quad i \leq t < i+1, i \in Z$$

and

$$\Phi(t) = \phi(t_i) + (t-i)\big[\phi(t_{i+1}) - \phi(t_i)\big], \quad i \leq t < i+1, i \in Z.$$

Since $\{I_i(x) : i \in Z\}$ is a square-mean almost automorphic sequence, by Lemma 2.1, $I \in AA(R \times L^2(P,H), L^2(P,H))$, $\Phi \in AA(R, L^2(P,H))$.

For any $t \in R$, there exists $i \in Z$ such that $|t-i| \leq 1$, then

$$\begin{aligned}
&E\big\|I(t,x) - I(t,y)\big\|^2 \\
&\quad = E\big\|I_i(x) + (t-i)\big[I_{i+1}(x) - I_i(x)\big] - I_i(y) - (t-i)\big[I_{i+1}(y) - I_i(y)\big]\big\|^2 \\
&\quad \leq E\big\|I_i(x) - I_i(y) + |t-i|\big[I_{i+1}(x) - I_{i+1}(y)\big] + |t-i|\big[I_i(x) - I_i(y)\big]\big\|^2 \\
&\quad \leq 3E\big\|I_i(x) - I_i(y)\big\|^2 + 3|t-i|^2 E\big\|I_{i+1}(x) - I_{i+1}(y)\big\|^2 \\
&\qquad + 3|t-i|^2 E\big\|I_i(x) - I_i(y)\big\|^2 \\
&\quad \leq 3LE\|x-y\|^2 + 3|t-i|^2 LE\|x-y\|^2 + 3|t-i|^2 LE\|x-y\|^2 \\
&\quad \leq 9LE\|x-y\|^2.
\end{aligned}$$

By the composite property of the square-mean almost automorphic function, we have $I(\cdot, \Phi(\cdot)) \in AA(R, L^2(P,H))$.

Thus, $\{I(i, \Phi(i)) : i \in Z\}$ is a square-mean almost automorphic sequence, that is, $\{I_i(\phi(t_i)) : i \in Z\}$ is a square-mean almost automorphic sequence. □

We list the following result for a square-mean piecewise almost automorphic function, one may refer to [21] for more details.

Lemma 2.2 *Assume $f, g \in AA_T(R, L^2(P,H))$, the sequence $\{x_i\}_{i\in Z}$ is a square-mean almost automorphic, then, for any $\varepsilon > 0$ and $\{s_n'\} \subseteq R$, $\{\tau_n'\} \subseteq Z$, there exist subsequences $\{s_n\} \subseteq \{s_n'\}$, $\{\tau_n\} \subseteq \{\tau_n'\}$ and $\widetilde{f}, \widetilde{g} \in PC(R, L^2(P,H))$, $\{y_i\}_{i\in Z}$ such that*

- (i) *$E\|f(t+s_n) - \widetilde{f}(t)\|^2 < \varepsilon$ and $E\|\widetilde{f}(t-s_n) - f(t)\|^2 < \varepsilon$ for all $t \in R$, $|t-t_i| > \varepsilon$, $\{s_n\} \subseteq R$, $i \in Z$.*
- (ii) *$E\|g(t+s_n) - \widetilde{g}(t)\|^2 < \varepsilon$ and $E\|\widetilde{g}(t-s_n) - g(t)\|^2 < \varepsilon$ for all $t \in R$, $|t-t_i| > \varepsilon$, $\{s_n\} \subseteq R$, $i \in Z$.*
- (iii) *$E\|x_{i+\tau_n} - y_i\|^2 < \varepsilon$ and $E\|y_{i-\tau_n} - x_i\|^2 < \varepsilon$ for all $\{\tau_n\} \subseteq Z$, $i \in Z$.*
- (iv) *$E\|t_{i+\tau_n} - t_i - s_n\|^2 < \varepsilon$ for all $\{\tau_n\} \subseteq Z$, $\{s_n\} \subseteq R$, $i \in Z$.*

3 Square-mean piecewise almost automorphic mild solutions for impulsive stochastic evolution equations

In this part, we study the existence and stability of the square-mean piecewise almost automorphic mild solution for impulsive stochastic evolution equations.

3.1 Linear impulsive stochastic evolution equations

Consider the following linear impulsive stochastic evolution equations:

$$\begin{cases} dx(t) = [Ax(t) + f(t)]\,dt + g(t)\,dw(t), & t \in R, t \neq t_i, i \in Z, \\ \Delta x(t_i) = x(t_i^+) - x(t_i^-) = \beta_i, & i \in Z, \end{cases} \tag{6}$$

where A is an infinitesimal generator of C_0-semigroup $\{T(t) : t \geq 0\}$ such that, for all $t \geq 0$, $\|T(t)\| \leq Me^{-\delta t}$ with $M, \delta > 0$, and $w(t)$ is a two-sided standard one-dimensional Brownian motion, which is defined on the filtered probability space (Ω, F, P, F_σ) with $F_t = \sigma\{w(u) - w(v) : u, v \leq t\}$.

Definition 3.1 A function $x \in PC(R, L^2(P, H))$ is called a mild solution of linear impulsive stochastic evolution equations (6), if

$$x(t) = T(t-\sigma)x(\sigma) + \int_\sigma^t T(t-s)f(s)\,ds + \int_\sigma^t T(t-s)g(s)\,dw(s) + \sum_{\sigma < t_i < t} T(t-t_i)\beta_i,$$

where $t > \sigma$, $\sigma \neq t_i$, $i \in Z$. If $x \in AA_T(R, L^2(P, H))$, then x is called the square-mean piecewise almost automorphic mild solution of Eq. (6).

Theorem 3.1 *Assume $f, g \in AA_T(R, L^2(P, H))$, $\{\beta_i : i \in Z\}$ is a square-mean almost automorphic sequence, then Eq. (6) has a square-mean piecewise almost automorphic mild solution.*

Proof From semigroup theory, we know

$$x(t) = T(t-\sigma)x(\sigma) + \int_\sigma^t T(t-s)f(s)\,ds + \int_\sigma^t T(t-s)g(s)\,dw(s), \quad t > \sigma$$

is a mild solution to

$$dx(t) = \bigl[Ax(t) + f(t)\bigr]\,dt + g(t)\,dw(t).$$

For any $t \in (\sigma, t_i]$,

$$x\bigl(t_1^-\bigr) = T(t_1 - \sigma)x(\sigma) + \int_\sigma^{t_1} T(t_1 - s)f(s)\,ds + \int_\sigma^{t_1} T(t_1 - s)g(s)\,dw(s),$$

by using $\triangle x(t_i) = x(t_i^+) - x(t_i^-) = \beta_i$, we get

$$x\bigl(t_1^+\bigr) = T(t_1 - \sigma)x(\sigma) + \int_\sigma^{t_1} T(t_1 - s)f(s)\,ds + \int_\sigma^{t_1} T(t_1 - s)g(s)\,dw(s) + \beta_1.$$

If $t \in (t_1, t_2]$, then

$$\begin{aligned} x(t) &= T(t-t_1)x\bigl(t_1^+\bigr) + \int_{t_1}^t T(t-s)f(s)\,ds + \int_{t_1}^t T(t-s)g(s)\,dw(s) \\ &= T(t-t_1)\biggl[T(t_1 - \sigma)x(\sigma) + \int_\sigma^{t_1} T(t_1 - s)f(s)\,ds \end{aligned}$$

$$
\begin{aligned}
&+ \int_{\sigma}^{t_1} T(t_1 - s)g(s)\,dw(s) + \beta_1 \Bigg] \\
&+ \int_{t_1}^{t} T(t-s)f(s)\,ds + \int_{t_1}^{t} T(t-s)g(s)\,dw(s) \\
&= T(t-\sigma)x(\sigma) + \int_{\sigma}^{t_1} T(t-s)f(s)\,ds + \int_{\sigma}^{t_1} T(t-s)g(s)\,dw(s) \\
&+ T(t-t_1)\beta_1 + \int_{t_1}^{t} T(t-s)f(s)\,ds + \int_{t_1}^{t} T(t-s)g(s)\,dw(s) \\
&= T(t-\sigma)x(\sigma) + \int_{\sigma}^{t} T(t-s)f(s)\,ds + \int_{\sigma}^{t} T(t-s)g(s)\,dw(s) \\
&+ T(t-t_1)\beta_1.
\end{aligned}
$$

Since

$$
x\bigl(t_2^-\bigr) = T(t_2-\sigma)x(\sigma) + \int_{\sigma}^{t_2} T(t_2-s)f(s)\,ds + \int_{\sigma}^{t_2} T(t_2-s)g(s)\,dw(s) + T(t_2-t_1)\beta_1,
$$

by $\triangle x(t_i) = x(t_i^+) - x(t_i^-) = \beta_i$, we get

$$
\begin{aligned}
x\bigl(t_2^+\bigr) &= x\bigl(t_2^-\bigr) + \beta_2 \\
&= T(t_2-\sigma)x(\sigma) + \int_{\sigma}^{t_2} T(t_2-s)f(s)\,ds + \int_{\sigma}^{t_2} T(t_2-s)g(s)\,dw(s) \\
&\quad + T(t_2-t_1)\beta_1 + \beta_2.
\end{aligned}
$$

If $t \in (t_2, t_3]$, then

$$
\begin{aligned}
x(t) &= T(t-t_2)x\bigl(t_2^+\bigr) + \int_{t_2}^{t} T(t-s)f(s)\,ds + \int_{t_2}^{t} T(t-s)g(s)\,dw(s) \\
&= T(t-t_2)\Bigg[T(t_2-\sigma)x(\sigma) + \int_{\sigma}^{t_2} T(t_2-s)f(s)\,ds \\
&\quad + \int_{\sigma}^{t_2} T(t_2-s)g(s)\,dw(s) + T(t_2-t_1)\beta_1 + \beta_2 \Bigg] \\
&\quad + \int_{t_2}^{t} T(t-s)f(s)\,ds + \int_{t_2}^{t} T(t-s)g(s)\,dw(s) \\
&= T(t-\sigma)x(\sigma) + \int_{\sigma}^{t_2} T(t-s)f(s)\,ds + \int_{\sigma}^{t_2} T(t-s)g(s)\,dw(s) \\
&\quad + T(t-t_1)\beta_1 + T(t-t_2)\beta_2 + \int_{t_2}^{t} T(t-s)f(s)\,ds + \int_{t_2}^{t} T(t-s)g(s)\,dw(s) \\
&= T(t-\sigma)x(\sigma) + \int_{\sigma}^{t} T(t-s)f(s)\,ds + \int_{\sigma}^{t} T(t-s)g(s)\,dw(s) \\
&\quad + T(t-t_1)\beta_1 + T(t-t_2)\beta_2.
\end{aligned}
$$

Therefore, reiterating this procedure, we get

$$\begin{aligned} x(t) &= T(t-\sigma)x(\sigma) + \int_{\sigma}^{t} T(t-s)f(s)\,ds + \int_{\sigma}^{t} T(t-s)g(s)\,dw(s) \\ &\quad + \sum_{\sigma<t_i<t} T(t-t_i)\beta_i. \end{aligned} \tag{7}$$

By Definition 3.1, (7) is a mild solution of Eq. (6), therefore, we only need to prove the above (7) is a square-mean piecewise almost automorphic process.

Let $\sigma \to -\infty$, then $\|T(t-\sigma)\| \le Me^{-\delta(t-\sigma)} = Me^{-\delta t}e^{\delta\sigma} \to 0$, by Definition 2.1, $x(\sigma)$ is stochastically bounded, so (7) can be defined as

$$\begin{aligned} x(t) &= \int_{-\infty}^{t} T(t-s)f(s)\,ds + \int_{-\infty}^{t} T(t-s)g(s)\,dw(s) + \sum_{t_i<t} T(t-t_i)\beta_i \\ &\triangleq x_1(t) + x_2(t) + x_3(t). \end{aligned}$$

Next we show that $x \in AA_T(R, L^2(P,H))$. The following verification procedure is divided into three steps.

Step 1. $x_1 \in AA_T(R, L^2(P,H))$

Since $f \in AA_T(R, L^2(P,H))$, by Definition 2.3, for any sequence $\{s_n'\} \subseteq R$, there exist a subsequence $\{s_n\} \subseteq \{s_n'\}$ and $\widetilde{f} \in PC(R, L^2(P,H))$ such that

$$\lim_{n\to\infty} E\left\|f(t+s_n) - \widetilde{f}(t)\right\|^2 = 0$$

and

$$\lim_{n\to\infty} E\left\|\widetilde{f}(t-s_n) - f(t)\right\|^2 = 0$$

for every $t \in R$ and $t \neq t_i$.

Let $\widetilde{x_1}(t) = \int_{-\infty}^{t} T(t-s)\widetilde{f}(s)\,ds$, then

$$\begin{aligned} &E\left\|x_1(t+s_n) - \widetilde{x_1}(t)\right\|^2 \\ &\quad = E\left\|\int_{-\infty}^{t+s_n} T(t+s_n-s)f(s)\,ds - \int_{-\infty}^{t} T(t-s)\widetilde{f}(s)\,ds\right\|^2 \\ &\quad = E\left\|\int_{-\infty}^{t} T(t-s)f(s+s_n)\,ds - \int_{-\infty}^{t} T(t-s)\widetilde{f}(s)\,ds\right\|^2 \\ &\quad = E\left\|\int_{-\infty}^{t} T(t-s)\left[f(s+s_n) - \widetilde{f}(s)\right]ds\right\|^2 \\ &\quad \le E\left(\int_{-\infty}^{t} \left\|T(t-s)\right\| \left\|f(s+s_n) - \widetilde{f}(s)\right\| ds\right)^2 \\ &\quad \le E\left(\int_{-\infty}^{t} Me^{-\delta(t-s)} \left\|f(s+s_n) - \widetilde{f}(s)\right\| ds\right)^2 \\ &\quad \le E\left(\int_{-\infty}^{t} M^2 e^{-\delta(t-s)}\,ds \int_{-\infty}^{t} e^{-\delta(t-s)} \left\|f(s+s_n) - \widetilde{f}(s)\right\|^2 ds\right) \end{aligned}$$

$$= \int_{-\infty}^{t} M^2 e^{-\delta(t-s)}\,ds \int_{-\infty}^{t} e^{-\delta(t-s)} E\big\|f(s+s_n) - \widetilde{f}(s)\big\|^2\,ds$$
$$\le \frac{M^2}{\delta} \int_{-\infty}^{t} e^{-\delta(t-s)} E\big\|f(s+s_n) - \widetilde{f}(s)\big\|^2\,ds.$$

Similarly,

$$E\big\|\widetilde{x_1}(t-s_n) - x_1(t)\big\|^2 \le \frac{M^2}{\delta} \int_{-\infty}^{t} e^{-\delta(t-s)} E\big\|\widetilde{f}(s-s_n) - f(s)\big\|^2\,ds.$$

So, by Lebesgue's dominated convergence theorem, we get

$$\lim_{n\to\infty} E\big\|x_1(t+s_n) - \widetilde{x_1}(t)\big\|^2 \le \frac{M^2}{\delta} \int_{-\infty}^{t} e^{-\delta(t-s)} \lim_{n\to\infty} E\big\|f(s+s_n) - \widetilde{f}(s)\big\|^2\,ds$$

and

$$\lim_{n\to\infty} E\big\|\widetilde{x_1}(t-s_n) - x_1(t)\big\|^2 \le \frac{M^2}{\delta} \int_{-\infty}^{t} e^{-\delta(t-s)} \lim_{n\to\infty} E\big\|\widetilde{f}(s-s_n) - f(s)\big\|^2\,ds.$$

Since $\lim_{n\to\infty} E\|f(t+s_n) - \widetilde{f}(t)\|^2 = 0$ and $\lim_{n\to\infty} E\|\widetilde{f}(t-s_n) - f(t)\|^2 = 0$, $x_1 \in AA_T(R, L^2(P,H))$.

Step 2. $x_2 \in AA_T(R, L^2(P,H))$

Since $g \in AA_T(R, L^2(P,H))$, by Lemma 2.2, for the above sequence $\{s'_n\} \subseteq R$, there exist a subsequence $\{s_n\} \subseteq \{s'_n\}$ and $\widetilde{g} \in PC(R, L^2(P,H))$ such that

$$\lim_{n\to\infty} E\big\|g(t+s_n) - \widetilde{g}(t)\big\|^2 = 0$$

and

$$\lim_{n\to\infty} E\big\|\widetilde{g}(t-s_n) - g(t)\big\|^2 = 0$$

for every $t \in R$ and $t \neq t_i$.

Let $\widetilde{x_2}(t) = \int_{-\infty}^{t} T(t-s)\widetilde{g}(s)\,dw(s)$, by the Ito integral, then

$$\begin{aligned}
&E\big\|x_2(t+s_n) - \widetilde{x_2}(t)\big\|^2 \\
&\quad= E\bigg\|\int_{-\infty}^{t+s_n} T(t+s_n-s)g(s)\,dw(s) - \int_{-\infty}^{t} T(t-s)\widetilde{g}(s)\,dw(s)\bigg\|^2 \\
&\quad= E\bigg\|\int_{-\infty}^{t} T(t-s)g(s+s_n)\,dw(s+s_n) - \int_{-\infty}^{t} T(t-s)\widetilde{g}(s)\,dw(s)\bigg\|^2 \\
&\quad= E\bigg\|\int_{-\infty}^{t} T(t-s)\big[g(s+s_n) - \widetilde{g}(s)\big]\,d\widetilde{w}(s)\bigg\|^2 \\
&\quad\le \int_{-\infty}^{t} E\big\|T(t-s)\big[g(s+s_n) - \widetilde{g}(s)\big]\big\|^2\,ds \\
&\quad\le \int_{-\infty}^{t} M^2 e^{-2\delta(t-s)} E\big\|g(s+s_n) - \widetilde{g}(s)\big\|^2\,ds.
\end{aligned}$$

Similarly,

$$E\left\|\widetilde{x_2}(t-s_n)-x_2(t)\right\|^2 \le \int_{-\infty}^{t} M^2 e^{-2\delta(t-s)} E\left\|\widetilde{g}(s-s_n)-g(s)\right\|^2 ds.$$

So, by Lebesgue's dominated convergence theorem, we get

$$\lim_{n\to\infty} E\left\|x_2(t+s_n)-\widetilde{x_2}(t)\right\|^2 \le \int_{-\infty}^{t} M^2 e^{-2\delta(t-s)} \lim_{n\to\infty} E\left\|g(s+s_n)-\widetilde{g}(s)\right\|^2 ds$$

and

$$\lim_{n\to\infty} E\left\|\widetilde{x_2}(t-s_n)-x_2(t)\right\|^2 \le \int_{-\infty}^{t} M^2 e^{-2\delta(t-s)} \lim_{n\to\infty} E\left\|\widetilde{g}(s-s_n)-g(s)\right\|^2 ds.$$

Since $\lim_{n\to\infty} E\|g(t+s_n)-\widetilde{g}(t)\|^2 = 0$ and $\lim_{n\to\infty} E\|\widetilde{g}(t-s_n)-g(t)\|^2 = 0$, $x_2 \in AA_T(R, L^2(P,H))$.

Step 3. $x_3 \in AA_T(R, L^2(P,H))$

Since β_i is a square-mean almost automorphic sequence, by Lemma 2.2, for any sequence $\{\tau_n'\} \subseteq Z$, there exists a subsequence $\{\tau_n\} \subseteq \{\tau_n'\}$ and $\widetilde{\beta_i}$ is a stochastically bounded piecewise continuous function sequence such that

$$\lim_{n\to\infty} E\|\beta_{i+\tau_n} - \widetilde{\beta_i}\|^2 = 0$$

and

$$\lim_{n\to\infty} E\|\widetilde{\beta}_{i-\tau_n} - \beta_i\|^2 = 0$$

for every $i \in Z$.

For $t_i < t \le t_{i+1}$, $|t-t_i| > \varepsilon$, $|t-t_{i+1}| > \varepsilon$, $i \in Z$, by Lemma 2.2, we have

$$t+s_n > t_i + \varepsilon + s_n > t_{i+\tau_n}$$

and

$$t_{i+\tau_n+1} > t_{i+1} + s_n - \varepsilon > t + s_n.$$

Therefore,

$$t_{i+\tau_n+1} > t+s_n > t_{i+\tau_n}.$$

Let $\widetilde{x_3}(t) = \sum_{t_i<t} T(t-t_i)\widetilde{\beta_i}$, by Lemma 2.2, then

$$\begin{aligned} E\left\|x_3(t+s_n)-\widetilde{x_3}(t)\right\|^2 &= E\left\|\sum_{t_i<t+s_n} T(t+s_n-t_i)\beta_i - \sum_{t_i<t} T(t-t_i)\widetilde{\beta_i}\right\|^2 \\ &= E\left\|\sum_{t_i-s_n<t} T\big(t-(t_i-s_n)\big)\beta_i - \sum_{t_i<t} T(t-t_i)\widetilde{\beta_i}\right\|^2 \end{aligned}$$

$$
\begin{aligned}
&= E\left\| \sum_{t_j<t} T(t-t_j)\beta_{j+\tau_n} - \sum_{t_i<t} T(t-t_i)\widetilde{\beta}_i \right\|^2 \\
&= E\left\| \sum_{t_i<t} T(t-t_i)\beta_{i+\tau_n} - \sum_{t_i<t} T(t-t_i)\widetilde{\beta}_i \right\|^2 \\
&= E\left\| \sum_{t_i<t} T(t-t_i)(\beta_{i+\tau_n} - \widetilde{\beta}_i) \right\|^2 \\
&\leq E\left(\sum_{t_i<t} \| T(t-t_i) \| \|\beta_{i+\tau_n} - \widetilde{\beta}_i\| \right)^2 \\
&\leq E\left(\sum_{t_i<t} Me^{-\delta(t-t_i)} \|\beta_{i+\tau_n} - \widetilde{\beta}_i\| \right)^2 \\
&\leq E\left(\sum_{t_i<t} M^2 e^{-\delta(t-t_i)} \sum_{t_i<t} e^{-\delta(t-t_i)} \|\beta_{i+\tau_n} - \widetilde{\beta}_i\|^2 \right) \\
&\leq \frac{M^2}{1-e^{-\delta\gamma}} \sum_{t_i<t} e^{-\delta(t-t_i)} E\|\beta_{i+\tau_n} - \widetilde{\beta}_i\|^2.
\end{aligned}
$$

So, by Lebesgue's dominated convergence theorem, we get

$$
\lim_{n\to\infty} E\left\| x_3(t+s_n) - \widetilde{x}_3(t) \right\|^2 \leq \frac{M^2}{1-e^{-\delta\gamma}} \sum_{t_i<t} e^{-\delta(t-t_i)} \lim_{n\to\infty} E\|\beta_{i+\tau_n} - \widetilde{\beta}_i\|^2
$$

and

$$
\lim_{n\to\infty} E\left\| \widetilde{x}_3(t-s_n) - x_3(t) \right\|^2 \leq \frac{M^2}{1-e^{-\delta\gamma}} \sum_{t_i<t} e^{-\delta(t-t_i)} \lim_{n\to\infty} E\|\widetilde{\beta}_{i-\tau_n} - \beta_i\|^2.
$$

Since $\lim_{n\to\infty} E\|\beta_{i+\tau_n} - \widetilde{\beta}_i\|^2 = 0$ and $\lim_{n\to\infty} E\|\widetilde{\beta}_{i-\tau_n} - \beta_i\|^2 = 0$, $x_3 \in AA_T(R, L^2(P,H))$. Thus, $x \in AA_T(R, L^2(P,H))$. □

3.2 Nonlinear impulsive stochastic evolution equations

Consider the following nonlinear impulsive stochastic evolution equation:

$$
\begin{cases}
dx(t) = [Ax(t) + f(t, x(t))]\,dt + g(t, x(t))\,dw(t), & t \in R, t \neq t_i, i \in Z, \\
\Delta x(t_i) = x(t_i^+) - x(t_i^-) = I_i(x(t_i)), & i \in Z,
\end{cases}
\tag{8}
$$

where $f, g : R \times L^2(P,H) \to L^2(P,H)$, $I_i : L^2(P,H) \to L^2(P,H)$, $i \in Z$, and $w(t)$ is a two-sided standard one-dimensional Brownian motion defined on the filtered probability space (Ω, F, P, F_σ) with $F_t = \sigma\{w(u) - w(v) : u, v \leq t\}$.

Definition 3.2 A function $x \in PC(R, L^2(P,H))$ is called a mild solution of Eq. (8), if x satisfies

$$
\begin{aligned}
x(t) = {} & T(t-\sigma)x(\sigma) + \int_\sigma^t T(t-s) f\big(s, x(s)\big)\,ds + \int_\sigma^t T(t-s) g\big(s, x(s)\big)\,dw(s) \\
& + \sum_{\sigma<t_i<t} T(t-t_i) I_i\big(x(t_i)\big),
\end{aligned}
$$

where $t > \sigma$, $\sigma \neq t_i$, $i \in Z$. If $x \in AA_T(R, L^2(P,H))$, then x is called a square-mean piecewise almost automorphic mild solution of Eq. (8).

Theorem 3.2 *Suppose Eq.* (8) *satisfies the following conditions*:

(i) *The operator* $A : D(A) \subseteq L^2(P,H) \to L^2(P,H)$ *is the infinitesimal generator of a* C_0*-semigroup* $\{T(t) : t \geq 0\}$, *that is, there exist* $M, \delta > 0$ *such that* $\|T(t)\| \leq Me^{-\delta t}$, $t \geq 0$.

(ii) *The functions* $f, g \in AA_T(R \times L^2(P,H), L^2(P,H))$, $\{I_i(\cdot) : i \in Z\}$ *is a square-mean almost automorphic function sequence, and there exist positive numbers* L_1, L_2, L *such that*

$$E\|f(t,x) - f(t,y)\|^2 \leq L_1 E\|x - y\|^2,$$
$$E\|g(t,x) - g(t,y)\|^2 \leq L_2 E\|x - y\|^2$$

and

$$E\|I_i(x) - I_i(y)\|^2 \leq LE\|x - y\|^2.$$

If $\frac{3M^2}{\delta^2}L_1 + \frac{3M^2}{2\delta}L_2 + \frac{3M^2}{(1-e^{-\delta\gamma})^2}L < 1$, *then Eq.* (8) *has a square-mean piecewise almost automorphic mild solution.*

Proof Let

$$\begin{aligned} \Gamma\varphi(t) = {} & \int_{-\infty}^t T(t-s)f(s,\varphi(s))\,ds + \int_{-\infty}^t T(t-s)g(s,\varphi(s))\,dw(s) \\ & + \sum_{t_i<t} T(t-t_i)I_i(\varphi(t_i)). \end{aligned}$$

For any $\varphi \in AA_T(R, L^2(P,H))$, by (ii) and Theorem 2.2, we have $f(\cdot,\varphi(\cdot)), g(\cdot,\varphi(\cdot)) \in AA_T(R, L^2(P,H))$, by Theorem 2.3, $\{I_i(\varphi(t_i)) : i \in Z\}$ is a square-mean almost automorphic sequence. Similar to the proof of Theorem 3.1, we have $\Gamma\varphi(t) \in AA_T(R, L^2(P,H))$.

For any $\varphi, \psi \in AA_T(R, L^2(P,H))$, by $(a + b + c)^2 \leq 3(a^2 + b^2 + c^2)$ and Cauchy–Schwarz inequality, we have

$$\begin{aligned} & E\|\Gamma\varphi(t) - \Gamma\psi(t)\|^2 \\ & \quad = E\bigg\| \int_{-\infty}^t T(t-s)\big[f(s,\varphi(s)) - f(s,\psi(s))\big]\,ds \\ & \qquad + \int_{-\infty}^t T(t-s)\big[g(s,\varphi(s)) - g(s,\psi(s))\big]\,dw(s) \\ & \qquad + \sum_{t_i<t} T(t-t_i)\big[I_i(\varphi(t_i)) - I_i(\psi(t_i))\big]\bigg\|^2 \\ & \quad \leq 3E\bigg\| \int_{-\infty}^t T(t-s)\big[f(s,\varphi(s)) - f(s,\psi(s))\big]\,ds\bigg\|^2 \\ & \qquad + 3E\bigg\| \int_{-\infty}^t T(t-s)\big[g(s,\varphi(s)) - g(s,\psi(s))\big]\,dw(s)\bigg\|^2 \end{aligned}$$

$$
\begin{aligned}
&+ 3E\left\| \sum_{t_i<t} T(t-t_i)\bigl[I_i\bigl(\varphi(t_i)\bigr) - I_i\bigl(\psi(t_i)\bigr)\bigr] \right\|^2 \\
&\le 3E\left[\int_{-\infty}^{t} \|T(t-s)\| \bigl\|f\bigl(s,\varphi(s)\bigr) - f\bigl(s,\psi(s)\bigr)\bigr\|\,ds\right]^2 \\
&+ 3\int_{-\infty}^{t} E\bigl\|T(t-s)\bigl[g\bigl(s,\varphi(s)\bigr) - g\bigl(s,\psi(s)\bigr)\bigr]\bigr\|^2\,ds \\
&+ 3E\left[\sum_{t_i<t} \|T(t-t_i)\| \bigl\|I_i\bigl(\varphi(t_i)\bigr) - I_i\bigl(\psi(t_i)\bigr)\bigr\|\right]^2 \\
&\le 3E\left[\int_{-\infty}^{t} Me^{-\delta(t-s)} \bigl\|f\bigl(s,\varphi(s)\bigr) - f\bigl(s,\psi(s)\bigr)\bigr\|\,ds\right]^2 \\
&+ 3\int_{-\infty}^{t} M^2e^{-2\delta(t-s)} E\bigl\|g\bigl(s,\varphi(s)\bigr) - g\bigl(s,\psi(s)\bigr)\bigr\|^2\,ds \\
&+ 3E\left[\sum_{t_i<t} Me^{-\delta(t-t_i)} \bigl\|I_i\bigl(\varphi(t_i)\bigr) - I_i\bigl(\psi(t_i)\bigr)\bigr\|\right]^2 \\
&\le 3\int_{-\infty}^{t} M^2e^{-\delta(t-s)}\,ds \int_{-\infty}^{t} e^{-\delta(t-s)} E\bigl\|f\bigl(s,\varphi(s)\bigr) - f\bigl(s,\psi(s)\bigr)\bigr\|^2\,ds \\
&+ 3\int_{-\infty}^{t} M^2e^{-2\delta(t-s)} E\bigl\|g\bigl(s,\varphi(s)\bigr) - g\bigl(s,\psi(s)\bigr)\bigr\|^2\,ds \\
&+ 3\left(\sum_{t_i<t} M^2e^{-\delta(t-t_i)}\right)\left(\sum_{t_i<t} e^{-\delta(t-t_i)} E\bigl\|I_i\bigl(\varphi(t_i)\bigr) - I_i\bigl(\psi(t_i)\bigr)\bigr\|^2\right) \\
&\le \frac{3M^2}{\delta}\int_{-\infty}^{t} e^{-\delta(t-s)} L_1 E\bigl\|\varphi(s) - \psi(s)\bigr\|^2\,ds \\
&+ 3M^2\int_{-\infty}^{t} e^{-2\delta(t-s)} L_2 E\bigl\|\varphi(s) - \psi(s)\bigr\|^2\,ds \\
&+ \frac{3M^2}{1-e^{-\delta\gamma}} \sum_{t_i<t} e^{-\delta(t-t_i)} LE\bigl\|\varphi(t_i) - \psi(t_i)\bigr\|^2 \\
&\le \left[\frac{3M^2}{\delta}\int_{-\infty}^{t} e^{-\delta(t-s)} L_1\,ds + 3M^2\int_{-\infty}^{t} e^{-2\delta(t-s)} L_2\,ds \right.\\
&\left.+ \frac{3M^2}{1-e^{-\delta\gamma}} \sum_{t_i<t} e^{-\delta(t-t_i)} L\right] E\|\varphi - \psi\|^2 \\
&\le \left[\frac{3M^2}{\delta^2}L_1 + \frac{3M^2}{2\delta}L_2 + \frac{3M^2}{(1-e^{-\delta\gamma})^2}L\right] E\|\varphi - \psi\|^2.
\end{aligned}
$$

Since $\frac{3M^2}{\delta^2}L_1 + \frac{3M^2}{2\delta}L_2 + \frac{3M^2}{(1-e^{-\delta\gamma})^2}L < 1$, Γ is a contraction. Therefore, Eq. (8) has a square-mean piecewise almost automorphic mild solution. □

Lemma 3.1 (Generalized Gronwall–Bellman inequality) *Assume $u \in PC(R,R)$ satisfies the following inequality*:

$$
0 \le u(t) \le C + \int_{t'}^{t} \nu(\tau)u(\tau)\,d\tau + \sum_{t'<t_i<t} \beta_i u(t_i), \quad t \ge t',
$$

where $C \geq 0$, $\beta_i \geq 0$, $\nu(\tau) > 0$, *then the following estimate holds for the function* $u(t)$:

$$u(t) \leq C \prod_{t'<t_i<t} (1+\beta_i) e^{\int_{t'}^{t} \nu(\tau)\,d\tau}.$$

Theorem 3.3 *Suppose the conditions of Theorem 3.2 hold, if further that*

$$\frac{1}{\gamma} \ln\left(1 + \frac{4M^2}{1-e^{-\delta\gamma}} L\right) + \frac{4M^2}{\delta} L_1 + 4M^2 L_2 - \delta < 0,$$

then Eq. (8) *has an exponentially stable square-mean piecewise almost automorphic mild solution.*

Proof By Theorem 3.2, $\varphi(t)$ is a solution of Eq. (8), then

$$\begin{aligned}\varphi(t) &= T(t-\sigma)\varphi(\sigma) + \int_\sigma^t T(t-s) f\bigl(s,\varphi(s)\bigr)\,ds + \int_\sigma^t T(t-s) g\bigl(s,\varphi(s)\bigr)\,dw(s) \\ &\quad + \sum_{\sigma<t_i<t} T(t-t_i) I_i\bigl(\varphi(t_i)\bigr).\end{aligned}$$

Let $\psi(t)$ be a solution of Eq. (8), then

$$\begin{aligned}\psi(t) &= T(t-\sigma)\psi(\sigma) + \int_\sigma^t T(t-s) f\bigl(s,\psi(s)\bigr)\,ds + \int_\sigma^t T(t-s) g\bigl(s,\psi(s)\bigr)\,dw(s) \\ &\quad + \sum_{\sigma<t_i<t} T(t-t_i) I_i\bigl(\psi(t_i)\bigr).\end{aligned}$$

Thus, by Cauchy–Schwarz inequality and the Ito integral, we have

$$\begin{aligned}&E\bigl\|\varphi(t)-\psi(t)\bigr\|^2 \\ &\quad = E\biggl\| T(t-\sigma)\bigl[\varphi(\sigma)-\psi(\sigma)\bigr] + \int_\sigma^t T(t-s)\bigl[f\bigl(s,\varphi(s)\bigr) - f\bigl(s,\psi(s)\bigr)\bigr]\,ds \\ &\qquad + \int_\sigma^t T(t-s)\bigl[g\bigl(s,\varphi(s)\bigr) - g\bigl(s,\psi(s)\bigr)\bigr]\,dw(t) \\ &\qquad + \sum_{\sigma<t_i<t} T(t-t_i)\bigl[I_i\bigl(\varphi(t_i)\bigr) - I_i\bigl(\psi(t_i)\bigr)\bigr]\biggr\|^2 \\ &\quad \leq 4E\bigl\| T(t-\sigma)\bigl[\varphi(\sigma)-\psi(\sigma)\bigr]\bigr\|^2 \\ &\qquad + 4E\biggl\|\int_\sigma^t T(t-s)\bigl[f\bigl(s,\varphi(s)\bigr) - f\bigl(s,\psi(s)\bigr)\bigr]\,ds\biggr\|^2 \\ &\qquad + 4E\biggl\|\int_\sigma^t T(t-s)\bigl[g\bigl(s,\varphi(s)\bigr) - g\bigl(s,\psi(s)\bigr)\bigr]\,dw(t)\biggr\|^2 \\ &\qquad + 4E\biggl\|\sum_{\sigma<t_i<t} T(t-t_i)\bigl[I_i\bigl(\varphi(t_i)\bigr) - I_i\bigl(\psi(t_i)\bigr)\bigr]\biggr\|^2 \\ &\quad \leq 4M^2 e^{-2\delta(t-\sigma)} E\bigl\|\varphi(\sigma)-\psi(\sigma)\bigr\|^2\end{aligned}$$

$$
\begin{aligned}
&+ 4E\left[\int_\sigma^t Me^{-\delta(t-s)}\big\|f\big(s,\varphi(s)\big)-f\big(s,\psi(s)\big)\big\|\,ds\right]^2 \\
&+ 4\int_\sigma^t E\big\|T(t-s)\big[g\big(s,\varphi(s)\big)-g\big(s,\psi(s)\big)\big]\big\|^2\,ds \\
&+ 4E\left[\sum_{\sigma<t_i<t} Me^{-\delta(t-t_i)}\big\|I_i\big(\varphi(t_i)\big)-I_i\big(\psi(t_i)\big)\big\|\right]^2 \\
\leq{}& 4M^2e^{-2\delta(t-\sigma)}E\big\|\varphi(\sigma)-\psi(\sigma)\big\|^2 \\
&+ 4E\left[\int_\sigma^t M^2e^{-\delta(t-s)}\,ds\int_\sigma^t e^{-\delta(t-s)}\big\|f\big(s,\varphi(s)\big)-f\big(s,\psi(s)\big)\big\|^2\,ds\right] \\
&+ 4\int_\sigma^t M^2e^{-2\delta(t-s)}E\big\|g\big(s,\varphi(s)\big)-g\big(s,\psi(s)\big)\big\|^2\,ds \\
&+ 4E\left[\left(\sum_{\sigma<t_i<t} M^2e^{-\delta(t-t_i)}\right)\sum_{\sigma<t_i<t} e^{-\delta(t-t_i)}\big\|I_i\big(\varphi(t_i)\big)-I_i\big(\psi(t_i)\big)\big\|^2\right] \\
\leq{}& 4M^2e^{-\delta(t-\sigma)}E\big\|\varphi(\sigma)-\psi(\sigma)\big\|^2 \\
&+ \frac{4M^2}{\delta}\int_\sigma^t e^{-\delta(t-s)}E\big\|f\big(s,\varphi(s)\big)-f\big(s,\psi(s)\big)\big\|^2\,ds \\
&+ 4M^2\int_\sigma^t e^{-2\delta(t-s)}E\big\|g\big(s,\varphi(s)\big)-g\big(s,\psi(s)\big)\big\|^2\,ds \\
&+ \frac{4M^2}{1-e^{-\delta\gamma}}\sum_{\sigma<t_i<t} e^{-\delta(t-t_i)}E\big\|I_i\big(\varphi(t_i)\big)-I_i\big(\psi(t_i)\big)\big\|^2 \\
\leq{}& 4M^2e^{-\delta(t-\sigma)}E\big\|\varphi(\sigma)-\psi(\sigma)\big\|^2 \\
&+ \left[\frac{4M^2}{\delta}L_1+4M^2L_2\right]\int_\sigma^t e^{-\delta(t-s)}E\big\|\varphi(s)-\psi(s)\big\|^2\,ds \\
&+ \frac{4M^2}{1-e^{-\delta\gamma}}L\sum_{\sigma<t_i<t} e^{-\delta(t-t_i)}E\big\|\varphi(t_i)-\psi(t_i)\big\|^2.
\end{aligned}
$$

So,

$$
\begin{aligned}
&e^{\delta t}E\big\|\varphi(t)-\psi(t)\big\|^2 \\
&\quad\leq 4M^2e^{\delta\sigma}E\big\|\varphi(\sigma)-\psi(\sigma)\big\|^2+\left[\frac{4M^2}{\delta}L_1+4M^2L_2\right]\int_\sigma^t e^{\delta s}E\big\|\varphi(s)-\psi(s)\big\|^2\,ds \\
&\qquad+\frac{4M^2}{1-e^{-\delta\gamma}}L\sum_{\sigma<t_i<t} e^{\delta t_i}E\big\|\varphi(t_i)-\psi(t_i)\big\|^2.
\end{aligned}
$$

Let $\Upsilon(t)=e^{\delta t}E\|\varphi(t)-\psi(t)\|^2$, then

$$
\Upsilon(t)\leq 4M^2\Upsilon(\sigma)+\left[\frac{4M^2}{\delta}L_1+4M^2L_2\right]\int_\sigma^t \Upsilon(s)\,ds+\frac{4M^2}{1-e^{-\delta\gamma}}L\sum_{\sigma<t_i<t}\Upsilon(t_i).
$$

By Lemma 3.1, we get

$$
\begin{aligned}
\Upsilon(t) &\le 4M^2\Upsilon(\sigma) + \left[\frac{4M^2}{\delta}L_1 + 4M^2L_2\right]\int_\sigma^t \Upsilon(s)\,ds + \frac{4M^2}{1-e^{-\delta\gamma}}L\sum_{\sigma<t_i<t}\Upsilon(t_i) \\
&\le 4M^2\Upsilon(\sigma)\prod_{\sigma<t_i<t}\left(1+\frac{4M^2}{1-e^{-\delta\gamma}}L\right)e^{\int_\sigma^t(\frac{4M^2}{\delta}L_1+4M^2L_2)\,ds} \\
&= 4M^2\Upsilon(\sigma)\prod_{\sigma<t_i<t}\left(1+\frac{4M^2}{1-e^{-\delta\gamma}}L\right)e^{(\frac{4M^2}{\delta}L_1+4M^2L_2)(t-\sigma)} \\
&= 4M^2\Upsilon(\sigma)\left(1+\frac{4M^2}{1-e^{-\delta\gamma}}L\right)^{\frac{t-\sigma}{\gamma}}e^{(\frac{4M^2}{\delta}L_1+4M^2L_2)(t-\sigma)} \\
&= 4M^2\Upsilon(\sigma)e^{[\frac{1}{\gamma}\ln(1+\frac{4M^2}{1-e^{-\delta\gamma}}L)+\frac{4M^2}{\delta}L_1+4M^2L_2](t-\sigma)},
\end{aligned}
$$

that is,

$$
\Upsilon(t) \le 4M^2\Upsilon(\sigma)e^{[\frac{1}{\gamma}\ln(1+\frac{4M^2}{1-e^{-\delta\gamma}}L)+\frac{4M^2}{\delta}L_1+4M^2L_2](t-\sigma)}.
$$

Therefore

$$
\begin{aligned}
&e^{\delta t}E\|\varphi(t)-\psi(t)\|^2 \\
&\quad\le 4M^2e^{\delta\sigma}E\|\varphi(\sigma)-\psi(\sigma)\|^2e^{[\frac{1}{\gamma}\ln(1+\frac{4M^2}{1-e^{-\delta\gamma}}L)+\frac{4M^2}{\delta}L_1+4M^2L_2](t-\sigma)},
\end{aligned}
$$

that is,

$$
\begin{aligned}
&E\|\varphi(t)-\psi(t)\|^2 \\
&\quad\le 4M^2E\|\varphi(\sigma)-\psi(\sigma)\|^2e^{[\frac{1}{\gamma}\ln(1+\frac{4M^2}{1-e^{-\delta\gamma}}L)+\frac{4M^2}{\delta}L_1+4M^2L_2-\delta](t-\sigma)}.
\end{aligned}
$$

Since

$$
\frac{1}{\gamma}\ln\left(1+\frac{4M^2}{1-e^{-\delta\gamma}}L\right)+\frac{4M^2}{\delta}L_1+4M^2L_2-\delta<0,
$$

Equation (8) has an exponentially stable square-mean piecewise almost automorphic mild solution. □

4 Applications

Consider the following impulsive stochastic evolution equation:

$$
\begin{cases}
du(t,x) = [\frac{\partial^2u(t,x)}{\partial x^2} + \frac{1}{6}\sin\frac{1}{2+\cos t+\cos\sqrt{2}t}\sin u(t,x)]\,dt \\
\qquad\qquad + \frac{1}{3}\sin\frac{1}{2+\cos t+\cos\sqrt{2}t}\sin u(t,x)\,dw(t), & t\in R, t\neq t_i, i\in Z, x\in[0,\pi], \\
\Delta u(t_i,x) = \beta_i\sin u(t_i,x), & i\in Z, x\in[0,\pi], \\
u(t,0) = u(t,\pi) = 0, & t\in R,
\end{cases}
\tag{9}
$$

where $w(t)$ is a two-sided standard one-dimensional Brownian motion defined on the filtered probability space (Ω, F, P, F_t), $\beta_i = \frac{1}{6}\sin\frac{1}{2+\cos i+\cos\sqrt{2}i}$, $t_i = i + \frac{1}{3}|\sin\frac{1}{2+\cos i+\cos\sqrt{2}i}|$ $(\varphi(i) = \frac{1}{2+\cos i+\cos\sqrt{2}i})$.

Let $X = L^2(0,\pi)$, define the operators $A : D(A) \subseteq X \to X$ by $Au = u''$. It is well known that A is the infinitesimal generator of a semigroup $\{T(t) : t \geq 0\}$ on X and $\|T(t)\| \leq e^{-t}$ for $t \geq 0$ with $M = \delta = 1$, then condition (i) of Theorem 3.2 is satisfied. By Definition 2.3, $\{t_i^j : i \in Z\}_{j\in Z}$ are equipotentially almost automorphic sequence and

$$\begin{aligned}
t_i^1 &= t_{i+1} - t_i \\
&= 1 + \frac{1}{3}\left|\sin\frac{1}{2+\cos(i+1)+\cos\sqrt{2}(i+1)}\right| \\
&\quad - \frac{1}{3}\left|\sin\frac{1}{2+\cos i+\cos\sqrt{2}i}\right| \\
&\geq 1 - \frac{1}{3}\left|\sin\frac{1}{2+\cos(i+1)+\cos\sqrt{2}(i+1)} - \sin\frac{1}{2+\cos i+\cos\sqrt{2}i}\right| \\
&\geq 1 - \frac{2}{3}\left|\sin\frac{\varphi(i+1)-\varphi(i)}{2}\cos\frac{\varphi(i+1)+\varphi(i)}{2}\right| \\
&> 1 - \frac{2}{3} = \frac{1}{3}.
\end{aligned}$$

Hence, $\gamma = \inf_{i\in Z}(t_{i+1} - t_i) > \frac{1}{3} > 0$.

Let $f(t,u) = \frac{1}{6}\sin\frac{1}{2+\cos t+\cos\sqrt{2}t}\sin u$, $g(t,u) = \frac{1}{3}\sin\frac{1}{2+\cos t+\cos\sqrt{2}t}\sin u$ and $I_i(u) = \beta_i \sin u$, then $f, g \in AA_T(R \times L^2(P,H), L^2(P,H))$ and $\{I_i(\cdot) : i \in Z\}$ is a square-mean almost automorphic function sequence.

For any u, v, we have

$$\begin{aligned}
&E\|f(t,u) - f(t,v)\|^2 \\
&\quad = E\left\|\frac{1}{6}\sin\frac{1}{2+\cos t+\cos\sqrt{2}t}\sin u - \frac{1}{6}\sin\frac{1}{2+\cos t+\cos\sqrt{2}t}\sin v\right\|^2 \\
&\quad = E\left\|\frac{1}{6}\sin\frac{1}{2+\cos t+\cos\sqrt{2}t}(\sin u - \sin v)\right\|^2 \\
&\quad \leq \left[\frac{1}{6}\sin\frac{1}{2+\cos t+\cos\sqrt{2}t}\right]^2 E\|\sin u - \sin v\|^2 \\
&\quad \leq \frac{1}{36}E\|u - v\|^2.
\end{aligned}$$

Similarly, $E\|g(t,u) - g(t,v)\|^2 \leq \frac{1}{9}E\|u - v\|^2$, $E\|I_i(u) - I_i(v)\|^2 \leq \frac{1}{36}E\|u - v\|^2$, then $L_1 = \frac{1}{36}$, $L_2 = \frac{1}{9}$, $L = \frac{1}{36}$. Therefore, condition (ii) of Theorem 3.2 is satisfied.

Since $\frac{3M^2}{\delta^2}L_1 + \frac{3M^2}{2\delta}L_2 + \frac{3M^2}{(1-e^{-\delta\gamma})^2}L = 3 \times \frac{1}{36} + \frac{3}{2} \times \frac{1}{9} + \frac{3}{(1-e^{-1/3})^2} \times \frac{1}{36} < 1$, by Theorem 3.2, Eq. (9) has a square-mean piecewise almost automorphic mild solution.

Also since

$$\begin{aligned}
&\frac{1}{\gamma}\ln\left(1 + \frac{4M^2}{1-e^{-\delta\gamma}}L\right) + \frac{4M^2}{\delta}L_1 + 4M^2L_2 - \delta \\
&\quad = 3\ln\left(1 + \frac{4}{1-e^{-1/3}} \times \frac{1}{36}\right) + 4 \times \frac{1}{36} + 4 \times \frac{1}{9} - 1 < 0,
\end{aligned}$$

by Theorem 3.3, Eq. (9) has an exponentially stable square-mean piecewise almost automorphic mild solution.

5 Conclusion

In this paper, we mainly construct the square-mean piecewise almost automorphic function, and the existence and exponential stability of square-mean piecewise almost automorphic mild solutions for impulsive stochastic evolution equations is proved by the theory of semigroups of operators, the contraction mapping principle and the generalized Gronwall–Bellman inequality. Finally, an interesting example is given to illustrate our results.

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