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Certain properties of difference operator and stability of Fréchet functional equation

Teerapol Sukhonwimolmal[1*]

*Correspondence:
teerasu@kku.ac.th
[1]Department of Mathematics, Khon Kaen University, Khon Kaen, Thailand

Abstract

Let M be a commutative monoid, and let B be a Banach space. We give a new recursive method to obtain a Găvruţa-type stability result for the functional equation

$$\Delta_y^{n+1} f(x) := \sum_{k=0}^{n+1} (-1)^{n+1+k} \binom{n+1}{k} f(x+ky) = 0$$

via algebraic manipulations of the forward difference operator.

MSC: 39B52; 39B82

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1 Motivation

One of the best-known classes of functional equations, the Fréchet functional equations, consists of functional equations equivalent to the equation

$$\Delta_{y_1}\Delta_{y_2}\cdots\Delta_{y_{n+1}} f(0) = 0, \tag{1}$$

studied by Fréchet [1] in 1909. In this paper, we focus on the equation

$$\Delta_y^{n+1} f(x) = 0, \tag{2}$$

for which the stability result relies on its equivalence with the equation

$$\Delta_{y_1}\Delta_{y_2}\cdots\Delta_{y_{n+1}} f(x) = 0. \tag{3}$$

The stability problem of functional equations originated from a problem posed by S. Ulam regarding "almost additive" functions that satisfy

$$\left\| f(x+y) - f(x) - f(y) \right\| \le \epsilon$$

for a fixed ϵ. This inequality was studied (under stricter assumptions than those for the posed problem) by Hyers [2] in 1941. The result was that such a function can be approximated by an additive function. Hence the problems in this sense, where the assigned bound is a constant, became known as Hyers–Ulam-type stability. Years later, Aoki [3] and Rassias [4] presented stability results where the bound is a power function of x and y. Thus the stability in this sense became known as the Aoki–Rassias-type stability. Later Găvruța [5] generalized this so that the bound (which can also be called the control function) is a function with specific properties.

The solutions of (1), (2), and (3) are called generalized polynomials of degree at most n. It is well known that a generalized polynomial p is constructed from diagonalization of multiadditive functions [6]. The Hyers–Ulam stability of (2) and (3) has been studied in [6–9]. The Găvruța-type stability of (1) has been studied by Dăianu [10]. Dăianu used an equivalence theorem, which is more general than that presented by Kuczma [11], to obtain a stability result for (2) under the assumption that M is $(n+1)!$-divisible. In this paper, we present a result in which divisibility of M is not required. Instead, we require $(n+1)!$-divisibility of B, which is readily true since B is a Banach space.

2 The forward difference operator

Let M be a commutative monoid, let be B be a Banach space, and let $\mathbb{N}$ be the set of positive integers. For a function $f : M \to B$ and $y \in M$, define $\Delta_y f : M \to B$ by

$$\Delta_y f(x) = f(x+y) - f(x)$$

for $x \in M$. Also, define its iterations

$$\Delta_{y_1}\Delta_{y_2}\cdots\Delta_{y_n} f = \Delta_{y_1}(\Delta_{y_2}\Delta_{y_3}\cdots\Delta_{y_n} f) \quad \text{and} \quad \Delta_y^n f = \underbrace{\Delta_y\Delta_y\cdots\Delta_y}_{n \text{ terms}} f$$

for $y, y_1, y_2, \dots, y_n \in M$. Observe that $\Delta_{y_1}\Delta_{y_2} f = \Delta_{y_2}\Delta_{y_1} f$, so the ordering of y_i is interchangable, and

$$\Delta_y^n f(x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(x+ky)$$

and

$$\Delta_{y_1}\Delta_{y_2}\cdots\Delta_{y_n} f(x) = \sum_{\epsilon_1,\epsilon_2,\dots,\epsilon_n=0}^{1} (-1)^{n+\epsilon_1+\epsilon_2+\cdots+\epsilon_n} f\left(x + \sum_{i=1}^{n} \epsilon_i y_i\right).$$

We will establish some algebraic manipulation of the forward difference operator Δ. We begin with a modified version of Kuczma's theorem.

Lemma 2.1 *Let $n \in \mathbb{N}$ and $f : M \to B$. Then*

$$\Delta_{y_1}\Delta_{2y_2}\Delta_{3y_3}\cdots\Delta_{ny_n} f(x) = \sum_{\epsilon_1,\epsilon_2,\dots,\epsilon_n=0}^{1} (-1)^{\epsilon_1+\epsilon_2+\cdots+\epsilon_n} \Delta^n_{b_{\epsilon_1,\epsilon_2,\dots,\epsilon_n}} f(x + a_{\epsilon_1,\epsilon_2,\dots,\epsilon_n}) \tag{4}$$

for all $x, y_1, y_2, \ldots, y_n \in X$, *where*

$$a_{\epsilon_1,\epsilon_2,\ldots,\epsilon_n} = \sum_{i=1}^{n} i\epsilon_i y_i \quad \text{and} \quad b_{\epsilon_1,\epsilon_2,\ldots,\epsilon_n} = \sum_{i=1}^{n} (1-\epsilon_i) y_i.$$

Proof Recall that

$$\begin{aligned}
&\Delta^n_{b_{\epsilon_1,\epsilon_2,\ldots,\epsilon_n}} f(x + a_{\epsilon_1,\epsilon_2,\ldots,\epsilon_n}) \\
&\quad = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f\left(x + \sum_{i=1}^{n} i\epsilon_i y_i + k \sum_{i=1}^{n} (1-\epsilon_i) y_i\right).
\end{aligned}$$

We will show that each of these terms cancels out in the sum on the right-hand side of (4), except for $k = 0$. For $k \neq 0$, we observe that ϵ_k is absent in the term where $i = k$. So changing ϵ_k does not change the argument of this term.

Consider another term in the sum on the right-hand side of (4), where only ϵ_k differs from this term. Then the kth term in its expansion cancels the term we mentioned before.

Hence every term where $k \neq 0$ has its negative in the sum in (4). Then

$$\begin{aligned}
&\sum_{\epsilon_1,\epsilon_2,\ldots,\epsilon_n=0}^{1} (-1)^{\epsilon_1+\epsilon_2+\cdots+\epsilon_n} \Delta^n_{b_{\epsilon_1,\epsilon_2,\ldots,\epsilon_n}} f(x + a_{\epsilon_1,\epsilon_2,\ldots,\epsilon_n}) \\
&\quad = \sum_{\epsilon_1,\epsilon_2,\ldots,\epsilon_n=0}^{1} (-1)^{n+\epsilon_1+\epsilon_2+\cdots+\epsilon_n} f\left(x + \sum_{i=1}^{n} \epsilon_i i y_i\right) \\
&\quad = \Delta_{y_1} \Delta_{2y_2} \Delta_{3y_3} \cdots \Delta_{ny_n} f(x).
\end{aligned}$$

□

Lemma 2.2 *Let* $n, m \in \mathbb{N}$ *and* $f : M \to B$. *Then*

$$\Delta^n_y f(x + my) = \Delta^n_y f(x) + \sum_{k=0}^{m-1} \Delta^{n+1}_y f(x + ky)$$

for all $x, y \in X$.

Proof Since $\Delta^{n+1}_y f(x) = \Delta^n_y f(x+y) - \Delta^n_y f(x)$ for all $x, y \in M$,

$$\begin{aligned}
\Delta^n_y f(x + my) &= \Delta^n_y f\bigl(x + (m-1)y\bigr) + \Delta^{n+1}_y f\bigl(x + (m-1)y\bigr) \\
&= \Delta^n_y f\bigl(x + (m-2)y\bigr) + \Delta^{n+1}_y f\bigl(x + (m-2)y\bigr) + \Delta^{n+1}_y f\bigl(x + (m-1)y\bigr) \\
&\;\;\vdots \\
&= \Delta^n_y f(x) + \sum_{k=0}^{m-1} \Delta^{n+1}_y f(x + ky). \qquad (5)
\end{aligned}$$

□

In the following theorems, for convenience, we let $\sum_{i=0}^{-1} a_i = 0$ for any sequence (a_i).

Lemma 2.3 *Let $n \in \mathbb{N}$ and $f : M \to B$. Then*

$$n!\Delta_y^n f(x) = \Delta_{2y,3y,\ldots,(n+1)y} f(x) - \sum_{l_1=0}^{1}\sum_{l_2=0}^{2}\cdots\sum_{l_n=0}^{n}\sum_{s=0}^{l_2+l_3+\cdots+l_n-1} \Delta_y^{n+1} f(x+sy)$$

for all $x, y \in X$.

Proof By Lemma 2.2 it is sufficient to show that

$$\Delta_{2y,3y,\ldots,(n+1)y} f(x) = \sum_{l_1=0}^{1}\sum_{l_2=0}^{2}\cdots\sum_{l_n=0}^{n} \Delta_y^n f\bigl(x + (l_1 + l_2 + \cdots + l_n)y\bigr). \tag{6}$$

For the case $n = 1$, we have

$$\begin{aligned}\Delta_{2y} f(x) &= f(x+2y) - f(x)\\ &= f(x+2y) - f(x+y) + f(x+y) - f(x) = \Delta_y f(x+y) + \Delta_y f(x).\end{aligned}$$

Suppose that (6) is true for $n = k$. Since the order of Δ_{iy} is interchangable,

$$\begin{aligned}\Delta_{2y,3y,\ldots,(k+2)y} f(x) &= \Delta_{2y,3y,\ldots,(k+1)y}\bigl(\Delta_{(k+2)y} f\bigr)(x)\\ &= \sum_{l_1=0}^{1}\sum_{l_2=0}^{2}\cdots\sum_{l_k=0}^{k} \Delta_y^n\bigl(\Delta_{(k+2)y} f\bigr)\bigl(x + (l_1 + l_2 + \cdots + l_k)y\bigr)\\ &= \sum_{l_1=0}^{1}\sum_{l_2=0}^{2}\cdots\sum_{l_k=0}^{k} \Delta_{(k+2)y}\bigl(\Delta_y^k f\bigr)\bigl(x + (l_1 + l_2 + \cdots + l_k)y\bigr)\\ &= \sum_{l_1=0}^{1}\sum_{l_2=0}^{2}\cdots\sum_{l_k=0}^{k}\sum_{l_{k+1}=0}^{k+1} \Delta_y\bigl(\Delta_y^k f\bigr)\bigl(x + (l_1 + l_2 + \cdots + l_k)y + l_{k+1}y\bigr).\end{aligned}$$

□

The following theorem follows from Lemmas 2.3 and 2.1.

Theorem 2.4 *Let $n \in \mathbb{N}$ and $f : M \to B$. Then there exist $m \in \mathbb{N}$ and nonnegative integers a_i, b_i, c_i, d_i, e_i for $i \in \{1, 2, \ldots, m\}$ such that*

$$n!\Delta_{y_1}\Delta_{y_2}^n f(x) = \sum_{i=1}^{m} (-1)^{a_i} \Delta_{b_i y_1 + c_i y_2}^{n+1} f(x + d_i y_1 + e_i y_2)$$

for $x, y_1, y_2 \in M$. To be precise,

$$\begin{aligned}n!\Delta_{y_1}\Delta_{y_2}^n f(x) = &\sum_{\epsilon_1,\epsilon_2,\ldots,\epsilon_{n+1}=0}^{1} (-1)^{\epsilon_1+\epsilon_2+\cdots+\epsilon_{n+1}} \Delta_{b_{\epsilon_1,\epsilon_2,\ldots,\epsilon_{n+1}}}^{n+1} f(x + a_{\epsilon_1,\epsilon_2,\ldots,\epsilon_{n+1}})\\ &- \sum_{l_1=0}^{1}\sum_{l_2=0}^{2}\cdots\sum_{l_n=0}^{n}\sum_{s=0}^{l_1+l_2+\cdots+l_n-1} \Delta_{y_2}^{n+1} f(x + y_1 + sy_2)\\ &+ \sum_{l_1=0}^{1}\sum_{l_2=0}^{2}\cdots\sum_{l_n=0}^{n}\sum_{s=0}^{l_1+l_2+\cdots+l_n-1} \Delta_{y_2}^{n+1} f(x + sy_2),\end{aligned}$$

where

$$a_{\epsilon_1,\epsilon_2,\ldots,\epsilon_{n+1}} = \epsilon_1 y_1 + \sum_{i=2}^{n+1} i\epsilon_i y_2 \quad \textit{and} \quad b_{\epsilon_1,\epsilon_2,\ldots,\epsilon_{n+1}} = (1-\epsilon_1)y_1 + \sum_{i=2}^{n+1}(1-\epsilon_i)y_2.$$

3 The stability of $\Delta_y^{n+1} f(x) = 0$

We recall a theorem from the author's dissertation [12].

Theorem 3.1 *Let $n \in \mathbb{N}$ and $f : M \to B$. Then*

$$2^n \Delta_y^n f(x) = \Delta_{2y}^n f(x) - \sum_{i=1}^{n} \binom{n}{i} \sum_{k=0}^{i-1} \Delta_y^{n+1} f(x+ky)$$

for $x, y \in M$.

Next, we introduce the class of preferred control functions. Consider the following properties of a function $\varphi : M \to [0,\infty)$.

P1 $\varphi(x+y) \le \varphi(x) + \varphi(y)$ for all $x, y \in M$.

P2 For each $x \in M$, either there exists $N \in \mathbb{N}$ such that $\varphi(2^k x) = 0$ for every $k > N$, or $\lim_{k\to\infty} \frac{\varphi(2^{k+1}x)}{\varphi(2^k x)} < 2$.

We take the notation of limit in P2 more loosely than normal. We allow it to not actually converge, as long as every of its limit points are in $[0,2)$.

Also note that P2 implies that $\sum_{k=0}^{\infty} \frac{\varphi(2^k x)}{2^k}$ converges. The next proposition states that the set of these functions forms a convex cone under pointwise addition and scalar multiplication.

Proposition 3.2 *Let $\varphi_1, \varphi_2 : M \to [0,\infty)$ satisfy P1 and P2. Then, for all $c_1, c_2 \in [0,\infty)$, $c_1\varphi_1 + c_2\varphi_2$ also satisfies P1 and P2.*

Proof The case $c_1 c_2 = 0$ is straightforward, so we omit it. Firstly, it is clear that $c_1\varphi_1 + c_2\varphi_2$ satisfies P1. Let $x \in M$. We will consider the cases depending on whether N_1 and N_2 exist such that $\varphi_1(2^k x) = 0$ for $k > N_1$ and $\varphi_2(2^l x) = 0$ for $l > N_2$.

- If both such N_1 and N_2 exist, then $c_1\varphi_1(2^k x) + c_2\varphi_2(2^k x) = 0$ whenever $k > \max\{N_1, N_2\}$.
- If only one exists, then without loss of generality we assume that N_1 exists but N_2 does not. Then

$$\lim_{k\to\infty} \frac{c_1\varphi_1(2^{k+1}x) + c_2\varphi_2(2^{k+1}x)}{c_1\varphi_1(2^k x) + c_2\varphi_2(2^k x)} = \lim_{k\to\infty} \frac{c_2\varphi_2(2^{k+1}x)}{c_2\varphi_2(2^k x)} < 2.$$

- If there are no such N_1 and N_2, then there exist $N_1', N_2' \in \mathbb{N}$ and $r \in (0,2)$ such that $\varphi_1(2^{k+1}x) < r\varphi_1(2^k x)$ for $k > N_1'$ and $\varphi_2(2^{l+1}x) < r\varphi_2(2^l x)$ for $l > N_2'$. Thus

$$c_1\varphi_1\left(2^{k+1}x\right) + c_2\varphi_2\left(2^{k+1}x\right) < rc_1\varphi_1\left(2^k x\right) + rc_2\varphi_2\left(2^k x\right)$$

for $k > \max\{N_1', N_2'\}$. This implies that $\lim_{k\to\infty} \frac{c_1\varphi_1(2^{k+1}x)+c_2\varphi_2(2^{k+1}x)}{c_1\varphi_1(2^k x)+c_2\varphi_2(2^k x)} \le r < 2$,

and we conclude that $c_1\varphi_1 + c_2\varphi_2$ satisfies P2. ∎

Let $C = \{\varphi : M \to [0,\infty) | \varphi$ satisfy P1 and P2$\}$. For all $\varphi : M \to [0,\infty)$ and $n \in \mathbb{N}$, we define $\lambda_n \varphi : M \to [0,\infty)$ by

$$\lambda_n \varphi(x) = \sum_{k=0}^{\infty} \frac{\varphi(2^k x)}{2^{kn}}.$$

The next theorem shows that $\lambda_n(C) \subseteq C$.

Theorem 3.3 *Let $\varphi \in C$ and $n \in \mathbb{N}$. Then $\lambda_n \varphi \in C$.*

Proof It is clear that $\lambda_n \varphi$ satisfies P1. We will consider P2 for $\lambda_n \varphi$. Let $x \in M$.

If there exists $N \in \mathbb{N}$ such that $\varphi(2^k x) = 0$ for all $k > N$, then it is straightforward to show that $\lambda_n \varphi(2^k x) = 0$ for every $k > N$.

If no such N exists, then $\varphi(2^k x) > 0$ for all $k \in \mathbb{N}$. This implies that $\lambda_n \varphi(2^k x) > 0$ for all $k \in \mathbb{N}$. Since φ satisfies P2, there exist $N \in \mathbb{N}$ and $r \in (0,2)$ such that $\varphi(2^{k+1}x) < r\varphi(2^k x)$ whenever $k > N$. So,

$$\frac{\lambda_n \varphi(2^{k+1}x)}{\varphi(2^k x)} = \sum_{i=0}^{\infty} \frac{\varphi(2^{k+i+1}x)}{2^{in}\varphi(2^k x)} < \sum_{i=0}^{\infty} \frac{r^{i+1}}{2^{in}} = r\sum_{i=0}^{\infty} \left(\frac{r}{2^n}\right)^i = \frac{2^n r}{2^n - r}.$$

Also note that $\lambda_n \varphi(2^k x) = \varphi(2^k x) + \frac{1}{2^n}\lambda_n \varphi(2^{k+1}x)$. Thus

$$\frac{\lambda_n \varphi(2^k x)}{\lambda_n \varphi(2^{k+1}x)} = \frac{\varphi(2^k x)}{\lambda_n \varphi(2^{k+1}x)} + \frac{1}{2^n} > \frac{2^n - r}{2^n r} + \frac{1}{2^n} = \frac{1}{r}.$$

Hence we have $\frac{\lambda_n \varphi(2^{k+1}x)}{\lambda_n \varphi(2^k x)} \le r < 2$ whenever $k > N$. This completes the proof. □

Now we establish our main theorems.

Theorem 3.4 *Let $n \in \mathbb{N}$, $f : M \to B$, $\theta \in [0,\infty)$, and $\varphi_1, \varphi_2 \in C$. If*

$$\left\| \Delta_y^{n+1} f(x) \right\| \le \theta + \varphi_1(x) + \varphi_2(y)$$

for all $x, y \in M$, then there exists $\varphi_3 \in C$ such that

$$\left| \Delta_{y_1} \Delta_{y_2}^n f(x) \right| \le \frac{n2^n}{2^n - 1}\theta + \frac{n2^{n-1}\varphi_1(y_1)}{2^n - 1} + \frac{n2^n}{2^n - 1}\varphi_1(x) + \varphi_3(y_2)$$

for all $x, y_1, y_2 \in M$, where φ_3 is defined by

$$\varphi_3 := \frac{n(n-1)}{4}\lambda_n \varphi_1 + n\lambda_n \varphi_2.$$

Proof According to Theorem 2.4, there exist $m \in \mathbb{N}$ and nonnegative integers a_i, b_i, c_i, d_i, e_i for $i \in \{1, 2, \dots, m\}$ such that

$$\begin{aligned} \left\| \Delta_{y_1} \Delta_{y_2}^n f(x) \right\| &= \frac{1}{n!} \left\| \sum_{i=1}^{m} (-1)^{a_i} \Delta_{b_i y_1 + c_i y_2}^{n+1} f(x + d_i y_1 + e_i y_2) \right\| \\ &\le \frac{1}{n!} \sum_{i=1}^{m} \bigl(\theta + \varphi_1(x + d_i y_1 + e_i y_2) + \varphi_2(b_i y_1 + c_i y_2)\bigr). \end{aligned} \tag{7}$$

Denote the right-hand side of (7) by $\alpha_0(x,y_1,y_2)$. Since $\varphi_1,\varphi_2 \in C$,

$$\lim_{k\to\infty} \frac{\alpha_0(x,y_1,2^k y_2)}{2^{kn}} = 0.$$

For each nonnegative integer k, let

$$\begin{aligned}\alpha_{k+1}(x,y_1,y_2) &= \frac{\alpha_k(x,y_1,2y_2)}{2^n} \\ &\quad + \frac{1}{2^n}\sum_{i=1}^{n}\binom{n}{i}\sum_{k=0}^{i-1}\bigl(2\theta + 2\varphi_1(x) + \varphi_1(y_1) + 2k\varphi_1(y_2) + 2\varphi_2(y_2)\bigr).\end{aligned}$$

We can see that if $\|\Delta_{y_1}\Delta_{y_2}^n f(x)\| \le \alpha_k(x,y_1,y_2)$, then by Theorem 3.1

$$\begin{aligned}&\|\Delta_{y_1}\Delta_{y_2}^n f(x)\| \\ &= \left\| \frac{1}{2^n}\Delta_{y_1}\Delta_{2y_2}^n f(x) - \frac{1}{2^n}\sum_{i=1}^{n}\binom{n}{i}\sum_{k=0}^{i-1}\Delta_{y_1}\Delta_{y_2}^{n+1} f(x+ky_2)\right\| \\ &\le \frac{\alpha_k(x,y_1,2y_2)}{2^n} + \frac{1}{2^n}\sum_{i=1}^{n}\binom{n}{i}\sum_{k=0}^{i-1}\bigl\|\Delta_{y_2}^{n+1} f(x+y_1+ky_2) - \Delta_{y_2}^{n+1} f(x+ky_2)\bigr\| \\ &\le \frac{\alpha_k(x,y_1,2y_2)}{2^n} + \frac{1}{2^n}\sum_{i=1}^{n}\binom{n}{i}\sum_{k=0}^{i-1}\bigl(2\theta + \varphi_1(x+y_1+ky_2) + \varphi_1(x+ky_2) + 2\varphi_2(y_2)\bigr) \\ &\le \frac{\alpha_k(x,y_1,2y_2)}{2^n} + \frac{1}{2^n}\sum_{i=1}^{n}\binom{n}{i}\sum_{k=0}^{i-1}\bigl(2\theta + 2\varphi_1(x) + \varphi_1(y_1) + 2k\varphi_1(y_2) + 2\varphi_2(y_2)\bigr) \\ &= \alpha_{k+1}(x,y_1,y_2).\end{aligned}$$

Hence $\|\Delta_{y_1}\Delta_{y_2}^n f(x)\| \le \alpha_k(x,y_1,y_2)$ for any nonnegative integer k. Observe that, for $m \ge 1$,

$$\begin{aligned}\alpha_m(x,y_1,y_2) &= \frac{\alpha_0(x,y_1,2^m y_2)}{2^{mn}} \\ &\quad + \frac{1}{2^n}\sum_{j=0}^{m-1}\sum_{i=1}^{n}\binom{n}{i}\sum_{k=0}^{i-1}\left(\frac{2\theta + 2\varphi_1(x) + \varphi_1(y_1)}{2^{jn}} + \frac{2k\varphi_1(2^j y_2) + 2\varphi_2(2^j y_2)}{2^{jn}}\right) \\ &= \frac{\alpha_0(x,y_1,2^m y_2)}{2^{mn}} \\ &\quad + \frac{1}{2^n}\sum_{i=1}^{n}\binom{n}{i}\sum_{k=0}^{i-1}\sum_{j=0}^{m-1}\left(\frac{2\theta + 2\varphi_1(x) + \varphi_1(y_1)}{2^{jn}} + \frac{2k\varphi_1(2^j y_2) + 2\varphi_2(2^j y_2)}{2^{jn}}\right).\end{aligned}$$

It follows that

$$\begin{aligned}&\lim_{m\to\infty} a_m(a,y_1,y_2) \\ &= 0 + \frac{1}{2^n}\sum_{i=1}^{n}\binom{n}{i}\sum_{k=0}^{i-1}\sum_{j=0}^{\infty}\left(\frac{2\theta + 2\varphi_1(x) + \varphi_1(y_1)}{2^{jn}} + \frac{2k\varphi_1(2^j y_2) + 2\varphi_2(2^j y_2)}{2^{jn}}\right)\end{aligned}$$

$$= \frac{1}{2^n}\sum_{i=1}^{n}\binom{n}{i}\sum_{k=0}^{i-1}\left(\frac{2^n}{2^n-1}\big(2\theta+2\varphi_1(x)+\varphi_1(y_1)\big)+2k\lambda_n\varphi_1(y_2)+2\lambda_n\varphi_2(y_2)\right)$$
$$= \frac{n2^{n-1}}{2^n-1}\big(2\theta+2\varphi_1(x)+\varphi_1(y_1)\big)+\frac{n(n-1)}{4}\lambda_n\varphi_1(y_2)+n\lambda_n\varphi_2(y_2).$$

Hence

$$\begin{aligned}\big\|\Delta_{y_1}\Delta_{y_2}^n f(x)\big\| &\le \lim_{m\to\infty}\alpha_m(x,y_1,y_2)\\ &= \frac{n2^n}{2^n-1}\left(\theta+\varphi_1(x)+\frac{\varphi_1(y_1)}{2}\right)+\frac{n(n-1)}{4}\lambda_n\varphi_1(y_2)+n\lambda_n\varphi_2(y_2).\end{aligned}$$ □

Let $\Lambda_{n,k}\varphi = (\prod_{k+1}^{n}\frac{2^k}{2^k-1})\lambda_1\lambda_2\cdots\lambda_k\varphi$ for $k<n$ and $\Lambda_{n,n}=\lambda_1\lambda_2\cdots\lambda_n\varphi$. Using Theorem 3.4 inductively, we get the following theorem.

Theorem 3.5 *Let $n\in\mathbb{N}$, $\theta\in[0,\infty)$, $\varphi_1,\varphi_2\in C$, and $f:M\to B$. If $|\Delta_y^{n+1}f(x)|\le\theta+\varphi_1(x)+\varphi_2(y)$ for all $x,y\in M$, then there exists $\varphi_3\in C$ such that*

$$\big\|\Delta_{y_1}\Delta_{y_2}\cdots\Delta_{y_{n+1}}f(x)\big\|\le n!\left(\prod_{k=1}^{n}\frac{2^k}{2^k-1}\right)\left(\theta+\varphi_1(x)+\sum_{i=1}^{n}\frac{\varphi_1(y_i)}{2}\right)+\varphi_3(y_{n+1})$$

for all $x,y_1,y_2\in M$, where φ_3 is defined by

$$\varphi_3 := n!\Lambda_{n,n}\varphi_2+n!\sum_{i=1}^{n}\frac{i-1}{4}\Lambda_{n,i}\varphi_1.$$

Proof Let $y_1\in M$. By Theorem 3.4 there exists $\varphi_2'\in C$ such that

$$\big\|\Delta_{y_1}\Delta_{y_2}^n f(x)\big\|\le\frac{n2^n}{2^n-1}\left(\theta+\varphi_1(x)+\frac{\varphi_1(y_1)}{2}\right)+\frac{n(n-1)}{4}\lambda_n\varphi_1+n\lambda_n\varphi_2.$$

Let $f_{y_1}=\Delta_{y_1}f$, $\theta_{y_1}=\frac{n2^n}{2^n-1}(\theta+\frac{\varphi_1(y_1)}{2})$, $\psi_1=\frac{n2^n}{2^n-1}\varphi_1$, and $\psi_2=\frac{n(n-1)}{4}\lambda_n\varphi_1+n\lambda_n\varphi_2$. Since the order of Δ_{y_i} can be interchanged without affecting the value on the left-hand side, we have

$$\begin{aligned}\big\|\Delta_{y_2}^n f_{y_1}(x)\big\| &= \big\|\Delta_{y_2}^n\Delta_{y_1}f(x)\big\|\\ &\le\frac{n2^n}{2^n-1}\left(\theta+\varphi_1(x)+\frac{\varphi_1(y_1)}{2}\right)+\frac{n(n-1)}{4}\lambda_n\varphi_1(y_2)+n\lambda_n\varphi_2(y_2)\\ &=\theta_{y_1}'+\psi_1(x)+\psi_2(y_2).\end{aligned}$$

Since y_1 is currently fixed and $\psi_1,\psi_2\in C$, the theorem is true by induction on n. □

Now we apply this to the result of Theorem 4.4 in [10]. For all $\varphi:M^{n+1}\to[0,\infty)$ and $n\in\mathbb{N}$, define $r_n\varphi, R_n\varphi:M^n\to\mathbb{R}^*$ by

$$\begin{aligned}&r_n\varphi(x_1,x_2,\dots,x_n)\\ &\quad=\varphi(2x_1,2x_2,\dots,2x_{n-1},x_n,x_n)+2\varphi(2x_1,2x_2,\dots,2x_{n-2},x_n,x_{n-1},x_{n-1})+\cdots\end{aligned}$$

$$
\begin{aligned}
&+ 2^{n-2}\varphi(2x_1, x_n, x_{n-1}, \dots, x_3, x_2, x_2) + 2^{n-1}\varphi(x_n, x_{n-1}, \dots, x_2, x_1, x_1),\\
R_n\varphi(x_1, x_2, \dots, x_n) &= \sum_{k=0}^{\infty} \frac{r_n\varphi(2^k x_1, 2^k x_2, \dots, 2^k x_n)}{2^{n(k+1)}}.
\end{aligned}
$$

Also, let

$$
\begin{aligned}
D_n^+ = \Biggl\{ &\bigl(\varphi, \varphi'\bigr) | \varphi : M^{n+1} \to \mathbb{R}^*, \sum_{k=0}^{\infty} 2^{-n(k+1)}\varphi\bigl(2^k z\bigr) < \infty, z \in M^{n+1},\\
&\varphi' : M^n \to [0, \infty), \varphi'(y) \ge R_n\varphi(y), \text{ and } \lim_{k\to\infty} 2^{-nk}\varphi'\bigl(2^k y\bigr) = 0, y \in M^n \Biggr\}.
\end{aligned}
$$

We restate the theorem as follows.

Theorem 3.6 *Let $n \in \mathbb{N}$, and let $\varphi_1, \varphi_2, \dots, \varphi_{n+1} : M^i \to [0, \infty)$ for $i \in \{1, 2, \dots, n+1\}$ be such that $(\varphi_{i+1}, \varphi_i) \in D_i^+$ for $1 \le i \le n$. If $f : M \to B$ satisfies*

$$
\bigl\| \Delta_{y_1}\Delta_{y_2} \cdots \Delta_{y_{n+1}} f(0) \bigr\| \le \varphi_{n+1}(y_1, y_2, \dots, y_{n+1})
$$

for all $y_1, y_2, \dots, y_{n+1} \in M$, then there exists a generalized polynomial $p : M \to B$ of degree at most n such that

$$
\bigl\| f(x) - p(x) \bigr\| \le \varphi_1(x)
$$

for all $x \in M$ and $p(0) = f(0)$.

If we let

$$
\varphi(x_1, x_2, \dots, x_n, x_{n+1}) = \theta + \sum_{i=1}^{n+1} \varphi_i(x_i)
$$

with $\varphi_1, \varphi_2, \dots, \varphi_{n+1} \in C$, then

$$
\begin{aligned}
r_n(x_1, x_2, \dots, x_n) \le \bigl(2^n - 1\bigr)\theta &+ \bigl(\bigl(2^n - 2\bigr)\varphi_1(x_1) + 2^{n-1}\varphi_n(x_1) + 2^{n-1}\varphi_{n+1}(x_1)\bigr)\\
&+ \bigl(2^{n-1} - 2\bigr)\varphi_2(x_2) + 2^{n-1}\varphi_{n-1}(x_2) + 2^{n-2}\varphi_n(x_2) + 2^{n-2}\varphi_{n+1}(x_2)\\
&\vdots\\
&+ \varphi_{n+1}(x_n) + \sum_{i=1}^{n} 2^{n-i}\varphi_i(x_n).
\end{aligned}
$$

So

$$
\begin{aligned}
R_n\varphi(y_1, y_2, \dots, y_n) \le 2^n\theta &+ \left(\frac{2^n - 2}{2^n}\lambda_n\varphi_1(x_1) + \frac{1}{2}\lambda_n\varphi_n(x_1) + \frac{1}{2}\lambda_n\varphi_{n+1}(x_1) \right)\\
&+ \frac{2^{n-1} - 2}{2^n}\lambda_n\varphi_2(x_2) + \frac{1}{2}\lambda_n\varphi_{n-1}(x_2) + \frac{1}{4}\lambda_n\varphi_n(x_2) + \frac{1}{4}\lambda_n\varphi_{n+1}(x_2)
\end{aligned}
$$

$$\vdots$$

$$+\frac{1}{2^n}\lambda_n\varphi_{n+1}(x_n)+\sum_{i=1}^{n}\frac{1}{2^i}\lambda_n\varphi_i(x_n). \tag{8}$$

Let $\Psi(y_1,y_2,\dots,y_n)$ be the right-hand side of (8). Then $(\varphi,\Psi)\in D_n^+$ and Ψ can be used to produce the next pair, resulting in a stability chain. We have the following result.

Theorem 3.7 *Let $n\in\mathbb{N}$, $\theta\in[0,\infty)$, $\varphi_1,\varphi_2\in C$, and $f:M\to B$. If*

$$\left\|\Delta_y^{n+1}f(x)\right\|\le\theta+\varphi_1(x)+\varphi_2(y)$$

for all $x,y\in M$, then there exist a generalized polynomial $p:M\to B$ of degree at most n and $\varphi_3\in C$ such that

$$\left\|f(x)-p(x)\right\|\le\left(2^{\frac{n(n+1)}{2}}\right)\left(n!\prod_{i=1}^{n}\frac{2^n}{2^n-1}\right)\theta+\varphi_3(x)$$

for all $x\in M$.

A direct corollary of this theorem is the Aoki–Rassias stability:

$$\left\|\Delta^{n+1}f(x)\right\|\le\theta+c_1\left|x^p\right|+c_2|y|^p$$

for $0<p<1$ when M is either $\mathbb{N}\cup\{0\}$ or the set of all integers. In this case, $\varphi_1(x)=|x|^p$ and

$$\lambda_n\varphi_1(x)=\sum_{k=0}^{\infty}\frac{|2^kx|^p}{2^{kn}}=\sum_{k=0}^{\infty}\frac{|2^kx|^p}{2^{kn}}=|x|^p\sum_{k=0}^{\infty}\frac{1}{2^{k(n-p)}}=\frac{2^{n-p}}{2^{n-p}-1}|x|^p.$$

Theorem 3.8 *Let $n\in\mathbb{N}$, $\theta,c_1,c_2\in[0,\infty)$, $p\in(0,1)$, and $f:\mathbb{N}\cup\{0\}\to B$. If*

$$\left\|\Delta_y^{n+1}f(x)\right\|\le\theta+c_1|x|^p+c_2|y|^p$$

for all $x,y\in\mathbb{N}\cup\{0\}$, then there exist $M_n\in[0,\infty)$ and a polynomial $p:\mathbb{N}\cup\{0\}\to B$ of degree at most n such that

$$\left\|f(x)-p(x)\right\|\le\left(2^{\frac{n(n+1)}{2}}\right)\left(n!\prod_{i=1}^{n}\frac{2^n}{2^n-1}\right)\theta+M_n|x|^p$$

for all $x\in\mathbb{N}\cup\{0\}$.

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