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Impulsive quantum (p,q)-difference equations

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Abstract

In this paper we study quantum (p,q)-difference equations with impulse and initial or boundary conditions. We consider first order impulsive (p,q)-difference boundary value problems and second order impulsive (p,q)-difference initial value problems. Existence and uniqueness results are proved via Banach's fixed point theorem.

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1 Introduction and preliminaries

Let p, q be quantum constants satisfying $0 < q < p \leq 1$. The (p,q)-number, $[n]_{p,q}$, is defined by

$$[n]_{p,q} = \frac{p^n - q^n}{p - q}.$$

If n is a positive integer, then

$$[n]_{p,q} = p^{n-1} + p^{n-2}q + \cdots + pq^{n-2} + q^{n-1} \quad \text{and} \quad \lim_{(p,q)\to(1,1)} [n]_{p,q} = n.$$

The (p,q)-difference of a function f on $[0,\infty)$ is defined by

$$D_{p,q}f(t) = \frac{f(pt) - f(qt)}{(p-q)t}, \quad t \neq 0, \tag{1.1}$$

and $D_{p,q}f(0) = f'(0)$. If $f(t) = t^\alpha$, $\alpha \geq 0$, then we have

$$D_{p,q}t^\alpha = [\alpha]_{p,q}t^{\alpha-1}. \tag{1.2}$$

Note that if the function f is defined on $[0,T]$, then the function $D_{p,q}f(t)$ is defined on $[0,T/p]$. For some details of the shifting property and nonlocal boundary value problems for first-order (p,q)-difference equations, we refer the reader to [1]. In addition, in [2], the authors defined the second-order (p,q)-difference by

$$D_{p,q}^2 f(t) = \frac{qf(p^2t) - (p+q)f(pqt) + pf(q^2t)}{pq(p-q)^2t^2}.$$

Then we see that if $f(t)$ is defined on $[0,T]$ then the function $D_{p,q}^2 f(t)$ is defined on $[0,T/p^2]$.

The (p,q)-integral of a function f on $[0,\infty)$ is defined by

$$\int_0^t f(s)\,d_{p,q}s = (p-q)t\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}f\left(\frac{q^n}{p^{n+1}}t\right). \tag{1.3}$$

If $f(t)=t^\alpha$, $\alpha>0$, then we have the formula

$$\int_0^t s^\alpha\,d_{p,q}s = \frac{p-q}{p^{\alpha+1}-q^{\alpha+1}}t^{\alpha+1}. \tag{1.4}$$

Now we observe that if the function f is defined on a finite interval $[0,T]$ then the function $\int_0^t f(s)\,d_{p,q}s$ is defined on $[0,pT]$. In [1], the authors gave the formula of the double (p,q)-integral

$$\begin{aligned}\int_0^t\int_0^s f(r)\,d_{p,q}r\,d_{p,q}s &= \frac{1}{p}\int_0^t (t-qs)f\left(\frac{1}{p}s\right)d_{p,q}s \\ &= \frac{1}{p}(p-q)t^2\sum_{n=0}^{\infty}\frac{q^n}{p^{2n+2}}\left(p^{n+1}-q^{n+1}\right)f\left(\frac{q^n}{p^{n+2}}t\right),\end{aligned}$$

which implies that if f is defined on $[0,T]$, then the function $\int_0^t\int_0^s f(r)\,d_{p,q}r\,d_{p,q}s$ is defined on $[0,p^2T]$.

The (p,q)-calculus was introduced in [3]. For some recent results, see [4–10] and references cited therein. For $p=1$, the (p,q)-calculus is reduced to the classical q-calculus initiated by Jackson [11, 12]. See also [13, 14].

In [15, 16], M. Tunç and E. Göv defined the quantum (p,q)-difference of a function f on the finite interval $[a,b]$ by

$${}_aD_{p,q}f(t) = \frac{f(pt+(1-p)a) - f(qt+(1-q)a)}{(p-q)(t-a)}, \quad t\neq a, \tag{1.5}$$

and ${}_aD_{p,q}f(a) = f'(a)$. The (p,q)-difference of a power function $f(t)=(t-a)^\alpha$, $\alpha\geq 0$, is given by

$$D_{p,q}(t-a)^\alpha = [\alpha]_{p,q}(t-a)^{\alpha-1}. \tag{1.6}$$

Furthermore, they defined the (p,q)-integral of a function f on $[a,b]$ as

$$\int_a^t f(s)\,{}_ad_{p,q}s = (p-q)(t-a)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}f\left(\frac{q^n}{p^{n+1}}t + \left(1-\frac{q^n}{p^{n+1}}\right)a\right). \tag{1.7}$$

As is customary, we put the following relation:

$$\int_a^t (s-a)^\alpha \,{}_a d_{p,q} s = \frac{p-q}{(p^{\alpha+1}-q^{\alpha+1})}(t-a)^{\alpha+1}, \quad \alpha \geq 0. \tag{1.8}$$

It is obvious that if $a = 0$, then equations (1.5)–(1.8) are reduced to (1.1)–(1.4), respectively.

The domain-shift properties of the (p,q)-difference and (p,q)-integral operators for a function $f(t)$, $t \in [a,b]$ are respectively given by

$${}_aD_{p,q}f(t), \quad t \in \left[a, \frac{1}{p}(b-a)+a\right] \quad \text{and} \quad \int_a^t f(s)\,{}_a d_{p,q}s, \quad t \in \bigl[a, p(b-a)+a\bigr].$$

Also we remark that if $p = 1$, then both domains are reduced to $[a,b]$. For the shifting of the second order (p,q)-difference and integral domains, we consider the following result.

Lemma 1.1 *Let f be a function defined on an interval $[a,b]$ with $a \geq 0$. The domains of ${}_aD^2_{p,q}f$ and $\int_a^t \int_a^r f(s)\,{}_a d_{p,q}s\,{}_a d_{p,q}r$ are*

$$\left[a, \frac{1}{p^2}(b-a)+a\right] \quad \textit{and} \quad \bigl[a, p^2(b-a)+a\bigr],$$

respectively.

Proof We have

$$\begin{aligned}
{}_aD^2_{p,q}f(t) &= {}_aD_{p,q}({}_aD_{p,q}f)(t) = {}_aD_{p,q}\left(\frac{f(pt+(1-p)a)-f(qt+(1-q)a)}{(p-q)(t-a)}\right) \\
&= \left\{\frac{f(p(pt+(1-p)a)+(1-p)a)-f(q(pt+(1-p)a)+(1-q)a)}{(p-q)((pt+(1-p)a)-a)}\right. \\
&\quad \left. - \frac{f(p(qt+(1-q)a)+(1-p)a)-f(q(qt+(1-q)a)+(1-q)a)}{(p-q)((qt+(1-q)a)-a)}\right\} \\
&\quad /(p-q)(t-a) \\
&= \frac{qf(p^2t+(1-p^2)a)-(p+q)f(pqt+(1-pq)a)+pf(q^2t+(1-q^2)a)}{pq(p-q)^2(t-a)^2}.
\end{aligned}$$

Setting $p^2t+(1-p^2)a = b$, we have

$$t = \frac{1}{p^2}(b-a)+a.$$

Then ${}_aD^2_{p,q}f$ is defined on $[a,(b-a)/p^2+a]$.

Next we write the double (p,q)-integral in the form of an infinite sum of a function f defined on $[a,b]$. We have

$$\begin{aligned}
\int_a^t \int_a^s f(r)\,{}_a d_{p,q}r\,{}_a d_{p,q}s &= \int_a^t \left[(p-q)(s-a)\sum_{n=0}^\infty \frac{q^n}{p^{n+1}} f\left(\frac{q^n}{p^{n+1}}s + \left\{1-\frac{q^n}{p^{n+1}}\right\}a\right)\,{}_a d_{p,q}s\right] \\
&= (p-q)\sum_{n=0}^\infty \frac{q^n}{p^{n+1}}\left[\int_a^t (s-a) f\left(\frac{q^n}{p^{n+1}}s + \left\{1-\frac{q^n}{p^{n+1}}\right\}a\right)\,{}_a d_{p,q}s\right].
\end{aligned}$$

Now we consider

$$\begin{aligned}
&\int_a^t (s-a)f\left(\frac{q^n}{p^{n+1}}s+\left\{1-\frac{q^n}{p^{n+1}}\right\}a\right)\,{}_ad_{p,q}s \\
&\quad = (p-q)(t-a)\sum_{m=0}^{\infty}\frac{q^m}{p^{m+1}}\left(\frac{q^m}{p^{m+1}}t+\left\{1-\frac{q^m}{p^{m+1}}\right\}a-a\right) \\
&\qquad \times f\left(\frac{q^n}{p^{n+1}}\left[\frac{q^m}{p^{m+1}}t+\left\{1-\frac{q^m}{p^{m+1}}\right\}a\right]+\left\{1-\frac{q^n}{p^{n+1}}\right\}a\right) \\
&\quad = (p-q)(t-a)^2\sum_{m=0}^{\infty}\frac{q^{2m}}{p^{2m+2}}f\left(\frac{q^{m+n}}{p^{m+n+2}}t+\left\{1-\frac{q^{m+n}}{p^{m+n+2}}\right\}a\right),
\end{aligned}$$

which leads to the expression

$$\begin{aligned}
&\int_a^t\int_a^s f(r)\,{}_ad_{p,q}r\,{}_ad_{p,q}s \\
&\quad = (p-q)^2(t-a)^2\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q^{2m+n}}{p^{2m+n+3}}f\left(\frac{q^{m+n}}{p^{m+n+2}}t+\left\{1-\frac{q^{m+n}}{p^{m+n+2}}\right\}a\right).
\end{aligned} \tag{1.9}$$

For $m=n=0$ and setting

$$\frac{1}{p^2}t+\left\{1-\frac{1}{p^2}\right\}a=b,$$

we obtain $t=p^2(b-a)+a$, which implies that $\int_a^t\int_a^r f(s)\,{}_ad_{p,q}s\,{}_ad_{p,q}r$ is valid on $[a,p^2(b-a)+a]$. The proof is completed. □

Before going to the next result, we would like to recall the operator ${}_a\Phi_r$ defined by

$${}_a\Phi_r(m)=rm+(1-r)a,$$

where $m,a\in\mathbb{R}$ and $r\in[0,1]$. Some properties of this operator can be found in [17].

Lemma 1.2 *Let f be a function defined on $[a,b]$. Then the double (p,q)-integral of f can be written as a single one by*

$$\int_a^t\int_a^s f(r)\,{}_ad_{p,q}r\,{}_ad_{p,q}s=\frac{1}{p}\int_a^t\big(t-{}_a\Phi_q(s)\big)f\big({}_a\Phi_{\frac{1}{p}}(s)\big)\,{}_ad_{p,q}s,\quad t\in\big[a,p^2(b-a)+a\big]. \tag{1.10}$$

Proof The double summation in (1.9) can be formulated by a single summation as

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q^{2m+n}}{p^{2m+n+3}}f\left(\frac{q^{m+n}}{p^{m+n+2}}t+\left\{1-\frac{q^{m+n}}{p^{m+n+2}}\right\}a\right) \\
&\quad = \sum_{n=0}^{\infty}\left[\frac{q^n}{p^{n+3}}f\left(\frac{q^n}{p^{n+2}}t+\left\{1-\frac{q^n}{p^{n+2}}\right\}a\right)+\frac{q^{n+2}}{p^{n+5}}f\left(\frac{q^{n+1}}{p^{n+3}}t+\left\{1-\frac{q^{n+1}}{p^{n+3}}\right\}a\right)\right.
\end{aligned}$$

$$
\begin{aligned}
&\quad + \frac{q^{n+4}}{p^{n+7}} f\left(\frac{q^{n+2}}{p^{n+4}} t + \left\{1 - \frac{q^{n+2}}{p^{n+4}}\right\} a\right) + \frac{q^{n+6}}{p^{n+9}} f\left(\frac{q^{n+3}}{p^{n+5}} t + \left\{1 - \frac{q^{n+3}}{p^{n+5}}\right\} a\right) + \cdots \Bigg] \\
&= \frac{1}{p^3} f\left(\frac{1}{p^2} t + \left\{1 - \frac{1}{p^2}\right\} a\right) + \frac{q}{p^4}\left(1 + \frac{q}{p}\right) f\left(\frac{q}{p^3} t + \left\{1 - \frac{q}{p^3}\right\} a\right) \\
&\quad + \frac{q^2}{p^5}\left(1 + \frac{q}{p} + \frac{q^2}{p^2}\right) f\left(\frac{q^2}{p^4} t + \left\{1 - \frac{q^2}{p^4}\right\} a\right) + \cdots \\
&= \sum_{n=0}^{\infty} \frac{q^n}{p^{n+3}} \left(\frac{p^{n+1} - q^{n+1}}{p^n (p-q)}\right) f\left(\frac{q^n}{p^{n+2}} t + \left\{1 - \frac{q^n}{p^{n+2}}\right\} a\right) \\
&= \frac{1}{p-q} \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} \left(\frac{1}{p} - \frac{q^{n+1}}{p^{n+2}}\right) f\left(\frac{q^n}{p^{n+2}} t + \left\{1 - \frac{q^n}{p^{n+2}}\right\} a\right).
\end{aligned}
$$

Substituting into (1.9) yields

$$
\begin{aligned}
&\int_a^t \int_a^s f(r)\, {}_a d_{p,q} r \, {}_a d_{p,q} s \\
&= \frac{1}{p}(p-q)(t-a) \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} \left(t - \left[\frac{q^{n+1}}{p^{n+1}} t + \left\{1 - \frac{q^{n+1}}{p^{n+1}}\right\} a\right]\right) \\
&\quad \times f\left(\frac{q^n}{p^{n+2}} t + \left\{1 - \frac{q^n}{p^{n+2}}\right\} a\right) \\
&= \frac{1}{p}(p-q)(t-a) \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} \left(t - {}_a\Phi_q\left(\left[\frac{q^n}{p^{n+1}} t + \left\{1 - \frac{q^n}{p^{n+1}}\right\} a\right]\right)\right) \\
&\quad \times f\left({}_a\Phi_{\frac{1}{p}}\left(\frac{q^n}{p^{n+1}} t + \left\{1 - \frac{q^n}{p^{n+1}}\right\} a\right)\right) \\
&= \frac{1}{p} \int_a^t \left(t - {}_a\Phi_q(s)\right) f\left({}_a\Phi_{\frac{1}{p}}(s)\right) {}_a d_{p,q} s,
\end{aligned}
$$

which is completed the proof. □

Remark 1.3 If $a = 0$, then (1.10) is reduced to a result of Theorem 3 in [1].

The following theorem has been proved in [16].

Theorem 1.4 *The fundamental relations of (p,q)-calculus can be stated as*

(i) ${}_aD_{p,q} \int_a^t f(s)\, {}_a d_{p,q} s = f(t)$;
(ii) $\int_a^t {}_aD_{p,q} f(s)\, {}_a d_{p,q} s = f(t) - f(a)$.

In this paper we study the impulsive (p,q)-difference equations with initial and boundary conditions. We consider four types of problems, two impulsive (p,q)-difference equations of type I and two impulsive (p,q)-difference equations of type II (explained in the next section). Existence and uniqueness results are proved via Banach's contraction mapping principle. Examples illustrating the obtained results are also constructed.

2 Impulsive (p,q)-difference equations

In this section, we consider the first and second order (p,q)-difference equations with initial or boundary conditions and also prove the existence and uniqueness of solutions for impulsive problems. Firstly, let t_k, $k = 1,\dots,m$, be the impulsive points such that $0 = t_0 < t_1 < \cdots < t_k < \cdots < t_m < t_{m+1} = T$ and $J_k = (t_k, t_{k+1}]$, $k = 1,\dots,m$, $J_0 = [0,t_1]$ be the intervals such that $\bigcup_{k=0}^{m} J_k = [0,T] := J$. The investigations are based on (p,q)-calculus introduced in the previous section by replacing a point a by t_k, quantum numbers p by p_k and q by q_k, $k = 0,1,\dots,m$, and also applying the (p_k,q_k)-difference and (p_k,q_k)-integral operators only on a finite subinterval of J. In addition, the consecutive subintervals can be related with jump conditions which provide a meaning of quantum difference equations with impulse effects. There are two types of impulsive problems which will be established in the next two subsections. The consecutive domains of impulsive (p,q)-difference equations of type I are overlapped, while the unknown functions of impulsive equations of type II are defined on disconnected consecutive domains.

2.1 Impulsive (p,q)-difference equations of type I

Consider the first-order impulsive (p,q)-difference impulsive boundary value problem of the form

$$\begin{cases} {}_{t_k}D_{p_k,q_k}x(t) = f(t,x(t)), \quad t \in (t_k, \frac{1}{p_k}(t_{k+1}-t_k)+t_k], k = 0,1,\dots,m, \\ \Delta x(t_k) = \varphi_k(x(t_k)), \quad k = 1,2,\dots,m, \\ \alpha x(0) + \beta x(T) = \gamma, \end{cases} \tag{2.1}$$

where α, β, and γ are real constants with $\alpha \neq -\beta$, the quantum numbers p_k, q_k satisfy $0 < q_k < p_k \leq 1$, $k = 0,1,\dots,m$, $f : [0,((T-t_m)/p_m)+t_m] \times \mathbb{R} \to \mathbb{R}$ and $\varphi_k : \mathbb{R} \to \mathbb{R}$, $k = 1,2,\dots,m$, are given functions, and ${}_{t_k}D_{p_k,q_k}$ is the quantum (p_k,q_k)-difference operator starting at a point t_k, $k = 0,1,\dots,m$.

We remark that there are some overlapped intervals of domains of the first equation in (2.1). For example, if the unknown function $x(t)$ is defined on $J = [0,2]$ and if there is an impulse point $t_1 = 1$, that is, $x(1^+) \neq x(1^-)$, with $p_0 = 1/2$, $q_0 = 1/3$, $p_1 = 1/4$, and $q_1 = 1/5$. Then we have the (p,q)-difference equations

$${}_0D_{\frac{1}{2},\frac{1}{3}}x(t) = f\bigl(t,x(t)\bigr), \quad t \in (0,2] \quad \text{and} \quad {}_1D_{\frac{1}{4},\frac{1}{5}}x(t) = f\bigl(t,x(t)\bigr), \quad t \in (1,5].$$

However, by the shifting property of (p,q)-integration applied to the two above equations, we have

$$x(t) = x(0) + \int_0^t f\bigl(s,x(s)\bigr)\,{}_0d_{\frac{1}{2},\frac{1}{3}}s \quad t \in (0,1],$$

and

$$x(t) = x\bigl(1^+\bigr) + \int_1^t f\bigl(s,x(s)\bigr)\,{}_1d_{\frac{1}{4},\frac{1}{5}}s, \quad t \in (1,2],$$

respectively.

Theorem 2.1 *The nonlinear first-order (p,q)-difference boundary value problem* (2.1) *can be transformed into an integral equation*

$$x(t) = \frac{\gamma}{(\alpha+\beta)} - \frac{\beta}{(\alpha+\beta)}\left(\sum_{i=0}^{m}\int_{t_i}^{t_{i+1}} f\bigl(s,x(s)\bigr)\,{}_{t_i}d_{p_i,q_i}s + \sum_{j=1}^{m}\varphi_j\bigl(x(t_j)\bigr)\right)$$
$$+ \sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}} f\bigl(s,x(s)\bigr)\,{}_{t_i}d_{p_i,q_i}s + \sum_{j=1}^{k}\varphi_j\bigl(x(t_j)\bigr) + \int_{t_k}^{t} f\bigl(s,x(s)\bigr)\,{}_{t_k}d_{p_k,q_k}s, \quad t \in J, \quad (2.2)$$

with $\sum_a^b(\cdot) = 0$, *if* $b < a$.

Proof From ${}_{t_0}D_{p_0,q_0}x(t) = f(t,x(t))$, $t \in (t_0,(1/p_0)(t_1-t_0)+t_0]$, by taking the (p_0,q_0)-integral, we obtain

$$x(t) = x(0) + \int_{t_0}^{t} f\bigl(s,x(s)\bigr)\,{}_{t_0}d_{p_0,q_0}s, \quad t \in (t_0,t_1],$$

by using Theorem 1.4 and the shifting property. Next, for ${}_{t_1}D_{p_1,q_1}x(t) = f(t,x(t))$, $t \in (t_1,(1/p_1)(t_2-t_1)+t_1]$, where t_1 is the first impulsive point in J, we also obtain by applying the (p_1,q_1)-integration,

$$x(t) = x\bigl(t_1^+\bigr) + \int_{t_1}^{t} f\bigl(s,x(s)\bigr)\,{}_{t_1}d_{p_1,q_1}s, \quad t \in (t_1,t_2].$$

By the impulsive condition $x(t_1^+) = x(t_1) + \varphi_1(x(t_1))$, it follows, for $t \in (t_1,t_2]$, that

$$x(t) = x(0) + \int_{t_0}^{t_1} f\bigl(s,x(s)\bigr)\,{}_{t_0}d_{p_0,q_0}s + \varphi_1\bigl(x(t_1)\bigr) + \int_{t_1}^{t} f\bigl(s,x(s)\bigr)\,{}_{t_1}d_{p_1,q_1}s.$$

For ${}_{t_2}D_{p_2,q_2}x(t) = f(t,x(t))$, $t \in (t_2,(1/p_2)(t_3-t_2)+t_2]$, we get

$$x(t) = x\bigl(t_2^+\bigr) + \int_{t_2}^{t} f\bigl(s,x(s)\bigr)\,{}_{t_2}d_{p_2,q_2}s, \quad t \in (t_2,t_3],$$

by (p_2,q_2)-integration and

$$x(t) = x(0) + \int_{t_0}^{t_1} f\bigl(s,x(s)\bigr)\,{}_{t_0}d_{p_0,q_0}s + \int_{t_1}^{t_2} f\bigl(s,x(s)\bigr)\,{}_{t_1}d_{p_1,q_1}s$$
$$+ \varphi_1\bigl(x(t_1)\bigr) + \varphi_2\bigl(x(t_2)\bigr) + \int_{t_2}^{t} f\bigl(s,x(s)\bigr)\,{}_{t_2}d_{p_2,q_2}s, \quad t \in (t_2,t_3],$$

due to the impulsive condition $x(t_2^+) = x(t_2^-) + \varphi_2(x(t_2))$.

Repeating this process, we obtain, for $t \in J_k$, $k = 0,1,\dots,m$, that

$$x(t) = x(0) + \sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}} f\bigl(s,x(s)\bigr)\,{}_{t_i}d_{p_i,q_i}s + \sum_{j=1}^{k}\varphi_j\bigl(x(t_j)\bigr) + \int_{t_k}^{t} f\bigl(s,x(s)\bigr)\,{}_{t_k}d_{p_k,q_k}s. \quad (2.3)$$

After that from the boundary condition $\alpha x(0) + \beta x(T) = \gamma$, we have

$$x(0) = \frac{\gamma}{(\alpha+\beta)} - \frac{\beta}{(\alpha+\beta)}\left(\sum_{i=0}^{m}\int_{t_i}^{t_{i+1}} f\big(s,x(s)\big)\,{}_{t_i}d_{p_i,q_i}s + \sum_{j=1}^{m}\varphi_j\big(x(t_j)\big)\right).$$

Putting the value of $x(0)$ into (2.3), shows that (2.2) is true and the proof is completed. □

Remark 2.2 If $\alpha \neq 0$ and $\beta = 0$, then the boundary value problem (2.1) can be reduced to the initial value problem with initial condition $x(0) = \gamma/\alpha$.

Before going to the second-order impulsive problem, we define

$$\tau_k = \frac{1}{p_{k-1}}(t_k - t_{k-1}) + t_{k-1}, \quad k = 1,2,\dots,m,$$

which are impulsive shifting points of the (p_k,q_k)-derivative of the unknown function in our system. In addition, we introduce a notation

$$\langle t_{i+1}\rangle_k = \begin{cases} t_{i+1}, & t_{i+1} \le t_k, \\ t, & t_{i+1} > t_k. \end{cases}$$

For example,

$$\begin{aligned}\sum_{i=0}^{2}\big(\langle t_{i+1}\rangle_2 - t_i\big)K_i &= \big(\langle t_1\rangle_2 - t_0\big)K_0 + \big(\langle t_2\rangle_2 - t_2\big)K_1 + \big(\langle t_3\rangle_2 - t_2\big)K_2 \\ &= (t_1 - t_0)K_0 + (t_2 - t_1)K_1 + (t - t_2)K_2,\end{aligned}$$

where $K_i \in \mathbb{R}$, $i = 0,1,2$.

Now, we consider the second-order impulsive (p,q)-difference initial value problem of the form

$$\begin{cases} {}_{t_k}D^2_{p_k,q_k}x(t) = f(t,x(t)), & t \in (t_k, \frac{1}{p_k^2}(t_{k+1}-t_k)+t_k], k = 0,1,\dots,m, \\ \Delta x(t_k) = \varphi_k(x(t_k)), & k = 1,2,\dots,m, \\ {}_{t_k}D_{p_k,q_k}x(t_k^+) - {}_{t_{k-1}}D_{p_{k-1},q_{k-1}}x(\tau_k) = \varphi_k^*(x(t_k)), & k = 1,2,\dots,m, \\ x(0) = \lambda_1, & {}_{t_0}D_{p_0,q_0}x(0) = \lambda_2, \end{cases} \tag{2.4}$$

where $f : [0,((T-t_m)/p_m^2)+t_m] \times \mathbb{R} \to \mathbb{R}$, $\varphi_k : \mathbb{R} \to \mathbb{R}$ and $\varphi_k^* : \mathbb{R} \to \mathbb{R}$, are given functions, λ_1, λ_2 are given constants. Observe that the distance between the impulsive points t_k and τ_k in the third equation of (2.4) depends on the value of p_{k-1} for $k = 1,2,\dots,m$. Indeed,

$$\tau_k - t_k = \frac{1}{p_{k-1}}(t_k - t_{k-1}) + t_{k-1} - t_k = \frac{(1-p_{k-1})}{p_{k-1}}(t_k - t_{k-1}),$$

which has appeared by the shifting property of (p,q)-calculus as discussed in the previous section.

Theorem 2.3 *The impulsive initial value problem of type I given by the* (p,q)*-difference equation* (2.4) *can be expressed as an integral equation of the form*

$$\begin{aligned} x(t) &= \lambda_1 + \sum_{i=0}^{k}\bigl(\langle t_{i+1}\rangle_k - t_i\bigr)\Biggl[\lambda_2 + \sum_{j=0}^{i-1}\biggl\{\int_{t_j}^{\tau_{j+1}} f\bigl(s,x(s)\bigr)\,{}_{t_j}d_{p_j,q_j}s + \varphi_{j+1}^*\bigl(x(t_{j+1})\bigr)\biggr\}\Biggr] \\ &\quad + \sum_{r=0}^{k-1}\biggl\{\frac{1}{p_r}\int_{t_r}^{t_{r+1}}\bigl(t_{r+1} - {}_{t_r}\Phi_{q_r}(s)\bigr)f_x\bigl({}_{t_r}\Phi_{\frac{1}{p_r}}(s)\bigr)\,{}_{t_r}d_{p_r,q_r}s + \varphi_{r+1}\bigl(x(t_{r+1})\bigr)\biggr\} \\ &\quad + \frac{1}{p_k}\int_{t_k}^{t}\bigl(t - {}_{t_k}\Phi_{q_k}(s)\bigr)f_x\bigl({}_{t_k}\Phi_{\frac{1}{p_k}}(s)\bigr)\,{}_{t_k}d_{p_k,q_k}s, \quad t\in J_k, k=0,1,\ldots,m, \end{aligned} \tag{2.5}$$

where $f_x({}_{t_r}\Phi_{\frac{1}{p_r}}(s)) = f({}_{t_r}\Phi_{\frac{1}{p_r}}(s), x({}_{t_r}\Phi_{\frac{1}{p_r}}(s)))$, $r=0,1,\ldots,k$, *and* $\sum_a^b(\cdot)=0$, *when* $b<a$.

Proof By computing the (p_0,q_0)-integral of both sides of the first equation of (2.4), we get

$${}_{t_0}D_{p_0,q_0}x(t) = {}_{t_0}D_{p_0,q_0}x(0) + \int_{t_0}^{t} f\bigl(s,x(s)\bigr)\,{}_{t_0}d_{p_0,q_0}s, \quad t\in\biggl(0,\frac{1}{p_0}t_1\biggr].$$

Applying another (p_0,q_0)-integration, we obtain, for $t\in(0,t_1]$,

$$\begin{aligned} x(t) &= x(0) + t\,{}_{t_0}D_{p_0,q_0}x(0) + \int_{t_0}^{t}\int_{t_0}^{r} f\bigl(s,x(s)\bigr)\,{}_{t_0}d_{p_0,q_0}s\,{}_{t_0}d_{p_0,q_0}r \\ &= \lambda_1 + \lambda_2 t + \frac{1}{p_0}\int_{t_0}^{t}\bigl(t - {}_{t_0}\Phi_{q_0}(s)\bigr)f_x\bigl({}_{t_0}\Phi_{\frac{1}{p_0}}(s)\bigr)\,{}_{t_0}d_{p_0,q_0}s. \end{aligned}$$

For $t\in(t_1,((t_2-t_1)/p_1^2)+t_1]$, applying the double (p_1,q_1)-integration to both sides of the first equation of (2.4), we have

$$x(t) = x\bigl(t_1^+\bigr) + (t-t_1)\,{}_{t_1}D_{p_1,q_1}x\bigl(t_1^+\bigr) + \frac{1}{p_1}\int_{t_1}^{t}\bigl(t - {}_{t_1}\Phi_{q_1}(s)\bigr)f_x\bigl({}_{t_1}\Phi_{\frac{1}{p_1}}(s)\bigr)\,{}_{t_1}d_{p_1,q_1}s,$$

where $t\in(t_1,t_2]$. Due to the impulsive conditions

$$\begin{aligned} x\bigl(t_1^+\bigr) &= x(t_1) + \varphi_1\bigl(x(t_1)\bigr) \\ &= \lambda_1 + \lambda_2 t_1 + \frac{1}{p_0}\int_{t_0}^{t_1}\bigl(t_1 - {}_{t_0}\Phi_{q_0}(s)\bigr)f_x\bigl({}_{t_0}\Phi_{\frac{1}{p_0}}(s)\bigr)\,{}_{t_0}d_{p_0,q_0}s + \varphi_1\bigl(x(t_1)\bigr) \end{aligned}$$

and

$$\begin{aligned} {}_{t_1}D_{p_1,q_1}x\bigl(t_1^+\bigr) &= {}_{t_0}D_{p_0,q_0}x(\tau_1) + \varphi_1^*\bigl(x(t_1)\bigr) \\ &= \lambda_2 + \int_{t_0}^{\tau_1} f\bigl(s,x(s)\bigr)\,{}_{t_0}d_{p_0,q_0}s + \varphi_1^*\bigl(x(t_1)\bigr), \end{aligned}$$

we have

$$
\begin{aligned}
x(t) &= \lambda_1 + \lambda_2 t_1 + \frac{1}{p_0}\int_{t_0}^{t_1}\big(t_1 - {}_{t_0}\Phi_{q_0}(s)\big) f_x\big({}_{t_0}\Phi_{\frac{1}{p_0}}(s)\big)\,{}_{t_0}d_{p_0,q_0}s + \varphi_1\big(x(t_1)\big) \\
&\quad + (t-t_1)\bigg[\lambda_2 + \int_{t_0}^{\tau_1} f\big(s,x(s)\big)\,{}_{t_0}d_{p_0,q_0}s + \varphi_1^*\big(x(t_1)\big)\bigg] \\
&\quad + \frac{1}{p_1}\int_{t_1}^{t}\big(t - {}_{t_1}\Phi_{q_1}(s)\big) f_x\big({}_{t_1}\Phi_{\frac{1}{p_1}}(s)\big)\,{}_{t_1}d_{p_1,q_1}s, \quad t\in(t_1,t_2].
\end{aligned}
$$

Similarly, we deduce the integral equation (2.5), as desired. □

Now, the existence and uniqueness results for problems (2.1) and (2.4) will be proved by using the Banach's contraction mapping principle. Let us define the space $PC(J,\mathbb{R}) = \{x : J\to\mathbb{R}: x(t)$ is continuous everywhere except for some t_k in which $x(t_k^+)$ and $x(t_k^-)$ exist and $x(t_k^-) = x(t_k)$, $k = 1,2,\dots,m\}$. The set $PC(J,\mathbb{R})$ is a Banach space equipped with the norm $\|x\| = \sup\{|x(t)| : t\in J\}$. For convenience, we put

$$
\begin{aligned}
\Omega_1 &= \frac{|\beta| + |\alpha+\beta|}{|\alpha+\beta|}\sum_{i=0}^{m}(t_{i+1}-t_i), \\
\Omega_2 &= m\left(\frac{|\beta| + |\alpha+\beta|}{|\alpha+\beta|}\right), \\
\Omega_3 &= \sum_{i=0}^{m}\left\{(t_{i+1}-t_i)\sum_{j=0}^{i-1}(\tau_{j+1}-t_j)\right\} + \sum_{r=0}^{m}\frac{(t_{r+1}-t_r)^2}{p_r+q_r}, \\
\Omega_4 &= \sum_{i=0}^{m}(t_{i+1}-t_i)i.
\end{aligned}
$$

Theorem 2.4 *Let $f : [0,((T-t_m)/p_m)+t_m]\times\mathbb{R}\to\mathbb{R}$ and $\varphi_k:\mathbb{R}\to\mathbb{R}$, $k = 1,2,\dots,m$, be given functions satisfying*

(H_1) *There exist positive constants L_1 and L_2 such that*

$$
\big|f(t,x) - f(t,y)\big| \le L_1|x-y| \quad \text{and} \quad \big|\varphi_k(x) - \varphi_k(y)\big| \le L_2|x-y|,
$$

for all $t\in[0,((T-t_m)/p_m)+t_m]$, $x,y\in\mathbb{R}$ and $k = 1,2,\dots,m$.

If

$$
L_1\Omega_1 + L_2\Omega_2 < 1, \tag{2.6}
$$

then the boundary value problem (2.1) has a unique solution on J.

Proof In view of Theorem 2.1, we define the operator $\mathcal{A} : PC(J,\mathbb{R})\to PC(J,\mathbb{R})$ by

$$
\begin{aligned}
\mathcal{A}x(t) &= \frac{\gamma}{(\alpha+\beta)} - \frac{\beta}{(\alpha+\beta)}\left(\sum_{i=0}^{m}\int_{t_i}^{t_{i+1}} f\big(s,x(s)\big)\,{}_{t_i}d_{p_i,q_i}s + \sum_{j=1}^{m}\varphi_j\big(x(t_j)\big)\right) \\
&\quad + \sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}} f\big(s,x(s)\big)\,{}_{t_i}d_{p_i,q_i}s + \sum_{j=1}^{k}\varphi_j\big(x(t_j)\big) + \int_{t_k}^{t} f\big(s,x(s)\big)\,{}_{t_k}d_{p_k,q_k}s, \quad t\in J.
\end{aligned}
$$

Define the ball $B_{r_1} = \{x \in PC(J,\mathbb{R}) : \|x\| \leq r_1\}$ where the positive constant r_1 is defined by

$$r_1 > \frac{(|\gamma|/|\alpha+\beta|) + M_1\Omega_1 + M_2\Omega_2}{1-(L_1\Omega_1 + L_2\Omega_2)}.$$

The Banach contraction mapping principle is used to claim that there exists a unique fixed point of an operator equation $x = \mathcal{A}x$ in B_{r_1}. By setting $\sup_{t\in J}|f(t,0)| = M_1$, and $\sup\{|\varphi_i(0)|, i = 1,2,\ldots,m\} = M_2$ and using the inequalities $|f(t,x)| \leq |f(t,x) - f(t,0)| + |f(t,0)| \leq L_1r_1 + M_1$ and $|\varphi_i(x)| \leq |\varphi_i(x) - \varphi_i(0)| + |\varphi_i(0)| \leq L_1r_1 + M_2$, $i = 1,2,\ldots,m$, we have

$$\begin{aligned}
\bigl|\mathcal{A}x(t)\bigr| &\leq \frac{|\gamma|}{|\alpha+\beta|} + \frac{|\beta|}{|\alpha+\beta|}\left(\sum_{i=0}^{m}\int_{t_i}^{t_{i+1}}\bigl|f\bigl(s,x(s)\bigr)\bigr|\,{}_{t_i}d_{p_i,q_i}s + \sum_{j=1}^{m}\bigl|\varphi_j\bigl(x(t_j)\bigr)\bigr|\right)\\
&\quad + \sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}}\bigl|f\bigl(s,x(s)\bigr)\bigr|\,{}_{t_i}d_{p_i,q_i}s + \sum_{j=1}^{k}\bigl|\varphi_j\bigl(x(t_j)\bigr)\bigr| + \int_{t_k}^{t}\bigl|f\bigl(s,x(s)\bigr)\bigr|\,{}_{t_k}d_{p_k,q_k}s\\
&\leq \frac{|\gamma|}{|\alpha+\beta|} + \frac{|\beta|}{|\alpha+\beta|}\left(\sum_{i=0}^{m}(L_1r_1+M_1)\int_{t_i}^{t_{i+1}}(1)\,{}_{t_i}d_{p_i,q_i}s + (L_2r_1+M_2)\sum_{j=1}^{m}(1)\right)\\
&\quad + (L_1r_1+M_1)\sum_{i=0}^{m-1}\int_{t_i}^{t_{i+1}}(1)\,{}_{t_i}d_{p_i,q_i}s + (L_2r_1+M_2)\sum_{j=1}^{m}(1)\\
&\quad + (L_1r_1+M_1)\int_{t_m}^{t_{m+1}}(1)\,{}_{t_k}d_{p_k,q_k}s\\
&= \frac{|\gamma|}{|\alpha+\beta|} + \frac{|\beta|}{|\alpha+\beta|}\left((L_1r_1+M_1)\sum_{i=0}^{m}(t_{i+1}-t_i) + m(L_2r_1+M_2)\right)\\
&\quad + (L_1r_1+M_1)\sum_{i=0}^{m-1}(t_{i+1}-t_i) + m(L_2r_1+M_2) + (L_1r_1+M_1)(t_{m+1}-t_m)\\
&= \frac{|\gamma|}{|\alpha+\beta|} + L_1\Omega_1r_1 + L_2\Omega_2r_1 + M_1\Omega_1 + M_2\Omega_2 < r_1,
\end{aligned}$$

which leads to $\mathcal{A}B_{r_1} \subset B_{r_1}$. To prove that $\mathcal{A}$ is a contraction, we let $x, y \in B_{r_1}$. Then we have

$$\begin{aligned}
&\bigl|\mathcal{A}x(t) - \mathcal{A}y(t)\bigr|\\
&\quad\leq \frac{|\beta|}{|\alpha+\beta|}\left(\sum_{i=0}^{m}\int_{t_i}^{t_{i+1}}\bigl|f\bigl(s,x(s)\bigr) - f\bigl(s,y(s)\bigr)\bigr|\,{}_{t_i}d_{p_i,q_i}s + \sum_{j=1}^{m}\bigl|\varphi_j\bigl(x(t_j)\bigr) - \varphi_j\bigl(y(t_j)\bigr)\bigr|\right)\\
&\qquad + \sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}}\bigl|f\bigl(s,x(s)\bigr) - f\bigl(s,y(s)\bigr)\bigr|\,{}_{t_i}d_{p_i,q_i}s\\
&\qquad + \sum_{j=1}^{k}\bigl|\varphi_j\bigl(x(t_j)\bigr) - \varphi_j\bigl(y(t_j)\bigr)\bigr| + \int_{t_k}^{t}\bigl|f\bigl(s,x(s)\bigr) - f\bigl(s,y(s)\bigr)\bigr|\,{}_{t_k}d_{p_k,q_k}s\\
&\quad\leq \frac{|\beta|}{|\alpha+\beta|}\left(L_1\|x-y\|\sum_{i=0}^{m}(t_{i+1}-t_i) + mL_2\|x-y\|\right)
\end{aligned}$$

$$
\begin{aligned}
&+ L_1\|x-y\|\sum_{i=0}^{m}(t_{i+1}-t_i) + mL_2\|x-y\| \\
&= (L_1\Omega_1 + L_2\Omega_2)\|x-y\|.
\end{aligned}
$$

Therefore, $\|\mathcal{A}x - \mathcal{A}y\| \leq (L_1\Omega_1 + L_2\Omega_2)\|x-y\|$. By means of the Banach contraction mapping principle, the operator $\mathcal{A}$ has a unique fixed point in B_{r_1} which is a unique solution of boundary value problem (2.1). The proof is completed. □

Theorem 2.5 *Assume that the functions* $f : [0, ((T-t_m)/p_m^2) + t_m] \times \mathbb{R} \to \mathbb{R}$ *and* $\varphi_k : \mathbb{R} \to \mathbb{R}$, $k = 1, 2, \dots, m$, *satisfy* (H_1). *In addition, we suppose that the functions* $\varphi_k^* : \mathbb{R} \to \mathbb{R}$, $k = 1, 2, \dots, m$, *satisfy*

(H_2) *There exists a positive constant* L_3 *such that*

$$
\left|\varphi_k^*(x) - \varphi_k^*(y)\right| \leq L_3|x-y|,
$$

for all $x, y \in \mathbb{R}$.

If

$$
L_1\Omega_3 + L_2 m + L_3\Omega_4 < 1, \tag{2.7}
$$

then the boundary value problem (2.4) *has a unique solution on* $[0, T]$.

Proof The proof is similar to that of Theorem 2.4 and is omitted. □

Example 2.6 Consider the following first-order impulsive quantum (p,q)-difference equation of type I subject to the boundary condition of the form:

$$
\begin{cases}
{}_kD_{\frac{1}{k+2},\frac{1}{k+3}}x(t) = \frac{1}{18+t^2}\left(\frac{x^2(t)+2|x(t)|}{1+|x(t)|}\right) + \frac{3}{2}, & t \in (k, 2k+2], k = 0, 1, 2, \\
\Delta x(k) = \frac{1}{6k}\sin x(t_k), & k = 1, 2, \\
\frac{1}{2}x(0) + \frac{1}{3}x(3) = \frac{1}{4}.
\end{cases} \tag{2.8}
$$

Here $p_k = 1/(k+2)$, $q_k = 1/(k+3)$, $k = 0, 1, 2$, $\alpha = 1/2$, $\beta = 1/3$, $\gamma = 1/4$, $t_k = k$, $k = 1, 2$, $T = 3$, and $m = 2$. The given data leads to constants $\Omega_1 = 21/5$, $\Omega_2 = 14/5$. Setting

$$
f(t,x) = \frac{1}{18+t^2}\left(\frac{x^2+2|x|}{1+|x|}\right) + \frac{3}{2} \quad \text{and} \quad \varphi_k(x) = \frac{1}{6k}\sin x,
$$

we have $|f(t,x) - f(t,y)| \leq (1/9)|x-y|$ and $|\varphi_k(x) - \varphi_k(y)| \leq (1/6)|x-y|$ which satisfy Condition (H_1) in Theorem 2.4 with $L_1 = 1/9$ and $L_2 = 1/6$. Since $L_1\Omega_1 + L_2\Omega_2 = 14/15 < 1$, by Theorem 2.4, the boundary value problem (2.8) has a unique solution x on $[0, 3]$.

Example 2.7 Consider the following second-order impulsive quantum (p,q)-difference equation of type I with the initial conditions of the form:

$$\begin{cases} {}_kD^2_{\frac{1}{k+2},\frac{1}{k+3}}x(t) = \frac{1}{5(t+5)}\tan^{-1}|x(t)| + \frac{1}{2}, & t \in (k, k^2+5k+4], k = 0,1,2, \\ \Delta x(k) = \frac{|x(t_k)|}{10k(1+|x(t_k)|)}, & k = 1,2, \\ {}_kD_{\frac{1}{k+2},\frac{1}{k+3}}x(k) - {}_{(k-1)}D_{\frac{1}{k+1},\frac{1}{k+2}}x(2k) = \frac{1}{15k^2}\sin|x(t_k)|, & k = 1,2, \\ x(0) = \frac{3}{5}, \quad {}_0D_{\frac{1}{2},\frac{1}{3}}x(0) = \frac{5}{7}. \end{cases} \tag{2.9}$$

Here the quantum constants p_k, q_k and impulsive points t_k, are as in Example 2.6. In addition, $\tau_k = 2k$, $k = 1,2$, and initial constants $\lambda_1 = 3/5$, $\lambda_2 = 5/7$. Next, we can compute that $\Omega_3 = 12.1365$ and $\Omega_4 = 3$. Set

$$f(t,x) = \frac{1}{5(t+5)}\tan^{-1}|x| + \frac{1}{2}, \qquad \varphi_k(x) = \frac{|x|}{10k(1+|x|)}, \quad \text{and} \quad \varphi_k^*(x) = \frac{1}{15k^2}\sin|x|.$$

It is easy to see that f, φ_k, and φ_k^* satisfy (H_1) and (H_2) with $L_1 = 1/25$, $L_2 = 1/10$, and $L_3 = 1/15$. Therefore, we have $L_1\Omega_3 + L_2 m + L_3\Omega_4 = 0.8855 < 1$. Hence the boundary value problem (2.9) has a unique solution x on $[0,3]$ by Theorem 2.5.

2.2 Impulsive (p,q)-difference equations of type II

Now we study the first-order impulsive (p,q)-difference boundary value problem of the form

$$\begin{cases} {}_{t_k}D_{p_k,q_k}x(t) = f(t,x(t)), & t \in (t_k, t_{k+1}], k = 0,1,\dots,m, \\ x(t_k^+) - x(\rho_k) = \varphi_k(x(\rho_k)), & k = 1,2,\dots,m, \\ \alpha x(0) + \beta x(\rho_{m+1}) = \gamma, \end{cases} \tag{2.10}$$

where $f : J \times \mathbb{R} \to \mathbb{R}$ and the functions φ_k, $k = 1,2,\dots,m$, and constants α, β, γ are defined as in Sect. 2.1. The constant ρ_k is defined by

$$\rho_k = p_{k-1}(t_k - t_{k-1}) + t_{k-1}, \quad k = 1,2,\dots,m,m+1.$$

Then the lagging distance is $t_k - \rho_k = (1-p_{k-1})(t_k - t_{k-1})$ which depends on the value of $p_{k-1} \in (0,1]$.

To observe the special characteristic of this type, by the shifting property of the (p,q)-derivative, we see that the unknown function $x(t)$ is defined on $[t_0,\rho_1] \cup (t_k,\rho_{k+1}]$, $k = 1,2,\dots,m$.

Example 2.8 Let $J = [0,2]$ and $t_1 = 1$ be an impulsive point. Then

$${}_0D_{\frac{1}{2},\frac{1}{3}}x(t) = f\bigl(t,x(t)\bigr), \quad t \in [0,1],$$

and

$${}_1D_{\frac{1}{4},\frac{1}{5}}x(t) = f\bigl(t,x(t)\bigr), \quad t \in (1,2],$$

can be presented as

$$x(t) = x(0) + \int_0^t f\bigl(s, x(s)\bigr)\, {}_0d_{\frac{1}{2},\frac{1}{3}}, \quad t \in \left[0, \frac{1}{2}\right],$$

and

$$x(t) = x\bigl(1^+\bigr) + \int_1^t f\bigl(s, x(s)\bigr)\, {}_1d_{\frac{1}{4},\frac{1}{5}}, \quad t \in \left(1, \frac{5}{4}\right].$$

Theorem 2.9 *The first-order type II (p,q)-difference boundary value problem (2.10) can be expressed as an integral equation*

$$\begin{aligned} x(t) = {} & \frac{\gamma}{(\alpha+\beta)} - \frac{\beta}{(\alpha+\beta)} \left(\sum_{i=0}^{m} \int_{t_i}^{\rho_{i+1}} f\bigl(s, x(s)\bigr)\, {}_{t_i}d_{p_i,q_i}s + \sum_{j=1}^{m} \varphi_j\bigl(x(\rho_j)\bigr) \right) \\ & + \sum_{i=0}^{k-1} \int_{t_i}^{\rho_{i+1}} f\bigl(s, x(s)\bigr)\, {}_{t_i}d_{p_i,q_i}s + \sum_{j=1}^{k} \varphi_j\bigl(x(\rho_j)\bigr) + \int_{t_k}^{t} f\bigl(s, x(s)\bigr)\, {}_{t_k}d_{p_k,q_k}s, \end{aligned} \tag{2.11}$$

with $\sum_a^b(\cdot) = 0$, if $b < a$.

Proof Firstly, the (p_0, q_0)-integration of the first equation in (2.10) yields

$$x(t) = x(0) + \int_{t_0}^{t} f\bigl(s, x(s)\bigr)\, {}_{t_0}d_{p_0,q_0}s, \quad t \in (t_0, \rho_1].$$

In particular, for $t = \rho_1$, we have

$$x(\rho_1) = x(0) + \int_{t_0}^{\rho_1} f\bigl(s, x(s)\bigr)\, {}_{t_0}d_{p_0,q_0}s.$$

For $k = 1$, by (p_1, q_1)-integration, we obtain

$$x(t) = x\bigl(t_1^+\bigr) + \int_{t_1}^{t} f\bigl(s, x(s)\bigr)\, {}_{t_1}d_{p_1,q_1}s, \quad t \in (t_1, \rho_2],$$

which leads to

$$x(t) = x(0) + \int_{t_0}^{\rho_1} f\bigl(s, x(s)\bigr)\, {}_{t_0}d_{p_0,q_0}s + \varphi_1\bigl(x(\rho_1)\bigr) + \int_{t_1}^{t} f\bigl(s, x(s)\bigr)\, {}_{t_1}d_{p_1,q_1}s,$$

by using the impulse condition $x(t_1^+) = x(\rho_1) + \varphi_1(x(\rho_1))$.

Repeating the process for any $t \in (t_k, \rho_{k+1}]$, we get

$$x(t) = x(0) + \sum_{i=0}^{k-1} \int_{t_i}^{\rho_{i+1}} f\bigl(s, x(s)\bigr)\, {}_{t_i}d_{p_i,q_i}s + \sum_{j=1}^{k} \varphi_j\bigl(x(\rho_j)\bigr) + \int_{t_k}^{t} f\bigl(s, x(s)\bigr)\, {}_{t_k}d_{p_k,q_k}s.$$

Since

$$x(\rho_{m+1}) = x(0) + \sum_{i=0}^{m} \int_{t_i}^{\rho_{i+1}} f\bigl(s, x(s)\bigr)\, {}_{t_i}d_{p_i,q_i}s + \sum_{j=1}^{m} \varphi_j\bigl(x(\rho_j)\bigr),$$

by the boundary condition, we have

$$x(0) = \frac{\gamma}{(\alpha+\beta)} - \frac{\beta}{(\alpha+\beta)}\left(\sum_{i=0}^{m}\int_{t_i}^{\rho_{i+1}} f\bigl(s,x(s)\bigr)\,{}_{t_i}d_{p_i,q_i}s + \sum_{j=1}^{m}\varphi_j\bigl(x(\rho_j)\bigr)\right),$$

which implies that (2.11) holds. This completes the proof. □

Next we define the points $\rho_k^* = p_{k-1}^2(t_k - t_{k-1}) + t_{k-1}$, $k = 1,2,\dots,m,m+1$. Now we consider the second-order type II impulsive (p,q)-difference initial value problem of the form

$$\begin{cases} {}_{t_k}D^2_{p_k,q_k}x(t) = f(t,x(t)), & t \in (t_k,t_{k+1}], k = 0,1,\dots,m, \\ x(t_k^+) - x(\rho_k^*) = \varphi_k(x(\rho_k^*)), & k = 1,2,\dots,m, \\ {}_{t_k}D_{p_k,q_k}x(t_k^+) - {}_{t_{k-1}}D_{p_{k-1},q_{k-1}}x(\rho_k) = \varphi_k^*(x(\rho_k^*)), & k = 1,2,\dots,m, \\ x(0) = \lambda_1, \quad {}_{t_0}D_{p_0,q_0}x(0) = \lambda_2, \end{cases} \tag{2.12}$$

where $f : J \times \mathbb{R} \to \mathbb{R}$, while other functions and constants are defined as in Sect. 2.1. Since $0 < p_k \le 1$, we have $\rho_k^* \le t_k$, and consequently $(t_k,\rho_k^*] \subseteq (t_k,t_{k+1}]$ for all $k = 0,1,\dots,m$. By Lemma 1.1, the unknown function $x(t)$ of problem (2.12) is defined on $[t_0,\rho_1^*]\bigcup_{k=1}^{m}(t_k,\rho_{k+1}^*]$.

Theorem 2.10 *The initial value problem* (2.12) *of the impulsive* (p,q)*-difference equation of type II can be stated as an integral equation of the form*

$$\begin{aligned} x(t) = {}& \lambda_1 + \sum_{i=0}^{k}\bigl(\langle\rho_{i+1}^*\rangle_k - t_i\bigr)\left[\lambda_2 + \sum_{j=0}^{i-1}\left\{\int_{t_j}^{\rho_{j+1}} f\bigl(s,x(s)\bigr)\,{}_{t_j}d_{p_j,q_j}s + \varphi_{j+1}^*\bigl(x(\rho_{j+1}^*)\bigr)\right\}\right] \\ & + \sum_{r=0}^{k-1}\left\{\frac{1}{p_r}\int_{t_r}^{\rho_{r+1}^*}\bigl(\rho_{r+1}^* - {}_{t_r}\Phi_{q_r}(s)\bigr)f_x\bigl({}_{t_r}\Phi_{\frac{1}{p_r}}(s)\bigr)\,{}_{t_r}d_{p_r,q_r}s + \varphi_{r+1}\bigl(x(\rho_{r+1}^*)\bigr)\right\} \\ & + \frac{1}{p_k}\int_{t_k}^{t}\bigl(t - {}_{t_k}\Phi_{q_k}(s)\bigr)f_x\bigl({}_{t_k}\Phi_{\frac{1}{p_k}}(s)\bigr)\,{}_{t_k}d_{p_k,q_k}s, \quad t \in (t_k,\rho_{k+1}^*], k = 0,1,\dots,m. \end{aligned} \tag{2.13}$$

Proof The mathematical induction will be used to prove that (2.13) holds. To do this, by applying the double (p_0,q_0)-integration to the first equation of (2.12), we obtain

$$x(t) = \lambda_1 + \lambda_2 t + \frac{1}{p_0}\int_{t_0}^{t}\bigl(t - {}_{t_0}\Phi_{q_0}(s)\bigr)f_x\bigl({}_{t_0}\Phi_{\frac{1}{p_0}}(s)\bigr)\,{}_{t_0}d_{p_0,q_0}s, \quad t \in (t_0,\rho_1^*],$$

which implies that (2.13) is true for $k = 0$. In the next step, we suppose that (2.13) holds for $t \in (t_k,\rho_{k+1}^*]$. By mathematical induction, we shall show that (2.13) holds on $(t_{k+1},\rho_{k+2}^*]$. Now, the double (p_0,q_0)-integration of the first equation of (2.12) yields on $t \in (t_{k+1},\rho_{k+2}^*]$ that

$$\begin{aligned} x(t) = {}& x\bigl(t_{k+1}^+\bigr) + (t - t_{k+1})\,{}_{t_{k+1}}D_{p_{k+1},q_{k+1}}x\bigl(t_{k+1}^+\bigr) \\ & + \frac{1}{p_{k+1}}\int_{t_{k+1}}^{t}\bigl(t - {}_{t_{k+1}}\Phi_{q_{k+1}}(s)\bigr)f_x\bigl({}_{t_{k+1}}\Phi_{\frac{1}{p_{k+1}}}(s)\bigr)\,{}_{t_{k+1}}d_{p_{k+1},q_{k+1}}s. \end{aligned} \tag{2.14}$$

We have

$$\begin{aligned}
x\big(t_{k+1}^{+}\big) &= x\big(\rho_{k+1}^{*}\big) + \varphi_{k+1}\big(x\big(\rho_{k}^{*}\big)\big) \\
&= \lambda_1 + \sum_{i=0}^{k}\big(\rho_{i+1}^{*} - t_i\big)\left[\lambda_2 + \sum_{j=0}^{i-1}\left\{\int_{t_j}^{\rho_{j+1}} f\big(s,x(s)\big)\, {}_{t_j}d_{p_j,q_j}s + \varphi_{j+1}^{*}\big(x\big(\rho_{j+1}^{*}\big)\big)\right\}\right] \\
&\quad + \sum_{r=0}^{k}\left\{\frac{1}{p_r}\int_{t_r}^{\rho_{r+1}^{*}}\big(\rho_{r+1}^{*} - {}_{t_r}\Phi_{q_r}(s)\big) f_x\big({}_{t_r}\Phi_{\frac{1}{p_r}}(s)\big)\, {}_{t_r}d_{p_r,q_r}s + \varphi_{r+1}\big(x\big(\rho_{r+1}^{*}\big)\big)\right\}
\end{aligned}$$

and

$$\begin{aligned}
{}_{t_{k+1}}D_{p_{k+1},q_{k+1}}x\big(t_{k+1}^{+}\big) &= {}_{t_k}D_{p_k,q_k}x(\rho_{k+1}) + \varphi_{k+1}^{*}\big(x\big(\rho_{k}^{*}\big)\big) \\
&= \lambda_2 + \sum_{j=0}^{k-1}\left\{\int_{t_j}^{\rho_{j+1}} f\big(s,x(s)\big)\, {}_{t_j}d_{p_j,q_j}s + \varphi_{j+1}^{*}\big(x\big(\rho_{j+1}^{*}\big)\big)\right\} \\
&\quad + \int_{t_k}^{\rho_{k+1}} f\big(s,x(s)\big)\, {}_{t_k}d_{p_k,q_k}s + \varphi_{k+1}^{*}\big(x\big(\rho_{k+1}^{*}\big)\big) \\
&= \lambda_2 + \sum_{j=0}^{k}\left\{\int_{t_j}^{\rho_{j+1}} f\big(s,x(s)\big)\, {}_{t_j}d_{p_j,q_j}s + \varphi_{j+1}^{*}\big(x\big(\rho_{j+1}^{*}\big)\big)\right\}.
\end{aligned}$$

Substituting above two values into (2.14), we obtain

$$\begin{aligned}
x(t) &= \lambda_1 + \sum_{i=0}^{k}\big(\rho_{i+1}^{*} - t_i\big)\left[\lambda_2 + \sum_{j=0}^{i-1}\left\{\int_{t_j}^{\rho_{j+1}} f\big(s,x(s)\big)\, {}_{t_j}d_{p_j,q_j}s + \varphi_{j+1}^{*}\big(x\big(\rho_{j+1}^{*}\big)\big)\right\}\right] \\
&\quad + \sum_{r=0}^{k}\left\{\frac{1}{p_r}\int_{t_r}^{\rho_{r+1}^{*}}\big(\rho_{r+1}^{*} - {}_{t_r}\Phi_{q_r}(s)\big) f_x\big({}_{t_r}\Phi_{\frac{1}{p_r}}(s)\big)\, {}_{t_r}d_{p_r,q_r}s + \varphi_{r+1}\big(x\big(\rho_{r+1}^{*}\big)\big)\right\} \\
&\quad + (t - t_{k+1})\left(\lambda_2 + \sum_{j=0}^{k}\left\{\int_{t_j}^{\rho_{j+1}} f\big(s,x(s)\big)\, {}_{t_j}d_{p_j,q_j}s + \varphi_{j+1}^{*}\big(x\big(\rho_{j+1}^{*}\big)\big)\right\}\right) \\
&\quad + \frac{1}{p_{k+1}}\int_{t_{k+1}}^{t}\big(t - {}_{t_{k+1}}\Phi_{q_{k+1}}(s)\big) f_x\big({}_{t_{k+1}}\Phi_{\frac{1}{p_{k+1}}}(s)\big)\, {}_{t_{k+1}}d_{p_{k+1},q_{k+1}}s \\
&= \lambda_1 + \sum_{i=0}^{k+1}\big(\big\langle\rho_{i+1}^{*}\big\rangle_{k+1} - t_i\big)\left[\lambda_2 + \sum_{j=0}^{i-1}\left\{\int_{t_j}^{\rho_{j+1}} f\big(s,x(s)\big)\, {}_{t_j}d_{p_j,q_j}s + \varphi_{j+1}^{*}\big(x\big(\rho_{j+1}^{*}\big)\big)\right\}\right] \\
&\quad + \sum_{r=0}^{k}\left\{\frac{1}{p_r}\int_{t_r}^{\rho_{r+1}^{*}}\big(\rho_{r+1}^{*} - {}_{t_r}\Phi_{q_r}(s)\big) f_x\big({}_{t_r}\Phi_{\frac{1}{p_r}}(s)\big)\, {}_{t_r}d_{p_r,q_r}s + \varphi_{r+1}\big(x\big(\rho_{r+1}^{*}\big)\big)\right\} \\
&\quad + \frac{1}{p_{k+1}}\int_{t_{k+1}}^{t}\big(t - {}_{t_{k+1}}\Phi_{q_{k+1}}(s)\big) f_x\big({}_{t_{k+1}}\Phi_{\frac{1}{p_{k+1}}}(s)\big)\, {}_{t_{k+1}}d_{p_{k+1},q_{k+1}}s,
\end{aligned}$$

which holds for $(t_{k+1}, \rho_{k+2}^{*}]$. This completes the proof. □

To investigate the impulsive (p,q)-difference equations of type II, we define intervals of solutions as $\Lambda_1 = (\bigcup_{k=0}^{m}(t_k, \rho_{k+1}]) \cup \{0\}$ and $\Lambda_2 = (\bigcup_{k=0}^{m}(t_k, \rho_{k+1}^{*}]) \cup \{0\}$, and also the spaces

$PC_1(\Lambda_1,\mathbb{R}) = \{x : \Lambda_1 \to \mathbb{R}: x(t)$ is continuous everywhere on Λ_1 such that $x(t_k^+)$ and $x(\rho_{k+1})$ exist for all $k = 0,1,\dots,m\}$ and $PC_2(\Lambda_2,\mathbb{R}) = \{x : \Lambda_2 \to \mathbb{R}: x(t)$ is continuous everywhere on Λ_2 such that $x(t_k^+)$ and $x(\rho_{k+1}^*)$ exist for all $k = 0,1,\dots,m\}$. Both of them are Banach spaces equipped with the norms $\|x\|_1 = \sup\{|x(t)|, t \in \Lambda_1\}$ and $\|x\|_2 = \sup\{|x(t)|, t \in \Lambda_2\}$.

In proving our next results, we use the constants:

$$\begin{aligned}
\Omega_5 &= \frac{|\beta| + |\alpha+\beta|}{|\alpha+\beta|}\sum_{i=0}^{m}(\rho_{i+1} - t_i), \\
\Omega_6 &:= \sum_{i=0}^{m}\left\{(\rho_{i+1}^* - t_i)\sum_{j=0}^{i-1}(\rho_{j+1} - t_j)\right\} + \sum_{r=0}^{m}\frac{(\rho_{r+1}^* - t_r)^2}{p_r + q_r}, \\
\Omega_7 &:= \sum_{i=0}^{m}(\rho_{i+1}^* - t_i)i.
\end{aligned}$$

Applying Theorem 2.9 to define the operator on $PC_1(\Lambda_1,\mathbb{R})$ and following the method of Theorem 2.4, we can easily prove the existence of a unique solution of problem (2.10).

Theorem 2.11 *Assume that the functions $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ and $\varphi_k : \mathbb{R} \to \mathbb{R}$, $k = 1,2,\dots,m$, satisfy condition (H_1). If*

$$L_1\Omega_5 + L_2\Omega_2 < 1, \tag{2.15}$$

then the boundary value problem of type II (2.10) has a unique solution on Λ_1.

Theorem 2.12 *Assume that the functions $f : [0,T] \times \mathbb{R} \to \mathbb{R}$, $\varphi_k : \mathbb{R} \to \mathbb{R}$ and $\varphi_k^* : \mathbb{R} \to \mathbb{R}$, $k = 1,2,\dots,m$, satisfy (H_1)–(H_2). If*

$$L_1\Omega_6 + L_2 m + L_3\Omega_7 < 1, \tag{2.16}$$

then the problem of type II (2.12) has a unique solution on Λ_2.

Proof To show the technique of computation of constants Ω_6 and Ω_7, we give a short proof. Now we prove that the operator equation $x = \mathcal{B}x$ has a unique fixed point, where the operator $\mathcal{B} : PC_2(\Lambda_2,\mathcal{R}) \to PC_2(\Lambda_2,\mathcal{R})$ is defined, in view of Theorem 2.10, by

$$\begin{aligned}
\mathcal{B}x(t) &= \lambda_1 + \sum_{i=0}^{k}\left(\langle\rho_{i+1}^*\rangle_k - t_i\right)\left[\lambda_2 + \sum_{j=0}^{i-1}\left\{\int_{t_j}^{\rho_{j+1}} f\big(s,x(s)\big)\,{}_{t_j}d_{p_j,q_j}s + \varphi_{j+1}^*\big(x(\rho_{j+1}^*)\big)\right\}\right] \\
&\quad + \sum_{r=0}^{k-1}\left\{\frac{1}{p_r}\int_{t_r}^{\rho_{r+1}^*}\left(\rho_{r+1}^* - {}_{t_r}\Phi_{q_r}(s)\right)f_x\big({}_{t_r}\Phi_{\frac{1}{p_r}}(s)\big)\,{}_{t_r}d_{p_r,q_r}s + \varphi_{r+1}\big(x(\rho_{r+1}^*)\big)\right\} \\
&\quad + \frac{1}{p_k}\int_{t_k}^{t}\left(t - {}_{t_k}\Phi_{q_k}(s)\right)f_x\big({}_{t_k}\Phi_{\frac{1}{p_k}}(s)\big)\,{}_{t_k}d_{p_k,q_k}s, \quad t \in (t_k,\rho_{k+1}^*], k = 0,1,\dots,m.
\end{aligned}$$

By a similar method as in Theorem 2.4, we can show that the operator $\mathcal{B}$ maps a subset of $PC_2(\Lambda_2,\mathcal{R})$ into subset of $PC_2(\Lambda_2,\mathcal{R})$. Next, we will prove that $\mathcal{B}$ is a contraction. Let

$x, y \in PC_2(\Lambda_2, \mathcal{R})$. Then we have

$$\begin{aligned}
&\big|\mathcal{B}x(t) - \mathcal{B}y(t)\big| \\
&\quad\le \sum_{i=0}^{k}\big(\langle\rho_{i+1}^*\rangle_k - t_i\big)\Bigg[\sum_{j=0}^{i-1}\bigg\{\int_{t_j}^{\rho_{j+1}}\big|f\big(s,x(s)\big) - f\big(s,y(s)\big)\big|\,{}_{t_j}d_{p_j,q_j}s \\
&\qquad + \big|\varphi_{j+1}^*\big(x\big(\rho_{j+1}^*\big)\big) - \varphi_{j+1}^*\big(y\big(\rho_{j+1}^*\big)\big)\big|\bigg\}\Bigg] \\
&\qquad + \sum_{r=0}^{k-1}\bigg\{\frac{1}{p_r}\int_{t_r}^{\rho_{r+1}^*}\big(\rho_{r+1}^* - {}_{t_r}\Phi_{q_r}(s)\big)\big|f_x\big({}_{t_r}\Phi_{\frac{1}{p_r}}(s)\big) - f_y\big({}_{t_r}\Phi_{\frac{1}{p_r}}(s)\big)\big|\,{}_{t_r}d_{p_r,q_r}s \\
&\qquad + \big|\varphi_{r+1}\big(x\big(\rho_{r+1}^*\big)\big) - \varphi_{r+1}\big(y\big(\rho_{r+1}^*\big)\big)\big|\bigg\} \\
&\qquad + \frac{1}{p_k}\int_{t_k}^{t}\big(t - {}_{t_k}\Phi_{q_k}(s)\big)\big|f_x\big({}_{t_k}\Phi_{\frac{1}{p_k}}(s)\big) - f_y\big({}_{t_k}\Phi_{\frac{1}{p_k}}(s)\big)\big|\,{}_{t_k}d_{p_k,q_k}s \\
&\le \sum_{i=0}^{m}\big(\rho_{i+1}^* - t_i\big)\Bigg[\sum_{j=0}^{i-1}\bigg\{L_1\|x-y\|_2\int_{t_j}^{\rho_{j+1}}(1)\,{}_{t_j}d_{p_j,q_j}s + L_3\|x-y\|_2\bigg\}\Bigg] \\
&\qquad + \sum_{r=0}^{m-1}\bigg\{\frac{1}{p_r}L_1\|x-y\|_2\int_{t_r}^{\rho_{r+1}^*}\big(\rho_{r+1}^* - {}_{t_r}\Phi_{q_r}(s)\big)(1)\,{}_{t_r}d_{p_r,q_r}s + L_2\|x-y\|_2\bigg\} \\
&\qquad + \frac{1}{p_m}L_1\|x-y\|_2\int_{t_m}^{\rho_{m+1}^*}\big(\rho_{m+1}^* - {}_{t_m}\Phi_{q_m}(s)\big)(1)\,{}_{t_m}d_{p_m,q_m}s \\
&\quad= \sum_{i=0}^{m}\big(\rho_{i+1}^* - t_i\big)\Bigg[\sum_{j=0}^{i-1}\big\{L_1\|x-y\|_2(\rho_{j+1} - t_j) + L_3\|x-y\|_2\big\}\Bigg] \\
&\qquad + \sum_{r=0}^{m-1}\bigg\{L_1\|x-y\|_2\frac{(\rho_{r+1}^* - t_r)^2}{p_r + q_r} + L_2\|x-y\|_2\bigg\} + L_1\|x-y\|_2\frac{(\rho_{m+1}^* - t_m)^2}{p_m + q_m} \\
&\quad= (L_1\Omega_6 + L_2 m + L_3\Omega_7)\|x-y\|_2,
\end{aligned}$$

which implies that $\|\mathcal{B}x - \mathcal{B}y\|_2 \le (L_1\Omega_6 + L_2 m + L_3\Omega_7)\|x-y\|_2$. Condition (2.16) and the Banach contraction mapping principle guarantee that the impulsive (p,q)-difference initial value problem of type II (2.12) has a unique solution on Λ_2. The proof is completed. □

Example 2.13 Consider the following first-order impulsive (p,q)-difference equation of type *II* subject to the boundary condition of the form:

$$\begin{cases} {}_kD_{\frac{k+1}{k+2},\frac{k+1}{k+3}}x(t) = \frac{5}{6(3+t)^2}\big(\frac{x^2(t)+2|x(t)|}{1+|x(t)|}\big) + \frac{3}{4}, & t \in (k, k+1], k = 0,1,2, \\ x(k) - x\big(\frac{k^2+k-1}{k+1}\big) = \frac{1}{6k}\tan^{-1}\big(x\big(\frac{k^2+k-1}{k+1}\big)\big), & k = 1,2, \\ \frac{1}{2}x(0) + \frac{1}{3}x\big(\frac{11}{4}\big) = \frac{1}{4}. \end{cases} \tag{2.17}$$

Here the quantum numbers are $p_k = (k+1)/(k+2)$, $q_k = (k+1)/(k+3)$, $k = 0,1,2$, $J = [0,3]$, $t_k = k$, $k = 1,2$, $\alpha = 1/2$, $\beta = 1/3$, $\gamma = 1/4$, and $\rho_k = (k^2 + k - 1)/(k+1)$. We can find that

$\Omega_2 = 2.8000$, $\Omega_5 = 2.6833$, and

$$\Lambda_1 = \left[0, \frac{1}{2}\right] \cup \left(1, \frac{5}{3}\right] \cup \left(2, \frac{11}{4}\right].$$

By setting

$$f(t,x) = \frac{5}{6(3+t)^2}\left(\frac{x^2 + 2|x|}{1+|x|}\right) + \frac{3}{4} \quad \text{and} \quad \varphi_k(x) = \frac{1}{6k}\tan^{-1}(x),$$

we see that the functions f and φ_k satisfy (H_1) with $L_1 = 5/27$ and $L_2 = 1/6$, respectively. Then we get $L_1\Omega_5 + L_2\Omega_2 = 0.9543 < 1$. Therefore, by Theorem 2.11, the boundary value problem (2.17) has a unique solution x on Λ_1.

Example 2.14 Consider the following second-order impulsive (p,q)-difference equation of type *II* with the initial conditions of the form:

$$\begin{cases} {}_kD^2_{\frac{k+1}{k+2},\frac{k+1}{k+3}}x(t) = \frac{1}{10(t+6)}\sin|x(t)| + \frac{5}{6}, \quad t \in (k, k+1], k = 0,1,2, \\ x(k^+) - x(\frac{k^3+2k^2-k-1}{(k+1)^2}) = \frac{3}{5(k+1)^2}\tan^{-1}(x(\frac{k^3+2k^2-k-1}{(k+1)^2})), \quad k = 1,2, \\ {}_kD_{\frac{k+1}{k+2},\frac{k+1}{k+3}}x(k^+) - {}_{(k-1)}D_{\frac{k}{k+1},\frac{k}{k+2}}x(\frac{k^2+k-1}{k+1}) = \frac{1}{5k^3}|x(\frac{k^3+2k^2-k-1}{(k+1)^2})|, \quad k = 1,2, \\ x(0) = \frac{3}{5}, \qquad {}_0D_{\frac{1}{2},\frac{1}{3}}x(0) = \frac{5}{7}. \end{cases} \tag{2.18}$$

The quantum numbers p_k, q_k, impulsive points t_k, ρ_k, and interval J are defined the same as in Example 2.13. We have the constants $\lambda_1 = 3/5$, $\lambda_2 = 5/7$, and points $\rho_k^* = (k^3 + 2k^2 - k - 1)/(k+1)^2$. Next we can find that $\Omega_6 = 18.4273$, $\Omega_7 = 1.5694$, and

$$\Lambda_2 = \left[0, \frac{1}{4}\right] \cup \left(1, \frac{13}{9}\right] \cup \left(2, \frac{41}{16}\right].$$

By setting

$$f(t,x) = \frac{1}{10(t+6)}\sin|x| + \frac{5}{6}, \qquad \varphi_k(x) = \frac{3}{5(k+1)^2}\tan^{-1}(x), \quad \text{and} \quad \varphi_k^*(x) = \frac{1}{5k^3}|x|,$$

we deduce that (H_1)–(H_2) are fulfilled with $L_1 = 1/60$, $L_2 = 3/20$, and $L_3 = 1/5$. Hence, it follows that $L_1\Omega_6 + L_2 m + L_3\Omega_7 = 0.9210 < 1$. Therefore, by applying Theorem 2.12, the boundary value problem (2.18) has a unique solution x on Λ_2.

3 Conclusion

In this research, we initiated the study of the first and second order (p,q)-difference equations with initial or boundary conditions. Firstly, we let t_k, $k = 1, \dots, m$, be the impulsive points such that $0 = t_0 < t_1 < \cdots < t_k < \cdots < t_m < t_{m+1} = T$ and $J_k = (t_k, t_{k+1}]$, $k = 1, \dots, m$, $J_0 = [0, t_1]$ be the intervals such that $\bigcup_{k=0}^m J_k = [0,T] := J$. The investigations were based on (p,q)-calculus introduced in the first section of this paper, by replacing a point a by t_k, quantum numbers p by p_k and q by q_k, $k = 0, 1, \dots, m$, and also applying the (p_k, q_k)-difference and (p_k, q_k)-integral operators only on a finite subinterval of J. In addition, the consecutive subintervals could be related with jump conditions which led to a meaning of quantum difference equations with impulse effects. There are two types of impulsive problems. The consecutive domains of impulsive (p,q)-difference equations of type I are overlapped, while the unknown functions of impulsive equations of type II are defined on disjoint consecutive domains. Four types of problems were considered, two impulsive (p,q)-difference equations of type I and two impulsive (p,q)-difference equations of type II. Existence and uniqueness results were proved via Banach's contraction mapping principle. Examples illustrating the obtained results were also presented.

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References

1. Kamsrisuk, N., Promsakon, C., Ntouyas, S.K., Tariboon, J.: Nonlocal boundary value problems for (p,q)-difference equations. Differ. Equ. Appl. **10**(2), 183–195 (2018)
2. Promsakon, C., Kamsrisuk, N., Ntouyas, S.K., Tariboon, J.: On the second-order quantum (p,q)-difference equations with separated boundary conditions. Adv. Math. Phys. **2018**, Article ID 9089865 (2018)
3. Chakrabarti, R., Jagannathan, R.: A (p,q)-oscillator realization of two-parameter quantum algebras. J. Phys. A, Math. Gen. **24**, L711–L718 (1991)
4. Jagannathan, R., Srinivasa Rao, K.: Two-parameter quantum algebras, twin-basic number, and associated generalized hypergeometric series. arXiv:math/0602613v1
5. Sahai, V., Yadav, S.: Representations of two parameter quantum algebras and (p,q)-special functions. J. Math. Anal. Appl. **335**, 268–279 (2007)
6. Burban, I.: Two-parameter deformation of the oscillator algebra and (p,q)-analog of two-dimensional conformal field theory. J. Nonlinear Math. Phys. **2**(3–4), 384–391 (2013)
7. Njionou Sadjang, P.: On the fundamental theorem of (p,q)-calculus and some (p,q)-Taylor formulas. arXiv:1309.3934v1
8. Hounkonnou, M.N., Bukweli Kyemba, J.D.: $R(p,q)$-Calculus differentiation and integration. J. Math. **49**(2), 145–167 (2013)
9. Mursaleen, M., Ansari, K.J., Khan, A.: On (p,q)-analogue of Bernstein operators. Appl. Math. Comput. **266**, 874–882 (2015)
10. Araci, S., Duran, U., Acikgoz, M., Srivastava, H.M.: A certain (p,q)-derivative operator and associated divided differences. J. Inequal. Appl. **2016**, Article ID 301 (2016)
11. Jackson, H.F.: On q-definite integral. Q. J. Pure Appl. Math. **41**, 193–203 (1910)
12. Jackson, H.F.: q-Difference equations. Am. J. Math. **32**, 305–314 (1910)
13. Kac, V., Cheung, P.: Quantum Calculus. Springer, New York (2002)
14. Ahmad, B., Ntouyas, S.K., Tariboon, J.: Quantum Calculus: New Concepts, Impulsive IVPs and BVPs, Inequalities. Trends in Abstract and Applied Analysis, vol. 4. World Scientific, Singapore (2016)
15. Tunç, M., Göv, E.: (p,q)-Integral inequalities. RGMIA Res. Rep. Collect. **19**, Article ID 97 (2016)
16. Tunç, M., Göv, E.: Some integral inequalities via (p,q)-calculus on finite intervals. RGMIA Res. Rep. Collect. **19**, Article ID 95 (2016)
17. Tariboon, J., Ntouyas, S.K., Agarwal, P.: New concepts of fractional quantum calculus and applications to impulsive fractional q-difference equations. Adv. Differ. Equ. **2015**, Article ID 18 (2015)