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Finite-time synchronization criterion of graph theory perspective fractional-order coupled discontinuous neural networks

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Abstract

In this research work, the finite-time synchronization and adaptive finite-time synchronization criterion of graph theory perspective fractional-order coupled discontinuous neural networks (FCDNNs) are investigated under two different control strategies. By utilizing differential inclusion theory, Filippov framework, suitable Lyapunov functional, and graph theory approach, several sufficient criteria based on discontinuous state feedback control protocol and discontinuous adaptive feedback control protocol are established for ensuring the finite-time synchronization and adaptive finite-time synchronization of FCDNNs. Finally, two numerical cases illustrate the efficiency of the proposed finite-time synchronization results.

Keywords: Discontinuous fractional-order neural networks; Coupled systems; Finite time synchronization

1 Introduction and modeling

In recent times, differential equation and fractional differential equation models have found their applications in a variety of fields including biology [1–6], physics [7–10], engineering [11–14], mathematics [15–18], information technology, and so on [19–27]. They are also one of the most rudimentary tools for neural networks. They were first and foremost conferred by de Leibnitz and Gottfried Wilhelm Leibnitz in 1695 (see [28, 29]). Currently fractional-order calculus (FOC) has been considered predominantly due to its extensive applications in several fields, for instance, biology, control, optics, viscoelasticity, and signal processing (see [30–34]). As is known, FOC is an expansion of conventional integer-order calculus. At present, fractional-order differential techniques are employed widely to explore the dynamical behaviors of the networks, especially neural networks (NNs) and complex networks (CNs) (see [35–38]). The fractional-order differential systems possess unlimited memory property and more degrees of freedom in contrast to the conventional integer-order differential systems, which is the main benefit of the FOC. As a consequence of these benefits, some researchers have shown their keen interest to integrate the FOC into NNs to make fractional-order neural networks (FONNs) models. Among others, the dynamical behaviors of FONNs have already become a hot research topic, and lots of scientific results have been well published in the literature (see [39–43]).

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Synchronization, which defines the dynamical behaviors of coupled systems achieving the same spatial state at the same time, has become an important research topic, and it has been successfully applied in image processing, secure communication, optimization, and so on. The synchronization is majorly segregated into two classifications based on the existing synchronization results and convergence time such as infinite-time synchronization (IFTS), exponential synchronization, lag synchronization, asymptotical synchronization and Mittag-Leffler synchronization, and finite-time synchronization (FTS). Generally, asymptotic synchronization reaches an infinite time, it becomes infinite synchronization. In realistic engineering applications, humans continuously like to obtain synchronization in a finite convergence time, which is known as FTS. Moreover, time delays are inevitable in nearly all dynamical systems including neural network, chemical process, and nuclear reactors, which may lead to system oscillation, instability behaviors, and divergence because of the limited switching speed of amplifier circuits (see [44–48]). In recent decades, an increasing interest in the field of finite-time synchronization criterion of FONNs with time delays has attracted many scientific communities, which has given rise to some meaningful and significant outcomes (see [49–51]).

During the last decades, complex dynamical networks (CDNs) have obtained the attention of many researchers owing to their wide application in diverse areas like global economic markets, traffic control networks, communication networks, and synchronization results have been discussed in the existing literature (see [37, 38, 52, 53]). Coupled neural networks (CONNs) can be recognized as a significant improvement of complex dynamical networks [54, 55]. In CONNs, the coupling term depends on the following perspectives: (1) The relation and control between at least two neurons (see [56, 57]); (2) The amount of information that transfers from one neuron to another neuron (see [58]); (3) The complex structure between at least two neurons (see [59]). CONNs have been applied in numerous fields like harmonic oscillation generation, image encryption, and classification (see [60, 61]). In recent years, some significant related results of the fractional-order CONNs can be witnessed in the previous literature (see [62, 63]). Even though there exist numerous fractional-order CONNs, so far, most of them have considered the case of continuous neuron activations only. But in general, signal transmission between neurons and signal output of neuron are all discontinuous. The activation functions of CONNs are not generally continuous. As a result, the classical solution for fractional-order differential equations is not suitable to consider FCDNNs. Thus the investigation of FCDNNs is significant and much more challenging. Unfortunately, according to our literature survey, no one has investigated the FTS and adaptive FTS criterion of graph theory perspective FCDNNs. This situation stimulates the interest towards the further investigation for FCDNNs.

Motivated by the above conversation, we aim to analyze the FTS and adaptive FTS criterion of graph theory perspective FCDNNs with time delays using graph theory techniques. The main novelty of this research work is outlined in detail as follows:

1. For the first time the algebraic graph theory technique is incorporated into FTS and adaptive FTS criterion of FCDNNs with time delays.
2. These theoretical results and techniques can be extended to FTS of fractional-order CONNs and fractional-order discontinuous neural networks.
3. Two kinds of different control strategies, such as discontinuous state feedback control and discontinuous adaptive feedback control, are designed respectively to achieve the FTS and adaptive FTS of a class of FCDNNs.

4. By using the discontinuous FONNs results, coupling terms are added to discontinuous FONNs, and these results are established by using graph theoretical concepts.
5. Moreover, the proposed results in this paper are also still valid for FTS and adaptive FTS criterion for both integer-order and fractional-order CONNs with continuous activations, respectively, and these results do not exist in the previous works of literature.

Section 2 contains basic results on a graph-theoretical concept, fractional-order calculus, and formulations for FCDNNs systems. Section 3 is formulated for two different control strategies. Here we also derive sufficient criteria for the FTS and adaptive FTS criterion of FCDNNs based on algebraic graph theory techniques. Two numerical cases with simulations are established to demonstrate the efficiency of the obtained synchronization results in Sect. 4. At last, Sect. 5 terminates with conclusions.

2 Basic knowledge and model description

In this section, some basic concepts of graph theoretical results, fractional-order calculus, problem statement, and some necessary assumptions are given.

A directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ consists of vertices or nodes $\mathcal{V} = \{1, 2, \dots, N\}$ and a set \mathcal{E} of arcs (k, l) leading from k th node to l th node. A directed path \mathbb{D} is a subgraph of \mathcal{G} with distinct vertices $\{1, 2, \dots, p\}$ such that its set of arcs is $\{(k_x, k_{x+1}) : x = 1, 2, \dots, p - 1\}$. If the first and last nodes are similar, \mathbb{D} is a directed cycle. A graph is strongly connected if there exists a directed path from k to l in \mathcal{G} . A graph \mathcal{G} with weight matrix $A = (a_{jk})_{N \times N}$ is represented as (\mathcal{G}, A) , here $a_{jk} > 0$ equals the weight of arc (j, k) if it exists, and zero otherwise. The Laplacian matrix of (\mathcal{G}, A) is described as follows:

$$\mathcal{L} = (l_{jk})_{N \times N} = \begin{cases} -a_{jk} & \text{if } j \neq k, \\ \sum_{j \neq p} a_{jp} & \text{if } j = k. \end{cases}$$

Lemma 2.1 ([64]) *Let the k th diagonal element of cofactor of the Laplacian matrix of (\mathcal{G}, A) be represented by γ_k and assume $n \geq 2$. Then the following relationship holds:*

$$\sum_{k,l} \gamma_k a_{kl} \Omega_{ki}(q_k, q_l) = \sum_{B \in \mathcal{B}} \mathcal{W}(B) \sum_{(i,m) \in \mathcal{E}(\mathcal{A}_B)} \Omega_{m,i}(q_m, q_i),$$

where $\Omega_{ki}(q_k, q_i), k, i \in \{1, 2, \dots, N\}$ are arbitrary functions, \mathcal{B} denotes the set of all spanning unicyclic graphs of (\mathcal{G}, A) , $\mathcal{W}(B)$ and \mathcal{A}_B respectively the weights and directed cycle of B . Moreover, $\gamma_k > 0$ if (\mathcal{G}, A) is strongly connected for $k \in \{1, 2, \dots, N\}$.

In order to describe our system, some basic definitions and important lemmas with respect to fractional-order calculus are presented.

Definition 2.2 ([29, 65]) The fractional-order integral of $q(t)$ is described as follows:

$$D^{-h}q(t) = \frac{1}{\Gamma(h)} \int_0^t (t - \varsigma)^{h-1} y(\varsigma) d\varsigma,$$

where $h > 0$ and $\Gamma(\cdot)$ is the gamma function.

Definition 2.3 ([29, 65]) The Caputo fractional integral of $q(t)$ is described as follows:

$$D^h q(t) = \begin{cases} D^{-(n-h)}\left(\frac{d^n}{dt^n} q(t)\right) & \text{if } h \in (n - 1, n), \\ \left(\frac{d^n}{dt^n} q(t)\right) & \text{if } h = n, \end{cases}$$

where $h \in \mathbb{R}^+, n \in \mathbb{Z}^+$.

In this paper, we consider an array of fractional-order coupled discontinuous neural networks (FCDNNs) consisting of N identical nodes with each isolated node network being an n -dimensional dynamical system, which is presented by

$$D^h p_k(t) = -U p_k(t) + V f(p_k(t)) + W f(p_k(t - \tau)) + \sum_{l=1}^N a_{kl} E(q_l(t) - q_k(t)) + y_k(t), \tag{1}$$

in which $k = 1, 2, \dots, N$, N is the total number of nodes in the networks, D^h signifies the Caputo fractional-order derivative with order $0 < h < 1$, $p_k(t) = (p_{k1}(t), p_{k2}(t), \dots, p_{kn}(t))^T$ is the state vector of the k th dynamical node, $U = \text{diag}\{u_1, \dots, u_n\}$ is the self inhibition, $y_k(t) = (y_{k1}(t), y_{k2}(t), \dots, y_{kn}(t))^T$ is the control inputs, $f(p_k(t)) = (f_1(p_{k1}(t)), \dots, f_n(p_{kn}(t)))^T$ signifies activation functions of the neurons at time t , τ is a positive constant, and $V = [v_{kl}]_{n \times n}$ and $W = [w_{kl}]_{n \times n}$ represent the connection weights of the k th neuron to l th neuron, $E = \text{diag}\{e_1, \dots, e_n\}$ represents the inner coupling matrices between two nodes k and l at the time t , and $A = (a_{kl})_{N \times N}$ is the topological structure of the networks (1), in which a_{kl} is defined as follows: if there are links from node k to node l ($k \neq l$), then $a_{kl} > 0$, otherwise $a_{kl} = 0$, and we assume that $a_{kk} = 0, k \in \{1, 2, \dots, N\}$.

Let $q(t)$ be an isolated node, the dynamics of which is given by

$$D^h q(t) = -U q(t) + V f(q(t)) + W f(q(t - \tau)). \tag{2}$$

Since the neuron activation function $f(\cdot)$ is a discontinuous function, the traditional solutions are not suitable to considered system (1) and (2). To this conclusion, we introduce the concept of Filippov solution (see [66]).

Our main aim is to solve the synchronization problem of FCDNNs. Before that, we make the following definitions, assumptions, and lemmas.

Definition 2.4 A vector-valued function $p(t)$ is said to be a Filippov solution for fractional differential systems $D^h p(t) = f(t, p)$, and it is defined on a degenerate interval $\mathcal{I} \subseteq \mathbb{R}$. The Filippov solution for fractional differential systems with initial values $p(0) = p_0$ is absolutely continuous on any compact subinterval $[t_1, t_2]$ of \mathcal{I} and for almost all $t \in \mathcal{I}$. In addition, $p(0) = p_0$ and functional differential inclusions

$$D^h p(t) \in \mathbb{F}(t, p),$$

where $\mathbb{F}(t, p)$ is the set-valued map of $f(t, p)$, hold.

Suppose that the neuron activation function $f(\cdot)$ satisfies the following conditions.

Assumption $[\mathcal{H}_1]$. For every $k = 1, 2, \dots, n$, suppose that the discontinuous activations $f_k : \mathbb{R} \rightarrow \mathbb{R}$ are bounded and continuous functions except on a finite number of jump discontinuities σ_l on every bounded interval. Furthermore, there exist left limits $f_k(\sigma_l^-)$ and right limits $f_k(\sigma_l^+)$, respectively.

Based on Definition 2.4, if the activation function satisfies Assumption $[\mathcal{H}_1]$, one can obtain that

$$\begin{aligned} \mathbb{F}[p_k(t)] &= \tilde{c}\tilde{o}[f(p_k(t))] = (\tilde{c}\tilde{o}[f_{k1}(p_{k1}(t))], \dots, \tilde{c}\tilde{o}[f_{kn}(p_{kn}(t))])^T \quad \text{and} \\ \tilde{c}\tilde{o}[f_i(p_{ki}(t))] &= [\min\{\tilde{c}\tilde{o}[f_i(p_{ki}^-(t))], \tilde{c}\tilde{o}[f_i(p_{ki}^+(t))]\}, \max\{\tilde{c}\tilde{o}[f_i(p_{ki}^-(t))], \tilde{c}\tilde{o}[f_i(p_{ki}^+(t))]\}]. \end{aligned}$$

Assumption $[\mathcal{H}_2]$. For every $k = 1, 2, \dots, N, l = 1, 2, \dots, n$, there exist positive constants $b_l, d_l > 0$ such that, for every $p_{kl}(t) \in \mathbb{R}, q_l(t) \in \mathbb{R}, \tilde{\mu}_l(t) \in \tilde{c}\tilde{o}[f_i(p_{kl}(t))]$, and $\mu_l(t) \in \tilde{c}\tilde{o}[f_l(q_l(t))]$, the following inequality holds:

$$|\tilde{\mu}_{kl}(t) - \mu_l(t)| \leq b_l |p_{kl}(t) - q_l(t)| + d_l.$$

Furthermore, $0 \in \tilde{c}\tilde{o}[f_i(0)]$.

Define the error signal: $\alpha_k(t) = p_k(t) - q(t)$, then the synchronization error system can be obtained from (1) and (2) as follows:

$$\begin{aligned} D^h \alpha_k(t) &= -U \alpha_k(t) + V \Psi(\alpha_k(t)) \\ &\quad + W \Psi(\alpha_k(t - \tau)) + \sum_{l=1}^N a_{kl} E(\alpha_l(t) - \alpha_k(t)) + y_k(t), \end{aligned} \tag{3}$$

where $\Psi(\alpha_k(t)) = f(p_k(t)) - f(q(t)), k = 1, 2, \dots, N$. Let the initial values of error system (3) be given as

$$\alpha_k(t) = \psi_k(t) \in C([-\tau, 0], \mathbb{R}^n), \quad k = 1, 2, \dots, N,$$

where $C([-\tau, 0], \mathbb{R}^n)$ represents the set of all continuous differential functions from $[-\tau, 0]$ into \mathbb{R}^n .

Definition 2.5 FCDNNs (1) is said to be finite-time synchronized with isolated networks (2) if there exists a settling time $t_1 > 0$, which is a real number if

$$\lim_{t \rightarrow t_1} \|p_k(t) - q(t)\| = 0 \quad \text{and} \quad \|p_k(t) - q(t)\| = 0 \quad \text{for } t > t_1, k \in \{1, 2, \dots, N\}.$$

Lemma 2.6 ([67]) Let $q(t) \in \mathbb{R}^n$ be a continuously derivable vector-valued function, then

$$D^h \{q^T(t)q(t)\} \leq 2q^T(t)\{D^h q(t)\}, \quad h \in (0, 1).$$

Lemma 2.7 ([68]) If $\Phi_1, \dots, \Phi_n \geq 0, 0 < \vartheta < \epsilon$, then the following inequality is established:

$$\left[\sum_{i=1}^n \Phi_i^\epsilon \right]^{\frac{1}{\epsilon}} \leq \left[\sum_{i=1}^n \Phi_i^\vartheta \right]^{\frac{1}{\vartheta}}.$$

Lemma 2.8 ([52]) *Assume that the positive definite function $X(t)$ is a continuous function, and it satisfies the following differential inequality:*

$$D^h X(t) \leq -\delta X^\nu(t),$$

where $\delta > 0$, $0 < \nu < h$ are all constants. Then $X(t)$ satisfies the following differential inequality:

$$X^{h-\nu}(t) \leq X^{h-\nu}(t_0) - \frac{\delta \Gamma(1+h-\nu)(t-t_0)^h}{\Gamma(1+h)\Gamma(1-\nu)}, \quad t \in [t_0, t_1],$$

and $X(t) = 0, \forall t \geq t_1$, where t_1 is denoted by

$$t_1 = t_0 + \left[\frac{\Gamma(1+h)\Gamma(1-\nu)X^{h-\nu}(t_0)}{\delta \Gamma(1+h-\nu)} \right]^{\frac{1}{h}}.$$

Remark 2.9 In [69, 70], the authors demonstrated the synchronization criterion of discontinuous fractional-order neural networks by using state feedback control law. The global synchronization criterion of FOCDNs by using graph theory techniques was analyzed in [53]. In [62, 63], the authors investigated the synchronization criterion of FOCONNs by using the LMI technique and Kronecker product technique. Besides, it is helpful for us to demonstrate our required finite-time synchronization and adaptive finite-time synchronization criterion from the results obtained in the aforementioned references [53, 62, 63, 69, 70].

3 Main results

In this section, we demonstrate the finite-time synchronization and adaptive finite-time synchronization criterion of FCDNNs (1) and the isolated networks (2) by using graph theory techniques, discontinuous state feedback control, and discontinuous adaptive feedback control.

3.1 Finite-time synchronization under discontinuous state feedback control

First, we design discontinuous feedback control protocol as follows:

$$\begin{aligned} y_k(t) = & -\beta \alpha_k(t) - \zeta \|\alpha_k(t-\tau)\|_1 \text{sign}(\alpha_k(t)) \\ & - \eta \text{sign}(\alpha_k(t)) - \xi \text{sign}(\alpha_k(t)) |\alpha_k(t)|^\theta, \end{aligned} \quad (4)$$

where $0 < \theta < 1$, $\beta > 0$, $\zeta > 0$, $\eta > 0$, $\xi > 0$ are control gains, they are properly selected in the following main theorem. According to the Filippov framework, FCDNNs (3) can be written as follows:

$$\begin{aligned} D^h \alpha_k(t) \in & -U \alpha_k(t) + V \tilde{c} \circ \{\Psi(\alpha_k(t))\} + W \tilde{c} \circ \{\Psi(\alpha_k(t-\tau))\} \\ & + \sum_{l=1}^N a_{kl} E(\alpha_l(t) - \alpha_k(t)) - \beta \alpha_k(t) \\ & - \zeta \|\alpha_k(t-\tau)\|_1 \tilde{c} \circ (\text{sign}(\alpha_k(t))) - \eta \tilde{c} \circ (\text{sign}(\alpha_k(t))) \end{aligned}$$

$$\begin{aligned}
 & -\xi \tilde{c}o(\text{sign}(\alpha_k(t)))|\alpha_k(t)|^\theta \\
 \subseteq & -U\alpha_k(t) + V[\tilde{c}o(p_k(t)) - \tilde{c}o(q(t))] \\
 & + W[\tilde{c}o(p_k(t - \tau)) - \tilde{c}o(q(t - \tau))] \\
 & + \sum_{l=1}^N a_{kl}E(\alpha_l(t) - \alpha_k(t)) \\
 & - \beta\alpha_k(t) - \zeta \|\alpha_k(t - \tau)\|_1 \tilde{c}o(\text{sign}(\alpha_k(t))) \\
 & - \eta \tilde{c}o(\text{sign}(\alpha_k(t))) - \xi \tilde{c}o(\text{sign}(\alpha_k(t)))|\alpha_k(t)|^\theta. \tag{5}
 \end{aligned}$$

Then there exist a measurable function $\tilde{\mu}_k(t) \in \tilde{c}o[f(p_k(t))]$ and $\mu(t) \in \tilde{c}o[f(q(t))]$ for a.e. $t \in [-\tau, t_1]$ such that

$$\begin{aligned}
 D^h \alpha_k(t) = & -U\alpha_k(t) + V[\tilde{\mu}_k(t) - \mu(t)] \\
 & + W[\tilde{\mu}_k(t - \tau) - \mu(t - \tau)] + \sum_{l=1}^N a_{kl}E(\alpha_l(t) - \alpha_k(t)) \\
 & - \beta\alpha_k(t) - \zeta \|\alpha_k(t - \tau)\|_1 \text{SIGN}(\alpha_k(t)) \\
 & - \eta \text{SIGN}(\alpha_k(t)) - \xi \text{SIGN}(\alpha_k(t - \tau))|\alpha_k(t)|^\theta, \tag{6}
 \end{aligned}$$

where $\text{SIGN}(\alpha_k(t)) = [\text{SIGN}(\alpha_{k1}(t)), \dots, \text{SIGN}(\alpha_{kn}(t))]^T$ with

$$\text{SIGN}(\alpha) = \begin{cases} -1, & \alpha < 0, \\ [-1, 1], & \alpha = 0, \\ 1, & \alpha > 0. \end{cases}$$

Before starting the finite-time synchronization results, we introduce the following notations: $b^{\max} = \max_{1 \leq l \leq n} b_l$, $d^{\max} = \max_{1 \leq l \leq n} d_l$, $v^{\max} = \max_{1 \leq l, j \leq n} |v_{lj}|$, $w^{\max} = \max_{1 \leq l, j \leq n} |w_{lj}|$, and $\hat{\gamma}_{\min} = \min_{1 \leq k \leq N} \gamma_k$.

Theorem 3.1 *Suppose that Assumptions $[\mathcal{H}_1]$ – $[\mathcal{H}_3]$ hold. Then FCDNNs (1) and the isolated networks (2) are finite-time synchronized under the discontinuous feedback control protocol (4), if*

$$\beta > \Lambda_{\min}(U) - nb^{\max}v^{\max}, \tag{7}$$

$$\zeta > b^{\max}w^{\max}, \tag{8}$$

$$\eta > nd^{\max}(v^{\max} + w^{\max}). \tag{9}$$

Furthermore, the settling time is evaluated as follows:

$$t_1 = t_0 + \left[\frac{\Gamma(1+h)\Gamma(\frac{1-\theta}{2})X^{\frac{2h-1-\theta}{2}}(t_0)}{\delta\Gamma(\frac{2h+1-\theta}{2})} \right]^{\frac{1}{h}},$$

where $\delta = \xi 2^{\frac{\theta+1}{2}} \hat{\gamma}_{\min}^{\frac{1-\theta}{2}}$, $X(t_0) = \sum_{k=1}^N \frac{\gamma_k}{2} \alpha_k^T(t_0)\alpha_k(t_0)$, γ_k signifies the cofactor of the i th diagonal elements of \mathcal{L} , and $\Lambda_{\min}(U)$ signifies the minimal eigenvalue of U .

Proof For the FCDNNs error system (6), construct the following Lyapunov functional:

$$X(t) = \sum_{k=1}^N \frac{\gamma_k}{2} \alpha_k^T(t) \alpha_k(t). \tag{10}$$

Since $(\mathcal{G}, \mathcal{A})$ is strongly connected, by using Lemma 2.1, we get that $\gamma_k > 0$ for $k \in \{1, 2, \dots, N\}$. According to Lemma 2.6, one can get

$$\begin{aligned} D^h X(t) &\leq \sum_{k=1}^N \gamma_k \alpha_k^T(t) D^h \{ \alpha_k(t) \} \\ &= \sum_{k=1}^N \gamma_k \alpha_k^T(t) \left[-U \alpha_k(t) + V [\tilde{\mu}_k(t) - \mu(t)] \right. \\ &\quad + W [\tilde{\mu}_k(t - \tau) - \mu(t - \tau)] + \sum_{l=1}^N a_{kl} E (\alpha_l(t) - \alpha_k(t)) \\ &\quad - \beta \alpha_k(t) - \zeta \|\alpha_k(t - \tau)\|_1 \text{SIGN}(\alpha_k(t)) \\ &\quad \left. - \eta \text{SIGN}(\alpha_k(t)) - \xi \text{SIGN}(\alpha_k(t)) |\alpha_k(t)|^\theta \right]. \end{aligned} \tag{11}$$

Based on Assumption $[\mathcal{H}_2]$, one has

$$\begin{aligned} \sum_{k=1}^N \gamma_k \alpha_k^T(t) V [\tilde{\mu}_k(t) - \mu(t)] &= \sum_{k=1}^N \sum_{l=1}^n \sum_{j=1}^n \gamma_k \alpha_{kl}(t) v_{lj} [\tilde{\mu}_{kj}(t) - \mu_j(t)] \\ &\leq \sum_{k=1}^N \sum_{l=1}^n \sum_{j=1}^n \gamma_k |\alpha_{kl}(t)| |v_{lj}| |\tilde{\mu}_{kj}(t) - \mu_j(t)| \\ &\leq \sum_{k=1}^N \sum_{l=1}^n \sum_{j=1}^n \gamma_k |\alpha_{kl}(t)| |v_{lj}| [b_j |\alpha_{kj}(t)| + d_j] \\ &\leq b^{\max} v^{\max} \sum_{k=1}^N \sum_{l=1}^n \sum_{j=1}^n \gamma_k |\alpha_{kl}(t)| |\alpha_{kj}(t)| \\ &\quad + v^{\max} \sum_{k=1}^N \sum_{l=1}^n \sum_{j=1}^n \gamma_k |\alpha_{kl}(t)| d_j \\ &\leq n b^{\max} v^{\max} \sum_{k=1}^N \gamma_k \alpha_k^T(t) \alpha_k(t) \\ &\quad + n v^{\max} d^{\max} \sum_{k=1}^N \gamma_k \|\alpha_k(t)\|_1. \end{aligned} \tag{12}$$

Similar to (12), one has

$$\sum_{k=1}^N \gamma_k \alpha_k^T(t) W [\tilde{\mu}_k(t - \tau) - \mu(t - \tau)]$$

$$\begin{aligned}
 &= \sum_{k=1}^N \sum_{l=1}^n \sum_{j=1}^n \gamma_k \alpha_{kl}(t) w_{lj} [\tilde{\mu}_{kj}(t - \tau) - \mu_j(t - \tau)] \\
 &\leq \sum_{k=1}^N \sum_{l=1}^n \sum_{j=1}^n \gamma_k |\alpha_{kl}(t)| |w_{lj}| |\tilde{\mu}_{kj}(t - \tau) - \mu_j(t - \tau)| \\
 &\leq \sum_{k=1}^N \sum_{l=1}^n \sum_{j=1}^n \gamma_k |\alpha_{kl}(t)| |w_{lj}| [b_j |\alpha_{kj}(t - \tau)| + d_j] \\
 &\leq b^{\max} w^{\max} \sum_{k=1}^N \sum_{l=1}^n \sum_{j=1}^n \gamma_k |\alpha_{kl}(t)| |\alpha_{kj}(t - \tau)| \\
 &\quad + n d^{\max} w^{\max} \sum_{k=1}^N \sum_{l=1}^n \gamma_k |\alpha_{kl}(t)| \\
 &\leq b^{\max} w^{\max} \sum_{k=1}^N \gamma_k \|\alpha_k(t)\|_1 \|\alpha_k(t - \tau)\|_1 \\
 &\quad + n d^{\max} w^{\max} \sum_{k=1}^N \gamma_k \|\alpha_k(t)\|_1.
 \end{aligned} \tag{13}$$

From Eqs. (11)–(13), one has

$$\begin{aligned}
 D^h X(t) &\leq -[A_{\min}(U) + \beta - n b^{\max} v^{\max}] \sum_{k=1}^N \gamma_k \alpha_k^T(t) \alpha_k(t) \\
 &\quad - [\zeta - b^{\max} w^{\max}] \sum_{k=1}^N \gamma_k \|\alpha_k(t)\|_1 \|\alpha_k(t - \tau)\|_1 \\
 &\quad - [\eta - n d^{\max} (v^{\max} + w^{\max})] \sum_{k=1}^N \gamma_k \|\alpha_k(t)\|_1 \\
 &\quad + \sum_{k=1}^N \sum_{l=1}^N \gamma_k \alpha_k^T(t) a_{kl} E(\alpha_l(t) - \alpha_k(t)) \\
 &\quad - \xi \sum_{k=1}^N \sum_{l=1}^n \gamma_k |\alpha_{kl}(t)|^{\theta+1}.
 \end{aligned} \tag{14}$$

Next, our aim is to prove

$$\sum_{k=1}^N \sum_{l=1}^N \gamma_k \alpha_k^T(t) a_{kl} E(\alpha_l(t) - \alpha_k(t)) - \xi \sum_{k=1}^N \sum_{l=1}^n \gamma_k |\alpha_{kl}(t)|^{\theta+1} \leq -\delta X^{\frac{1+\theta}{2}}(t). \tag{15}$$

On the one side, we show that

$$\sum_{k=1}^N \sum_{l=1}^N \gamma_k \alpha_k^T(t) a_{kl} E(\alpha_l(t) - \alpha_k(t)) \leq 0. \tag{16}$$

To this conclusion, let $\tilde{\alpha}_k(t) = \sqrt{E}\alpha_k(t)$, then we can obtain from Lemma 2.1 that

$$\begin{aligned} & \sum_{k=1}^N \sum_{l=1}^N \gamma_k \alpha_k^T(t) a_{kl} E(\alpha_l(t) - \alpha_k(t)) \\ &= \sum_{k=1}^N \sum_{l=1}^N \gamma_k a_{kl} \tilde{\alpha}_k^T(t) (\tilde{\alpha}_l(t) - \tilde{\alpha}_k(t)) \\ &\leq \frac{1}{2} \sum_{k=1}^N \sum_{l=1}^N \gamma_k a_{kl} (\tilde{\alpha}_l^T(t) \tilde{\alpha}_l(t) - \tilde{\alpha}_k^T(t) \tilde{\alpha}_k(t)) \\ &= \frac{1}{2} \sum_{B \in \mathcal{B}} \mathcal{W}(B) \sum_{(i,m) \in \mathcal{E}(\mathcal{A}_B)} (\tilde{\alpha}_i^T(t) \tilde{\alpha}_i(t) - \tilde{\alpha}_m^T(t) \tilde{\alpha}_m(t)). \end{aligned} \tag{17}$$

For any directed cycle \mathcal{A}_B , the set $\mathcal{E}(\mathcal{A}_B)$ can be regarded as

$$\mathcal{E}(\mathcal{A}_B) = \{(k_x, k_{x+1}) / x = 1, 2, \dots, p-1, p \leq N, k_p = k_1\}. \tag{18}$$

It follows from (18) that

$$\begin{aligned} & \sum_{(i,m) \in \mathcal{E}(\mathcal{A}_B)} (\tilde{\alpha}_i^T(t) \tilde{\alpha}_i(t) - \tilde{\alpha}_m^T(t) \tilde{\alpha}_m(t)) \\ &= \tilde{\alpha}_{k_1}^T(t) \tilde{\alpha}_{k_1}(t) - \tilde{\alpha}_{k_2}^T(t) \tilde{\alpha}_{k_2}(t) \\ &\quad + \tilde{\alpha}_{k_2}^T(t) \tilde{\alpha}_{k_2}(t) - \tilde{\alpha}_{k_3}^T(t) \tilde{\alpha}_{k_3}(t) \\ &\quad + \tilde{\alpha}_{k_3}^T(t) \tilde{\alpha}_{k_3}(t) - \dots \\ &\quad + \tilde{\alpha}_{k_{p-1}}^T(t) \tilde{\alpha}_{k_{p-1}}(t) - \tilde{\alpha}_{k_p}^T(t) \tilde{\alpha}_{k_p}(t) \\ &\quad + \tilde{\alpha}_{k_p}^T(t) \tilde{\alpha}_{k_p}(t) - \tilde{\alpha}_{k_1}^T(t) \tilde{\alpha}_{k_1}(t) \\ &= 0. \end{aligned} \tag{19}$$

From (17) and (19), one obtains

$$\sum_{k=1}^N \sum_{l=1}^N \gamma_k \alpha_k^T(t) a_{kl} E(\alpha_l(t) - \alpha_k(t)) \leq 0. \tag{20}$$

On the other side, we show that

$$-\xi \sum_{k=1}^N \sum_{l=1}^n \gamma_k |\alpha_{kl}(t)|^{\theta+1} \leq -\delta X^{\frac{1+\theta}{2}}(t). \tag{21}$$

According to Lemma 2.7, one obtains

$$\begin{aligned} -\sum_{k=1}^N \sum_{l=1}^n \gamma_k |\alpha_{kl}(t)|^{\theta+1} &= -\sum_{k=1}^N \sum_{l=1}^n \left[\gamma_k^{\frac{2}{1+\theta}} |\alpha_{kl}(t)|^2 \right]^{\frac{\theta+1}{2}} \\ &\leq -\sum_{k=1}^N \sum_{l=1}^n \left[2 \gamma_k^{\frac{1-\theta}{1+\theta}} \frac{\gamma_k |\alpha_{kl}(t)|^2}{2} \right]^{\frac{\theta+1}{2}} \end{aligned}$$

$$\begin{aligned} &\leq -2^{\frac{\theta+1}{2}} \left(\widehat{\gamma}_{\min}^{\frac{1-\theta}{1+\theta}}\right)^{\frac{1+\theta}{2}} \left[\sum_{k=1}^N \sum_{l=1}^n \frac{\gamma_k}{2} |\alpha_{kl}(t)|^2 \right]^{\frac{\theta+1}{2}} \\ &= -2^{\frac{\theta+1}{2}} \widehat{\gamma}_{\min}^{\frac{1-\theta}{2}} X^{\frac{\theta+1}{2}}(t). \end{aligned} \tag{22}$$

From (22), we have

$$-\xi \sum_{k=1}^N \sum_{l=1}^n \gamma_k |\alpha_{kl}(t)|^{\theta+1} \leq -\delta X^{\frac{1+\theta}{2}}(t). \tag{23}$$

It follows from (14), (20), and (23) that

$$D^h X(t) \leq -\delta X^{\frac{1+\theta}{2}}(t). \tag{24}$$

According to Lemma 2.8, one has

$$X^{\frac{2h-\theta-1}{2}}(t) \leq X^{\frac{2h-\theta-1}{2}}(t_0) - \frac{\delta \Gamma(\frac{2h+1-\theta}{2})(t-t_0)^h}{\Gamma(1+h)\Gamma(\frac{1-\theta}{2})}, \quad t \in [t_0, t_1],$$

and $X(t) = 0, \forall t \geq t_1$, where t_1 is denoted by

$$t_1 = t_0 + \left[\frac{\Gamma(1+h)\Gamma(\frac{1-\theta}{2}) X^{\frac{2h-1-\theta}{2}}(t_0)}{\delta \Gamma(\frac{2h+1-\theta}{2})} \right]^{\frac{1}{h}}.$$

Based on Definition 2.5, FCDNNs (1) and the isolated networks (2) are finite-time synchronized under the discontinuous feedback control protocol (4).

The following kinds of fractional-order coupled neural networks (FCNNs) are also a very interesting problem. The following assumption is needed to derive the FTS criterion of FCNNs with continuous activations. \square

Assumption $[\mathcal{H}_4]$. For every $k = 1, 2, \dots, N, l = 1, 2, \dots, n$, there exist positive constants $b_l > 0$ such that, for every $p_{kl}(t) \in \mathbb{R}, q_l(t) \in \mathbb{R}$, the following inequality holds:

$$|f_{kl}(t) - q_l(t)| \leq b_l |p_{kl}(t) - q_l(t)|.$$

Corollary 3.2 *Suppose that Assumption $[\mathcal{H}_4]$ and conditions (7) and (8) of Theorem 3.1 hold. Then FCDNNs (1) and the isolated networks (2) with continuous activations are finite-time synchronized under the following feedback control protocol:*

$$y_k(t) = -\beta \alpha_k(t) - \zeta \|\alpha_k(t - \tau)\|_1 \text{sign}(\alpha_k(t)) - \xi \text{sign}(\alpha_k(t)) |\alpha_k(t)|^\theta,$$

where $0 < \theta < 1, \beta > 0, \zeta > 0, \xi > 0$ are control gains, and γ_k signifies the cofactor of the i th diagonal elements of \mathcal{L} .

Proof The proof of the corollary is similar to that of Theorem 3.1. Hence the proof is omitted here. \square

3.2 Finite-time synchronization under discontinuous adaptive feedback control

Next, we design discontinuous adaptive feedback control protocol as follows:

$$\begin{cases} y_k(t) = -\beta_k(t)\alpha_k(t) - \zeta \|\alpha_k(t - \tau)\|_1 \text{sign}(\alpha_k(t)) \\ \quad - (\xi_k(t) + \eta) \text{sign}(\alpha_k(t)) \\ D^h \beta_k(t) = \lambda |\alpha_k(t)|^2 - \phi \text{sign}[\beta_k(t) - \beta] + \frac{\varpi |\alpha_k(t)|}{|\beta_k(t) - \beta|} \\ D^h \xi_k(t) = \mu |\alpha_k(t)| - \sigma \text{sgn}(\xi_k(t) - \chi), \end{cases} \tag{25}$$

where $\zeta > 0, \eta > 0, \lambda > 0, \mu > 0, \sigma > 0, \beta > 0, \varpi > 0, \chi > 0$ are all constants, $\beta_k(t)$ and $\xi_k(t)$ are adaptive control gains. According to the Filippov framework and set-valued map analysis, FCDNNs (3) can be written as follows:

$$\begin{aligned} D^h \alpha_k(t) &\in -U\alpha_k(t) + V\tilde{c}\mathcal{O}\{\Psi(\alpha_k(t))\} + W\tilde{c}\mathcal{O}\{\Psi(\alpha_k(t - \tau))\} \\ &\quad + \sum_{l=1}^N a_{kl}E(\alpha_l(t) - \alpha_k(t)) - \zeta \|\alpha_k(t - \tau)\|_1 \tilde{c}\mathcal{O}(\text{sign}(\alpha_k(t))) \\ &\quad - \beta_k(t)\alpha_k(t) - (\xi_k(t) + \eta)\tilde{c}\mathcal{O}(\text{sign}(\alpha_k(t))) \\ &\subseteq -U\alpha_k(t) + V[\tilde{c}\mathcal{O}(p_k(t)) - \tilde{c}\mathcal{O}(q(t))] \\ &\quad + W[\tilde{c}\mathcal{O}(p_k(t - \tau)) - \tilde{c}\mathcal{O}(q(t - \tau))] \\ &\quad + \sum_{l=1}^N a_{kl}E(\alpha_l(t) - \alpha_k(t)) - \beta_k(t)\alpha_k(t) \\ &\quad - \zeta \|\alpha_k(t - \tau)\|_1 \tilde{c}\mathcal{O}(\text{sign}(\alpha_k(t))) \\ &\quad - (\xi_k(t) + \eta)(\text{sign}(\alpha_k(t))). \end{aligned} \tag{26}$$

Then there exist a measurable function $\tilde{\mu}_k(t) \in \tilde{c}\mathcal{O}[f(p_k(t))]$ and $\mu(t) \in \tilde{c}\mathcal{O}[f(q(t))]$ for a.e. $t \in [-\tau, t_1]$ such that

$$\begin{aligned} D^h \alpha_k(t) &= -U\alpha_k(t) + V[\tilde{\mu}_k(t) - \mu(t)] + W[\tilde{\mu}_k(t - \tau) - \mu(t - \tau)] \\ &\quad + \sum_{l=1}^N a_{kl}E(\alpha_l(t) - \alpha_k(t)) - \beta_k(t)\alpha_k(t) \\ &\quad - \zeta \|\alpha_k(t - \tau)\|_1 \text{SIGN}(\alpha_k(t)) - (\xi_k(t) + \eta)(\text{SIGN}(\alpha_k(t))). \end{aligned} \tag{27}$$

Theorem 3.3 *Suppose that Assumptions $[\mathcal{H}_1]$ – $[\mathcal{H}_3]$ and conditions (7)–(9) of Theorem 3.1 hold. Then FCDNNs (1) and the isolated networks (2) are finite-time synchronized under the discontinuous adaptive feedback control protocol (25), if*

$$\chi = 1 + \frac{\varpi}{\lambda}. \tag{28}$$

Furthermore, the settling time is evaluated as follows:

$$t_1 = t_0 + \left[\frac{X^{\frac{1}{h}}(t_0)\Gamma(1+h)\Gamma(\frac{1}{2})}{\delta\Gamma(\frac{2h+1}{2})} \right]^{\frac{1}{h}},$$

where

$$X(t_0) = \sum_{k=1}^N \frac{\gamma_k}{2} \alpha_k^T(t_0) \alpha_k(t_0) + \sum_{k=1}^N \frac{\gamma_k}{2\lambda} [\beta_k(t_0) - \beta]^2 + \sum_{k=1}^N \frac{\gamma_k}{2\mu} [\xi_k(t_0) - \chi]^2,$$

$$\delta = \sqrt{2} \min \left\{ \min\{\sqrt{\gamma_k}\}, \min\left\{\frac{\sqrt{\gamma_k \phi}}{\sqrt{\lambda}}\right\}, \min\left\{\frac{\sqrt{\gamma_k \sigma}}{\sqrt{\mu}}\right\} \right\},$$

and γ_k signifies the cofactor of the i th diagonal elements of \mathcal{L} .

Proof For the FCDNNs error system (6), construct the following Lyapunov functional:

$$X(t) = \sum_{k=1}^N \frac{\gamma_k}{2} \alpha_k^T(t) \alpha_k(t) + \sum_{k=1}^N \frac{\gamma_k}{2\lambda} [\beta_k(t) - \beta]^2 + \sum_{k=1}^N \frac{\gamma_k}{2\mu} [\xi_k(t) - \chi]^2. \tag{29}$$

Since (\mathcal{G}, A) is strongly connected, by using Lemma 2.1 that $\gamma_k > 0$ for $k \in \{1, 2, \dots, N\}$, and Lemma 2.6, one can get

$$\begin{aligned} D^h X(t) &\leq \sum_{k=1}^N \gamma_k \alpha_k^T(t) D^h \{ \alpha_k(t) \} \\ &\quad + \sum_{k=1}^N \frac{\gamma_k}{\lambda} [\beta_k(t) - \beta] D^h \beta_k(t) + \sum_{k=1}^N \frac{\gamma_k}{\mu} [\xi_k(t) - \chi] D^h \xi_k(t) \\ &= \sum_{k=1}^N \gamma_k \alpha_k^T(t) \left[-U \alpha_k(t) + V [\tilde{\mu}_k(t) - \mu(t)] \right. \\ &\quad \left. + W [\tilde{\mu}_k(t - \tau) - \mu(t - \tau)] + \sum_{l=1}^N a_{kl} E(\alpha_l(t) - \alpha_k(t)) \right. \\ &\quad \left. - \beta_k(t) \alpha_k(t) - \zeta \|\alpha_k(t - \tau)\|_1 \text{SIGN}(\alpha_k(t)) \right. \\ &\quad \left. - (\xi_k(t) + \eta) (\text{SIGN}(\alpha_k(t))) \right] + \sum_{k=1}^N \frac{\gamma_k}{\lambda} [\beta_k(t) - \beta] \\ &\quad \times \left[\lambda |\alpha_k(t)|^2 - \phi \text{sign}[\beta_k(t) - \beta] + \frac{\varpi |\alpha_k(t)|}{[\beta_k(t) - \beta]} \right] \\ &\quad + \sum_{k=1}^N \frac{\gamma_k}{\mu} [\xi_k(t) - \chi] [\mu |\alpha_k(t)| - \sigma \text{sgn}(\xi_k(t) - \chi)]. \tag{30} \end{aligned}$$

By using Theorem 3.1, Eq. (12), and Eq. (10), one has

$$\begin{aligned} D^h X(t) &\leq -[\Lambda_{\min}(U) - nb^{\max} v^{\max}] \sum_{k=1}^N \gamma_k \alpha_k^T(t) \alpha_k(t) \\ &\quad - [\zeta - b^{\max} w^{\max}] \sum_{k=1}^N \gamma_k \|\alpha_k(t)\|_1 \|\alpha_k(t - \tau)\|_1 \\ &\quad - [\eta - na^{\max} (v^{\max} + w^{\max})] \sum_{k=1}^N \gamma_k \|\alpha_k(t)\|_1 \end{aligned}$$

$$\begin{aligned}
 & - \sum_{k=1}^N \gamma_k \beta_k(t) |\alpha_k(t)|^2 - \sum_{k=1}^N \gamma_k \xi_k(t) |\alpha_k(t)| \\
 & + \sum_{k=1}^N \gamma_k [\beta_k(t) - \beta] |\alpha_k(t)|^2 - \sum_{k=1}^N \frac{\gamma_k |\beta_k(t) - \beta| \phi}{\lambda} \\
 & + \sum_{k=1}^N \frac{\gamma_k |\alpha_k(t)| \varpi}{\lambda} - \sum_{k=1}^N \frac{\gamma_k |\xi_k(t) - \chi| \sigma}{\mu} \\
 & + \sum_{k=1}^N \gamma_k [\xi_k(t) - \chi] |\alpha_k(t)| \\
 \leq & - [A_{\min}(U) + \beta - nb^{\max} v^{\max}] X(t) \\
 & - [\zeta - b^{\max} w^{\max}] \sum_{k=1}^N \gamma_k \|\alpha_k(t)\|_1 \|\alpha_k(t - \tau)\|_1 \\
 & - [\eta - na^{\max} (v^{\max} + w^{\max})] \sum_{k=1}^N \gamma_k \|\alpha_k(t)\|_1 \\
 & + \sum_{k=1}^N \gamma_k \left[\frac{\varpi}{\lambda} - \chi \right] |\alpha_k(t)| - \sum_{k=1}^N \frac{\gamma_k |\beta_k(t) - \beta| \phi}{\lambda} \\
 & - \sum_{k=1}^N \frac{\gamma_k |\xi_k(t) - \chi| \sigma}{\mu}. \tag{31}
 \end{aligned}$$

By virtue of Lemma 2.7, we obtain

$$\begin{aligned}
 & \sqrt{\sum_{k=1}^N \gamma_k \left(\frac{1}{2} |\alpha_k(t)|^2 + \frac{1}{2\lambda} [\beta_k(t) - \beta]^2 + \frac{1}{2\mu} [\xi_k(t) - \chi]^2 \right)} \\
 & \leq \sum_{k=1}^N \sqrt{\gamma_k} \left(\frac{1}{\sqrt{2}} |\alpha_k(t)| + \frac{1}{\sqrt{2\lambda}} [\beta_k(t) - \beta] + \frac{1}{\sqrt{2\mu}} [\xi_k(t) - \chi] \right). \tag{32}
 \end{aligned}$$

Then, from Eq. (7)–Eq. (9), Eq. (20), and Eq. (30)–Eq. (31), we can obtain

$$\begin{aligned}
 D^h X(t) & \leq - \sum_{k=1}^N \gamma_k \left[\chi - \frac{\varpi}{\lambda} \right] |\alpha_k(t)| \\
 & \quad - \sum_{k=1}^N \frac{\gamma_k |\beta_k(t) - \beta| \phi}{\lambda} - \sum_{k=1}^N \frac{\gamma_k |\xi_k(t) - \chi| \sigma}{\mu} \\
 & \leq -2 \left[\sum_{k=1}^N \frac{\gamma_k}{2} |\alpha_k(t)| + \sum_{k=1}^N \frac{\gamma_k |\beta_k(t) - \beta| \phi}{2\lambda} + \sum_{k=1}^N \frac{\gamma_k |\xi_k(t) - \chi| \sigma}{2\mu} \right] \\
 & \leq -\delta X^{\frac{1}{2}}(t). \tag{33}
 \end{aligned}$$

According to Lemma 2.8, one has

$$X^{\frac{2h-1}{2}}(t) \leq X^{\frac{1}{2}}(t_0) - \frac{\delta \Gamma(\frac{2h+1}{2})(t - t_0)^h}{\Gamma(1+h)\Gamma(\frac{1}{2})}, \quad t \in [t_0, t_1].$$

The settling time t_1 is estimated by

$$t_1 = t_0 + \left[\frac{X^{\frac{1}{2}}(t_0) \Gamma(1+h) \Gamma(\frac{1}{2})}{\delta \Gamma(\frac{2h+1}{2})} \right]^{\frac{1}{h}}.$$

Based on Definition 2.5, FCDNNs (1) and the isolated networks (2) are synchronized in finite time under the discontinuous adaptive feedback control protocol (25). \square

Corollary 3.4 *Suppose that Assumption $[\mathcal{H}_4]$ and conditions (7), (8), and (28) of Theorem 3.3 hold. Then system (1) and the isolated networks (2) with continuous activations are finite-time synchronized under the following adaptive feedback control protocol:*

$$\begin{cases} y_k(t) = -\beta_k(t)\alpha_k(t) - \zeta \|\alpha_k(t - \tau)\|_1 \text{sign}(\alpha_k(t)) - \xi_k(t) \text{sign}(\alpha_k(t)), \\ D^h \beta_k(t) = \lambda |\alpha_k(t)|^2 - \phi \text{sign}[\beta_k(t) - \beta] + \frac{\varpi |\alpha_k(t)|}{|\beta_k(t) - \beta|} \\ D^h \xi_k(t) = \mu |\alpha_k(t)| - \sigma \text{sgn}(\xi_k(t) - \chi), \end{cases} \quad (34)$$

where $\zeta > 0, \lambda > 0, \mu > 0, \sigma > 0, \beta > 0, \varpi > 0, \chi > 0$ are all constants, $\beta_k(t)$ and $\xi_k(t)$ are adaptive control gains, and γ_k signifies the cofactor of the i th diagonal elements of \mathcal{L} .

Proof The proof of the corollary is similar to that of Theorem 3.3. Hence the proof is omitted here. \square

Remark 3.5 It is the first time that the finite-time synchronization and adaptive finite-time synchronization criterion of FCDNNs have been investigated. In this paper, fractional order, discontinuations neuron activation, graph theory techniques, and coupling terms are taken into consideration, and their results are very complicated and not easy to calculate. The main innovation of this paper is to extend and to overcome this complication. Hence our proposed models are more general and advanced.

4 Computer simulations

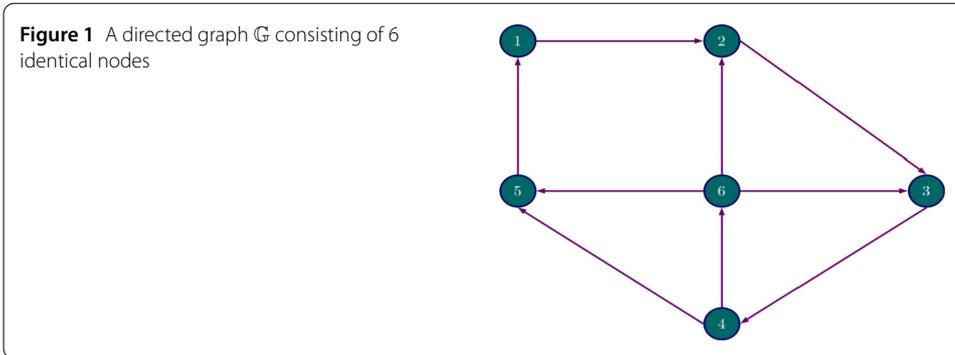
Here, two numerical cases are given to illustrate the efficiency of the proposed finite-time synchronization results.

Example 4.1 Consider a class of FCDNNs on a directed graph \mathbb{G} consisting of six identical nodes with every isolated node network being a 2-dimensional dynamical system, which is characterized by

$$\begin{aligned} D^h p_k(t) &= -U p_k(t) + V f(p_k(t)) + W f(p_k(t - \tau)) \\ &+ \sum_{l=1}^6 a_{kl} E(q_l(t) - q_k(t)) + y_k(t) \end{aligned} \quad (35)$$

with the isolated networks

$$D^h p_k(t) = -U p_k(t) + V f(p_k(t)) + W f(p_k(t - \tau)), \quad (36)$$



where $k = 1, 2, 3, 4, 5, 6$, $h = 0.98$, let $U = \text{diag}\{12, 12\}$, $\tau = 2.5$, $E = \text{diag}\{1, 1\}$, and

$$V = \begin{bmatrix} 0.5 & 2.8 \\ 1.4 & -1.3 \end{bmatrix}, \quad W = \begin{bmatrix} 0.6 & 1 \\ 1.3 & -1.5 \end{bmatrix}.$$

From Fig. 1, we can see that the coupling matrix A and the corresponding Laplacian matrix $\mathcal{L}(\mathbb{G}, A)$, respectively, are given as follows:

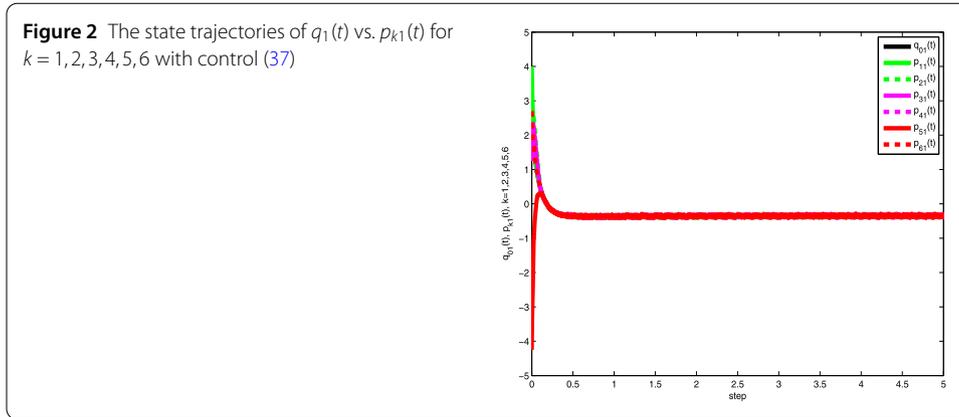
$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad \mathcal{L}(\mathbb{G}, A) = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ -1 & 2 & 0 & 0 & 0 & -1 \\ 0 & -1 & 2 & 0 & 0 & -1 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}.$$

The discontinuous neuron activation is selected as $f(p) = 0.5 + \text{sign}(p)$. Then one can obtain that $b^{\max} = 1$, $d^{\max} = 0.5$. The initial conditions are chosen as follows:

$$\begin{aligned} p_1(0) &= (p_{11}(0), p_{12}(0))^T = (4, -4)^T, \\ p_2(0) &= (p_{21}(0), p_{22}(0))^T = (2.5, -2.5)^T, \\ p_3(0) &= (p_{31}(0), p_{32}(0))^T = (1.3, -3.5)^T, \\ p_4(0) &= (p_{41}(0), p_{42}(0))^T = (1.25, -1.2)^T, \\ p_5(0) &= (p_{51}(0), p_{52}(0))^T = (-4.25, 3.25)^T, \\ p_6(0) &= (p_{61}(0), p_{62}(0))^T = (2.7, -4.8)^T, \\ q(0) &= (q_1(0), q_2(0))^T = (2.5, -4.5)^T. \end{aligned}$$

By simple calculation, we get $\gamma_1 = 2$, $\gamma_2 = 2$, $\gamma_3 = 4$, $\gamma_4 = 2$, $\gamma_5 = 1$, $\gamma_6 = 9$. The discontinuous feedback control protocol is designed by

$$\begin{aligned} y_k(t) &= -14\alpha_k(t) - 3\|\alpha_k(t - \tau)\|_1 \text{sign}(\alpha_k(t)) \\ &\quad - 6 \text{sign}(\alpha_k(t)) - 0.6 \text{sign}(\alpha_k(t))|\alpha_k(t)|^{0.6} \end{aligned} \tag{37}$$



for $k = 1, 2, 3, 4, 5, 6$. According to Theorem 3.1, it is simple to check

$$14 = \beta > \Lambda_{\min}(U) - nb^{\max}v^{\max} = 6.4,$$

$$3 = \zeta > b^{\max}w^{\max} = 1.5,$$

$$6 = \eta > nd^{\max}(v^{\max} + w^{\max}) = 4.3,$$

and all the conditions of Theorem 3.1 hold. Therefore, FCDNNs (35) and the isolated networks (36) are finite-time synchronized under the discontinuous feedback control protocol (37).

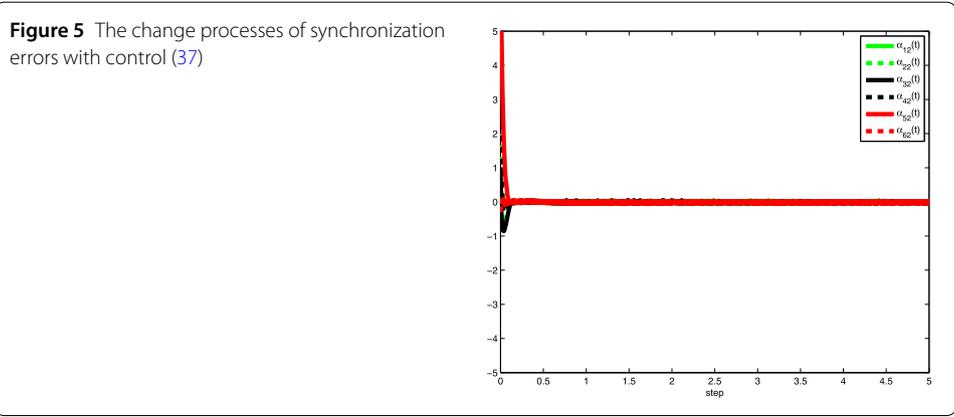
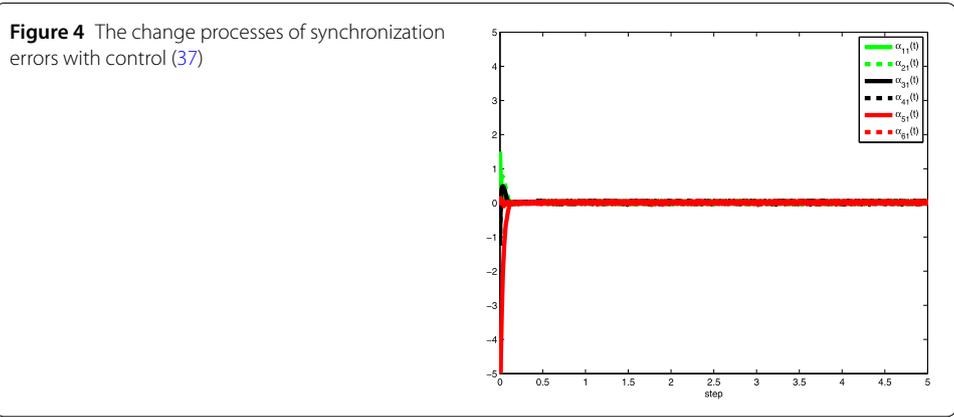
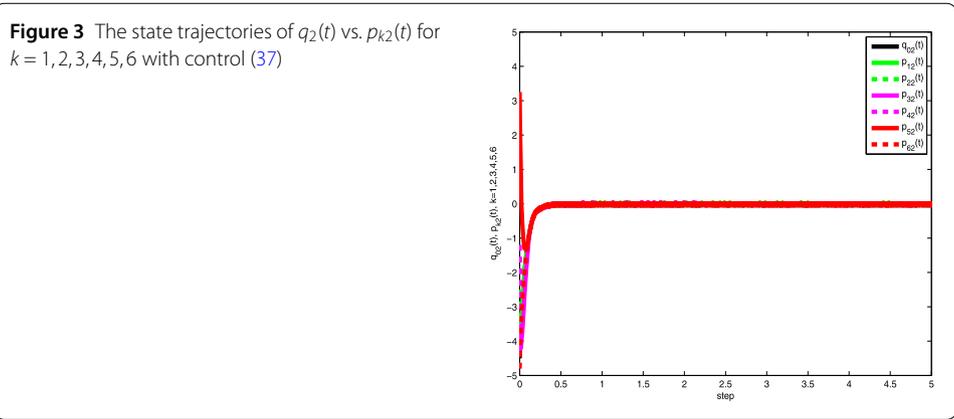
Furthermore, the settling time is evaluated as follows:

$$\begin{aligned} [H]t_1 &= t_0 + \left[\frac{\Gamma(1+h)\Gamma(\frac{1-\theta}{2})X^{\frac{2h-1-\theta}{2}}(t_0)}{\delta\Gamma(\frac{2h+1-\theta}{2})} \right]^{\frac{1}{h}} \\ &= 0 + \left[\frac{\Gamma(1+0.98)\Gamma(\frac{1-0.7}{2})X^{\frac{2(0.98)-1-0.9}{2}}(0)}{1.0815 \times \Gamma(\frac{2(0.98)+1-0.7}{2})} \right]^{\frac{1}{0.98}} \\ &= \left[\frac{\Gamma(1.98)\Gamma(0.15)(18.595)^{0.13}}{1.0815 \times \Gamma(1.13)} \right]^{\frac{1}{0.98}} \\ &= 8.916. \end{aligned}$$

Figures 2–5 demonstrate the numerical simulation results, which confirms the obtained theoretical findings. Figure 2 and Fig. 3 display the time response of considered systems (35) and (36) with control input (37), respectively, while the time response of synchronization errors $\alpha_{k1}(t)$ and $\alpha_{k2}(t)$ is displayed in Fig. 4 and Fig. 5, respectively.

Example 4.2 Consider a class of FCDNNs on directed graph \mathbb{G} consisting of four identical nodes characterized by the following form:

$$\begin{aligned} D^h p_k(t) &= -Up_k(t) + Vf(p_k(t)) \\ &\quad + Wf(p_k(t-\tau)) + \sum_{l=1}^4 a_{kl}E(q_l(t) - q_k(t)) + y_k(t) \end{aligned} \tag{38}$$

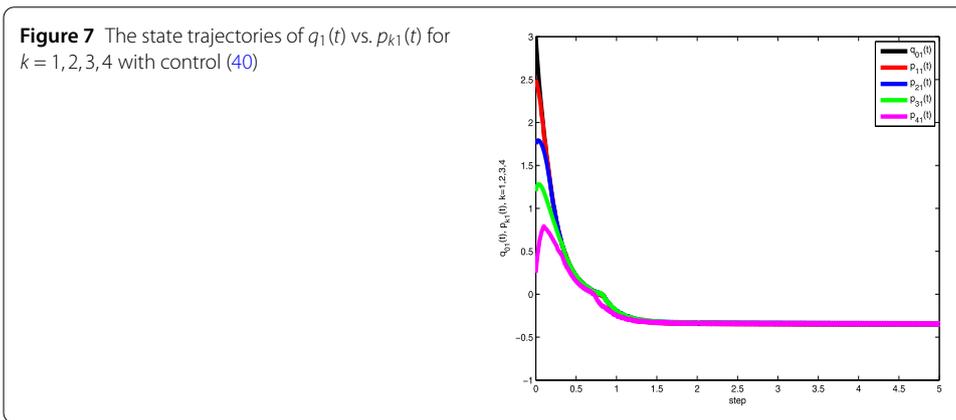
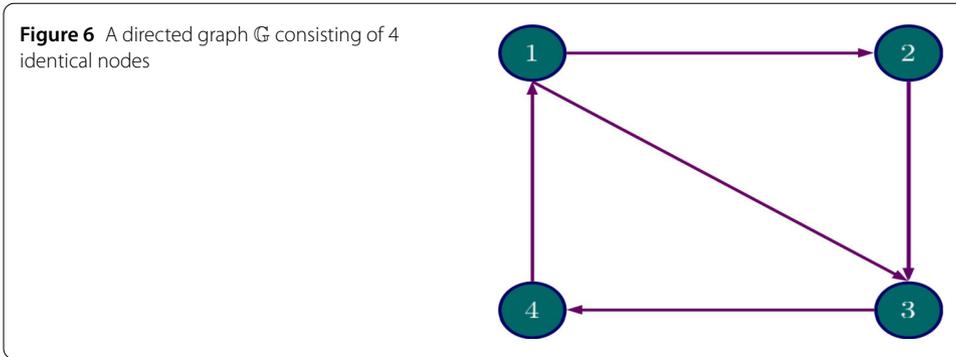


with the isolated networks

$$D^h p_k(t) = -U p_k(t) + V f(p_k(t)) + W f(p_k(t - \tau)), \tag{39}$$

where $k = 1, 2, 3, 4$, $h = 0.98$, let $U = \text{diag}\{10, 10\}$, $\tau = 1.5$, $E = \text{diag}\{1, 1\}$, and

$$V = \begin{bmatrix} 1.4 & -0.4 \\ -2.2 & 1.2 \end{bmatrix}, \quad W = \begin{bmatrix} 1 & 1 \\ -1 & 1.5 \end{bmatrix}.$$



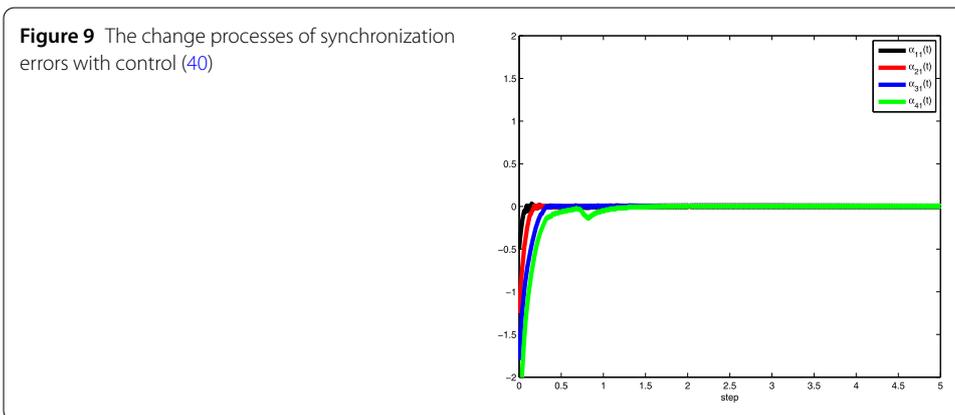
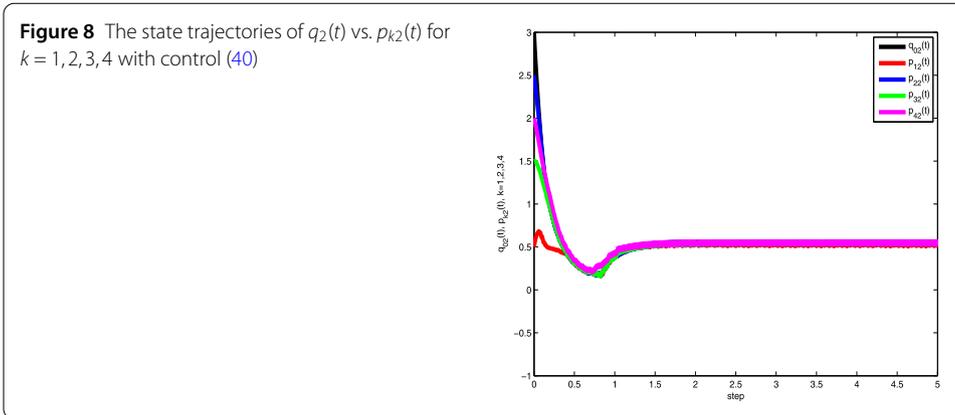
From Fig. 6, we can see that the coupling matrix A and the corresponding Laplacian matrix $\mathcal{L}(\mathbb{G}, A)$, respectively, are given as follows:

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad \mathcal{L}(\mathbb{G}, A) = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ -1 & 2 & 0 & 0 & 0 & -1 \\ 0 & -1 & 2 & 0 & 0 & -1 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}.$$

The discontinuous neuron activation is chosen as $f(p) = 0.1 + \text{sign}(p)$. Then one can obtain that $b^{\max} = 1.75, d^{\max} = 0.2$. The initial conditions of systems (38) and (39) are: $p_1(0) = (2.5, 2.5)^T, p_2(0) = (1.75, 2.5)^T, p_3(0) = (1.2, 1.5)^T, p_4(0) = (0.25, 2)^T, q(0) = (3, 3)^T$.

By simple calculation, we get $\gamma_1 = 2, \gamma_2 = 4, \gamma_3 = 2, \gamma_4 = 1$. The discontinuous adaptive feedback control protocol is designed by

$$\begin{cases} y_k(t) = -\beta_k(t)\alpha_k(t) - 3\|\alpha_k(t - \tau)\|_1 \text{sign}(\alpha_k(t)) \\ \quad - (\xi_k(t) + 2) \text{sign}(\alpha_k(t)), \\ D^h \beta_k(t) = |\alpha_k(t)|^2 - 0.5 \text{sign}[\beta_k(t) - 2.5] + \frac{0.004|\alpha_k(t)|}{[\beta_k(t) - \beta]} \\ D^h \xi_k(t) = 0.95|\alpha_k(t)| - \sigma \text{sgn}(\xi_k(t) - 1.004) \end{cases} \quad (40)$$



for $k = 1, 2, 3, 4$. According to Theorem 3.3, it is simple to check

$$2.5 = \beta > \Lambda_{\min}(U) - nb^{\max}v^{\max} = 2.3,$$

$$3 = \zeta > b^{\max}w^{\max} = 2.625,$$

$$2 = \eta > na^{\max}(v^{\max} + w^{\max}) = 1.48,$$

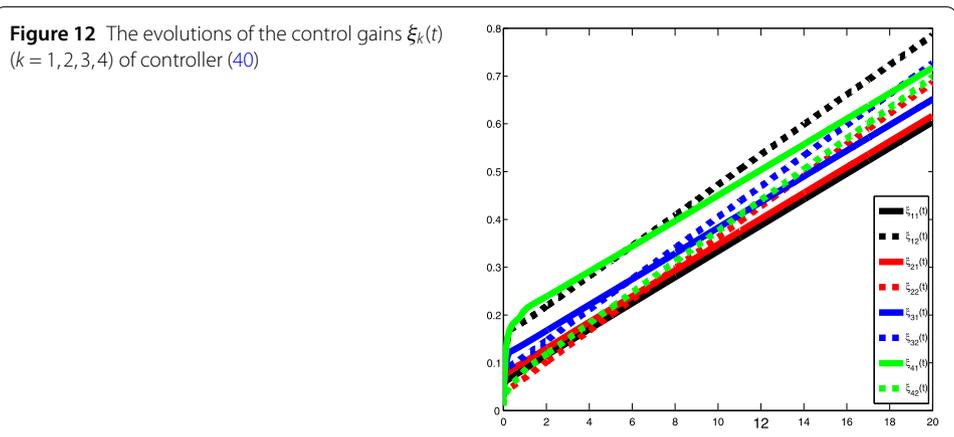
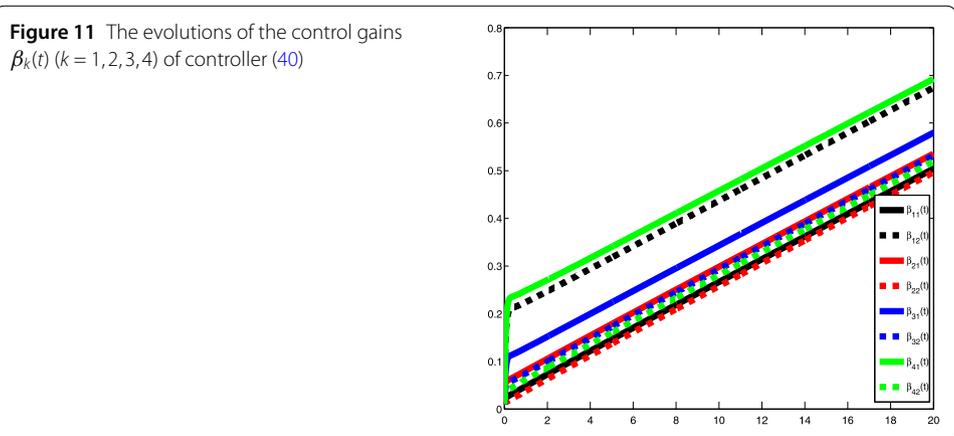
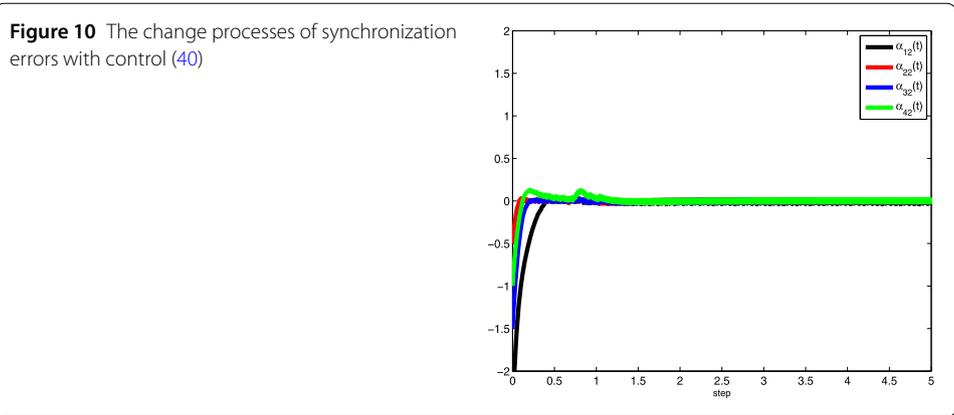
and all the conditions of Theorem 3.3 are satisfied. Therefore, FCDNNs (38) and the isolated networks (39) are finite-time synchronized under the discontinuous adaptive feedback control protocol (40). Next, we take the initial values of the discontinuous adaptive feedback control protocol (40) as follows: $\beta_1(0) = (0.02, 0.03)^T$, $\beta_2(0) = (0.03, 0.01)^T$, $\beta_3(0) = (0.02, 0.01)^T$, $\beta_4(0) = (0.01, 0.02)^T$, $\xi_1(0) = (0.05, 0.03)^T$, $\xi_2(0) = (0.04, 0.03)^T$, $\xi_3(0) = (0.03, 0.04)^T$, and $\xi_4(0) = (0.02, 0.01)^T$.

Furthermore, the settling time is evaluated as follows:

$$t_1 = t_1 = t_0 + \left[\frac{X^{\frac{1}{2}}(t_0)\Gamma(1+h)\Gamma(\frac{1}{2})}{\delta\Gamma(\frac{2h+1}{2})} \right]^{\frac{1}{h}}.$$

$$= 0 + \left[\frac{\sqrt{76.945} \times \Gamma(1+0.98)\sqrt{\pi}}{1.414 \times \Gamma(\frac{2(0.98)+1}{2})} \right]^{\frac{1}{0.98}}$$

$$= 12.95.$$



Figures 6–12 demonstrate the numerical simulation results, which confirms the accuracy of the theoretical results. Figure 7 and Fig. 8 show the state trajectories of considered systems (38) and (39) respectively. Figures 9–10 present the evaluations of synchronization errors between FCDNNs (38) and isolated networks (39) under the controller (40). Figure 11–12 demonstrate the adaptive feedback control gains (40), which shows that the adaptive control gains may go to some positive constants.

5 Conclusions

In this research paper, we have examined the finite-time synchronization and adaptive finite-time synchronization for graph theory perspective FCDNNs. By employing differential inclusion theory, Filippov framework, and designed discontinuous controllers, several finite-time synchronization criteria are established based on the graph theory approach. Numerical computer simulations are given to illustrate the accuracy of the proposed finite-time synchronization results. Our future work will be focused on finite-time synchronization criterion for graph theory perspective FCDNNs with coupling delays and impulses.

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Availability of data and materials

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The authors contributed equally to this work. All authors read and approved the final manuscript.

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