Liu et al. *Advances in Difference Equations* (2020) 2020:22
https://doi.org/10.1186/s13662-019-2478-5

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH **Open Access**

On the finite approximate controllability for Hilfer fractional evolution systems

Xianghu Liu[1*], Yanfang Li[1] and Guangjun Xu[1]

*Correspondence: liouxianghu04@126.com
[1] Mathematics Department, Zunyi Normal College, Zunyi, China

Abstract

In this paper, we consider the finite approximate controllability of some Hilfer fractional evolution systems. Using a variational approach and Schauder's fixed point theorem, we give sufficient conditions for finite approximate controllability of semilinear controlled systems. An example is given to illustrate our theory.

MSC: 93C10; 93C40

Keywords: Finite approximate controllability; Hilfer fractional derivative; Mild solutions; Variational approach; Schauder's fixed point theorem

1 Introduction

In this paper, we investigate the Hilfer fractional evolution system:

$$\begin{cases} D_{0^+}^{\nu,\mu}x(t) = Ax(t) + f(t,x(t)) + Bu(t), & t \in J' = (0,b], \\ I_{0^+}^{(1-\nu)(1-\mu)}x(0) = x_0, \end{cases} \tag{1.1}$$

where $D_{0^+}^{\nu,\mu}$ represents the Hilfer fractional derivative, $0 \le \nu \le 1$, $\frac{1}{2} < \mu < 1$, $x(\cdot)$ is assumed to be in a Hilbert space H, $I_{0^+}^{(1-\nu)(1-\mu)}$ is the Riemann–Liouville fractional integral of order $(1-\nu)(1-\mu)$, $A : D(A) \subseteq H \to H$ is the infinitesimal generator of a compact, uniformly bounded and C_0-semigroup $\{T(t), t \ge 0\}$ on a separable Hilbert space H. Now $f : J' \times H \to H$ is a given function that will be specified later. The control function u is taken in $L^2(J', U)$ and the admissible controls set U is a Hilbert space, B is a bounded linear operator from U into H, and finally, x_0 is an element of H.

Fractional calculus and fractional dynamic equations [1, 2] arise naturally in phenomena in engineering, physics, science and controllability. For recent work on the existence of mild solutions, controllability and optimal control for some fractional evolution systems we refer the reader to [3, 4], and for approximate controllability of some linear and nonlinear systems see [5–7] and the references therein. Hilfer [8] consider a generalized Riemann–Liouville fractional derivative called the Hilfer fractional derivative and in [9] the approximate controllability of Hilfer fractional differential inclusions with nonlocal conditions was investigated. Existence, nonexistence, uniqueness involving Hilfer fractional derivatives was discussed in [10–12] and in [13] the approximate controllability of fractional evolution equations involving Hilfer fractional derivatives was considered and in [14], we study the approximate controllability of Hilfer fractional evolution hemivariational inequalities by two resolvent operators and fixed point theorem. Compared with approximate controllability, finite approximate controllability is a stronger concept, and it is the consequence of approximate controllability in some linear heat equations. There are a number of papers on finite approximate controllability of differential systems. In [15], semilinear variational inequalities with distributed controls were studied, in [16] the author presented a finite-dimensional version of null controllability for the semilinear heat equation in bounded domains with Dirichlet boundary conditions, the author in [17] investigated finite approximate controllability for a nonlocal parabolic problem, and in [18, 19] the author considered approximate controllability and finite approximate controllability of some semilinear abstract equation, and finite approximate controllability for Sobolev-type nonlocal fractional semilinear evolution equations in Hilbert spaces.

There are only a few papers on finite approximate controllability of fractional evolution systems and motivated from the above (in particular [18, 19]), we will study the finite approximate controllability of some Hilfer fractional evolution systems. In Sect. 2, we present some preliminaries on fractional calculus and the definition of finite approximate controllability. In Sect. 3, sufficient conditions are given for the existence of mild solutions of system (1.1). In Sect. 4, by using the treatment in [14] and the variational method, the finite approximate controllability of system (1.1) is discussed. In Sect. 5, an example is given to illustrate the theory.

2 Preliminaries

Let $J = [0, b]$ and E be a Banach space with norm $\|\cdot\|_E$ (we usually write it as $\|\cdot\|$). Now E^* denotes its dual and $\langle\cdot,\cdot\rangle_E$ denotes the duality pairing between E^* and E. We use $L_b(E,E)$ to denote the space of bounded linear operators with the norm $\|\cdot\|_{L_b(E,E)}$. Let $C(J,E)$ be the Banach space of all continuous functions from J into E. Set $\gamma = \nu + \mu - \nu\mu$, $0 < \gamma < 1$, and then $1 - \gamma = (1 - \nu)(1 - \mu)$. Define

$$Y := C_{1-\gamma}\left(J', H\right) = \left\{x \in C\left(J', H\right) : \lim_{t\to 0^+} t^{1-\gamma}x(t) \text{ exist and finite}\right\}$$

endowed with the norm $\|x\|_Y = \sup_{t\in J'} \|t^{1-\gamma}x(t)\|_H$. Clearly, $(Y, \|\cdot\|_Y)$ is a Banach space.

For brevity, let $L^p_H = L^p(J,H)$, $L^p_{R^+} = L^p(J,R^+)$ and $L^p_U = L^p(J,U)$ for $1 \le p < \infty$.

We collect some definitions on fractional calculus of Riemann–Liouville type, Caputo type and Hilfer type; For more details, see [10, 12, 20–22].

Definition 2.1 For a given integral function $f : [a,\infty) \to E$, the integral

$$I^\alpha_{a^+}f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}f(s)\,ds, \quad t > a, \alpha > 0,$$

is called the right-side Riemann–Liouville fractional integral of order α, where Γ is the gamma function.

Definition 2.2 The right-side Riemann–Liouville fractional derivative of order $\alpha \in (n-1, n)$, $n \in Z^+$ for a function $f : [a, \infty) \to E$ is defined by

$$^{\mathrm{RL}}D^{\alpha}_{a^+}f(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^{(n)}\int_a^t (t-s)^{n-\alpha-1}f(s)\,dt, \quad t > a.$$

Definition 2.3 The right-side Hilfer fractional derivative of order ν, μ ($0 \le \nu \le 1$, $0 < \mu < 1$) for a function $f : [a, \infty) \to E$ is defined by

$$D^{\nu,\mu}_{a^+}f(t) = I^{\nu(1-\mu)}_{a^+}\left(\frac{d}{dt}I^{(1-\nu)(1-\mu)}_{a^+}f(t)\right), \quad t > a.$$

Definition 2.4 The right-side Caputo's fractional derivative of order $\alpha \in (n-1, n)$, $n \in Z^+$ for a function $f : [a, \infty) \to E$ is defined by

$$^{C}D^{\alpha}_{a^+}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-s)^{n-\alpha-1}f^{(n)}(s)\,dt, \quad t > a.$$

Remark 2.5

(i) When $\nu = 0$, $0 < \mu < 1$, and $a = 0$, the right-side Hilfer fractional derivative corresponds to the classical right-side Riemann–Liouville fractional derivative:

$$D^{0,\mu}_{0^+}f(t) = \frac{d}{dt}I^{(1-\mu)}_{a^+}f(t) = {}^{\mathrm{RL}}D^{\mu}_{0^+}f(t).$$

(ii) When $\nu = 1$, $0 < \mu < 1$, and $a = 0$, the right-side Hilfer fractional derivative corresponds to the classical right-side Caputo's fractional derivative:

$$D^{1,\mu}_{0^+}f(t) = I^{(1-\mu)}_{0^+}\frac{d}{dt}f(t) = {}^{C}D^{\mu}_{0^+}f(t).$$

Next we recall the definition of finite approximate controllability; see [15, 16]:

Definition 2.6 The system (1.1) is finite approximate controllable on J', if $x_b \in H$ and $\epsilon > 0$, there exists a control $u_\epsilon \in L^2_U$, such that the solution x_ϵ of system (1.1) satisfies the conditions:

$$\left\|x_\epsilon(b) - x_b\right\| < \epsilon \tag{2.1}$$

and

$$\Pi_{\mathcal{E}}x_\epsilon(b) = \Pi_{\mathcal{E}}x_b, \tag{2.2}$$

where $\mathcal{E}$ is a finite-dimensional subspace of H and $\Pi_{\mathcal{E}}$ is the orthogonal projection from H to $\mathcal{E}$.

The following definition is based on [12, Definition 2.3] and [13, Definition 5].

Definition 2.7 For each $u \in L_U^2$, a function $x \in Y$ is a mild solution of (1.1) if $I_{0^+}^{(1-\nu)(1-\mu)}x(0) = x_0$ and

$$x(t) = \mathcal{L}_{\nu,\mu}(t)x_0 + \int_0^t \mathcal{T}_\mu(t-s)\big[f\big(s,x(s)\big) + Bu(s)\big]\,ds, \quad t \in J', \tag{2.3}$$

where

$$\mathcal{P}_\mu(t) := \int_0^\infty \mu\theta M_\mu(\theta)T\big(t^\mu\theta\big)\,d\theta, \qquad \mathcal{T}_\mu(t) := t^{\mu-1}\mathcal{P}_\mu(t), \qquad \mathcal{L}_{\nu,\mu}(t) := I_{0^+}^{\nu(1-\mu)}\mathcal{T}_\mu(t),$$

and $M_\mu(\theta)$ is the M-Wright function defined by

$$M_\mu(\theta) = \sum_{n=1}^{\infty} \frac{(-\theta)^{n-1}}{(n-1)!\Gamma(1-\mu n)}, \quad 0 < \mu < 1, \theta \in \mathcal{C},$$

and it satisfies $M_\mu(\theta) > 0$, $\int_0^\infty M_\mu(\theta)\,d\theta = 1$ and $\int_0^\infty \theta^\delta M_\mu(\theta)\,d\theta = \frac{\Gamma(1+\delta)}{\Gamma(1+\mu\delta)}$, $\delta \in (-1,\infty)$.

We assume $T(t)$ $(t \geq 0)$ is uniformly bounded, so there exists $M > 1$ with $\sup_{t\in[0,\infty)} \|T(t)\| \leq M$.

Lemma 2.8 *The operators $\mathcal{T}_\mu(\cdot)$ and $\mathcal{L}_{\nu,\mu}(\cdot)$ have the following properties*:

(i) ([12, *Proposition* 2.16]) *For any fixed $t > 0$, $\mathcal{T}_\mu(t)$ and $\mathcal{L}_{\nu,\mu}(t)$ are linear and bounded operators, i.e., for any $x \in H$,*

$$\big\|\mathcal{T}_\mu(t)x\big\|_H \leq \frac{Mt^{\mu-1}}{\Gamma(\mu)}\|x\|_H \quad \textit{and} \quad \big\|\mathcal{L}_{\nu,\mu}(t)x\big\|_H \leq \frac{Mt^{\gamma-1}}{\Gamma(\gamma)}\|x\|_H, \quad \gamma = \nu + \mu - \nu\mu.$$

(ii) $\{\mathcal{T}_\mu(t), t > 0\}$ *and* $\{\mathcal{L}_{\nu,\mu}(t), t > 0\}$ *are compact if $T(t)$ is compact*, $t > 0$.

Remark 2.9 From Lemma 2.8(ii), we see that $\mathcal{T}_\mu(\cdot)$ and $\mathcal{L}_{\nu,\mu}(\cdot)$ are continuous in the uniform operator topology for $t > 0$, i.e.,

$$\big\|\mathcal{T}_\mu(t_2) - \mathcal{T}_\mu(t_1)\big\|_{L_b(H,H)} \to 0, \qquad \big\|\mathcal{L}_{\nu,\mu}(t_2) - \mathcal{L}_{\nu,\mu}(t_1)\big\|_{L_b(H,H)} \to 0$$

as $t_2 \to t_1$.

3 Existence of mild solutions

Consider the following assumptions:

$H(f)$: $f : J' \times H \to H$ is a function such that:

(i) the function $t \mapsto f(t,x)$ is measurable for all $x \in H$;

(ii) the function $x \mapsto f(t,x)$ is continuous for $t \in J'$;

(iii) for each $r > 0$, there exists a positive integrable function $\Phi_r(t) : J' \to (0,+\infty)$ such that

$$\sup_{\|x\|_Y \leq r} \big\|f\big(t,x(t)\big)\big\| \leq \Phi_r(t) \quad \text{for a.e. } t \in J' \text{ (here } x \in Y)$$

and

$$\lim_{r\to+\infty}\inf\frac{\|\Phi_r\|_{L^2_{R^+}}}{r}=\rho<+\infty;$$

(iv) the inequality

$$\frac{Mb^{\frac{1}{2}+\mu-\gamma}}{\sqrt{2\mu-1}\Gamma(\mu)}\rho<1$$

holds.

$H(B)$: the linear fractional control system

$$\begin{cases} D_{0^+}^{\nu,\mu}x(t)=Ax(t)+Bu(t),\\ I_{0^+}^{(1-\nu)(1-\mu)}x(0)=x_0,\end{cases}$$

is approximately controllable on J'.

Next, take into account two relevant operators:

$$\Gamma_0^b=\int_0^b\mathcal{T}_\mu(b-s)BB^*\mathcal{T}_\mu^*(b-s)\,ds,$$

and

$$R_\epsilon^b=\left(\epsilon I+\Gamma_0^b\right)^{-1},\quad \epsilon>0,$$

where I denotes the identity operator, B^* denotes the adjoint of B and $\mathcal{T}_\mu^*(\cdot)$ is the adjoint of $\mathcal{T}_\mu(\cdot)$.

Let $\epsilon>0$, $y\in Y$ and $x_b\in H$. We consider the functional $\mathcal{J}_\epsilon(\cdot;y):H\to R$ defined by

$$\mathcal{J}_\epsilon(\Psi;y)=\epsilon\left\|(I-\Pi_{\mathcal{E}})R_\epsilon^b\Psi\right\|_H+\frac{1}{2}\int_0^b\left\|B^*\mathcal{T}_\mu^*(b-t)R_\epsilon^b\Psi\right\|_H^2dt-\left\langle\mathcal{H}(y),R_\epsilon^b\Psi\right\rangle, \tag{3.1}$$

where

$$\mathcal{H}(y)=x_b-\mathcal{L}_{\nu,\mu}(b)x_0-\int_0^b\mathcal{T}_\mu(b-s)f\left(s,y(s)\right)ds.$$

We claim (and we will prove it after Lemma 3.3) that, for any $y\in Y$, the functional $\mathcal{J}_\epsilon(\cdot;y)$ admits a unique minimum $\widehat{\Psi_\epsilon}$ which defines a map $\mathcal{F}_\epsilon:Y\to H$ given by $\mathcal{F}_\epsilon:y\to\widehat{\Psi_\epsilon}$. Now let (here $x\in Y$)

$$u_\epsilon(s,x)=B^*\mathcal{T}_\mu^*(b-s)R_\epsilon^b\mathcal{F}_\epsilon(x)$$

and

$$(F_\epsilon x)(t)=\mathcal{L}_{\nu,\mu}(t)x_0+\int_0^t\mathcal{T}_\mu(t-s)\left[f\left(s,x(s)\right)+Bu_\epsilon(s,x)\right]ds.$$

For $r>0$, let $B_r^{(1-\gamma)}(J')=\{x\in Y:\|x\|_Y\le r\}$ and $B_r(J)=\{x\in C(J,H):\|x\|_C\le r\}$.

Lemma 3.1 *The set $\mathcal{H} = \{\mathcal{H}(y) : y \in B_r^{(1-\gamma)}(J')\}$ is relatively compact in Y*

Proof The proof is similar to that in step 4 in the proof of Theorem 3.6. □

Lemma 3.2 *$\mathcal{H} : B_r^{(1-\gamma)}(J') \to H$ is a continuous function.*

Proof The proof is similar to step 2 in the proof of Theorem 3.6. □

Lemma 3.3 *Let $\epsilon > 0$ and $r > 0$. Then with $\epsilon_1 = R_\epsilon^b \epsilon$, we have*

$$\varliminf_{\|\Psi\|_H \to +\infty} \inf_{y \in B_r^{(1-\gamma)}(J')} \frac{\mathcal{J}_\epsilon(\Psi; y)}{\|\Psi\|_H} \geq \epsilon_1.$$

Proof We follow the argument in [16, 18]. Suppose it is false. Then there exist sequences $\{\Psi_n\} \subset H$, $\{y_n\} \subset B_r^{(1-\gamma)}(J')$ with $\|\Psi_n\| \to +\infty$ and

$$\varliminf_{n \to +\infty} \frac{\mathcal{J}_\epsilon(\Psi_n; y_n)}{\|\Psi_n\|_H} < \epsilon_1. \tag{3.2}$$

Normalize with $\widehat{\Psi}_n = \frac{\Psi_n}{\|\Psi_n\|}$ (note $\|\widehat{\Psi}_n\| = 1$). The set $\{\mathcal{H}(y) : y \in B_r^{(1-\gamma)}(J')\}$ is relatively compact in H (see the argument later in Theorem 3.6), so without loss of generality assume $\mathcal{H}(y_n) \xrightarrow{\text{strongly}} h$ in H for some $h \in H$. Choose a subsequence which we will still denote by $\widehat{\Psi}_n$ with $\widehat{\Psi}_n \xrightarrow{\text{weakly}} \widehat{\Psi}$ for $\widehat{\Psi} \in H$. From the compactness of $\mathcal{T}_\mu(t)$ one has

$$B^* \mathcal{T}_\mu^*(b-t) R_\epsilon^b \widehat{\Psi}_n \xrightarrow{\text{strongly}} B^* \mathcal{T}_\mu^*(b-t) R_\epsilon^b \widehat{\Psi}.$$

From (3.1) we get

$$\begin{aligned} \frac{\mathcal{J}_\epsilon(\Psi_n; y_n)}{\|\Psi_n\|_H^2} &= \frac{\epsilon}{\|\Psi_n\|_H} \big\|(I - \Pi_{\mathcal{E}}) R_\epsilon^b \widehat{\Psi}_n\big\|_H + \frac{1}{2} \int_0^b \big\|B^* \mathcal{T}_\mu^*(b-t) R_\epsilon^b \widehat{\Psi}_n\big\|_H^2 \,dt \\ &\quad - \frac{1}{\|\Psi_n\|_H} \big\langle \mathcal{H}(y_n), R_\epsilon^b \widehat{\Psi}_n \big\rangle, \end{aligned}$$

and from Fatou's lemma, as $\|\Psi_n\| \to +\infty$, we have

$$\int_0^b \big\|B^* \mathcal{T}_\mu^*(b-t) R_\epsilon^b \widehat{\Psi}\big\|_H^2 \,dt \leq \varliminf_{n \to +\infty} \inf \int_0^b \big\|B^* \mathcal{T}_\mu^*(b-t) R_\epsilon^b \widehat{\Psi}_n\big\|_H^2 \,dt = 0,$$

which implies $\widehat{\Psi}_n \xrightarrow{\text{weakly}} 0$ in H. Since $\mathcal{E}$ is finite dimensional and we obtain $\Pi_{\mathcal{E}} R_\epsilon^b \widehat{\Psi}_n \xrightarrow{\text{strongly}} 0$ in H and so

$$\begin{aligned} &\big\|(I - \Pi_{\mathcal{E}}) R_\epsilon^b \widehat{\Psi}_n\big\|_H = \sqrt{\big\|R_\epsilon^b \widehat{\Psi}_n\big\|_H^2 + \big\|\Pi_{\mathcal{E}} R_\epsilon^b \widehat{\Psi}_n\big\|_H^2} \to \big\|R_\epsilon^b \widehat{\Psi}\big\|_H, \\ &\frac{\mathcal{J}_\epsilon(\Psi_n; y_n)}{\|\Psi_n\|_H} = \epsilon \big\|(I - \Pi_{\mathcal{E}}) R_\epsilon^b \widehat{\Psi}_n\big\|_H + \frac{\|\Psi_n\|_H}{2} \int_0^b \big\|B^* \mathcal{T}_\mu^*(b-t) R_\epsilon^b \widehat{\Psi}_n\big\|_H^2 \,dt \\ &\qquad - \big\langle \mathcal{H}(y_n), R_\epsilon^b \widehat{\Psi}_n \big\rangle, \end{aligned}$$

and as a result

$$\lim_{n\to+\infty} \frac{\mathcal{J}_\epsilon(\Psi_n;y_n)}{\|\Psi_n\|_H} \geq \lim_{n\to+\infty} \left(\epsilon\left\|(I-\Pi_{\mathcal{E}})R_\epsilon^b\widehat{\Psi}_n\right\|_H - \left\langle\mathcal{H}(y_n), R_\epsilon^b\widehat{\Psi}_n\right\rangle\right) = R_\epsilon^b\epsilon = \epsilon_1,$$

which contradicts (3.2). □

For $y \in B_r^{(1-\gamma)}(J')$ we have

$$\lim_{\|\psi\|_H\to+\infty} \frac{\mathcal{J}_\epsilon(\Psi;y)}{\|\Psi\|_H} \geq \epsilon_1.$$

Also note for any $y \in Y$ the map $\Psi \to \mathcal{J}_\epsilon(\Psi;y)$ is continuous and strictly convex. Now for $y \in B_r^{(1-\gamma)}(J')$ let $\{\Psi_{\epsilon,n}\}$ be a minimizing sequence of $\mathcal{J}_\epsilon(\cdot;y)$ and we suppose without loss of generality (note from the above this sequence is bounded) $\Psi_{\epsilon,n}$ converges weakly to $\widehat{\Psi_\epsilon}$ in H. Now

$$\mathcal{J}_\epsilon(\widehat{\Psi_\epsilon};y) \leq \underline{\lim}_{n\to+\infty} \mathcal{J}_\epsilon(\Psi_{\epsilon,n};y) = \inf_{\Psi\in H} \mathcal{J}_\epsilon(\Psi:y).$$

Thus $\widehat{\Psi_\epsilon}$ is a minimum and from the convexity of $\mathcal{J}_\epsilon(\cdot;y)$ the minimum is unique. Now we define a map $\mathcal{F}_\epsilon : y \to \widehat{\Psi_\epsilon}$ (which is the proof of the claim after (3.1)).

Lemma 3.4 *For all $y \in B_r^{(1-\gamma)}(J')$, there exists $R_\epsilon(r) > 0$, such that $\|\mathcal{F}_\epsilon(y)\| \leq R_\epsilon(r)$.*

Proof From Lemma 3.3, we see that there exists $R_\epsilon(r) > 0$ such that

$$\|\Psi\|_H > R_\epsilon(r), \qquad \inf_{y\in B_r^{(1-\gamma)}(J')} \frac{\mathcal{J}_\epsilon(\Psi;y)}{\|\Psi\|_H} \geq \epsilon.$$

If $\mathcal{F}_\epsilon$ is not bounded, we may as well suppose $\|\mathcal{F}_\epsilon\|_H \geq R_\epsilon(r)$, such that

$$\inf_{y\in B_r^{(1-\gamma)}(J')} \frac{\mathcal{J}_\epsilon(\mathcal{F}_\epsilon;y)}{\|\mathcal{F}_\epsilon\|_H} \geq \epsilon. \tag{3.3}$$

But from the definition of the map $\mathcal{F}_\epsilon$, we know

$$\mathcal{J}_\epsilon\left(\mathcal{F}_\epsilon(y);y\right) \leq \mathcal{J}_\epsilon\left(0^+;y\right) = 0,$$

which contradicts (3.3), thus, for all $y \in B_r^{(1-\gamma)}(J')$, we have $\|\mathcal{F}_\epsilon(y)\| \leq R_\epsilon(r)$. □

Lemma 3.5 *Suppose for any $y, y_n \in B_r^{(1-\gamma)}(J')$, $y_n \to y$ in Y. Then*

$$\mathcal{F}_\epsilon(y_n) \xrightarrow{strongly} \mathcal{F}_\epsilon(y).$$

Proof Assume that $y_n \in B_r^{(1-\gamma)}(J')$ be a subsequence and $y_n \to y$ as $n \to +\infty$. By the boundedness of $\mathcal{F}_\epsilon(y_n)$ denoted by $\widehat{\Psi_{\epsilon,n}}$, one can suppose $\widehat{\Psi_{\epsilon,n}}$ converges weakly to $\widehat{\Psi}_\epsilon$, then

$$\begin{aligned}\mathcal{J}_\epsilon\left(\widetilde{\Psi_\epsilon(y)};y\right) &\leq \mathcal{J}_\epsilon(\widehat{\Psi}_\epsilon;y_n) \leq \underline{\lim}_{n\to+\infty} \mathcal{J}_\epsilon\left(\widehat{\Psi_{\epsilon,n}}(y);y_n\right) \leq \overline{\lim}_{n\to+\infty} \mathcal{J}_\epsilon\left(\widehat{\Psi_{\epsilon,n}}(y);y_n\right)\\ &\leq \lim_{x\to+\infty} \mathcal{J}_\epsilon\left(\widetilde{\Psi_\epsilon(y)};y_n\right) = \mathcal{J}_\epsilon\left(\widetilde{\Psi_\epsilon(y)};y\right).\end{aligned}$$

From the above section, we know that $\widetilde{\Psi_\epsilon(y)}$ is the unique minimum point, thus $\widetilde{\Psi_\epsilon(y)} = \widehat{\Psi}_\epsilon(y)$. Combining the compactness of $B^*\mathcal{T}_\mu^*(b-t)R_\epsilon^b$, the continuity of function $H(y)$ and $\widehat{\Psi_{\epsilon,n}} \xrightarrow{w} \widehat{\Psi}_\epsilon$, we have

$$\begin{aligned}
&\lim_{n\to+\infty} \mathcal{J}_\epsilon(\widehat{\Psi_{\epsilon,n}}; y_n) = \mathcal{J}_\epsilon(\widehat{\Psi}_\epsilon; y) = \mathcal{J}_\epsilon(\widetilde{\Psi}_\epsilon; y),\\
&\lim_{n\to+\infty} \int_0^b \big\| B^*\mathcal{T}_\mu^*(b-t)R_\epsilon^b\widehat{\Psi_{\epsilon,n}}\big\|_H^2\,dt = \int_0^b \big\| B^*\mathcal{T}_\mu^*(b-t)R_\epsilon^b\widehat{\Psi}_\epsilon\big\|_H^2\,dt,\\
&\lim_{n\to+\infty} \big\langle H(y_n), R_\epsilon^b\widehat{\Psi_{\epsilon,n}},\big\rangle = \big\langle H(y), R_\epsilon^b\widehat{\Psi}_\epsilon\big\rangle,
\end{aligned}$$

thus, it follows that

$$\lim_{n\to+\infty} \epsilon\big\|(I-\Pi_{\mathcal{E}})R_\epsilon^b\widehat{\Psi_{\epsilon,n}}\big\|_H = \epsilon\big\|(I-\Pi_{\mathcal{E}})R_\epsilon^b\widehat{\Psi}_\epsilon\big\|_H.$$

Using the compactness of $\Pi_{\mathcal{E}}$, we infer that

$$\lim_{n\to+\infty} \|\widehat{\Psi_{\epsilon,n}}\|_H = \|\widehat{\Psi}_\epsilon\|_H,$$

which implies that $\mathcal{F}_\epsilon(y_n) \xrightarrow{\text{strongly}} \mathcal{F}_\epsilon(y)$. □

For any $x \in Y \subset L^2(J', H)$, we consider the map $\digamma : Y \to Y$

$$\begin{aligned}
\digamma(x) = \Big\{ g \in Y :&\\
&g(t) = \mathcal{L}_{\nu,\mu}(t)x_0 + \int_0^t \mathcal{T}_\mu(t-s)f\big(s, x(s)\big)\,ds + \int_0^t \mathcal{T}_\mu(t-s)Bu(s)\,ds, t \in J'\Big\}.
\end{aligned}$$

Clearly, $\lim_{t\to 0^+} t^{1-\gamma}g(t) = \frac{x_0}{\Gamma(\gamma)}$.

We will work with the operator $P := \cdot^{1-\gamma}\digamma\cdot^{\gamma-1}$ from $B_r(J)$ to $B_r(J)$ (i.e. for $y \in B_r(J)$, $Py(t) = t^{1-\gamma}\digamma(t^{\gamma-1}y(t))$). If we prove that P has a fixed point y^*, then $\digamma$ has a fixed point $x^* = \cdot^{\gamma-1}y^*$.

In our next result let $r > 0$ be such that

$$\frac{M}{\Gamma(\gamma)}\|x_0\|_H + \frac{Mb^{\frac{1}{2}+\mu-\gamma}}{\sqrt{2\mu-1}\,\Gamma(\mu)}\big(\|\Phi_r\|_{L^2_{R^+}} + \|B\|_{L_b(U,H)}\|u\|_{L^2_U}\big) \le r.$$

Theorem 3.6 *Assume that condition $H(f)$ holds. Then* (1.1) *has a mild solution in* $B_r^{(1-\gamma)}(J')$.

Proof We prove $\digamma$ has a fixed point in $B_r^{(1-\gamma)}(J')$ (i.e. P has a fixed point in $B_r(J)$). We divide the proof into four steps.

Step 1: $P : B_r(J) \to B_r(J)$.

Let $z \in B_r(J)$ (and $x = \cdot^{\gamma-1}z$ so $x \in B_r^{(1-\gamma)}(J')$). Now

$$Pz(t) = t^{1-\gamma}\mathcal{L}_{\nu,\mu}(t)x_0 + t^{1-\gamma}\int_0^t \mathcal{T}_\mu(t-s)\big[f\big(s, s^{\gamma-1}z(s)\big) + Bu(s)\big]\,ds, \quad t \in J,$$

so

$$Pz(t) = t^{1-\gamma}\mathcal{L}_{\nu,\mu}(t)x_0 + t^{1-\gamma}\int_0^t \mathcal{T}_\mu(t-s)\big[f\big(s, x(s)\big) + Bu(s)\big]\,ds, \quad t \in J. \tag{3.4}$$

From $H(f)$(iii), Lemma 2.8(i) and Hölder's inequality we have for $t \in J$

$$\begin{aligned}\|Pz(t)\| &\le \frac{M}{\Gamma(\gamma)}\|x_0\|_H + \frac{Mt^{1-\gamma}}{\Gamma(\mu)}\int_0^t (t-s)^{\mu-1}\big[\Phi_r(s) + \|B\|_{L_b(U,H)}\|u(s)\|_U\big]\,ds \\ &\le \frac{M}{\Gamma(\gamma)}\|x_0\|_H + \frac{Mb^{\frac{1}{2}+\mu-\gamma}}{\sqrt{2\mu-1}\Gamma(\mu)}\big(\|\Phi_r\|_{L^2_{R^+}} + \|B\|_{L_b(U,H)}\|u\|_{L^2_U}\big) \le r.\end{aligned}$$

Thus $P : B_r(J) \to B_r(J)$.

Step 2: $P : B_r(J) \to B_r(J)$ is continuous.

Let $\{z_n\}$ be the sequence in $B_r(J)$ with $z_n \to z$ in $B_r(J)$ as $n \to +\infty$ (note $\{x_n := \cdot^{\gamma-1}z_n\}$ is a sequence in $B_r^{(1-\gamma)}(J')$ and $x_n \to x$ as $n \to +\infty$; here $x = \cdot^{\gamma-1}z$). Now

$$\|Pz_n(t) - Pz(t)\| \le t^{1-\gamma}\int_0^t \mathcal{T}_\mu(t-s)\|f(s,x_n(s)) - f(s,x(s))\|\,ds,$$

and a standard argument using the Lebesgue dominated convergence theorem guarantees that $P : B_r(J) \to B_r(J)$ is continuous.

Step 3: $\{P(z) : z \in B_r(J)\}$ is equicontinuous.

Let $z \in B_r(J)$ (so $x = \cdot^{\gamma-1}z$), $0 \le \tau_1 < \tau_2 \le b$ and $\delta := \tau_2 - \tau_1 > 0$. Then

$$\|Pz(\tau_1) - Pz(\tau_2)\| \le Q_1 + Q_2 + Q_3 + Q_4,$$

where

$$\begin{aligned}Q_1 &= \big\|\tau_1^{1-\gamma}\mathcal{L}_{\nu,\mu}(\tau_1)x_0 - \tau_2^{1-\gamma}\mathcal{L}_{\nu,\mu}(\tau_2)x_0\big\|_H, \\ Q_2 &= (\tau_2-\tau_1)^{1-\gamma}\int_0^{\tau_1}\|\mathcal{T}_\mu(\tau_1-s)\|\big(\|f(s,x(s))\|_H + \|B\|_{L_b(U,H)}\|u(s)\|_U\big)\,ds, \\ Q_3 &= \tau_2^{1-\gamma}\sup_{s\in[0,\tau_1]}\|\mathcal{T}_\mu(\tau_1-s) - \mathcal{T}_\mu(\tau_2-s)\|_{L_b(H,H)} \\ &\quad \times \int_0^{\tau_1}\big(\|f(s,x(s))\|_H + \|B\|_{L_b(U,H)}\|u(s)\|_U\big)\,ds, \\ Q_4 &= \tau_2^{1-\gamma}\int_{\tau_1}^{\tau_2}\|\mathcal{T}_\mu(\tau_2-s)\|\big(\|f(s,x(s))\|_H + \|B\|_{L_b(U,H)}\|u(s)\|_U\big)\,ds.\end{aligned}$$

Clearly,

$$\begin{aligned}Q_1 &\le \big\|\tau_1^{1-\gamma}\mathcal{L}_{\nu,\mu}(\tau_1)x_0 - \tau_2^{1-\gamma}\mathcal{L}_{\nu,\mu}(\tau_1)x_0\big\|_H + \big\|\tau_2^{1-\gamma}\mathcal{L}_{\nu,\mu}(\tau_1)x_0 - \tau_2^{1-\gamma}\mathcal{L}_{\nu,\mu}(\tau_2)x_0\big\|_H \\ &\le (\tau_2-\tau_1)^{1-\gamma}\frac{Mb^{\gamma-1}}{\Gamma(\gamma)}\|x_0\|_H + \tau_2^{1-\gamma}\big\|\mathcal{L}_{\nu,\mu}(\tau_1)x_0 - \mathcal{L}_{\nu,\mu}(\tau_2)x_0\big\|_{L_b(H,H)} \\ &:= Q_{11} + Q_{12},\end{aligned}$$

Q_{11} tends to zero as $\delta \to 0$ and Q_{12} tends to zero as $\delta \to 0$ from Lemma 2.8(ii) and Remark 2.9. Thus, Q_1 tends to zero as $\delta \to 0$.

Next,

$$
\begin{aligned}
Q_2 &\le (\tau_2-\tau_1)^{1-\gamma}\int_0^{\tau_1}\big\|\mathcal{T}_\mu(\tau_1-s)\big\|\Phi_r(s)+\|B\|_{L_b(U,H)}\big\|u(s)\big\|_U\big)\,ds\\
&\le \delta^{1-\gamma}\int_0^{\tau_1}\frac{M(\tau_1-s)^{\mu-1}}{\Gamma(\mu)}\big(\Phi_r(s)+\|B\|_{L_b(U,H)}\big\|u(s)\big\|_U\big)\,ds\\
&\le \delta^{1-\gamma}\bigg(\frac{M\tau_1^{\mu-\frac{1}{2}}}{\sqrt{2\mu-1}\Gamma(\mu)}\big(\|\Phi_r\|_{L^2_{R^+}}+\|B\|_{L_b(U,H)}\|u\|_{L^2_U}\big)\bigg),
\end{aligned}
$$

which tends to zero as $\delta\to 0$.

In addition,

$$
\begin{aligned}
Q_3 &\le \tau_2^{1-\gamma}\sup_{s\in[0,\tau_1]}\big\|\mathcal{T}_\mu(\tau_1-s)-\mathcal{T}_\mu(\tau_2-s)\big\|_{L_b(H,H)}\int_0^{\tau_1}\big(\Phi_r(s)+\|B\|_{L_b(U,H)}\big\|u(s)\big\|_U\big)\,ds\\
&\le \tau_2^{1-\gamma}\sup_{s\in[0,\tau_1]}\big\|\mathcal{T}_\mu(\tau_1-s)-\mathcal{T}_\mu(\tau_2-s)\big\|_{L_b(H,H)}\\
&\quad\times\bigg(\frac{1}{\sqrt{\tau_1}}\|\Phi_r\|_{L^2_{R^+}}+\frac{1}{\sqrt{\tau_1}}\|B\|_{L_b(U,H)}\|u\|_{L^2_U}\bigg),
\end{aligned}
$$

which tends to zero as $\delta\to 0$ via Lemma 2.8(ii) and Remark 2.9 (so $\sup_{s\in[0,\tau_1]}\|\mathcal{T}_\mu(\tau_1-s)-\mathcal{T}_\mu(\tau_2-s)\|_{L_b(H,H)}\to 0$ as $\delta\to 0$).

Finally,

$$
\begin{aligned}
Q_4 &\le \tau_2^{1-\gamma}\int_{\tau_1}^{\tau_2}\frac{M(\tau_2-s)^{\mu-1}}{\Gamma(\mu)}\big(\Phi_r(s)+\|B\|_{L_b(U,H)}\big\|u(s)\big\|_U\big)\,ds\\
&\le \tau_2^{1-\gamma}\bigg(\frac{M\delta^{\mu-\frac{1}{2}}}{\sqrt{2\mu-1}\Gamma(\mu)}\big(\|\Phi_r\|_{L^2_{R^+}}+\|B\|_{L_b(U,H)}\|u\|_{L^2_U}\big)\bigg),
\end{aligned}
$$

which tends to zero as $\delta\to 0$.

Thus $\{P(z):z\in B_r(J)\}$ is an equicontinuous family of functions.

Step 4: $\forall t\in J$, the set $\Pi(t)=\{Pz(t):z\in B_r(J)\}$ is relatively compact in H.

For each $\epsilon\in(0,t)$, $t\in J$, $z\in B_r$ (and $x=\cdot^{\gamma-1}z$) and any $\delta>0$, we let

$$
Pz(t)=J_1(t)+J_2(t),\qquad Pz^{\epsilon,\delta}(t)=\tilde{J}_1(t)+\tilde{J}_2(t),
$$

where

$$
\begin{aligned}
J_1(t)&:=t^{1-\gamma}\frac{x_0}{\Gamma(\nu(1-\mu))}\int_0^t\int_0^\infty(t-s)^{\nu(1-\mu)-1}s^{\mu-1}\mu\theta M_\mu(\theta)T\big(s^\mu\theta\big)\,d\theta\,ds,\\
J_2(t)&:=t^{1-\gamma}\int_0^t\int_0^\infty(t-s)^{\mu-1}\mu\theta M_\mu(\theta)T\big((t-s)^\mu\theta\big)\big[f\big(s,x(s)\big)+Bu(s)\big]\,d\theta\,ds,\\
\tilde{J}_1(t)&:=t^{1-\gamma}\frac{x_0}{\Gamma(\nu(1-\mu))}\int_0^{t-\epsilon}\int_\delta^\infty(t-s)^{\nu(1-\mu)-1}s^{\mu-1}\mu\theta M_\mu(\theta)T\big(s^\mu\theta\big)\,d\theta\,ds,\\
\tilde{J}_2(t)&:=t^{1-\gamma}\int_0^{t-\epsilon}\int_\delta^\infty(t-s)^{\mu-1}\mu\theta M_\mu(\theta)T\big((t-s)^\mu\theta\big)\big[f\big(s,x(s)\big)+Bu(s)\big]\,d\theta\,ds.
\end{aligned}
$$

From Lemma 2.8(ii) we see that the set

$$\Pi_{\epsilon,\delta}(t) = \{Pz^{\epsilon,\delta}(t) : z \in B_r(J)\}$$

is relatively compact in H for each $\epsilon \in (0,t)$ and $\delta > 0$.

Moreover, we have

$$\begin{aligned}
&\|J_1(t) - \tilde{J}_1(t)\|_H \\
&= t^{1-\gamma}\frac{x_0}{\Gamma(\nu(1-\mu))}\bigg\| \int_0^t \int_0^\infty (t-s)^{\nu(1-\mu)-1}s^{\mu-1}\mu\theta M_\mu(\theta)T(s^\mu\theta)\,d\theta\,ds \\
&\quad - \int_0^t \int_\delta^\infty (t-s)^{\nu(1-\mu)-1}s^{\mu-1}\mu\theta M_\mu(\theta)T(s^\mu\theta)\,d\theta\,ds \\
&\quad + \int_{t-\epsilon}^t \int_\delta^\infty (t-s)^{\nu(1-\mu)-1}s^{\mu-1}\mu\theta M_\mu(\theta)T(s^\mu\theta)\,d\theta\,ds\bigg\|_H \\
&\le \sup_{t\in[0,b]} t^{1-\gamma}\frac{x_0 M\mu}{\Gamma(\nu(1-\mu))}\bigg\{\bigg\|\int_0^t (t-s)^{\nu(1-\mu)-1}s^{\mu-1}\,ds\int_0^\delta \theta M_\mu(\theta)\,d\theta\bigg\|_H \\
&\quad + \bigg\|\int_{t-\epsilon}^t (t-s)^{\nu(1-\mu)-1}s^{\mu-1}\,ds\bigg(\frac{1}{\Gamma(1+\mu)} - \int_0^\delta \theta M_\mu(\theta)\,d\theta\bigg)\bigg\|_H\bigg\} \\
&\le \frac{x_0 M\Gamma(\mu+1)}{\Gamma(\nu(1-\mu)+\mu)}\int_0^\delta \theta M_\mu(\theta)\,d\theta \\
&\quad + \frac{x_0 M\mu b^{1-\gamma}}{\Gamma(\nu(1-\mu)+\mu)}\bigg[\frac{\epsilon^{\nu(1-\mu)-\frac{1}{2}}}{\sqrt{2\nu(1-\mu)-1}} - \frac{\epsilon^{\mu-\frac{1}{2}}}{\sqrt{2\mu-1}}\bigg]\bigg[\frac{1}{\Gamma(1+\mu)} - \int_0^\delta \theta M_\mu(\theta)\,d\theta\bigg]
\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}
&\|J_2(t) - \tilde{J}_2(t)\|_H \\
&= t^{1-\gamma}\bigg\|\int_0^t\int_0^\infty (t-s)^{\mu-1}\mu\theta M_\mu(\theta)T\big((t-s)^\mu\theta\big)\big[f\big(s,x(s)\big)+Bu(s)\big]\,d\theta\,ds \\
&\quad - \int_0^t\int_\delta^\infty (t-s)^{\mu-1}\mu\theta M_\mu(\theta)T\big((t-s)^\mu\theta\big)\big[f\big(s,x(s)\big)+Bu(s)\big]\,d\theta\,ds \\
&\quad + \int_{t-\epsilon}^t\int_\delta^\infty (t-s)^{\mu-1}\mu\theta M_\mu(\theta)T\big((t-s)^\mu\theta\big)\big[f\big(s,x(s)\big)+Bu(s)\big]\,d\theta\,ds\bigg\|_H \\
&\le \sup_{t\in[0,b]} t^{1-\gamma}M\bigg\{\bigg\|\int_0^t (t-s)^{\mu-1}\mu\big[f\big(s,x(s)\big)+Bu(s)\big]\,ds\int_0^\delta \theta M_\mu(\theta)\,d\theta\bigg\|_H \\
&\quad + \bigg\|\int_{t-\epsilon}^t (t-s)^{\mu-1}\mu\big[f\big(s,x(s)\big)+Bu(s)\big]\,ds\bigg(\frac{1}{\Gamma(1+\mu)} - \int_0^\delta \theta M_\mu(\theta)\,d\theta\bigg)\bigg\|_H\bigg\} \\
&\le \frac{\mu M b^{2-2\gamma}}{\sqrt{2\mu-1}}\big(\|\Phi_r\|_{L^2_{R^+}} + \|B\|\|u\|_{L^2_U}\big)\bigg[b^{\mu-\frac{1}{2}}\int_0^\delta \theta M_\mu(\theta)\,d\theta + \frac{\epsilon^{\mu-\frac{1}{2}}}{\Gamma(1+\mu)}\bigg].
\end{aligned} \tag{3.6}$$

Since $0 \le \int_0^\delta \theta M_\mu(\theta)\,d\theta \le \int_0^\infty \theta M_\mu(\theta)\,d\theta = \frac{1}{\Gamma(1+\mu)}$, (3.5) and (3.6) tend to zero when $\epsilon \to 0$ and $\delta \to 0$. Therefore the set $\{\Pi(t), t > 0\}$ is relatively compact in H.

Schauder's fixed point theorem guarantees that P has a fixed point $z^* \in B_r(J)$. Let $x^* = \cdot^{\gamma-1}z^* \in B_r^{(1-\gamma)}(J')$ and then $\mathcal{F}(x)$ has a fixed point $x^* \in B_r^{(1-\gamma)}(J')$ (i.e. a mild solution of system (1.1)). The proof of Theorem 3.6 is complete. □

Theorem 3.7 *Assume that condition $H(f)$ holds. Then there exists a fixed point of $\mathcal{F}_\epsilon$ in Y.*

Proof The proof is similar to Theorem 3.6, so we omit it here. □

In fact, for any $\epsilon > 0$, there exist $x_\epsilon \in Y$ and a map $\boldsymbol{\Psi}(x) : Y \to H$ which related to a unique minimum $\boldsymbol{\Psi}_\epsilon$ of the functional $\mathcal{J}_\epsilon$ such that

$$x_\epsilon(t) = \mathcal{L}_{\nu,\mu}(t)x_0 + \int_0^t \mathcal{T}_\mu(t-s)\big[f\big(s, x_\epsilon(s)\big) + Bu_\epsilon(s,x)\big]\,ds,$$

where

$$u_\epsilon(s,x) = B^*\mathcal{T}_\mu^*(b-s)R_\epsilon^b\boldsymbol{\Psi}_\epsilon(x).$$

4 Finite approximate controllability for the semilinear case

In this section, we study the finite approximate controllability of system (1.1).

Theorem 4.1 *Assume that assumptions $H(f)$ and $H(B)$ hold. Then system (1.1) is finite approximately controllable on J'.*

Proof From (3.1) we know that the functional $\mathcal{J}_\epsilon(\Psi; x_\epsilon)$ is strictly convex, so we assume $\widetilde{\Psi}_\epsilon$ be the unique critical point which minimizes $\mathcal{J}_\epsilon(\Psi; x_\epsilon)$, that is,

$$\mathcal{J}_\epsilon(\widetilde{\Psi}_\epsilon; x_\epsilon) = \min_{\Psi\in H} \mathcal{J}_\epsilon(\Psi; x_\epsilon).$$

Because $\mathcal{J}_\epsilon(\Psi; x_\epsilon)$ is Gateaux differentiable at $\widetilde{\Psi}_\epsilon$, then, for any $\Psi_0 \in H$ and $\theta > 0$, we get

$$\begin{aligned}
&\mathcal{J}_\epsilon(\widetilde{\Psi}_\epsilon + \theta\Psi_0; x_\epsilon) - \mathcal{J}_\epsilon(\widetilde{\Psi}_\epsilon; x_\epsilon) \\
&\quad= \epsilon\big\|(I-\Pi_{\mathcal{E}})R_\epsilon^b(\widetilde{\Psi}_\epsilon + \theta\Psi_0)\big\|_H + \frac{1}{2}\int_0^b \big\|B^*\mathcal{T}_\mu^*(b-t)R_\epsilon^b(\widetilde{\Psi}_\epsilon + \theta\Psi_0)\big\|_H^2\,dt \\
&\qquad- \big\langle \mathcal{H}(x_\epsilon), R_\epsilon^b(\widetilde{\Psi}_\epsilon + \theta\Psi_0)\big\rangle \\
&\qquad- \epsilon\big\|(I-\Pi_{\mathcal{E}})R_\epsilon^b\widetilde{\Psi}_\epsilon\big\|_H - \frac{1}{2}\int_0^b \big\|B^*\mathcal{T}_\mu^*(b-t)R_\epsilon^b\widetilde{\Psi}_\epsilon\big\|_H^2\,dt + \big\langle \mathcal{H}(x_\epsilon), R_\epsilon^b\widetilde{\Psi}_\epsilon\big\rangle \\
&\quad= \epsilon\theta\big\|(I-\Pi_{\mathcal{E}})R_\epsilon^b\Psi_0\big\|_H + \theta\int_0^b \big\langle B^*\mathcal{T}_\mu^*(b-t)R_\epsilon^b\widetilde{\Psi}_\epsilon, B^*\mathcal{T}_\mu^*(b-t)R_\epsilon^b\Psi_0\big\rangle\,dt \\
&\qquad+ \frac{\theta^2}{2}\int_0^b \big\|B^*\mathcal{T}_\mu^*(b-t)R_\epsilon^b\Psi_0\big\|_H^2\,dt - \theta\big\langle \mathcal{H}(x_\epsilon), \Psi_0\big\rangle,
\end{aligned}$$

such that

$$
\begin{aligned}
0 &= \lim_{\theta\to 0^+} \frac{\mathcal{J}_\epsilon(\widetilde{\Psi_\epsilon}+\theta\Psi_0;x_\epsilon)-\mathcal{J}_\epsilon(\widetilde{\Psi_\epsilon};x_\epsilon)}{\theta} \\
&= \lim_{\theta\to 0^+}\biggl(\epsilon\bigl\|(I-\Pi_{\mathcal{E}})R_\epsilon^b\Psi_0\bigr\|_H + \int_0^b \bigl\langle B^*\mathcal{T}_\mu^*(b-t)R_\epsilon^b\widetilde{\Psi_\epsilon}, B^*\mathcal{T}_\mu^*(b-t)R_\epsilon^b\Psi_0\bigr\rangle\,dt \\
&\quad + \frac{\theta}{2}\int_0^b \bigl\|B^*\mathcal{T}_\mu^*(b-t)R_\epsilon^b\Psi_0\bigr\|_H^2\,dt - \bigl\langle\mathcal{H}(x_\epsilon), R_\epsilon^b\Psi_0\bigr\rangle\biggr) \\
&= \epsilon\bigl\|(I-\Pi_{\mathcal{E}})R_\epsilon^b\Psi_0\bigr\|_H + \int_0^b \bigl\langle B^*\mathcal{T}_\mu^*(b-t)R_\epsilon^b\widetilde{\Psi_\epsilon}, B^*\mathcal{T}_\mu^*(b-t)R_\epsilon^b\Psi_0\bigr\rangle\,dt \\
&\quad - \bigl\langle\mathcal{H}(x_\epsilon), R_\epsilon^b\Psi_0\bigr\rangle\,dt,
\end{aligned}
$$

as is well known

$$
\begin{aligned}
&\int_0^b \bigl\langle B^*\mathcal{T}_\mu^*(b-t)R_\epsilon^b\widetilde{\Psi_\epsilon}, B^*\mathcal{T}_\mu^*(b-t)R_\epsilon^b\Psi_0\bigr\rangle\,dt \\
&\quad = \int_0^b \bigl\langle \mathcal{T}_\mu(b-t)BB^*\mathcal{T}_\mu^*(b-t)R_\epsilon^b\widetilde{\Psi_\epsilon}, R_\epsilon^b\Psi_0\bigr\rangle\,dt \\
&\quad = \int_0^b \bigl\langle \mathcal{T}_\mu(b-t)Bu_\epsilon(s,x), R_\epsilon^b\Psi_0\bigr\rangle\,dt,
\end{aligned}
$$

thus

$$
\begin{aligned}
&\bigl\langle\mathcal{H}(x_\epsilon), R_\epsilon^b\Psi_0\bigr\rangle \\
&\quad = \epsilon\bigl\|(I-\Pi_{\mathcal{E}})R_\epsilon^b\Psi_0\bigr\|_H + \int_0^b \bigl\langle \mathcal{T}_\mu(b-t)BB^*\mathcal{T}_\mu^*(b-t)R_\epsilon^b\widetilde{\Psi_\epsilon}, R_\epsilon^b\Psi_0\bigr\rangle\,dt \\
&\quad = \epsilon\bigl\|(I-\Pi_{\mathcal{E}})R_\epsilon^b\Psi_0\bigr\|_H + \int_0^b \bigl\langle \mathcal{T}_\mu(b-t)Bu_\epsilon(s,x), R_\epsilon^b\Psi_0\bigr\rangle\,dt.
\end{aligned}
$$

From the definition of $\mathcal{H}(x_\epsilon)$, $x_\epsilon(b)$, one can get

$$
\mathcal{H}(x_\epsilon) = x_b - x_\epsilon(b) + \int_0^t \mathcal{T}_\mu(t-s)Bu_\epsilon(s,x)\,ds,
$$

then

$$
\bigl|\bigl\langle x_b - x_\epsilon(b), \Psi_0\bigr\rangle\bigr| = \epsilon\bigl\|(I-\Pi_{\mathcal{E}})\Psi_0\bigr\|_H \le \epsilon\|\Psi_0\|_H,
$$

which is equivalent to

$$
\bigl\|x_b - x_\epsilon(b)\bigr\|_H \le \epsilon.
$$

On the other hand, if $\theta < 0$, we can get the same argument.

Thus given $\Psi_0 \in H$, we conclude that system (1.1) is approximately controllable on J', and if $\Psi_0 \in \mathcal{E}$, system (1.1) is finite approximately controllable on J', that is, $\Pi_{\mathcal{E}} x_\varepsilon(b) = \Pi_{\mathcal{E}} x_b$. □

5 An example

As an application of our result, consider the Hilfer fractional partial equation:

$$\begin{cases} D_{0^+}^{\nu,\mu}x(t,y) = x_{yy}(t,y) + Bu(t,y) + \int_0^t e^{-s}\frac{|x(s,y)|}{1+|x(s,y)|}\,ds, & 0 < t \le 1, 0 \le y \le \pi, \\ x(t,0) = x(t,\pi) = 0, & 0 < t \le 1, \\ I_{0^+}^{(1-\nu)(1-\mu)}x(0,y) = x_0(y), & 0 \le y \le \pi, \end{cases} \tag{5.1}$$

where $\nu = 1/2$, $\mu = 3/4$, and $x(t,y)$ represents the temperature function at the point $y \in [0,\pi]$ and time $t \in (0,1]$. Now, set $H = L^2[0,\pi]$ and $e_n(y) = \sqrt{2/\pi}\sin(ny)$, $n = 1,2,\ldots$. Then $\{e_n(y)\}$ is an orthonormal basis on H. Define $A : D(A) \subset H \to H$ by $Ax = x_{yy}$ with domain

$$\bigl\{x \in H : x, x' \text{ are absolutely continuous}, x'' \in H, x(0) = x(\pi) = 0\bigr\}.$$

Then

$$Ax = \sum_{n=1}^{\infty}\bigl(-n^2\bigr)\langle x, e_n\rangle e_n, \quad x \in D(A),$$

one can see that A generates a compact semigroup $T(t)$ $(t > 0)$ on H and

$$T(t)x = \sum_{n=1}^{\infty} e^{-n^2 t}\langle x, e_n\rangle e_n, \quad x \in H.$$

Hence $T(t)$ is compact and $\|T(t)\| \le 1$.

The infinite-dimensional Hilbert space U is

$$U := \left\{u : u = \sum_{n=2}^{\infty} u_n e_n, \sum_{n=2}^{\infty} u_n^2 < \infty\right\},$$

with the norm $\|u\|_U = (\sum_{n=2}^{\infty} u_n^2)^{1/2}$. We define a mapping $B \in \mathcal{L}(U,H)$ by

$$Bu = 4u_2 e_1 + 3u_2 e_2 + \sum_{n=3}^{\infty} u_n e_n \quad \text{for } u = \sum_{n=2}^{\infty} u_n e_n \in U,$$

and for $v = \sum_{n=1}^{\infty} v_n e_n \in H$, the inner product $\langle Bu, v\rangle = \langle u, B^* v\rangle$, and thus

$$B^* v = (4v_1 + 3v_2)e_2 + \sum_{n=3}^{\infty} v_n e_n$$

and

$$B^* T^*(t)x = \bigl(4x_1 e^{-t} + 3x_2 e^{-4t}\bigr)e_2 + \sum_{n=3}^{\infty} e^{-n^2 t} x_n e_n.$$

Assume $\|B^* T^*(t)x\| = 0$ for some $t \in J'$, and it follows that

$$\bigl\|4x_1 e^{-t} + 3x_2 e^{-4t}\bigr\|^2 + \sum_{n=3}^{\infty}\bigl\|e^{-n^2 t} x_n\bigr\|^2 = 0,$$

which implies that $x = 0$, and thus the linear part of system (5.1) is approximately controllable on J' (see Theorem 4.1.7 of [23]). Now

$$f\bigl(t,x(t,y)\bigr) = \int_0^t e^{-s}\frac{|x(s,y)|}{1+|x(s,y)|}\,ds \le \int_0^t e^{-s}\,ds = 1-e^{-t},$$

so the conditions of $H(f)$ hold. Thus system (5.1) is finite approximately controllable on J'.

Acknowledgements
Not applicable.

Funding
The work is supported by NSF of China (No. 11661084), Guizhou Province Science and technology fund [2016]1160, [2017]1201, Guizhou Province Innovative talents fund [2016]046, Zunyi Science and technology talents fund [2016]15, Sci-Tec Innovative Talents of Guizhou Province (No. [2015]502), Qian Ke He Ping Tai Ren Cai [2018]5784-08.

Availability of data and materials
All data generated or analyzed during this study are included in this published article.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
XL researched and analyzed the model, and was a major contributor in writing the manuscript. GX analyzed the data. YL reviewed the format of this manuscript. All authors read and approved the final manuscript.

Publisher's Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 2 May 2019 Accepted: 20 December 2019 Published online: 14 January 2020

References
1. Zhou, Y., Wang, J., Zhang, L.: Basic Theory of Fractional Differential Equations, 2nd edn. World Scientific, Singapore (2016)
2. Diethelm, K.: The Analysis of Fractional Differential Equations. Springer, Berlin (2010)
3. Hernández, E., O'Regan, D., Balachandran, K.: On recent developments in the theory of abstract differential equations with fractional derivatives. Nonlinear Anal. TMA **73**, 3462–3471 (2010)
4. Wang, J., Fečkan, M., Zhou, Y.: On the new concept of solutions and existence results for impulsive fractional evolution equations. Dyn. Partial Differ. Equ. **8**, 345–361 (2011)
5. Mahmudov, N.I.: Approximate controllability of semilinear deterministic and stochastic evolution equations in abstract spaces. SIAM J. Control Optim. **42**, 1604–1622 (2003)
6. Mahmudov, N.I.: Approximate controllability of evolution systems with nonlocal conditions. Nonlinear Anal. **68**, 536–546 (2008)
7. Mahmudov, N.I., Zorlu, S.: On the approximate controllability of fractional evolution equations with compact analytic semigroup. J. Comput. Appl. Math. **259**, 194–204 (2014)
8. Hilfer, R.: Applications of Fractional Calculus in Physics. World Scientific, Singapore (2000)
9. Yang, M., Wang, Q.: Approximate controllability of Hilfer fractional differential inclusions with nonlocal conditions. Math. Methods Appl. Sci. **40**, 1126–1138 (2007)
10. Furati, K.M., Kassim, M.D., Tatar, N.: Existence and uniqueness for a problem involving Hilfer fractional derivative. Comput. Math. Appl. **64**, 1616–1626 (2012)
11. Furati, K.M., Kassim, M.D., Tatar, N.: Non-existence of global solutions for a differential equation involving Hilfer fractional derivative. Electron. J. Differ. Equ. **235**, 1 (2013)
12. Gu, H., Trujillo, J.J.: Existence of mild solution for evolution equation with Hilfer fractional derivative. Appl. Math. Comput. **257**, 344–354 (2015)
13. Mahmudov, N.I., McKibben, M.A.: On the approximate controllability of fractional evolution equations with generalized Riemann–Liouville fractional derivative. J. Funct. Spaces **2015**, Article ID 263823 (2015)
14. Wang, J., Liu, X., O'Regan, D.: On the approximate controllability for Hilfer fractional evolution hemivariational inequalities. Numer. Funct. Anal. Optim. **40**, 743–762 (2019)
15. Li, X., Yong, J.: Optimal Control Theory for Infinite Dimensional Systems. Birkhäuser, Boston (1994)
16. Zuazua, E.: Finite dimensional null controllability for the semilinear heat equation. J. Math. Pures Appl. **76**, 237–264 (1997)
17. de Menezes, S.B.: Finite-dimensional approximate controllability for a nonlocal parabolic problem. Appl. Math. Sci. **2**, 1307–1326 (2008)
18. Mahmudov, N.I.: Finite-approximate controllability of evolution equations. Appl. Comput. Math. **16**, 159–167 (2017)
19. Mahmudov, N.I.: Variational approach to finite-approximate controllability of Sobolev-type fractional systems. J. Optim. Theory Appl. (in press)
20. Miller, K.S., Ross, B.: An Introduction to the Fractional Calculus and Differential Equations. Wiley, New York (1993)

21. Podlubny, I.: Fractional Differential Equations. Academic Press, San Diego (1999)
22. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations (2006)
23. Curtain, R.F., Zwart, H.: An Introduction to Infinite Dimensional Linear Systems Theory. Springer, New York (1995)