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Asymptotical stability of Runge–Kutta methods for nonlinear impulsive differential equations

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Abstract

In this paper, asymptotical stability of the exact solutions of nonlinear impulsive ordinary differential equations is studied under Lipschitz conditions. Under these conditions, asymptotical stability of Runge–Kutta methods is studied by the theory of Padé approximation. And two simple examples are given to illustrate the conclusions.

Keywords: Impulsive differential equation; Runge–Kutta method; Lipschitz condition; Asymptotical stability

1 Introduction

The impulsive differential equations (IDEs) are widely applied in numerous fields of science and technology: theoretical physics, mechanics, population dynamics, pharmacokinetics, industrial robotics, chemical technology, biotechnology, economics, etc. Recently, the theory of IDEs has been an object of active research. Especially, stability of the exact solutions of IDEs has been widely studied (see [1, 2, 9, 16, 18] and the references therein). However, many IDEs cannot be solved analytically or their solving is more complicated. Hence taking numerical methods is a good choice.

In recent years, the stability of numerical methods for IDEs has attracted more and more attention (see [11, 12, 15, 17, 22, 29] etc.). Stability of Runge–Kutta methods with the constant stepsize for scalar linear IDEs has been studied by [17]. Runge–Kutta methods with variable stepsizes for multidimensional linear IDEs has been investigated in [12]. Collocation methods for linear nonautonomous IDEs has been considered in [29]. An improved linear multistep method for linear IDEs has been investigated in [13]. Stability of the exact and numerical solutions of nonlinear IDEs has been studied by the Lyapunov method in [11]. Stability of Runge–Kutta methods for a special kind of nonlinear IDEs has been investigated by the properties of the differential equations without impulsive perturbations in [15]. Stability and asymptotic stability of implicit Euler method for stiff IDEs in Banach space has been studied by [22]. There is a lot of significant work on the numerical solution of impulsive differential equations, for example [6, 7, 10, 14, 23–27]. However, in this work the authors did not investigate the stability of the numerical methods for non-stiff

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nonlinear IDEs under Lipschitz conditions. Consider the equation of the form

$$\begin{cases} x'(t) = f(t, x(t)), & t > t_0, t \neq \tau_k, k = 1, 2, \dots, \\ x(\tau_k^+) = I_k(x(\tau_k)), & k = 1, 2, \dots, \\ x(t_0^+) = x_0, \end{cases} \tag{1}$$

where $x(t^+)$ is the right limit of $x(t)$, $t_0 = \tau_0 < \tau_1 < \tau_2 < \dots$, $\lim_{k \rightarrow \infty} \tau_k = \infty$, the function $f : [t_0, +\infty) \times \mathbb{C}^d \rightarrow \mathbb{C}^d$ is continuous in t and Lipschitz continuous with respect to the second variable in the following sense: there is a positive real constant α such that

$$\|f(t, x_1) - f(t, x_2)\| \leq \alpha \|x_1 - x_2\| \tag{2}$$

for arbitrary $t \in [t_0, \infty)$, $x_1, x_2 \in \mathbb{C}^d$, where $\|\cdot\|$ is any convenient norm on \mathbb{C}^d . And also assume that each function I_k , $k = 1, 2, \dots$ is Lipschitz continuous i.e. there is a positive constant β_k such that

$$\|I_k(x) - I_k(y)\| \leq \beta_k \|x - y\|, \quad \text{for } \forall x, y \in \mathbb{C}^d. \tag{3}$$

Definition 1.1 (See [1]) A function $x : [t_0, \infty) \rightarrow \mathbb{C}^d$ is said to be a solution of (1), if

- (i) $\lim_{t \rightarrow t_0^+} x(t) = x_0$,
- (ii) for $t \in (t_0, +\infty)$, $t \neq \tau_k$, $k = 1, 2, \dots$, $x(t)$ is differentiable and $x'(t) = f(t, x(t))$,
- (iii) $x(t)$ is left continuous in $(t_0, +\infty)$ and $x(\tau_k^+) = I_k(x(\tau_k))$, $k = 1, 2, \dots$.

2 Asymptotical stability of the exact solution

In this section, we study the asymptotical stability of the exact solution of (1). In order to investigate the asymptotical stability of $x(t)$, consider Eq. (1) with another initial data:

$$\begin{cases} y'(t) = f(t, y(t)), & t > t_0, t \neq \tau_k, k \in \mathbb{Z}^+, \\ y(\tau_k^+) = I_k(y(\tau_k)), & k \in \mathbb{Z}^+, \\ y(t_0^+) = y_0, \end{cases} \tag{4}$$

where $\mathbb{Z}^+ = \{1, 2, \dots\}$.

Definition 2.1 ([1, 18]) The exact solution $x(t)$ of (1) is said to be

- 1 stable if, for an arbitrary $\epsilon > 0$, there exists a positive number $\delta = \delta(\epsilon)$ such that, for any other solution $y(t)$ of (4), $\|x_0 - y_0\| < \delta$ implies

$$\|x(t) - y(t)\| < \epsilon, \quad \forall t > t_0;$$

- 2 asymptotically stable, if it is stable and $\lim_{t \rightarrow \infty} \|x(t) - y(t)\| = 0$.

Theorem 2.2 Assume that there exists a positive constant γ such that $\tau_k - \tau_{k-1} \leq \gamma$, $k \in \mathbb{Z}^+$. The exact solution of (1) is asymptotically stable if there is a positive constant C such that

$$\beta_k e^{\alpha(\tau_k - \tau_{k-1})} \leq C < 1 \tag{5}$$

for arbitrary $k \in \mathbb{Z}^+$.

Proof For arbitrary $t \in (\tau_k, \tau_{k+1}]$, $k = 0, 1, 2, \dots$, we obtain

$$\begin{aligned} \|x(t) - y(t)\| &= \left\| x(\tau_k^+) - y(\tau_k^+) + \int_{\tau_k}^t (f(s, x(s)) - f(s, y(s))) \, ds \right\| \\ &\leq \|x(\tau_k^+) - y(\tau_k^+)\| + \int_{\tau_k}^t \|f(s, x(s)) - f(s, y(s))\| \, ds \\ &\leq \|x(\tau_k^+) - y(\tau_k^+)\| + \alpha \int_{\tau_k}^t \|x(s) - y(s)\| \, ds. \end{aligned}$$

By the Gronwall theorem, for arbitrary $t \in (\tau_k, \tau_{k+1}]$, $k = 0, 1, 2, \dots$, we have

$$\|x(t) - y(t)\| \leq \|x(\tau_k^+) - y(\tau_k^+)\| e^{\alpha(t-\tau_k)},$$

which implies, by Definition 2.1(iii),

$$\|x(\tau_{k+1}) - y(\tau_{k+1})\| = \lim_{t \rightarrow \tau_{k+1}^-} \|x(t) - y(t)\| \leq \|x(\tau_k^+) - y(\tau_k^+)\| e^{\alpha(\tau_{k+1}-\tau_k)},$$

which also implies

$$\begin{aligned} &\|x(\tau_{k+1}^+) - y(\tau_{k+1}^+)\| \\ &= \|I_{k+1}(x(\tau_{k+1})) - I_{k+1}(y(\tau_{k+1}))\| \\ &\leq \beta_{k+1} \|x(\tau_{k+1}) - y(\tau_{k+1})\| \\ &\leq \|x(\tau_k^+) - y(\tau_k^+)\| \beta_{k+1} e^{\alpha(\tau_{k+1}-\tau_k)}. \end{aligned}$$

Therefore, by the method of introduction and the conditions (3) and (5), for arbitrary $t \in (\tau_k, \tau_{k+1}]$, $k = 0, 1, 2, \dots$, we obtain

$$\begin{aligned} &\|x(t) - y(t)\| \\ &\leq \|x_0 - y_0\| (\beta_1 e^{\alpha(\tau_1-\tau_0)}) (\beta_2 e^{\alpha(\tau_2-\tau_1)}) (\beta_k e^{\alpha(\tau_k-\tau_{k-1})}) e^{\alpha(t-\tau_k)} \\ &\leq C^k \|x_0 - y_0\| e^{\alpha(t-\tau_k)} \\ &\leq C^k \|x_0 - y_0\| e^{\alpha(\tau_{k+1}-\tau_k)} \\ &\leq C^k \|x_0 - y_0\| e^{\alpha\gamma}, \end{aligned}$$

which implies $\|x(\tau_{k+1}) - y(\tau_{k+1})\| \leq C^k \|x_0 - y_0\| e^{\alpha\gamma}$ and $\|x(\tau_{k+1}^+) - y(\tau_{k+1}^+)\| \leq \|x_0 - y_0\| C^{k+1}$. Hence for an arbitrary $\epsilon > 0$, there exists $\delta = e^{-\alpha\gamma} \epsilon$ such that $\|x_0 - y_0\| < \delta$ implies

$$\|x(t) - y(t)\| \leq C^k \|x_0 - y_0\| e^{\alpha\gamma} \leq \|x_0 - y_0\| e^{\alpha\gamma} < \epsilon$$

for arbitrary $t \in (\tau_k, \tau_{k+1}]$, $k = 0, 1, 2, \dots$, i.e.

$$\|x(t) - y(t)\| < \epsilon, \quad \forall t > t_0.$$

So the exact solution of (1) is stable. Obviously, for arbitrary $t \in (\tau_k, \tau_{k+1}]$, $k = 0, 1, 2, \dots$,

$$\|x(t) - y(t)\| \leq C^k \|x_0 - y_0\| e^{\alpha\gamma} \rightarrow 0, \quad k \rightarrow \infty.$$

Similarly, we also obtain

$$\|x(\tau_{k+1}) - y(\tau_{k+1})\| \leq C^k \|x_0 - y_0\| e^{\alpha\gamma} \rightarrow 0, \quad k \rightarrow \infty,$$

and

$$\|x(\tau_{k+1}^+) - y(\tau_{k+1}^+)\| \leq C^{k+1} \|x_0 - y_0\| \rightarrow 0, \quad k \rightarrow \infty.$$

Consequently, the exact solution of (1) is asymptotically stable. □

From the proof of Theorem 2.2, we can obtain the following result.

Remark 2.3 If the condition (5) of Theorem 2.2 is changed into the weaker condition

$$\beta_k e^{\alpha(\tau_k - \tau_{k-1})} \leq 1, \quad \forall k \in \mathbb{Z}^+, \tag{6}$$

then the exact solution of (1) is stable.

3 Runge–Kutta methods

In this section, Runge–Kutta methods for (1) can be constructed as follows:

$$\begin{cases} X_{k,l}^i = x_{k,l} + h_k \sum_{j=1}^s a_{ij} f(t_{k,l}^j, X_{k,l}^j), & k \in \mathbb{N}, i = 1, 2, \dots, s, \\ x_{k,l+1} = x_{k,l} + h_k \sum_{i=1}^s b_i f(t_{k,l}^i, X_{k,l}^i), & l = 0, 1, 2, \dots, m - 1, \\ x_{k+1,0} = I_{k+1}(x_{k,m}), \\ x_{0,0} = x_0, \end{cases} \tag{7}$$

where $h_k = \frac{\tau_{k+1} - \tau_k}{m}$, $t_{k,l} = \tau_k + lh_k$, $t_{k,l}^i = t_{k,l} + c_i h_k$, $x_{k,l}$ is an approximation to the exact solution $x(t_{k,l})$ and $X_{k,l}^i$ is an approximation to the exact solution $x(t_{k,l}^i)$, $k \in \mathbb{N} = \{0, 1, 2, \dots\}$, $l = 0, 1, \dots, m - 1$, $i = 1, 2, \dots, s$, s is referred to as the number of stages. The weights b_i , the abscissae $c_i = \sum_{j=1}^s a_{ij}$ and the matrix $A = [a_{ij}]_{i,j=1}^s$ will be denoted by (A, b, c) . Similarly, the Runge–Kutta methods for (4) can be constructed as follows:

$$\begin{cases} Y_{k,l}^i = y_{k,l} + h_k \sum_{j=1}^s a_{ij} f(t_{k,l}^j, Y_{k,l}^j), & k \in \mathbb{N}, i = 1, 2, \dots, s, \\ y_{k,l+1} = y_{k,l} + h_k \sum_{i=1}^s b_i f(t_{k,l}^i, Y_{k,l}^i), & l = 0, 1, 2, \dots, m - 1, \\ y_{k+1,0} = I_{k+1}(y_{k,m}), \\ y_{0,0} = y_0. \end{cases} \tag{8}$$

Definition 3.1 The Runge–Kutta method (7) for impulsive differential equation (1) is said to be

- 1 stable, if $\exists M > 0$, $m \geq M$, $h_k = \frac{\tau_{k+1} - \tau_k}{m}$, $k \in \mathbb{N}$,
- (i) $I - zA$ is invertible for all $z = \alpha h_k$,

(ii) for an arbitrary $\epsilon > 0$, there exists such a positive number $\delta = \delta(\epsilon)$ that, for any other numerical solutions of (8), $\|x_0 - y_0\| < \delta$ implies

$$\| |X_k - Y_k| \| < \epsilon, \quad \forall k \in \mathbb{N},$$

where $X_k = (x_{k,0}, x_{k,1}, \dots, x_{k,m})^T$, $Y_k = (y_{k,0}, y_{k,1}, \dots, y_{k,m})^T$ and

$$\| |X_k - Y_k| \| = \max_{0 \leq l \leq m} \{ \|x_{k,l} - y_{k,l}\| \};$$

2 asymptotically stable, if it is stable and if $\exists M_1 > 0$, for any $m \geq M_1$, $h_k = \frac{\tau_{k+1} - \tau_k}{m}$, $k \in \mathbb{N}$, the following holds:

$$\lim_{k \rightarrow \infty} \| |X_k - Y_k| \| = 0.$$

Lemma 3.2 ([3, 5, 8, 21]) *The (j, k)-Padé approximation to e^z is given by*

$$R(z) = \frac{P_j(z)}{Q_k(z)}, \tag{9}$$

where

$$P_j(z) = 1 + \frac{j}{j+k} \cdot z + \frac{j(j-1)}{(j+k)(j+k-1)} \cdot \frac{z^2}{2!} + \dots + \frac{j!k!}{(j+k)!} \cdot \frac{z^j}{j!},$$

$$Q_k(z) = 1 - \frac{k}{j+k} \cdot z + \frac{k(k-1)}{(j+k)(j+k-1)} \cdot \frac{z^2}{2!} + \dots + (-1)^k \cdot \frac{k!j!}{(j+k)!} \cdot \frac{z^k}{k!},$$

with error

$$e^z - R(z) = (-1)^k \cdot \frac{j!k!}{(j+k)!(j+k+1)!} \cdot z^{j+k+1} + O(z^{j+k+2}).$$

It is the unique rational approximation to e^z of order $j + k$, such that the degrees of numerator and denominator are j and k , respectively.

Lemma 3.3 ([19, 20, 28]) *Assume that $R(z)$ is the (j, k)-Padé approximation to e^z . Then $R(z) < e^z$ for all $z > 0$ if and only if k is even, when $z > 0$.*

Theorem 3.4 *Assume that $R(z)$ is the stability function of Runge–Kutta method (7) i.e.*

$$R(z) = 1 + zb^T(I - zA)^{-1}e = \frac{P_j(z)}{Q_k(z)}.$$

Under the conditions of Theorem 2.2, Runge–Kutta method (7) with nonnegative coefficients ($a_{ij} \geq 0$ and $b_i \geq 0$, $1 \leq i \leq s$, $1 \leq j \leq s$) for (1) is asymptotically stable for $h_k = \frac{\tau_{k+1} - \tau_k}{m}$, $k \in \mathbb{N}$, $m \in \mathbb{Z}^+$ and $m \geq M$, if k is even, where $M = \inf\{m : I - zA \text{ is invertible and } (I - zA)^{-1}e \geq 0, z = \alpha h_k, k \in \mathbb{N}, m \in \mathbb{Z}^+\}$. (The last inequality should be interpreted entry-wise.)

Proof Because $a_{ij} \geq 0$ and $b_i \geq 0, 1 \leq i \leq s, 1 \leq j \leq s$, we obtain

$$\begin{aligned} & \|X_{k,l}^i - Y_{k,l}^i\| \\ &= \left\| x_{k,l} - y_{k,l} + h_k \sum_{j=1}^s a_{ij} (f(t_{k,l}^j, X_{k,l}^j) - f(t_{k,l}^j, Y_{k,l}^j)) \right\| \\ &\leq \|x_{k,l} - y_{k,l}\| + h_k \sum_{j=1}^s a_{ij} \|f(t_{k,l}^j, X_{k,l}^j) - f(t_{k,l}^j, Y_{k,l}^j)\| \\ &\leq \|x_{k,l} - y_{k,l}\| + \alpha h_k \sum_{j=1}^s a_{ij} \|X_{k,l}^j - Y_{k,l}^j\|. \end{aligned}$$

And when $m \geq M, (I - zA)^{-1}e \geq 0, z = \alpha h_k, k \in \mathbb{Z}^+, so$

$$\left[\|X_{k,l}^i - Y_{k,l}^i\| \right] \leq (I - \alpha h_k A)^{-1} e \|x_{k,l} - y_{k,l}\|,$$

where $\left[\|X_{k,l}^i - Y_{k,l}^i\| \right] = (\|X_{k,l}^1 - Y_{k,l}^1\|, \|X_{k,l}^2 - Y_{k,l}^2\|, \dots, \|X_{k,l}^s - Y_{k,l}^s\|)^T$. By Lemma 3.2 and Lemma 3.3, we can obtain

$$\begin{aligned} & \|x_{k,l+1} - y_{k,l+1}\| \\ &= \left\| x_{k,l} - y_{k,l} + h_k \sum_{j=1}^s b_j (f(t_{k,l}^j, X_{k,l}^j) - f(t_{k,l}^j, Y_{k,l}^j)) \right\| \\ &\leq \|x_{k,l} - y_{k,l}\| + h_k \sum_{j=1}^s b_j \|f(t_{k,l}^j, X_{k,l}^j) - f(t_{k,l}^j, Y_{k,l}^j)\| \\ &\leq \|x_{k,l} - y_{k,l}\| + \alpha h_k \sum_{j=1}^s b_j \|X_{k,l}^j - Y_{k,l}^j\| \\ &= \|x_{k,l} - y_{k,l}\| + \alpha h_k b^T \left[\|X_{k,l}^i - Y_{k,l}^i\| \right] \\ &\leq (1 + \alpha h_k b^T (I - \alpha h_k A)^{-1} e) \|x_{k,l} - y_{k,l}\| \\ &= \mathbf{R}(\alpha h_k) \|x_{k,l} - y_{k,l}\| \\ &\leq e^{\alpha h_k} \|x_{k,l} - y_{k,l}\|. \end{aligned}$$

Hence for arbitrary $k = 0, 1, 2, \dots$ and $l = 0, 1, \dots, m$, we have

$$\|x_{k,l} - y_{k,l}\| \leq \|x_{k,0} - y_{k,0}\| e^{\alpha l h_k}.$$

Therefore, by the method of the introduction and the condition (5), we obtain

$$\begin{aligned} & \|x_{k,l} - y_{k,l}\| \\ &\leq \|x_0 - y_0\| (\beta_1 e^{\alpha(\tau_1 - \tau_0)}) (\beta_2 e^{\alpha(\tau_2 - \tau_1)}) (\beta_k e^{\alpha(\tau_k - \tau_{k-1})}) e^{\alpha l h_k} \\ &\leq \|x_0 - y_0\| C^k e^{\alpha \gamma}, \end{aligned}$$

which implies that Runge–Kutta method for (1) is asymptotically stable for $h_k = \frac{\tau_{k+1}-\tau_k}{m}$, $k \in \mathbb{N}$, $m \in \mathbb{Z}^+$ and $m \geq M$. □

Remark 3.5

- (1) For z sufficiently close to zero, the matrix $I - zA$ is invertible and $(I - zA)^{-1}e \geq 0$. Therefore, taking stepsizes $h_k = \frac{\tau_{k+1}-\tau_k}{m}$, $k \in \mathbb{N}$, $m \in \mathbb{Z}^+$ and $m \geq M$ and $M = \inf\{m : I - zA \text{ is invertible, } (I - zA)^{-1}e \geq 0, z = \alpha h_k, k \in \mathbb{N}, m \in \mathbb{Z}^+\}$ in Theorem 3.4 is reasonable.
- (2) Under the conditions of Remark 2.3, Runge–Kutta method (7) with nonnegative coefficients ($a_{ij} \geq 0$ and $b_i \geq 0$, $1 \leq i \leq s$, $1 \leq j \leq s$) for (1) is stable for $h_k = \frac{\tau_{k+1}-\tau_k}{m}$, $k \in \mathbb{N}$, $m \in \mathbb{Z}^+$ and $m \geq M$, if \mathbf{k} is even, where $M = \inf\{m : I - zA \text{ is invertible and } (I - zA)^{-1}e \geq 0, z = \alpha h_k, k \in \mathbb{N}, m \in \mathbb{Z}^+\}$.

By Theorem 3.4 as $\mathbf{k} = 0$, we can obtain the following corollary.

Corollary 3.6 *Under the conditions of Theorem 2.2, the following p -stage p th order explicit Runge–Kutta methods with nonnegative coefficients ($a_{ij} \geq 0$ and $b_i \geq 0$, $1 \leq j < i$, $1 \leq i \leq p$) for (1) are asymptotically stable for $h_k = \frac{\tau_{k+1}-\tau_k}{m}$, $k \in \mathbb{N}$, $m \in \mathbb{Z}^+$, when $p \leq 4$.*

- (1) *Explicit Euler method;*
- (2) *2-stage second order explicit Runge–Kutta methods*

0	0	0	0	0	0	0	0	0
$\frac{1}{2}$	$\frac{1}{2}$	0	1	1	0	$\frac{3}{4}$	$\frac{3}{4}$	0
	0	1		$\frac{1}{2}$	$\frac{1}{2}$		$\frac{1}{3}$	$\frac{2}{3}$

Modified Euler method Heun’s method, order 2 Ralston’s method

- (3) *3-stage third order explicit Runge–Kutta methods*

0	0	0	0	0	0	0	0
$\frac{1}{3}$	$\frac{1}{3}$	0	0	$\frac{2}{3}$	$\frac{2}{3}$	0	0
$\frac{2}{3}$	0	$\frac{2}{3}$	0	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0
	$\frac{1}{4}$	0	$\frac{3}{4}$		$\frac{1}{4}$	0	$\frac{3}{4}$

Heun’s method, order 3 Runge–Kutta method, order 3

- (4) *The classical 4-stage fourth order explicit Runge–Kutta method*

0	0	0	0	0
$\frac{1}{2}$	$\frac{1}{2}$	0	0	0
$\frac{1}{2}$	0	$\frac{1}{2}$	0	0
1	0	0	1	0
	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{6}$

Unfortunately, we cannot obtain the p -stage explicit Runge–Kutta methods of order p for $p \geq 5$, because of the Butcher barriers (see [4, Theorem 370B, pp. 259] or [8, Theorem 5.1 pp. 173]).

In the following of this section, we will consider the θ -method for (1):

$$\begin{cases} x_{k,l+1} = x_{k,l} + h_k(1 - \theta)f(t_{k,l}, x_{k,l}) + h_k\theta f(t_{k,l+1}, x_{k,l+1}), \\ x_{k+1,0} = I_{k+1}(x_{k,m}), \\ x_{0,0} = x_0, \end{cases} \tag{10}$$

where $h_k = \frac{\tau_{k+1} - \tau_k}{m}$, $m \geq 1$, m is an integer, $k = 0, 1, 2, \dots$

Lemma 3.7 (See [19]) *For $m > \sup\{\alpha\tau_k - \tau_{k-1}\}$ and $z_k = h_k\alpha$, $h = \frac{\tau_k - \tau_{k-1}}{m}$, $m, k \in \mathbb{Z}^+$, then*

$$\left(1 + \frac{z_k}{1 - z_k\theta}\right)^m \leq e^{h_k\alpha}$$

if and only if $0 \leq \theta \leq \varphi(1)$, where $\varphi(x) = \frac{1}{x} - \frac{1}{e^x - 1}$.

Theorem 3.8 *Under the conditions of Theorem 2.2, if $0 \leq \theta \leq \varphi(1)$, there is a positive M such that θ method for (1) is asymptotically stable for $h_k = \frac{\tau_{k+1} - \tau_k}{m}$, $k \in \mathbb{N}$, $m \in \mathbb{Z}^+$ and $m \geq M$.*

Proof Obviously, we can obtain

$$\begin{aligned} & \|x_{k,l+1} - y_{k,l+1}\| \\ & \leq \|x_{k,l} - y_{k,l} + (1 - \theta)h_k(f(t_{k,l}, x_{k,l}) - f(t_{k,l}, y_{k,l}))\| \\ & \quad + \theta h_k \|f(t_{k,l+1}, x_{k,l+1}) - f(t_{k,l+1}, y_{k,l+1})\| \\ & \leq (1 + (1 - \theta)\alpha h_k) \|x_{k,l} - y_{k,l}\| + \theta\alpha h_k \|x_{k,l+1} - y_{k,l+1}\|, \end{aligned}$$

which implies

$$\|x_{k,l+1} - y_{k,l+1}\| \leq \frac{1 + (1 - \theta)\alpha h_k}{1 - \theta\alpha h_k} \cdot \|x_{k,l} - y_{k,l}\|.$$

Therefore, by Lemma 3.7 and the method of introduction, we obtain

$$\|x_{k,l+1} - y_{k,l+1}\| \leq e^{\alpha h_k} \|x_{k,l} - y_{k,l}\|.$$

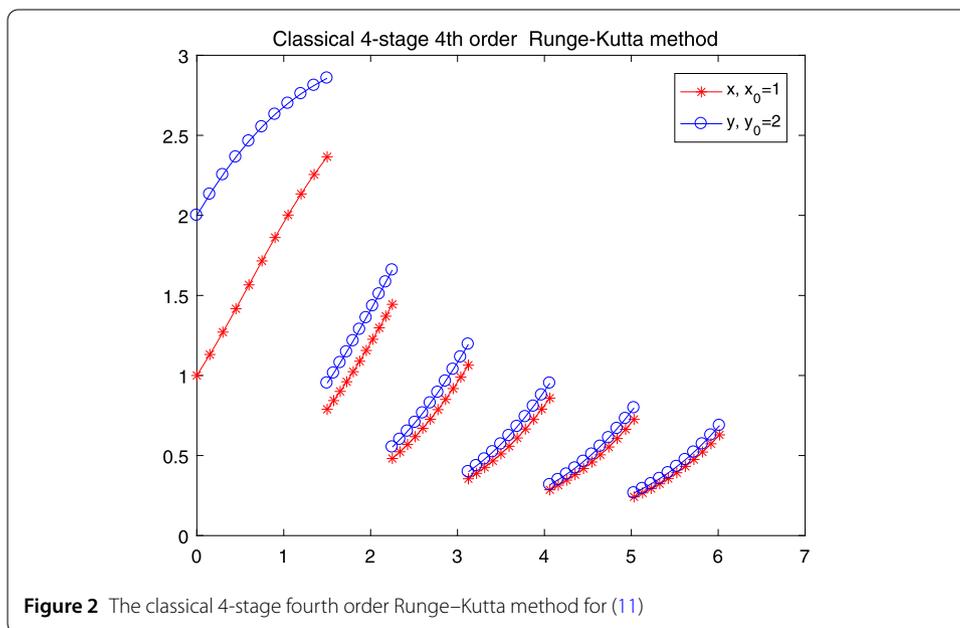
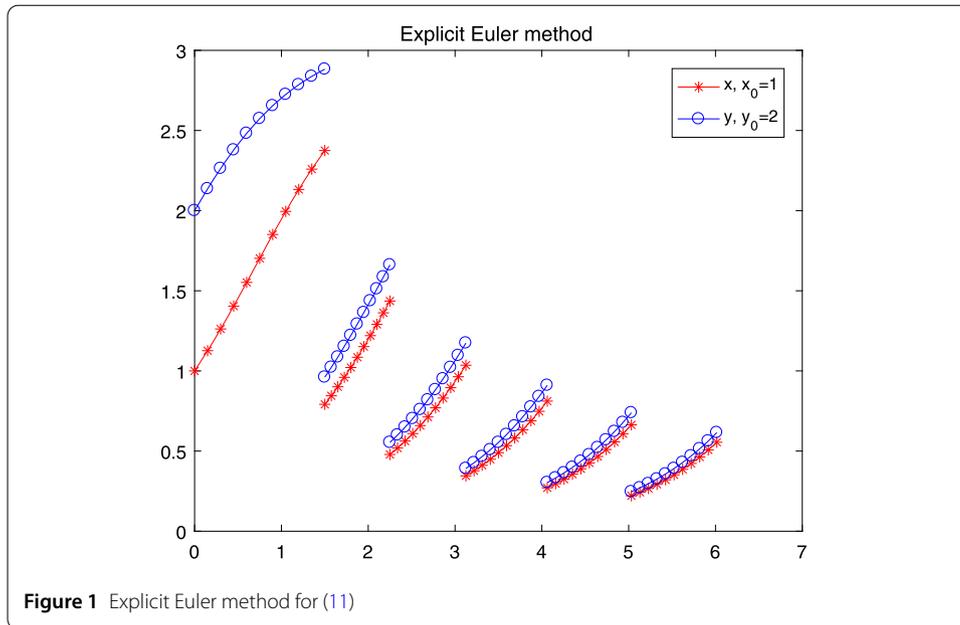
So θ -method for (1) is asymptotically stable for $h_k = \frac{\tau_{k+1} - \tau_k}{m}$, $k \in \mathbb{N}$, $m \in \mathbb{Z}^+$ and $m > \sup\{\alpha(\tau_{k+1} - \tau_k)\}$, if $0 \leq \theta \leq \varphi(1)$. □

4 Numerical experiments

In this section, two simple numerical examples in real space are given.

Example 4.1 Consider the following scalar impulsive differential equation:

$$\begin{cases} x'(t) = \sin(x(t)), & t > 0, t \neq \tau_k, \tau_k = k + 2^{-k}, k = 1, 2, \dots, \\ x(\tau_k^+) = \frac{x(\tau_k)}{3}, & k = 1, 2, \dots, \\ x(0^+) = x_0. \end{cases} \tag{11}$$



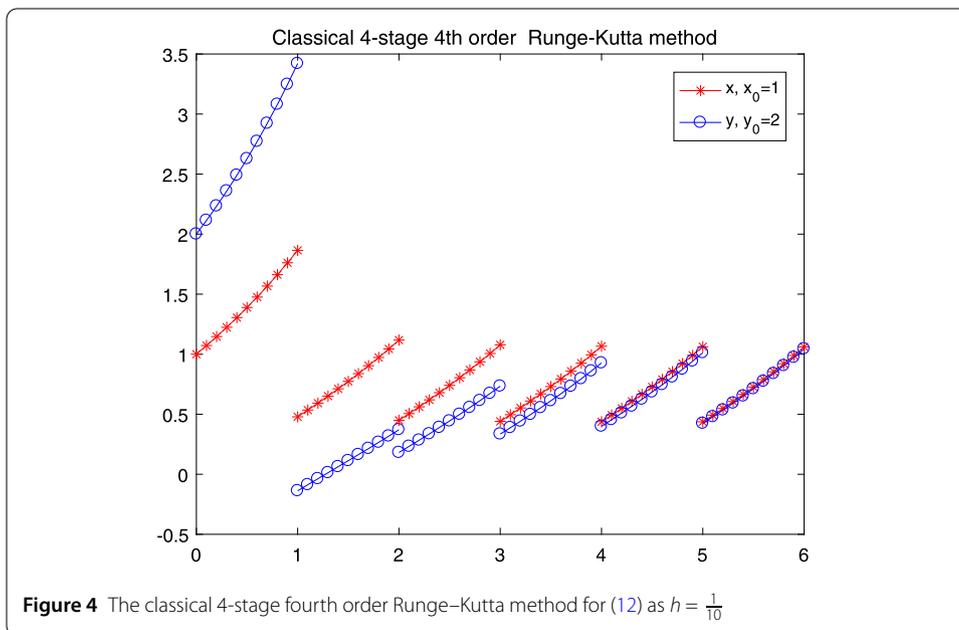
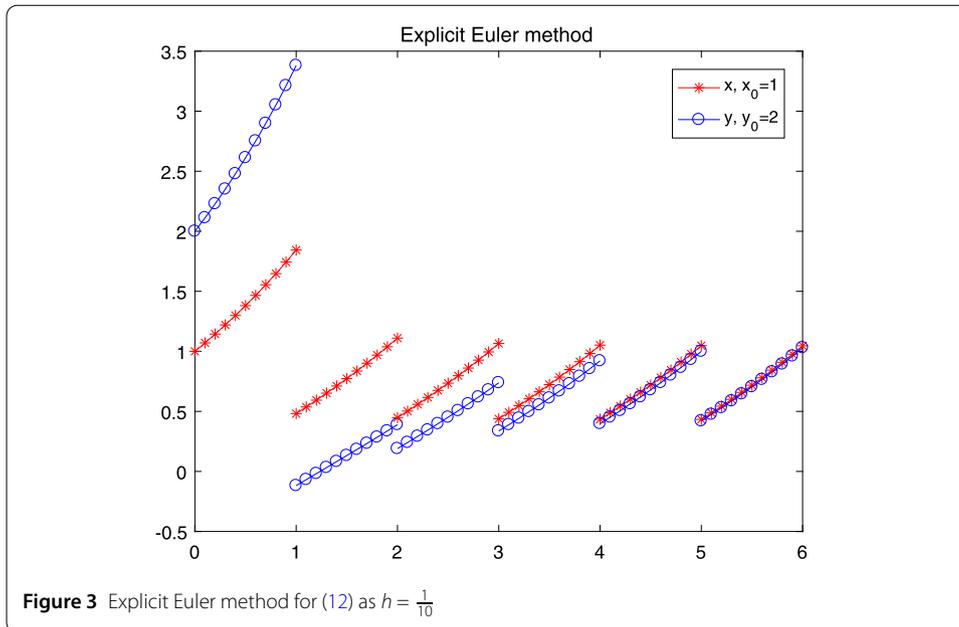
Obviously, for arbitrary $x, y \in \mathbb{R}$, we obtain

$$|\sin(x) - \sin(y)| = \left| 2 \cos\left(\frac{x+y}{2}\right) \sin\left(\frac{x-y}{2}\right) \right| \leq 2 \left| \frac{x-y}{2} \right| = |x-y|,$$

which implies the Lipschitz coefficient $\alpha = 1$. Hence, for $k \geq 2$,

$$\beta_k e^{\alpha(\tau_k - \tau_{k-1})} = \frac{e^{k+2-k-(k-1)-2^{-(k-1)}}}{3} < \frac{e}{3} < 1.$$

Therefore, by Theorem 2.2, the exact solution of (11) is asymptotically stable.



By Corollary 3.6, the explicit Euler method (see Fig. 1) and classical 4-stage fourth order explicit Runge–Kutta method (see Fig. 2) for (11) are asymptotically stable for $h_0 = \frac{3}{2m}$ and $h_k = \frac{1+2^{-(k+1)}-2^{-k}}{m}$, $k \in \mathbb{Z}^+$, $m \in \mathbb{Z}^+$ and $m \geq 2$.

Example 4.2 Consider the following scalar nonlinear impulsive differential equation:

$$\begin{cases} x'(t) = \frac{\sqrt{1+x^2(t)}}{2}, & t \geq 0, t \neq k, k = 1, 2, \dots, \\ x(t^+) = \frac{\sin(x(t))}{2}, & t = k, k = 1, 2, \dots, \\ x(0^+) = x_0. \end{cases} \tag{12}$$

Table 1 The errors Runge–Kutta methods for (11)

m	Explicit Euler method		Ralston's method		Classical fourth-order method	
	AE	RE	AE	RE	AE	RE
10	0.02781	0.01871	7.58538e-04	5.10274e-04	5.35551e-07	3.60269e-07
20	0.01375	0.00925	1.94278e-04	1.30692e-04	3.36904e-08	2.26638e-08
40	0.00683	0.00460	4.91483e-05	3.30624e-05	2.11236e-09	1.42100e-09
80	0.00341	0.00229	1.23593e-05	8.31421e-06	1.32228e-10	8.89510e-11
160	0.00170	0.00114	3.09886e-06	2.08462e-06	8.26939e-12	5.56287e-12
Ratio	2.01119	2.01119	3.95556	3.95556	15.95266	15.95266

Table 2 The errors Runge–Kutta methods for (12)

m	Explicit Euler method		Ralston's method		Classical fourth-order method	
	AE	RE	AE	RE	AE	RE
10	0.00710	0.00635	3.89245e-05	3.48363e-05	3.99870e-09	3.57871e-09
20	0.00356	0.00319	9.81334e-06	8.78265e-06	2.61135e-10	2.33708e-10
40	0.00178	0.00160	2.46328e-06	2.20456e-06	1.66902e-11	1.49372e-11
80	8.92885e-04	7.99105e-04	6.17041e-07	5.52233e-07	1.05427e-12	9.43538e-13
160	4.46619e-04	3.99711e-04	1.54411e-07	1.38194e-07	6.55032e-14	5.86234e-14
Ratio	1.99666	1.99666	3.98463	3.98463	15.72119	15.72119

Obviously, for arbitrary $x, y \in \mathbb{R}$, we have

$$\left| \frac{\sqrt{1+x^2}}{2} - \frac{\sqrt{1+y^2}}{2} \right| \leq \frac{1}{2}|x-y|,$$

which implies the Lipschitz constant $\alpha = \frac{1}{2}$. So

$$\beta_k e^{\alpha(\tau_k - \tau_{k-1})} = \left(\frac{1}{2}\right) e^{\frac{1}{2}(k-(k-1))} = \frac{\sqrt{e}}{2} < 1.$$

Therefore, by Theorem 2.2, the exact solution of (12) is asymptotically stable.

By Corollary 3.6, the explicit Euler method (see Fig. 3) and classical 4-stage fourth order explicit Runge–Kutta methods (see Fig. 4) for (12) are asymptotically stable for $h_k = \frac{1}{m}$, $k \in \mathbb{N}$, m is an arbitrary positive integer.

From Tables 1 and 2, we can see that the Runge–Kutta methods conserve their orders of convergence.

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Availability of data and materials

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Competing interests

The author declares that there is no conflict of interests regarding the publication of this paper.

Authors' contributions

The author read and approved the current manuscript.

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