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Oscillation results for nonlinear second order difference equations with mixed neutral terms

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Abstract

In this paper, we establish new oscillation criteria for nonlinear second order difference equations with mixed neutral terms. The key idea of our approach is to compare with first order equations whose oscillatory behaviors are already known. The obtained results not only improve and extend existing results reported in the literature but also provide a new platform for the investigation of a wide class of nonlinear second order difference equations. The results are supported by examples to demonstrate the validity of the theoretical findings.

Keywords: Oscillation criteria; Comparison techniques; Nonlinear second order difference equations; Mixed neutral terms

1 Introduction

The study of differential/difference equations has been the object of many researchers over the last decades. Different approaches and various techniques have been adopted to investigate the qualitative properties of their solutions. Recently and driven by their widespread applications, the investigation of differential/difference equations of fractional order has drawn significant attention. The existence–uniqueness, stability and oscillation of solutions have been the main features that have attracted consideration [1–18].

In spite of the increasing interest in the study of differential/difference equations of fractional order, the oscillation and nonoscillation of solutions for integer order second order difference equations are still considered as an open area to investigate [19–21]. Equations with neutral terms are of particular significance as they arise in many applications including systems of control, electrodynamics, mixing liquids, neutron transportation, networks and population models. In the qualitative analysis of such systems, indeed, the oscillatory behavior of solutions of equations, where the rate of the growth depends not only on the current and the past states but also on rate of change in the past, play an important role [22–24]. In the light of this motivation and justification, different results have been reported regarding the asymptotic behavior of second order difference equations with neutral terms [25–40]. For relevant results on the application of oscillation theory, the reader can consult [18, 41, 42].

Following this bias in this paper, we establish new oscillation criteria for all solutions of nonlinear second order difference equations with mixed neutral terms of the form

$$\Delta\big(a(t)\big(\Delta y(t)\big)^{\alpha}\big) + q(t)x^{\gamma}(t-m+1) + c(t)x^{\mu}\big(t+m^{*}+1\big) = 0 \tag{1}$$

and

$$\Delta\big(a(t)\big(\Delta y(t)\big)^{\alpha}\big) = q(t)x^{\gamma}(t-m+1) + c(t)x^{\mu}\big(t+m^{*}+1\big), \tag{2}$$

where $y(t) = x(t) + p_1(t)x^{\beta}(t-k) - p_2x^{\delta}(t-k)$ and under the conditions:

(i) α, β, γ, μ and δ are the ratios of positive odd integers, $\alpha \geq 1$,

(ii) $\{p_1(t)\}$, $\{p_2(t)\}$, $\{q(t)\}$ and $\{c(t)\}$ are sequences of positive real numbers,

(iii) k, m, m^{*} are positive real numbers with $h(t) = t-m+k+1$ and $h^{*}(t) = t+m^{*}+k+1$.

Let $\theta = \max\{k, m-1, m^{*}+1\}$. By a solution of Eq. (1) (respectively, (2)), we mean a real sequence $\{x(t)\}$ defined for all $t \geq t_0 - \theta$ and satisfies Eq. (1) (respectively, (2)) for all $t \geq t_0$.

A solution of Eq. (1) (respectively, (2)) is called oscillatory if its terms are neither eventually positive nor eventually negative. Otherwise, it is called nonoscillatory. If all solutions of the equation are oscillatory, then we say the equation itself is called oscillatory.

The objective of the present paper is to provide sufficient conditions for the oscillation of Eqs. (1) and (2) whenever $\beta < 1$ and $\delta > 1$ and subject to the assumption

$$A(v,u) = \sum_{s=u}^{v-1} \frac{1}{a^{\frac{1}{\alpha}}(s)} \quad \text{and} \quad A(t,t_1) = \sum_{s=t_1}^{t-1} \frac{1}{a^{\frac{1}{\alpha}}(s)} \to \infty \quad \text{as } t \to \infty \text{ for } t \geq t_1 \geq t_0. \tag{3}$$

The key idea of our approach is to conduct a comparison with first order equations whose oscillatory behaviors are already known. In view of the theorems established in the literature, the results of this paper are new and have merit in the sense that no existing results can provide criteria which ensure the oscillation of all solutions of Eq. (1) or (2). Thus, we claim that the obtained results not only improve and extend existing results reported in the literature but also provide a new platform for the investigation of a wide class of nonlinear second order difference equations.

The rest of the paper is organized as follows: Sect. 2 is devoted to the main results of the paper. We present our investigations in two folds for Eqs. (1) and (2). Meanwhile, a relevant result on the existence of positive solutions for first order difference equations is stated. The proofs rely on some mathematical inequalities which are given for the sake of completeness. In Sect. 3, we provide two examples with specific parameters to illustrate the applicability of our theorems. We end the paper by a concluding remark.

2 Main results

For the sake of convenience, we use the notations

$$g_1(t) := (1-\beta)\beta^{\frac{\beta}{1-\beta}} p^{\frac{\beta}{\beta-1}}(t) p_1^{\frac{1}{1-\beta}}(t), \qquad g_2(t) := (\delta-1)\delta^{\frac{\delta}{1-\delta}} p_2^{\frac{1}{1-\delta}}(t) p^{\frac{\delta}{\delta-1}}(t),$$

$$C(t) := \frac{c(t)}{(p_2(h^{*}(t)))^{\frac{\mu}{\delta}}}, \qquad Q(t) := \frac{q(t)}{(p_2(h(t)))^{\frac{\gamma}{\delta}}},$$

for $t \geq t_1$ for some $t_1 \geq t_0$ where $\{p(t)\}$ is a sequence of positive real numbers.

Prior to proceeding to the main results, we start by stating two fundamental lemmas.

Lemma 1 *Let $\{q(t)\}$ be a sequence of positive real numbers, m and m^* are positive real numbers and $f : \mathbb{R} \to \mathbb{R}$ is a continuous nondecreasing function such that $xf(x) > 0$ for $x \neq 0$.*

(I) *The first order delay difference inequality*

$$\Delta y(t) + q(t)f\bigl(y(t-m+1)\bigr) \le 0$$

has an eventually positive solution, so does the delay difference equation

$$\Delta y(t) + q(t)f\bigl(y(t-m+1)\bigr) = 0.$$

(II) *The first order advanced difference inequality*

$$\Delta y(t) - q(t)f\bigl(y\bigl(t+m^*+1\bigr)\bigr) \ge 0$$

has an eventually positive solution, and so does the advanced difference equation

$$\Delta y(t) - q(t)f\bigl(y\bigl(t+m^*+1\bigr)\bigr) = 0.$$

The above statement is the discrete analog of Lemma 6.2.2 in [21] and Corollary 1 in [35, 36, 43]. The proof is straightforward and hence is omitted.

Lemma 2 ([44]) *If X and Y are nonnegative, then*

$$X^\lambda + (\lambda-1)Y^\lambda - \lambda XY^{\lambda-1} \ge 0, \quad \text{for } \lambda > 1, \tag{4}$$

$$X^\lambda - (1-\lambda)Y^\lambda - \lambda XY^{\lambda-1} \le 0, \quad \text{for } 0 < \lambda < 1, \tag{5}$$

where equalities hold if and only if $X = Y$.

2.1 Oscillation of Eq. (1) when $\beta < 1$ and $\delta > 1$

In what follows, we present our first oscillation result.

Theorem 1 *Let $\beta < 1$, $\delta > 1$, conditions* (i)–(iv) *and* (3) *hold. Assume that there exist a positive sequence $\{p(t)\}$ and positive real numbers k_1 and k_2 such that $k_1 < m-k-1$ and $k_2 < m^*+k+1$ where*

$$\lim_{t\to\infty}\bigl[g_1(t)+g_2(t)\bigr] = 0. \tag{6}$$

If the first order advanced equation

$$\Delta z(t) - \left(\frac{1}{a(t)}\sum_{s=t-k_2}^{t-1} C(s)\right)^{\frac{1}{\alpha}} z^{\frac{\mu}{\alpha\delta}}\bigl(\rho(t)\bigr) = 0, \tag{7}$$

where $\rho(t) = t + m^* + k + 1 - k_2$, *is oscillatory and and we may assume that there exists a number* $\theta \in (0,1)$ *such that the two delay equations*

$$\Delta Z(t) + \theta q(t)A^{\gamma}(t-m+1,t_1)Z^{\frac{\gamma}{\alpha}}(t-m+1) = 0, \quad \textit{for some } t_1 \geq t_0, \tag{8}$$

and

$$\Delta W(t) + Q(t)A^{\frac{\gamma}{\delta}}\big(\xi(t),h(t)\big)W^{\frac{\gamma}{\alpha\delta}}\big(\xi(t)\big) = 0, \quad \textit{where } \xi(t) = t - m + k_1 + 1, \tag{9}$$

are oscillatory, then Eq. (1) *is oscillatory.*

Proof Let $\{x(t)\}$ be a nonoscillatory solution of Eq. (1) say $x(t) > 0, x(t-k) > 0, x(t-m+1) > 0$ and $x(t+m^*+1) > 0$, for $t \geq t_1$ for some $t_1 \geq t_0$. It follows from Eq. (1) that

$$\Delta\big(a(t)\big(\Delta y(t)\big)^{\alpha}\big) = -q(t)x^{\gamma}(t-m+1) \leq 0. \tag{10}$$

Hence $a(t)(\Delta y(t))^{\alpha}$ is nonincreasing and of one sign. That is, there exists $t_2 \geq t_1$ such that $\Delta y(t) > 0$ or $\Delta y(t) < 0$ for $t \geq t_2$. From this, we shall consider the following four cases:

(I) $y(t) > 0$ and $\Delta y(t) < 0$, *(II)* $y(t) > 0$ and $\Delta y(t) > 0$,

(III) $y(t) < 0$ and $\Delta y(t) > 0$, *(IV)* $y(t) < 0$ and $\Delta y(t) < 0$.

Case (I): Since $\Delta y(t) < 0$ for $t \geq t_2$,

$$a(t)\big(\Delta y(t)\big)^{\alpha} \leq -c < 0, \quad \text{or} \quad \Delta y(t) \leq \left(\frac{-c}{a(t)}\right)^{\frac{1}{\alpha}}, \quad \text{for } t \geq t_2.$$

By condition (3), we conclude that $\lim_{t\to\infty} y(t) = -\infty$. This is a contradiction to the fact that y is eventually positive.

Case (II): From the definition of y, we get

$$y(t) = x(t) + \big(p(t)x(t-k) - p_2(t)x^{\delta}(t-k)\big) + \big(p_1(t)x^{\beta}(t-k) - p(t)x(t-k)\big),$$

or

$$x(t) = y(t) - \big(p(t)x(t-k) - p_2(t)x^{\delta}(t-k)\big) - \big(p_1(t)x^{\beta}(t-k) - p(t)x(t-k)\big).$$

If we apply (4) with $\lambda = \delta > 1$, $X = p_2^{\frac{1}{\delta}}(t)x(t)$ and $Y = (\frac{1}{\delta}p(t)p_2^{\frac{-1}{\delta}}(t))^{\frac{1}{\delta-1}}$, we have

$$\big(p(t)x(t-k) - p_2(t)x^{\delta}(t-k)\big) \leq (\delta-1)\delta^{\frac{\delta}{1-\delta}}p_2^{\frac{1}{1-\delta}}(t)p^{\frac{\delta}{\delta-1}}(t) := g_2(t).$$

If we apply (5) with $\lambda = \beta < 1$, $X = p_1^{\frac{1}{\beta}}(t)x(t)$ and $Y = (\frac{1}{\beta}p(t)p_1^{\frac{-1}{\beta}}(t))^{\frac{1}{\beta-1}}$, we have

$$\big(p_1(t)x^{\beta}(t-k) - p(t)x(t-k)\big) \leq (1-\beta)\beta^{\frac{\beta}{1-\beta}}p^{\frac{\beta}{\beta-1}}(t)p_1^{\frac{1}{1-\beta}}(t) := g_1(t).$$

Thus, we see that

$$x(t) \geq \left[1 - \frac{g_1(t) + g_2(t)}{y(t)}\right] y(t). \tag{11}$$

Since y is nondecreasing, there exists a constant $C > 0$ such that $y(t) \geq C$. Thus, we have

$$x(t) \geq \left[1 - \frac{g_1(t) + g_2(t)}{C}\right] y(t).$$

Now, there exists a constant $c_1 \in (0,1)$ such that

$$x(t) \geq c_1 y(t). \tag{12}$$

Thus, we have

$$\Delta\big(a(t)\big(\Delta y(t)\big)^{\alpha}\big) + c_1^{\gamma} q(t) y^{\gamma}(t - m + 1) \leq 0. \tag{13}$$

Clearly, we see that

$$y(t) \geq \sum_{s=t_1}^{t-1} a^{-\frac{1}{\alpha}}(s)\big(a^{\frac{1}{\alpha}}(s)\Delta y(s)\big) \geq A(t,t_1)\big(a^{\frac{1}{\alpha}}(t)\Delta y(t)\big).$$

If we let $w(t) = a(t)(\Delta y(t))^{\alpha}$, then $\Delta y(t) = (\frac{w(t)}{a(t)})^{\frac{1}{\alpha}}$. The above inequality becomes

$$y(t) \geq A(t,t_1) w^{\frac{1}{\alpha}}(t). \tag{14}$$

Using (14) in (13), we get

$$\Delta w(t) + c_1^{\gamma} q(t) A^{\gamma}(t - m + 1, t_1) w^{\frac{\gamma}{\alpha}}(t - m + 1) \leq 0.$$

It follows from Lemma 1(I) that the corresponding difference equation (8) has a positive solution. This is a contradiction.

Let

$$z(t) = -y(t) = -x(t) - p_1(t)x^{\beta}(t-k) + p_2(t)x^{\delta}(t-k) \leq p_2(t)x^{\delta}(t-k),$$

or

$$x(t-k) \geq \left(\frac{z(t)}{p_2(t)}\right)^{\frac{1}{\delta}}, \quad \text{or} \quad x(t) \geq \left(\frac{z(t+k)}{p_2(t+k)}\right)^{\frac{1}{\delta}}.$$

Case (III): From the above arguments, we have $\Delta z(t) = -\Delta y(t) < 0$ for $t \geq t_1$. We have

$$\begin{aligned}\Delta\big(a(t)\big(\Delta z(t)\big)^{\alpha}\big) &= q(t)x^{\gamma}(t-m+1) + c(t)x^{\mu}\big(w(t)\big) \\ &\geq \frac{q(t)}{p_2^{\frac{\gamma}{\delta}}(h(t))} z^{\frac{\gamma}{\delta}}\big(h(t)\big) + \frac{c(t)}{p_2^{\frac{\mu}{\delta}}(h^*(t))} z^{\frac{\mu}{\delta}}\big(h^*(t)\big).\end{aligned} \tag{15}$$

In this case, we consider

$$\Delta\big(a(t)\big(\Delta z(t)\big)^{\alpha}\big) \geq Q(t)z^{\frac{\gamma}{\delta}}\big(h(t)\big). \tag{16}$$

For $t_1 \leq u \leq v$, we may write

$$z(u) - z(v) = -\sum_{s=u}^{v-1} a^{-\frac{1}{\alpha}}(s)\big(a(s)\big(\Delta z(s)\big)^{\alpha}\big)^{\frac{1}{\alpha}} \geq A(v,u)\big(-a^{\frac{1}{\alpha}}(v)\big(\Delta z(v)\big)\big).$$

If we let $u = h(t)$ and $v = \xi(t)$ in the above inequality, we obtain

$$z\big(h(t)\big) \geq A\big(\xi(t), h(t)\big)\big(-a^{\frac{1}{\alpha}}\big(\xi(t)\big)\big)\big(\Delta z\big(\xi(t)\big)\big). \tag{17}$$

Using (17) in (16), we have

$$\Delta\big(a(t)\big(\Delta z(t)\big)^{\alpha}\big) \geq Q(t)A\big(\xi(t), h(t)\big)\big(-a^{\frac{1}{\alpha}}\big(\xi(t)\big)\big)\big(\Delta z\big(\xi(t)\big)\big)^{\frac{\gamma}{\delta}}.$$

Setting $W(t) = a(t)(-\Delta z(t))^{\alpha}$, we get

$$\Delta W(t) + Q(t)A^{\frac{\gamma}{\delta}}\big(\xi(t), h(t)\big)W^{\frac{\gamma}{\alpha\delta}}\big(\xi(t)\big) \leq 0.$$

The rest of the proof is similar to that of Case (I) and hence is omitted.

Case (IV): From (15), we have the inequality

$$\Delta\big(a(t)\big(\Delta z(t)\big)^{\alpha}\big) \geq C(t)z^{\frac{\mu}{\delta}}\big(h^*(t)\big). \tag{18}$$

Summing (18) from $t - k_2$ to $t - 1$, one can easily get

$$\Delta z(t) \geq \left(\frac{1}{a(t)}\sum_{s=t-k_2}^{t-1} C(s)\right)^{\frac{1}{\alpha}} Z^{\frac{\mu}{\delta\alpha}}\big(\rho(t)\big). \tag{19}$$

It follows from Lemma 1(II) that the corresponding differential equation (7) also has a positive solution. This contradiction completes the proof. □

Corollary 1 *Let $\beta < 1$, $\delta > 1$, conditions* (i)–(iv) *and* (3) *hold. Assume that there exist a positive sequence $\{p(t)\}$ and positive real numbers k_1 and k_2 such that $k_1 < m - k - 1$ and $k_2 < m^* + k + 1$ and* (7) *hold. If*

$$\liminf_{t\to\infty}\sum_{u=t}^{\rho(t)-1}\left(\frac{1}{a(u)}\sum_{s=u-k_2}^{u-1} C(s)\right)^{\frac{1}{\alpha}} \begin{cases} = \infty, & \text{when } \mu > \alpha\delta, \\ > \frac{(\rho(t)-1)^{\rho(t)}}{\rho^{\rho(t)}(t)}, & \text{when } \mu = \alpha\delta, \end{cases} \tag{20}$$

$$\liminf_{t\to\infty}\sum_{u=\xi(t)}^{t-1} Q(s)A^{\frac{\gamma}{\delta}}\big(\xi(s), h(s)\big) \begin{cases} = \infty, & \text{when } \gamma < \alpha\delta, \\ > \frac{\xi^{\xi(t)}(t)}{(\xi(t)+1)^{\xi(t)+1}}, & \text{when } \gamma = \alpha\delta, \end{cases} \tag{21}$$

and

$$\liminf_{t\to\infty}\sum_{s=t-m+1}^{t-1} q(s)A^{\gamma}(s-m+1, t_1) = \infty, \quad \text{when } \gamma \leq \alpha, \tag{22}$$

for some $t_1 \geq t_0$, then Eq. (1) *is oscillatory.*

Corollary 2 *Let $\beta < 1$, $\delta > 1$, conditions* (i)–(iv) *and* (3) *hold. Assume that there exist a positive sequence $\{p(t)\}$ and positive real numbers k_1 and k_2 such that $k_1 < m - k - 1$ and $k_2 < m^* + k + 1$,* (7) *hold and conditions* (20) *and* (22) *are satisfied. If*

$$\limsup_{t\to\infty}\left(A(t-m+1,t_1)\sum_{s=t}^{\infty}q(s)\right)=\infty, \quad \textit{when } \gamma = 1, \tag{23}$$

and

$$\limsup_{t\to\infty}\sum_{u=t_1}^{\infty}q(u)\left(A(u-m+1,t_1)\sum_{s=u}^{\infty}q(s)\right)^{\frac{\gamma}{1-\gamma}}=\infty, \quad \textit{when } \gamma = \pm 1, \tag{24}$$

for some $t_1 \ge t_0$, then Eq. (1) *is oscillatory.*

Proof Let $\{x(t)\}$ be a nonoscillatory solution of Eq. (1), say $x(t) > 0$, $x(t-k) > 0$ and $y(t) > 0$ for $t \ge t_1$ for some $t_1 \ge t_0$. It is easy to see that $\Delta y(t) > 0$, $t \ge t_1$. Proceeding as in the proof of Theorem 1, one conclude that Cases (I), (III) and (IV) are invalid.

For Case (II), we refer to (13) and (14). Summing (13) from t to u and letting $u \to \infty$, we have

$$a(t-m+1)\big(\Delta y(t-m+1)\big)^{\alpha} \ge a(t)\big(\Delta y(t)\big)^{\alpha} \ge c^{\gamma}y^{\gamma}(t-m+1)\sum_{s=t}^{\infty}q(s).$$

Using (14) in the above inequality, we get

$$\begin{aligned} y(t-m+1) &\ge A(t-m+1,t_1)\big(a^{\frac{1}{\alpha}}(t-m+1)\Delta y(t-m+1)\big) \\ &\ge c^{\gamma}\left(A(t-m+1,t_1)\sum_{s=t}^{\infty}q(s)\right)y^{\gamma}(t-m+1) \end{aligned}$$

or

$$y^{1-\gamma}(t-m+1) \ge c^{\gamma}A(t-m+1,t_1)\sum_{s=t}^{\infty}q(s),$$

or

$$y(t-m+1) \ge \left(c^{\gamma}A(t-m+1,t_1)\sum_{s=t}^{\infty}q(s)\right)^{\frac{1}{1-\gamma}}.$$

Using this inequality in (13), we have

$$\Delta\big(a(t)\big(\Delta y(t)\big)^{\alpha}\big) + q(t)c^{\gamma}\left(c^{\gamma}A(t-m+1,t_1)\sum_{s=t}^{\infty}q(s)\right)^{\frac{\gamma}{1-\gamma}} \le 0.$$

The remaining part of the proof follows by adopting similar arguments as in the proof of Theorem 1. □

2.2 Oscillation of Eq. (2) when $\beta < 1$ and $\delta > 1$

Unlike the previous subsection, the main theorem herein provides a criterion for the oscillation as well as the oscillatory behavior of Eq. (2).

Theorem 2 *Let $\beta < 1$, $\delta > 1$, conditions* (i)–(iv) *and* (3) *hold. Assume that there exist a positive sequence $\{p(t)\}$, a constant $\theta \in (0,1)$ and positive real numbers k_1 and k_2 such that $k_1 < m - k - 1$ and $k_2 < m^* + k + 1$ and* (7) *holds. Further, assume the following condition*:

$$\sum_{s=t}^{\infty} q(s) = \infty, \quad \text{or} \quad \sum_{u=t}^{\infty} \left(\frac{\sum_{s=u}^{\infty} q(s)}{a(u)} \right)^{\frac{1}{\alpha}} = \infty. \tag{25}$$

If the first order advanced equation

$$\Delta z(t) - \theta \left(\frac{1}{a(t)} \sum_{s=t-k_2}^{t-1} C(s) \right)^{\frac{1}{\alpha}} z^{\frac{\mu}{\alpha\delta}} \bigl(\rho(t)\bigr) = 0, \quad \rho(t) = t + m^* + 1 - k_2 > t, \tag{26}$$

and the equation

$$\Delta Z(t) + Q(t) A^{\gamma} \bigl(h(t), t_1\bigr) Z^{\frac{\gamma}{\alpha}} \bigl(h(t)\bigr) = 0, \quad \text{for some } t_1 \geq t_0, \tag{27}$$

are oscillatory, then either Eq. (2) *is oscillatory or all solutions converge to zero.*

Proof Let $\{x(t)\}$ be a nonoscillatory solution of Eq. (2), say $x(t) > 0, x(t-k) > 0, x(t-m+1) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$. It follows from Eq. (2) that

$$\Delta\bigl(a(t)\bigl(\Delta y(t)\bigr)^{\alpha}\bigr) = q(t) x^{\gamma}(t - m + 1) + p(t) x^{\mu}\bigl(t + m^* + 1\bigr) \geq 0. \tag{28}$$

Hence $a(t)(\Delta y(t))^{\alpha}$ is of one sign. That is, there exists a $t_2 \geq t_1$ such that $\Delta y(t) > 0$ or $\Delta y(t) < 0$ for $t \geq t_2$. We shall study the following four cases:

$$\textit{(I)} \quad y(t) > 0 \quad \text{and} \quad \Delta y(t) < 0, \qquad \textit{(II)} \quad y(t) > 0 \quad \text{and} \quad \Delta y(t) > 0,$$
$$\textit{(III)} \quad y(t) < 0 \quad \text{and} \quad \Delta y(t) > 0, \qquad \textit{(IV)} \quad y(t) < 0 \quad \text{and} \quad \Delta y(t) < 0.$$

Case (I): We claim that $\lim_{t\to\infty} x(t) = 0$. To prove this, we assume that there exists a constant $b > 0$ such that $x(t - m + 1) > b$. Using this in (28), we get

$$\Delta\bigl(a(t)\bigl(\Delta y(t)\bigr)^{\alpha}\bigr) \geq q(t) b^{\gamma}.$$

Summing up the above inequality and using the first part of (25), we reach a contradiction. On the other hand, if we sum the above inequality from t to u and let $u \to \infty$, we get

$$-\Delta y(t) \geq \left(b^{\gamma} \frac{1}{a(t)} \sum_{s=t}^{\infty} q(s) \right)^{\frac{1}{\alpha}}.$$

Summing up again and using the second part of (25), we arrive at the desired contradiction.

Case (II): We proceed exactly as in Case (II) in the proof of Theorem 1 to obtain (12) and from Eq. (28) one can easily see that

$$\Delta\big(a(t)\big(\Delta y(t)\big)^{\alpha}\big) \geq c_1^{\mu} c(t) y^{\mu}\big(t+m^*+1\big).$$

The rest of the proof is similar to that of Case (IV) of Theorem 1 and hence is omitted.

Let

$$z(t) = -y(t) = -x(t) - p_1(t)x^{\beta}(t-k) + p_2(t)x^{\delta}(t-k) \leq p_2(t)x^{\delta}(t-k).$$

Therefore, we have

$$x(t-k) \geq \left(\frac{z(t)}{p_2(t)}\right)^{\frac{1}{\delta}}, \quad \text{or} \quad x(t) \geq \left(\frac{z(t+k)}{p_2(t+k)}\right)^{\frac{1}{\delta}}.$$

Thus, we get

$$-\Delta\big(a(t)\big(\Delta z(t)\big)^{\alpha}\big) \geq q(t)x^{\gamma}(t-m+1) \geq \frac{q(t)}{(p_2(h(t)))^{\frac{\gamma}{\delta}}} z^{\frac{\gamma}{\delta}}\big(h(t)\big),$$

or

$$\Delta\big(a(t)\big(\Delta z(t)\big)^{\alpha}\big) + Q(t)z^{\frac{\gamma}{\delta}}\big(h(t)\big) \leq 0. \tag{29}$$

Case (III): Clearly, we see that $\Delta z(t) = -\Delta y(t) < 0$ for $t \geq t_1$. However, this is impossible due to condition (3).

Case (IV): It follows that

$$z(t) \geq \sum_{s=t_1}^{t-1} a^{-\frac{1}{\alpha}}(s)\big(a^{\frac{1}{\alpha}}(s)\Delta(s)\big) \geq A(t,t_1)\big(a^{\frac{1}{\alpha}}(t)\Delta z(t)\big).$$

If we let $w(t) = a(t)(\Delta z(t))^{\alpha}$, then we obtain $\Delta z(t) = (\frac{w(t)}{a(t)})^{\frac{1}{\alpha}}$ and thus the above inequality becomes

$$z(t) \geq A(t,t_1)w^{\frac{1}{\alpha}}(t). \tag{30}$$

Using (30) in (29), we have

$$\Delta w + Q(t)A^{\gamma}\big(h(t),t_1\big)w^{\frac{\gamma}{\alpha}}\big(h(t)\big) \leq 0.$$

It follows from Lemma 1(I) that the corresponding difference equation (27) also has a positive solution, which is a contradiction. This completes the proof. □

Corollary 3 *Let $\beta < 1$, $\delta > 1$, conditions* (i)–(iv), (3) *and* (25) *hold. Assume that there exist a positive sequence $\{p(t)\}$, a constant $\theta \in (0,1)$ and positive real numbers k_1 and k_2 such that $k_1 < m-k-1$ and $k_2 < m^*+k+1$ and* (7) *hold. If*

$$\liminf_{t\to\infty} \sum_{u=t}^{\rho(t)-1} \left(\frac{1}{a(u)} \sum_{s=u-k_2}^{u-1} C(s)\right)^{\frac{1}{\alpha}} = \infty, \quad \textit{when } \mu \geq \alpha\delta, \tag{31}$$

and

$$\liminf_{t\to\infty} \sum_{s=h(t)}^{t-1} Q(s)A^{\gamma}\bigl(h(s),t_1\bigr) = \infty, \quad \textit{when } \gamma \le \alpha, \textit{ for some } t_1 \ge t_0, \tag{32}$$

then either Eq. (2) *is oscillatory or all solutions converge to zero.*

3 Examples and concluding remark

Two numerical examples are illustrated in this section to demonstrate the consistency to the theoretical findings. We end the paper by a concluding remark.

Example 1 Corresponding to (1), we consider the equation

$$\Delta^2\left(x(t) + \frac{1}{t}x^{\frac{1}{3}}(t-k) - x^3(t-k)\right) + x^3(t-m+1) + x^3\bigl(t+m^*+1\bigr) = 0, \tag{33}$$

where $a(t) = t$, $p_1(t) = \frac{1}{t} \to 0$ as $t \to \infty$ and $p_2(t) = 1$. Let $p(t) = 1$, $\alpha = 1$, $\beta = \frac{1}{3}$ and $\delta = 3 = \gamma = \mu$. Assume that there exist positive real numbers, k_1 and k_2 such that $k_1 < m-k-1$ and $k_2 < m^*+k+1$ and let $\rho(t)$ and $\xi(t)$ be as in Theorem 1.

It is easy to see that all conditions of Corollary 1 are satisfied if

$$k_1(m-k-k_1-1) > \frac{\xi^{\xi(t)}(t)}{(\xi(t)+1)^{\xi(t)+1}} \quad \text{and} \quad k_2\bigl(m^*+k-k_2+1\bigr) > \frac{(\rho(t)-1)^{\rho(t)}}{\rho^{\rho(t)}(t)}.$$

Hence, Eq. (33) is oscillatory.

Example 2 Corresponding to (2), we consider the equation

$$\Delta^2\left(x(t) + \frac{1}{t}x^{\frac{1}{3}}(t-k) - x^3(t-k)\right) = x(t-m+1) + x^3\bigl(t+m^*+1\bigr), \tag{34}$$

where $a(t) = t$, $p_1(t) = \frac{1}{t} \to 0$ as $t \to \infty$ and $p_2(t) = 1$. Let $p(t) = 1$, $\alpha = 1$, $\beta = \frac{1}{3}$ and $\delta = 3 = \gamma = \mu$. Assume that there exist positive real numbers k_1 and k_2 such that $k_1 < m-k-1$ and $k_2 < m^*+k+1$ and let $\rho(t)$ and $\xi(t)$ be as in Theorem 2.

It is easy to see that all conditions of Corollary 3 are satisfied if the advanced equation

$$\Delta z(t) - \theta\bigl(m^*-k_2\bigr)z\bigl(t+m^*+k-k_2+1\bigr) = 0$$

is oscillatory for $\theta \in (0,1)$ and hence either Eq. (34) is oscillatory or all solutions converge to zero.

Remark 1 In this paper, we study the oscillation of two classes of nonlinear second order difference equations involving nonlinear mixed neutral terms. The investigations are carried on under the canonical form of the equations, that is, when the main equations are subject to condition (3). Unlike the techniques most used in the literature, we employ a novel comparison technique that is based on comparing with the oscillatory behavior of first order delay difference equations.

The paper is presented under high degree of generality. Thus, it will be of interest to study its results for higher order nonlinear difference equations with mixed neutral terms of the form

$$\Delta\left(a(t)\left(\Delta^{n-1}y(t)\right)^{\alpha}\right) + q(t)x^{\gamma}(t-m+1) + c(t)x^{\mu}\left(t+m^{*}+1\right) = 0 \tag{35}$$

and

$$\Delta\left(a(t)\left(\Delta^{n-1}y(t)\right)^{\alpha}\right) = q(t)x^{\gamma}(t-m+1) + c(t)x^{\mu}\left(t+m^{*}+1\right) = 0. \tag{36}$$

In addition, the results of this paper can be easily obtained for a dynamic equation on time scales. We leave this for interested researchers as future work.

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Authors' contributions
The main idea of this paper was initially proposed by SRG who with JA performed all steps of the proofs of the main results. The two authors read and approved the final version of the manuscript.

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