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On weighted Atangana–Baleanu fractional operators

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Abstract
In this paper, we define the weighted Atangana–Baleanu fractional operators of Caputo sense. We obtain the solution of a related linear fractional differential equation in a closed form, and use the result to define the weighted Atangana–Baleanu fractional integral. We then express the weighted Atangana–Baleanu fractional derivative in a convergent series of Riemann–Liouville fractional integrals, and establish commutative results of the weighted Atangana–Baleanu fractional operators.

Keywords: Weighted fractional derivatives; Fractional derivatives with nonsingular kernels; Fractional differential equations

1 Introduction

The qualitative study of fractional differential equations depends on the type of the implemented fractional derivative. Mainly, there are two types of nonlocal fractional derivatives; the classical ones with singular kernels such as the Riemann–Liouville and Caputo derivatives, and the ones with nonsingular kernels, which have been introduced recently, such as the Atangana–Baleanu and Caputo–Fabrizio derivatives [14, 22]. Even though that there are no strong mathematical justifications of the new types of fractional derivatives, they got the interests of many researchers because of their appearance in different applications; see [7, 9, 10, 12, 13, 15–17, 19, 23, 24, 27–29, 35, 39]. For recent developments of fractional derivatives with nonsingular kernels we refer the reader to [4, 5, 8, 20, 21, 36].

The complexity of applications advises researchers to extend the definitions of fractional derivatives. Therefore, the weighted fractional derivatives have been introduced. The theory and applications of the weighted Caputo and Riemann–Liouville derivatives were discussed in [2, 11, 30–34]. Also, several types of integral equations are solved in an elegant way using the weighted fractional derivatives; see [2, 6]. Recently in [6], we introduced the weighted Caputo–Fabrizio fractional operators and studied related linear and nonlinear fractional differential equations. In this paper, we aim to extend the study to the Atangana–Baleanu fractional operators. We introduce the weighted Atangana–Baleanu fractional operators and study their properties. In Sect. 2, we present the definition of the weighted Atangana–Baleanu fractional derivative in Caputo sense and use the Laplace transform to solve an associated linear fractional differential equation. We then use the result to define the weighted Atangana–Baleanu fractional integral. In Sect. 3, we present the weighted Atangana–Baleanu operators in terms of the well-known Riemann–Liouville fractional integral, and investigate several properties of them. Finally, we end with some concluding remarks in Sect. 4.

2 Weighted Atangana–Baleanu operators

In the classical operators, the fractional integral is introduced and then used to define the fractional derivative. While, in the new types of fractional operators with nonsingular kernels, the fractional derivative is introduced and then implemented to define the fractional integrals. We follow the new approach and start with the definition of the left Atangana–Baleanu fractional derivative of a function $f(t)$ with respect to the weight function $w(t)$. We have

Definition 2.1 For $0 < \alpha < 1$, the weighted Atangana–Baleanu fractional derivative of Caputo sense of a function $f(t) \in W^1(0,T]$ with respect to the weight function $w(t)$ is defined by

$$\left({}_cD^{\alpha}_{w}f\right)(t) = \frac{M(\alpha)}{1-\alpha}\frac{1}{w(t)}\int_0^t E_\alpha\left[-\mu_\alpha(t-s)^\alpha\right]\frac{d}{ds}(wf)(s)\,ds, \quad t > 0. \tag{2.1}$$

Here $w \in C^1[0,T]$, $w, w' > 0$ on $[0,T]$, $M(\alpha)$ is a normalization function satisfying $M(0) = M(1) = 1$, $E_\alpha(t)$ is the well-known Mittag-Lefler function, $W^1(0,T]$ denotes the space of functions $f \in C^1(0,T]$ such that $f' \in L^1[0,T]$, and

$$\mu_\alpha = \frac{\alpha}{1-\alpha}. \tag{2.2}$$

The above integral-differential operator can be written as

$$\left({}_cD^{\alpha}_{w}f\right)(t) = \frac{M(\alpha)}{1-\alpha}\frac{1}{w(t)}\left(E_\alpha\left[-\mu_\alpha t^\alpha\right] * \frac{d}{dt}(wf)(t)\right), \quad t > 0. \tag{2.3}$$

Theorem 2.1 *Let $u \in W^1(0,T]$, if $g(0) = 0$, then the unique solution of the fractional differential equation,*

$$\left({}_cD^{\alpha}_{w}u\right)(t) = g(t), \quad t > 0, 0 < \alpha < 1, \tag{2.4}$$

is given by

$$u(t) = \frac{(wu)(0)}{w(t)} + \frac{1-\alpha}{M(\alpha)}g(t) + \frac{\alpha}{M(\alpha)\Gamma(\alpha)}\frac{1}{w(t)}\int_0^t (t-s)^{\alpha-1}w(s)g(s)\,ds.$$

Proof Because $f \in W^1(0,T]$, we have

$$\lim_{t\to 0^+}\int_0^t E_\alpha\left[-\mu_\alpha(t-s)^\alpha\right]\frac{d}{ds}(wu)(s)\,ds = 0,$$

and $({}_cD^{\alpha}_{w}u)(0^+) = 0$. Thus, a necessary condition for the existence of a solution to the problem (2.4) is that $g(0) = 0$. We have

$$\frac{M(\alpha)}{1-\alpha}\frac{1}{w(t)}\left(E_\alpha\left[-\mu_\alpha t^\alpha\right] * \frac{d}{dt}(wu)(t)\right) = g(t),$$

or

$$E_\alpha\left[-\mu_\alpha t^\alpha\right] * \frac{d}{dt}(wu)(t) = \frac{1-\alpha}{M(\alpha)} w(t)g(t).$$

Applying the Laplace transform to the above equation and using the convolution result, we have

$$L\left(E_\alpha\left[-\mu_\alpha t^\alpha\right]\right) L\left(\frac{d}{dt}(wu)(t)\right) = \frac{1-\alpha}{M(\alpha)} L\left(w(t)g(t)\right).$$

Since

$$L\left(E_\alpha\left[-\mu_\alpha t^\alpha\right]\right) = \frac{s^{\alpha-1}}{s^\alpha + \mu_\alpha}, \qquad \left|\frac{\mu_\alpha}{s^\alpha}\right| < 1,$$

we have

$$\frac{s^{\alpha-1}}{s^\alpha + \mu_\alpha}\left(sL(wu)(t) - (wu)(0)\right) = \frac{1-\alpha}{M(\alpha)} L\left(w(t)g(t)\right).$$

The above equation yields

$$\begin{aligned} L(wu)(t) &= (wu)(0)\frac{1}{s} + \frac{1-\alpha}{M(\alpha)}\left[L(wg)(t) + \frac{\mu_\alpha}{s^\alpha} L(wg)(t)\right] \\ &= (wu)(0)\frac{1}{s} + \frac{1-\alpha}{M(\alpha)}\left[L(wg)(t) + \frac{\mu_\alpha}{\Gamma(\alpha)} L\left(t^{\alpha-1}\right) L(wg)(t)\right]. \end{aligned}$$

Applying the inverse Laplace operator we have

$$\begin{aligned} (wu)(t) &= (wu)(0) + \frac{1-\alpha}{M(\alpha)}(wg)(t) + \frac{1-\alpha}{M(\alpha)} \frac{\mu_\alpha}{\Gamma(\alpha)}\left(t^{\alpha-1} * (wg)(t)\right) \\ &= (wu)(0) + \frac{1-\alpha}{M(\alpha)}(wg)(t) + \frac{\alpha}{M(\alpha)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} w(s)g(s)\,ds, \end{aligned}$$

which completes the proof. □

The result in Theorem 2.1 suggests to define the fractional integral operator $({}_cI_w^\alpha f)(t)$ as follows.

Definition 2.2 For $0 < \alpha < 1$, the weighted Atangana–Baleanu fractional integral of order α, of $f \in L^1(0,T)$ with respect to the weight function w is defined by

$$\left({}_cI_w^\alpha f\right)(t) = \frac{1-\alpha}{M(\alpha)} f(t) + \frac{\alpha}{M(\alpha)\Gamma(\alpha)} \frac{1}{w(t)} \int_0^t (t-s)^{\alpha-1} w(s)f(s)\,ds. \tag{2.5}$$

Remark 2.1 For $w(t) = 1$, the weighted Atangana–Baleanu fractional integral, coincides with the regular Atangana–Baleanu fractional integral; see [1, 3, 14].

The following result will be used throughout the text.

Proposition 2.1 *For* $0 < \alpha < 1$, *the following holds true.*

1.

$$L\left(\frac{d}{dt}\big(E_\alpha\big(at^\alpha\big)\big)\right) = \frac{a}{s^\alpha - a}, \qquad \left|\frac{a}{s^\alpha}\right| < 1. \tag{2.6}$$

2. *For arbitrary* $c_1, c_2 \in \mathbb{R}$, *and* $f, g \in W^1(0,T]$,

$$c_1\big(f' * g\big)(t) + c_2\big(f * g'\big)(t) = (c_1 + c_2)\big(f' * g\big)(t) + c_2\big(f(0)g(t) - f(t)g(0)\big). \tag{2.7}$$

Proof

1. We have

$$\begin{aligned}
L\left(\frac{d}{dt}\big(E_\alpha\big(at^\alpha\big)\big)\right) &= L\left(\frac{d}{dt}\sum_{n=0}^{\infty}\frac{a^n t^{n\alpha}}{\Gamma(\alpha n+1)}\right) \\
&= L\left(\sum_{n=1}^{\infty}\frac{a^n t^{\alpha n-1}}{\Gamma(\alpha n)}\right) \\
&= \sum_{n=1}^{\infty}\frac{a^n}{\Gamma(\alpha n)}L\big(t^{\alpha n-1}\big) \\
&= \sum_{n=1}^{\infty}\frac{a^n}{\Gamma(\alpha n)}\frac{\Gamma(\alpha n)}{s^{\alpha n}} \\
&= \sum_{n=1}^{\infty}\frac{a^n}{s^{\alpha n}} \\
&= \sum_{n=1}^{\infty}\left(\frac{a}{s^\alpha}\right)^n = \frac{\frac{a}{s^\alpha}}{1-\frac{a}{s^\alpha}} = \frac{a}{s^\alpha - a}, \qquad \left|\frac{a}{s^\alpha}\right| < 1,
\end{aligned}$$

which completes the proof.

2. The proof is straightforward using integration by parts.

□

Theorem 2.2 *Let* $u, g \in W^1(0,T]$, *if* $\lambda u(0) + g(0) = 0$, *then the unique solution of the linear fractional differential equation,*

$$\big({}_cD_w^\alpha u\big)(t) = \lambda u(t) + g(t), \quad t > 0, 0 < \alpha < 1, \tag{2.8}$$

is given by

$$\begin{aligned}
w(t)u(t) ={}& \frac{M(\alpha)(wu)(0)}{\delta_\alpha}E_\alpha\left(\frac{\lambda\alpha t^\alpha}{\delta_\alpha}\right) + \frac{1-\alpha}{\delta_\alpha}(wg)(t) \\
&+ \frac{M(\alpha)}{\lambda\delta_\alpha}\frac{d}{dt}E_\alpha\left(\frac{\lambda\alpha t^\alpha}{\delta_\alpha}\right) * (wg)(t),
\end{aligned} \tag{2.9}$$

where $\delta_\alpha = M(\alpha) - \lambda(1-\alpha) \neq 0$, $\lambda \neq 0$.

Proof We have

$$\lambda(wu)(t)+(wg)(t)=w(t)\big({}_cD_w^\alpha u\big)(t)=\frac{M(\alpha)}{1-\alpha}E_\alpha\big(-\mu_\alpha t^\alpha\big)*(wu)'(t).$$

Applying the Laplace transform to the above equation yields

$$\begin{aligned}L\big(\lambda(wu)(t)+(wg)(t)\big)&=\frac{M(\alpha)}{1-\alpha}L\big(E_\alpha\big(-\mu_\alpha t^\alpha\big)\big)L\big((wu)'(t)\big)\\&=\frac{M(\alpha)}{1-\alpha}\frac{s^{\alpha-1}}{s^\alpha+\mu_\alpha}\big(sL\big((wu)(t)\big)-(wu)(0)\big),\qquad\left|\frac{\mu_\alpha}{s^\alpha}\right|<1.\end{aligned}$$

Direct calculations lead to

$$\begin{aligned}L\big((wu)(t)\big)&=\frac{(1-\alpha)s^\alpha+\alpha}{(M(\alpha)-\lambda(1-\alpha))s^\alpha-\lambda\alpha}L\big((wg)(t)\big)\\&\quad+M(\alpha)(wu)(0)\frac{s^{\alpha-1}}{(M(\alpha)-\lambda(1-\alpha))s^\alpha-\lambda\alpha}\\&=\frac{(1-\alpha)s^\alpha+\alpha}{\delta_\alpha s^\alpha-\lambda\alpha}L\big((wg)(t)\big)+M(\alpha)(wu)(0)\frac{s^{\alpha-1}}{\delta_\alpha s^\alpha-\lambda\alpha}\\&=\frac{1-\alpha}{\delta_\alpha}\frac{s^{\alpha-1}}{s^\alpha-\frac{\lambda\alpha}{\delta_\alpha}}sL\big((wg)(t)\big)+\frac{\alpha}{\delta_\alpha}\frac{1}{s^\alpha-\frac{\lambda\alpha}{\delta_\alpha}}L\big((wg)(t)\big)\\&\quad+\frac{M(\alpha)(wu)(0)}{\delta_\alpha}\frac{s^{\alpha-1}}{s^\alpha-\frac{\lambda\alpha}{\delta_\alpha}}\\&=\frac{1-\alpha}{\delta_\alpha}L\left(E_\alpha\left(\frac{\lambda\alpha t^\alpha}{\delta_\alpha}\right)\right)\big(L\big((wg)'(t)\big)+(wg)(0)\big)\\&\quad+\frac{1}{\lambda}L\left(\frac{d}{dt}E_\alpha\left(\frac{\lambda\alpha t^\alpha}{\delta_\alpha}\right)\right)L\big((wg)(t)\big)\\&\quad+\frac{M(\alpha)(wu)(0)}{\delta_\alpha}L\left(E_\alpha\left(\frac{\lambda\alpha t^\alpha}{\delta_\alpha}\right)\right),\end{aligned}$$

hence,

$$\begin{aligned}(wu)(t)&=\frac{M(\alpha)(wu)(0)}{\delta_\alpha}E_\alpha\left(\frac{\lambda\alpha t^\alpha}{\delta_\alpha}\right)\\&\quad+\frac{1}{\lambda}\frac{d}{dt}E_\alpha\left(\frac{\lambda\alpha t^\alpha}{\delta_\alpha}\right)*(wg)(t)+\frac{1-\alpha}{\delta_\alpha}E_\alpha\left(\frac{\lambda\alpha t^\alpha}{\delta_\alpha}\right)*(wg)'(t).\end{aligned}\tag{2.10}$$

Using the result in Eq. (2.7) we have

$$\begin{aligned}&\frac{1}{\lambda}\frac{d}{dt}E_\alpha\left(\frac{\lambda\alpha t^\alpha}{\delta_\alpha}\right)*(wg)(t)+\frac{1-\alpha}{\delta_\alpha}E_\alpha\left(\frac{\lambda\alpha t^\alpha}{\delta_\alpha}\right)*(wg)'(t)\\&\quad=\frac{M(\alpha)}{\lambda\delta_\alpha}\frac{d}{dt}E_\alpha\left(\frac{\lambda\alpha t^\alpha}{\delta_\alpha}\right)*(wg)(t)+\frac{1-\alpha}{\delta_\alpha}\left((wg)(t)-(wg)(0)E_\alpha\left(\frac{\lambda\alpha t^\alpha}{\delta_\alpha}\right)\right),\end{aligned}\tag{2.11}$$

and hence the result is proved by substituting Eq. (2.11) in Eq. (2.10). □

3 Infinite series representation and properties of the weighted Atangana–Baleanu operators

The infinite series representation of the Atangana–Baleanu fractional derivative was introduced in [18] and has been used to establish several properties of the Atangana–Baleanu fractional operators. Given the Riemann–Liouville fractional integral

$$\big(I_0^\alpha f\big)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds,$$

and the infinite series representation of the Mittag-Leffler function

$$E_\alpha(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\alpha n+1)},$$

we have

$$\big({}_cI_w^\alpha f\big)(t) = \frac{1-\alpha}{M(\alpha)} f(t) + \frac{\alpha}{M(\alpha)} \frac{1}{w(t)} \big(I_0^\alpha wf\big)(t) \tag{3.1}$$

and

$$\begin{aligned}\big({}_cD_w^\alpha f\big)(t) &= \frac{M(\alpha)}{1-\alpha} \frac{1}{w(t)} \int_0^t E_\alpha\big[-\mu_\alpha (t-s)^\alpha\big] \frac{d}{ds}(wf)(s)\, ds \qquad (3.2)\\ &= \frac{M(\alpha)}{1-\alpha} \frac{1}{w(t)} \int_0^t \sum_{n=0}^{\infty} \frac{(-1)^n \mu_\alpha^n}{\Gamma(\alpha n+1)} (t-s)^{\alpha n} \frac{d}{ds}(wf)(s)\, ds \\ &= \frac{M(\alpha)}{1-\alpha} \frac{1}{w(t)} \sum_{n=0}^{\infty} \frac{(-1)^n \mu_\alpha^n}{\Gamma(\alpha n+1)} \int_0^t (t-s)^{\alpha n} \frac{d}{ds}(wf)(s)\, ds \\ &= \frac{M(\alpha)}{1-\alpha} \frac{1}{w(t)} \sum_{n=0}^{\infty} (-1)^n \mu_\alpha^n \left(I_0^{\alpha n+1} \frac{d}{dt}(wf)\right)(t). \qquad (3.3)\end{aligned}$$

Since $E_\alpha([-\mu_\alpha(t-s)^\alpha]$ is continuous, $w \in C^1[0,T]$ and $f \in W^1(0,T]$, the integral in Eq. (3.2) converges for a finite interval $[0,T]$, and hence the infinite series in Eq. (3.3) is convergent for all $t \in [0,T]$.

Theorem 3.1 *If $f \in W^1(0,T]$, then, for $0 < \alpha < 1$,*

1. $({}_cI_{w\,c}^\alpha D_w^\alpha f)(t) = f(t) - \frac{w(0)f(0)}{w(t)}$, *and*
2. $({}_cD_{w\,c}^\alpha I_w^\alpha f)(t) = f(t) - \frac{w(0)f(0)}{w(t)}$.

Proof

1. We have

$$\begin{aligned}\big({}_cI_{w\,c}^\alpha D_w^\alpha f\big)(t) &= \frac{1-\alpha}{M(\alpha)} \big({}_cD_w^\alpha f\big)(t) + \frac{\alpha}{M(\alpha)} \frac{1}{w(t)} \big(I_0^\alpha \big(w\,{}_cD_w^\alpha f\big)\big)(t) \\ &= \frac{1-\alpha}{M(\alpha)} \left[\frac{M(\alpha)}{1-\alpha} \frac{1}{w(t)} \sum_{n=0}^{\infty} (-1)^n \mu_\alpha^n \big(I_0^{\alpha n+1}(wf)'\big)(t)\right] \\ &\quad + \frac{\alpha}{M(\alpha)} \frac{1}{w(t)} \left[I_0^\alpha \left(\frac{M(\alpha)}{1-\alpha} \sum_{n=0}^{\infty} (-1)^n \mu_\alpha^n \big(I_0^{\alpha n+1}(wf)'\big)\right)(t)\right]\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{w(t)} \sum_{n=0}^{\infty} (-1)^n \mu_\alpha^n \bigl(I_0^{\alpha n+1}(wf)'\bigr)(t) \\
&\quad + \frac{\mu_\alpha}{w(t)} \sum_{n=0}^{\infty} (-1)^n \mu_\alpha^n \bigl(I_0^{\alpha(n+1)+1}(wf)'\bigr)(t) \\
&= \frac{1}{w(t)} \sum_{n=0}^{\infty} (-1)^n \mu_\alpha^n \bigl(I_0^{\alpha n+1}(wf)'\bigr)(t) \\
&\quad + \frac{\mu_\alpha}{w(t)} \sum_{n=1}^{\infty} (-1)^{n-1} \mu_\alpha^{n-1} \bigl(I_0^{\alpha n+1}(wf)'\bigr)(t) \\
&= \frac{1}{w(t)} \sum_{n=0}^{\infty} (-1)^n \mu_\alpha^n \bigl(I_0^{\alpha n+1}(wf)'\bigr)(t) \\
&\quad - \frac{1}{w(t)} \sum_{n=1}^{\infty} (-1)^n \mu_\alpha^n \bigl(I_0^{\alpha n+1}(wf)'\bigr)(t) \\
&= \frac{1}{w(t)} \bigl(I_0^1(wf)'\bigr)(t) = \frac{1}{w(t)} \bigl(w(t)f(t) - w(0)f(0)\bigr),
\end{aligned}
$$

which completes the proof.

2. We have

$$
\begin{aligned}
\bigl({}_cD_{wc}^{\alpha} I_w^{\alpha} f\bigr)(t) &= \frac{M(\alpha)}{1-\alpha} \frac{1}{w(t)} \sum_{n=0}^{\infty} (-1)^n \mu_\alpha^n \bigl(I_0^{\alpha n+1}\bigl(w_c I_w^{\alpha} f\bigr)'\bigr)(t) \\
&= \frac{M(\alpha)}{1-\alpha} \frac{1}{w(t)} \sum_{n=0}^{\infty} (-1)^n \mu_\alpha^n \biggl[I_0^{\alpha n+1}\biggl(\biggl(\frac{1-\alpha}{M(\alpha)}(wf)' \\
&\quad + \frac{\alpha}{M(\alpha)} \bigl(I_0^{\alpha}(wf)\bigr)'\biggr)(t)\biggr] \\
&= \frac{1}{1-\alpha} \frac{1}{w(t)} \sum_{n=0}^{\infty} (-1)^n \mu_\alpha^n \bigl[(1-\alpha)\bigl(\bigl(I_0^{\alpha n}(wf)\bigr)(t) - (wf)(0)\bigl(I_0^{\alpha n}1\bigr)(t)\bigr) \\
&\quad + \alpha\bigl(\bigl(I_0^{\alpha n}\bigl(I_0^{\alpha}(wf)\bigr)\bigr)(t) - \bigl(I_0^{\alpha}(wf)\bigr)(0)\bigl(I_0^{\alpha n}1\bigr)(t)\bigr)\bigr] \\
&= \frac{1}{w(t)} \sum_{n=0}^{\infty} (-1)^n \mu_\alpha^n \bigl[\bigl(I_0^{\alpha n}(wf)\bigr)(t) - (wf)(0)\bigl(I_0^{\alpha n}1\bigr)(t) \\
&\quad + \mu_\alpha\bigl(\bigl(I_0^{\alpha n+\alpha}(wf)\bigr)(t) - (wf)(0)\bigl(I_0^{\alpha n+\alpha}1\bigr)(t)\bigr)\bigr] \\
&= \frac{1}{w(t)} \sum_{n=0}^{\infty} (-1)^n \mu_\alpha^n \bigl[\bigl(I_0^{\alpha n}(wf)\bigr)(t) - (wf)(0)\bigl(I_0^{\alpha n}1\bigr)(t)\bigr] \\
&\quad + \frac{1}{w(t)} \sum_{n=0}^{\infty} (-1)^n \mu_\alpha^{\alpha(n+1)} \bigl[\bigl(I_0^{\alpha(n+1)}(wf)\bigr)(t) - (wf)(0)\bigl(I_0^{\alpha(n+1)}1\bigr)(t)\bigr] \\
&= \frac{1}{w(t)} \sum_{n=0}^{\infty} (-1)^n \mu_\alpha^n \bigl[\bigl(I_0^{\alpha n}(wf)\bigr)(t) - (wf)(0)\bigl(I_0^{\alpha n}1\bigr)(t)\bigr] \\
&\quad - \frac{1}{w(t)} \sum_{n=1}^{\infty} (-1)^n \mu_\alpha^{\alpha n} \bigl[\bigl(I_0^{\alpha n}(wf)\bigr)(t) - (wf)(0)\bigl(I_0^{\alpha n}1\bigr)(t)\bigr]
\end{aligned}
$$

$$= \frac{1}{w(t)}\big(\big(I_0^0(wf)\big)(t) - (wf)(0)\big(I_0^0 1\big)(t)\big)$$
$$= \frac{1}{w(t)}\big((wf)(t) - (wf)(0)\big), \tag{3.4}$$

which completes the proof. □

As a direct result of Theorem 3.1 we have the following.

Corollary 3.1 *If $f \in W^1(0,T]$, and $f(0) = 0$, then, for $0 < \alpha < 1$,*

1. $({}_cI_{w}^{\alpha}{}_cD_w^{\alpha} f)(t) = f(t)$, *and*
2. $({}_cD_{w}^{\alpha}{}_cI_w^{\alpha} f)(t) = f(t)$.

Theorem 3.2 *If $f \in W^1(0,T]$, then, for $\alpha, \beta \in (0,1)$,*

1. ${}_cD_w^{\alpha}({}_cD_w^{\beta} f)(t) = {}_cD_w^{\beta}({}_cD_w^{\alpha} f)(t)$, *and*
2. ${}_cI_w^{\alpha}({}_cI_w^{\beta} f)(t) = {}_cI_w^{\beta}({}_cI_w^{\alpha} f)(t)$.

That is, the weighted Atangana–Baleanu fractional operators in the Caputo sense are commutative operators.

Proof

1. We have

$${}_cD_w^{\alpha}\big({}_cD_w^{\beta} f\big)(t) = \frac{M(\alpha)}{1-\alpha}\frac{1}{w(t)}\sum_{n=0}^{\infty}(-1)^n \mu_\alpha^n\big(I_0^{\alpha n+1}\big(w_cD_w^{\beta} f\big)'\big)(t) \tag{3.5}$$

and

$$\big(w_cD_w^{\beta} f\big)(t) = \frac{M(\beta)}{1-\beta}\sum_{k=0}^{\infty}(-1)^k \mu_\beta^k\big(I_0^{\beta k+1}(wf)'\big)(t).$$

Thus,

$$\frac{d}{dt}\big(w_cD_w^{\beta} f\big)(t) = \frac{M(\beta)}{1-\beta}\sum_{k=0}^{\infty}(-1)^k \mu_\beta^k\big(I_0^{\beta k}(wf)'\big)(t). \tag{3.6}$$

By substituting Eq. (3.6) in Eq. (3.5) we have

$${}_cD_w^{\alpha}\big({}_cD_w^{\beta} f\big)(t) = \frac{M(\alpha)}{1-\alpha}\frac{1}{w(t)}\sum_{n=0}^{\infty}(-1)^n \mu_\alpha^n\left(I_0^{\alpha n+1}\frac{M(\beta)}{1-\beta}\sum_{k=0}^{\infty}(-1)^k \mu_\beta^k\big(I_0^{\beta k}(wf)'\big)\right)(t)$$
$$= \frac{1}{w(t)}\frac{M(\alpha)M(\beta)}{(1-\alpha)(1-\beta)}\sum_{n,k=0}(-\mu_\alpha)^n(-\mu_\beta)^k\big(I_0^{\alpha n+\beta k+1}(wf)'\big)(t), \tag{3.7}$$

and the result is proved since the last expression is symmetric in α and β.

2. We have

$${}_cI_w^{\alpha}\big({}_cI_w^{\beta} f\big)(t) = \frac{1-\alpha}{M(\alpha)}\big({}_cI_w^{\beta} f\big)(t) + \frac{\alpha}{M(\alpha)}\frac{1}{w(t)}I_0^{\alpha}\big(w_cI_w^{\beta} f\big)(t)$$
$$= \frac{1-\alpha}{M(\alpha)}\left(\frac{1-\beta}{M(\beta)}f(t) + \frac{\beta}{M(\beta)}\frac{1}{w(t)}I_0^{\beta}(wf)(t)\right)$$

$$
\begin{aligned}
&+ \frac{\alpha}{M(\alpha)} \frac{1}{w(t)} I_0^\alpha \left(\frac{1-\beta}{M(\beta)} (wf)(t) + \frac{\beta}{M(\beta)} I_0^\beta (wf)(t) \right) \\
&= \frac{(1-\alpha)(1-\beta)}{M(\alpha)M(\beta)} f(t) + \frac{\beta(1-\alpha)}{M(\alpha)M(\beta)} \frac{1}{w(t)} I_0^\beta (wf)(t) \\
&+ \frac{\alpha(1-\beta)}{M(\alpha)M(\beta)} \frac{1}{w(t)} I_0^\alpha (wf)(t) \\
&+ \frac{\alpha\beta}{M(\alpha)M(\beta)} \frac{1}{w(t)} I_0^{\alpha+\beta} (wf)(t),
\end{aligned}
$$

which proves the result as the last expression is symmetric in α and β. □

The weighted Atangana–Baleanu fractional operators are defined for $t > a$ and arbitrary $a \in \mathbb{R}^+$ as listed below. However, we started with $t > 0$, in order to apply the Laplace transform to define the weighted Atangana–Baleanu fractional integral.

Definition 3.1 For $0 < \alpha < 1$, the weighted Atangana–Baleanu fractional derivative of Caputo sense of a function $f(t) \in W^1(a, T]$, $a \in \mathbb{R}^+$ with respect to the weight function $w(t)$ is defined by

$$
\left({}_c D_{a,w}^\alpha f\right)(t) = \frac{M(\alpha)}{1-\alpha} \frac{1}{w(t)} \int_a^t E_\alpha\left[-\mu_\alpha (t-s)^\alpha\right] \frac{d}{ds} (wf)(s)\, ds, \quad t > a. \tag{3.8}
$$

Definition 3.2 For $0 < \alpha < 1$, the weighted Atangana–Baleanu fractional integral of order α, of $f \in L^1(a, T)$, $a \in \mathbb{R}^+$ with respect to the weight function w is defined by

$$
\left({}_c I_{a,w}^\alpha f\right)(t) = \frac{1-\alpha}{M(\alpha)} f(t) + \frac{\alpha}{M(\alpha)\Gamma(\alpha)} \frac{1}{w(t)} \int_a^t (t-s)^{\alpha-1} w(s) f(s)\, ds. \tag{3.9}
$$

By applying analogous steps in the previous section one can easily verify the following:

$$
\left({}_c I_{a,w}^\alpha f\right)(t) = \frac{1-\alpha}{M(\alpha)} f(t) + \frac{\alpha}{M(\alpha)} \frac{1}{w(t)} \left(I_a^\alpha wf\right)(t)
$$

and

$$
\left({}_c D_{a,w}^\alpha f\right)(t) = \frac{M(\alpha)}{1-\alpha} \frac{1}{w(t)} \sum_{n=0}^\infty (-1)^n \mu_\alpha^n \left(I_a^{\alpha n+1} \frac{d}{dt} (wf) \right)(t).
$$

The properties obtained in Theorems 3.1 and 3.2 will be valid for the $({}_c I_{a,w}^\alpha f)(t)$ and $({}_c D_{a,w}^\alpha f)(t)$ operators.

4 Concluding remarks

We have introduced the weighted Atangana–Baleanu fractional operators, and studied their properties. By means of the Laplace transform, we have obtained the solutions of related linear equations in closed forms. The weighted Atangana–Baleanu fractional integral is written in terms of the Riemann–Liouville integral, and the weighted Atangana–Baleanu fractional derivative is written in terms of an infinite series of Riemann–Liouville integrals. By means of these representations, we have established several properties of the weighted Atangana–Baleanu fractional operators. Because of the type of the kernel, it is well known that dealing with the Atangana–Baleanu fractional operators is more difficult than dealing with the Caputo–Fabrizio operators. Therefore, the problem of introducing and studying the weighted Atangana–Baleanu fractional operators with respect to another function $z(t)$ and weight function $w(t)$ with their properties is still open. Also, the question whether the new models in the paper can be solved by the available numerical techniques in the literature [25, 26, 37, 38] is a valid question, and this issue has to considered in a future research.

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