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Periodic averaging method for impulsive stochastic dynamical systems driven by fractional Brownian motion under non-Lipschitz condition

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Abstract

This paper presents the periodic averaging principle for impulsive stochastic dynamical systems driven by fractional Brownian motion (fBm). Under non-Lipschitz condition, we prove that the solutions to impulsive stochastic differential equations (ISDEs) with fBm can be approximated by the solutions to averaged SDEs without impulses both in the sense of mean square and probability. Finally, an example is provided to illustrate the theoretical results.

Keywords: Periodic averaging technique; Non-Lipschitz condition; Fractional Brownian motion; Impulsive dynamical systems; Stochastic differential equations

1 Introduction

In the past years, stochastic dynamical systems driven by fBm have became an active area of investigation due to their applications in telecommunications networks, finance markets, biology, and other fields [1–6]. The impulsive effects exist widely in many evolution processes in which states are changed abruptly at certain moments of time. Consequently, the impulsive differential equations have a wide range of applications in numerous branches of sciences such as finance, economics, medicine, biology, electronics, and telecommunications (see [7–10]).

On the other hand, it is well known that the averaging technique represents a good mathematical tool that approximates complicated time varying differential equations to autonomous differential equations. Since Krylov and Bogolyubov [11] put forward the cornerstone of the averaging principles for deterministic dynamical systems, averaging method has received considerable attention, and it has been found available and useful for exploring dynamical systems in many fields [12–16]. Up to now, there have been some works about stochastic averaging for dynamic problems with Gaussian random perturbation [17–19], Poisson noise [20, 21], Lévy motion [22–25], G-Brownian motion [26, 27], and fBm [28–31]. So far, no previous study has employed the periodic averaging technique to impulsive stochastic dynamical systems with fBm. Therefore, we make an attempt to establish the periodic averaging principle to ISDEs with fBm, which allows the averaged systems without impulses to replace the original ISDEs both in mean square sense and probability.

We consider a class of ISDEs with fBm of the form

$$\begin{aligned} &dx(t) = a\bigl(t,x(t)\bigr)\,dt + b\bigl(t,x(t)\bigr)\,dW^H(t), \quad t \neq t_j, \\ &\triangle x(t_j) = I_j\bigl(x\bigl(t_j^-\bigr)\bigr), \quad t = t_j, j \in \mathbb{N}, \\ &x(0) = x_0, \end{aligned} \tag{1}$$

where $\triangle x(t_j)$ denotes the jump of x at $t = t_j$, for $0 \leq t \leq T < \infty$, and $\triangle x(t_j) = x(t_j^+) - x(t_j^-)$, such that $x(t_j^+) = \lim_{t\to t_j^+} x(t)$ and $x(t_j^-) = \lim_{t\to t_j^-} x(t)$. x_0 represents the initial data of the system with $E|x_0|^2 < \infty$. The process $W^H(t)$ is fBm with Hurst index $H \in (\frac{1}{2}, 1)$ defined on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$. The coefficients $a(t,x(t)) : [0,T] \times R^n \to R^n$ and $b(t,x(t)) : [0,T] \times R^n \to R^{n\times m}$ are measurable functions.

The outline of this manuscript is as follows. In Sect. 2, we provide some background about stochastic integral with respect to fBm. Section 3 is devoted to establishing the stochastic periodic averaging approach to Eq. (1) under non-Lipschitz condition. Finally, an example is presented to demonstrate the theoretical results in Sect. 4.

2 Framework

In this section, we introduce some basic notions and preliminaries on path-wise integrals with respect to fBm, and for more detailed discussion, we refer the reader to [6, 32–35].

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete probability space equipped with a natural filtration $\{\mathcal{F}_t\}_{t\geq 0}$, where $\mathcal{F}_t$ is the σ-algebra generated by $\{W^H(t), t \in [0,T]\}$ and $\mathcal{F}_0$ contains all P-null sets.

Definition 2.1 The process $\{W^H(t), 1/2 < H < 1\}$ is said to be a centered self-similar fBm if the following properties are satisfied:

- $W^H(0) = 0$,
- $E[W^H(t)] = 0$, $t \in [0,T]$,
- $E[W^H(t)W^H(s)] = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H})$, $t,s \in [0,T]$.

Next, for the convenience of readers, we provide some basic properties on path-wise integrals. Firstly, we introduce the function $\varphi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ defined as

$$\varphi(t,s) = H(2H-1)|t-s|^{2H-2}, \quad t,s \in \mathbb{R}_+,$$

where $H \in (\frac{1}{2}, 1)$. Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a Borel measurable function and define the space

$$L^2_\varphi(\mathbb{R}_+) = \biggl\{f : \|f\|^2_\varphi = \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} f(t)f(s)\varphi(t,s)\,ds\,dt < \infty\biggr\},$$

which becomes a separable Hilbert space under the inner product

$$\langle f_1, f_2\rangle_\varphi = \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} f_1(t)f_2(s)\varphi(t,s)\,ds\,dt, \quad f_1, f_2 \in L^2_\varphi(\mathbb{R}_+).$$

Now, consider the set $\mathcal{E}$ of smooth and cylindrical random variables of the form

$$F(\omega) = g\left(\int_0^T \psi_1(t)\,dW^H(t),\dots,\int_0^T \psi_n(t)\,dW^H(t)\right),$$

where $n \geq 1$ and $g \in \mathcal{C}_b^\infty(\mathbb{R}^n)$ (i.e., g and its partial derivatives are bounded). Moreover, let $\mathcal{H}$ be the family of measurable functions such that, for $\psi_i \in \mathcal{H}$, $i = 1,\dots,n$, $n \in \mathbb{N}$, we have $\langle \psi_i, \psi_j\rangle_\varphi = \delta_{ij}$ and $\|\psi\|_\varphi^2 < \infty$. The elements of $\mathcal{H}$ may not be functions but distributions of negative order. Thanks to this reason, it is convenient to introduce the space $|\mathcal{H}|$ of measurable functions h on $[0,T]$ satisfying

$$\|h\|_{|\mathcal{H}|}^2 = \int_0^T\int_0^T \big|h(t)\big|\big|h(s)\big|\varphi(t,s)\,ds\,dt < \infty,$$

and it is easy to show that $|\mathcal{H}|$ is a Banach space under the norm $\|\cdot\|_{|\mathcal{H}|}$.

Definition 2.2 The Malliavin derivative D_t^H of a smooth and cylindrical random variable F is defined as an $\mathcal{H}$-valued random variable such that

$$D_t^H F = \sum_{i=1}^n \frac{\partial g}{\partial x_i}\left(\int_0^T \psi_1(t)\,dW^H(t),\dots,\int_0^T \psi_n(t)\,dW^H(t)\right)\psi_i(t),$$

hence, D_t^H represents a closable operator, so that $D_t^H : L^p(\Omega) \mapsto L^p(\Omega,\mathcal{H})$, $p \geq 1$. The iteration of Malliavin derivative is denoted by $D_t^{H,k}$, $k \geq 1$. For any $p \geq 1$, the Sobolev space $\mathbb{D}^{k,p}$ represents the closer of $\mathcal{E}$ with respect to the norm

$$\|F\|_{k,p}^p = E|F|^p + E\sum_{i=1}^k \big\|D_t^{H,i}F\big\|_{\mathcal{H}\otimes i}^p,$$

where $\otimes$ denotes the tensor product.

Similarly, for a Hilbert space U, we denote by $\mathbb{D}^{k,p}(U)$ the corresponding Sobolev space of U-valued random variables, and for $p > 0$, we denote by $\mathbb{D}^{1,p}(|\mathcal{H}|)$ the subspace of $\mathbb{D}^{1,p}(\mathcal{H})$ formed by the elements h of $|\mathcal{H}|$. According to [6], we introduce φ-derivative of F as follows:

$$D_t^\varphi F = \int_{\mathbb{R}_+} \varphi(t,s) D_s^H F\,ds.$$

Definition 2.3 The space $\mathcal{L}_\varphi[0,T]$ of integrals is defined as the family of stochastic processes $V(t)$ on $[0,T]$ such that $E\|V(t)\|_\varphi^2 < \infty$, $V(t)$ is φ-differentiable, the trace of the derivative $D_s^\varphi V(t)$ exists, and for $t,s \in [0,T]$,

$$E\left[\int_0^T\int_0^T \big|D_t^\varphi V(s)\big|^2\,ds\,dt\right] < \infty.$$

In addition, for each sequence of partitions $(\pi_n, n \in \mathbb{N})$ with $|\pi_n| \to 0$ as $n \to \infty$, the following are satisfied:

$$\sum_{i=0}^{n-1} E\left[\int_{t_i^{(n)}}^{t_{i+1}^{(n)}}\int_{t_j^{(n)}}^{t_{j+1}^{(n)}} \big|D_s^\varphi V^\pi\big(t_i^{(n)}\big) D_t^\varphi V^\pi\big(t_j^{(n)}\big) - D_s^\varphi V(t) D_t^\varphi V(s)\big|^2\,ds\,dt\right] \to 0$$

and

$$E\left\| V^{\pi} - V \right\|_{\varphi}^{2} \to 0,$$

as n tends to infinity, where $\pi_n = t_0^{(n)} < t_1^{(n)} < \cdots < t_{n-1}^{(n)} < t_n^{(n)} = T$, $|\pi| := \max_i (t_{i+1} - t_i)$ and $V^{\pi} = V_{t_i}$.

Now, define the space $\mathbb{H}_{\varphi}^{1,2}$, which represents the intersection of the spaces $\mathbb{D}^{1,2}(|\mathcal{H}|)$ and $\mathcal{L}_{\varphi}[0,T]$, such that $\mathbb{H}_{\varphi}^{1,2} = \mathbb{D}^{1,2}(|\mathcal{H}|) \cap \mathcal{L}_{\varphi}[0,T]$.

Definition 2.4 Let $V(t)$ be a stochastic process with integrable trajectories.

- The symmetric integral of $V(t)$ with respect to $W^H(t)$ is defined as follows:

  $$\lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_0^T V(s)\bigl[W^H(s+\epsilon) - W^H(s-\epsilon)\bigr]\,ds,$$

  provided that the limit exists in probability, the symmetric integral is denoted by

  $$\int_0^T V(s)\,d^{\circ} W^H(s).$$

- The forward integral of $V(t)$ with respect to $W^H(t)$ is defined as follows:

  $$\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_0^T V(s)\left[\frac{W^H(s+\epsilon) - W^H(s)}{\epsilon}\right] ds,$$

  provided that the limit exists in probability, the forward integral is denoted by

  $$\int_0^T V(s)\,d^{-} W^H(s).$$

- The backward integral of $V(t)$ with respect to $W^H(t)$ is defined as follows:

  $$\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_0^T V(s)\left[\frac{W^H(s-\epsilon) - W^H(s)}{\epsilon}\right] ds,$$

  provided that the limit exists in probability, the backward integral is denoted by

  $$\int_0^T V(s)\,d^{+} W^H(s).$$

In order to establish our results, we need to introduce some lemmas. The next lemma follows (Remark 1 in [35]) and (Proposition 6.2.3 in [6]).

Lemma 2.5 *If the stochastic process $V(t)$ satisfies*

$$\int_0^T \int_0^T \left|D_s^H V(t)\right| |t-s|^{2H-2}\,ds\,dt < \infty, \qquad V \in \mathbb{D}^{1,2}\bigl(|\mathcal{H}|\bigr),$$

then the symmetric integral coincides with the forward and backward integrals.

Since fBm is neither semi-martingale nor Markov process, we definitely lost the use of Burkholder–Davis–Gundy inequality and Ito-isometry. Therefore, there is a pressing need to use the following two lemmas from [6] and [28].

Lemma 2.6 *If $V(t)$ is a stochastic process on $\mathbb{H}_{\varphi}^{1,2}$, then the symmetric integral is well defined and*

$$\int_0^T V(s)\,d^{\circ}W^H(s) = \int_0^T V(s)\diamond dW^H(s) + \int_0^T D_s^{\varphi} V(s)\,ds,$$

where $\diamond$ denotes the Wick product.

We note that the forward and backward integrals are also well defined. Hence, by Lemma 2.5, the forward and backward integrals coincide with the symmetric integral under the condition of Lemma 2.6.

Lemma 2.7 *Let $W^H(t)$ be fBm with Hurst index $H \in (\frac{1}{2}, 1)$ and $V(t)$ be a stochastic process in $\mathbb{H}_{\varphi}^{1,2}$, then, for $0 \leq T < \infty$, there exists a constant $C > 0$ such that*

$$E\left|\int_0^T V(s)\,d^{\circ}W^H(s)\right|^2 \leq 2HT^{2H-1}E\int_0^T \left|V(s)\right|^2 ds + 4CT^2.$$

The following requisite lemma is taken from [36].

Lemma 2.8 *Let $T > 0$, $x_0 \geq 0$, and $x(t)$, $y(t)$ be two continuous functions on $[0, T]$. Assume that $\kappa : \mathbb{R}_+ \to \mathbb{R}_+$ is a concave continuous nondecreasing function such that $\kappa(v) > 0$ for $v > 0$. If we have*

$$x(t) \leq x_0 + \int_0^t y(s)\kappa\bigl(x(s)\bigr)\,ds \quad \forall t \in [0, T],$$

then

$$x(t) \leq G^{-1}\left(G(x_0) + \int_0^t y(s)\,ds\right) \quad \forall t \in [0, T],$$

where $(G(x_0) + \int_0^t y(s)\,ds) \in \mathrm{Dom}(G^{-1})$, $G(v) = \int_0^v \frac{ds}{\kappa(s)}\,ds$, $v > 0$. Moreover, if $x_0 = 0$ and $\int_{0^+} \frac{ds}{\kappa(s)}\,ds = \infty$, then $x(t) = 0$ for all $t \in [0, T]$.

Throughout this paper, the following assumptions are imposed.

Assumption A For all $x, y \in \mathbb{R}^n$, $t \in [0, T]$, and $a(t, \cdot), b(t, \cdot) \in \mathbb{H}_{\varphi}^{1,2}$, there exists a function $\kappa(\cdot)$ such that

$$\bigl|a(t,x) - a(t,y)\bigr|^2 + \bigl|b(t,x) - b(t,y)\bigr|^2 + \bigl|D_t^{\varphi}\bigl(b(t,x) - b(t,y)\bigr)\bigr|^2 \leq \kappa\bigl(|x-y|^2\bigr),$$

where $\kappa(\cdot)$ is a concave continuous nondecreasing function such that $\kappa(0) = 0$ and

$$\int_{0^+} \frac{1}{\kappa(x)}\,dx = \infty.$$

Moreover, since $\kappa(\cdot)$ is a concave continuous nondecreasing function, then there must exist two constants $\lambda_1 > 0$ and $\lambda_2 > 0$ such that

$$\kappa(x) \le \lambda_1 x + \lambda_2.$$

Remark 2.9 In view of Assumption A, we can see clearly, for a special case, if $\kappa(|x|) = K|x|$, then the Lipschitz condition is recovered. Therefore, Assumption A is much weaker than the usual Lipschitz condition.

Next, according to Lemma 3.1 in [37], the solution of impulsive stochastic dynamical system (1) can be given by the following integral equation:

$$x(t) = x_0 + \int_0^t a\big(s,x(s)\big)\,ds + \int_0^t b\big(s,x(s)\big)\,d^\circ W^H(s) + \sum_{0<t_j<t} I_j\big(x(t_j)\big). \tag{2}$$

Now, consider the standard ISDE with fBm

$$x_\epsilon(t) = x_0 + \epsilon^{2H}\int_0^t a\big(s,x_\epsilon(s)\big)\,ds + \epsilon^H\int_0^t b\big(s,x_\epsilon(s)\big)\,d^\circ W^H(s) + \epsilon^H\sum_{0<t_j<t} I_j\big(x_\epsilon(t_j)\big), \tag{3}$$

where $\epsilon \in (0,\epsilon_0]$ is a positive small parameter and ϵ_0 is a fixed number. Moreover, the averaged SDE of the standard ISDE (3) is

$$z_\epsilon(t) = x_0 + \epsilon^{2H}\int_0^t \big[\bar{a}\big(z_\epsilon(s)\big) + \bar{I}\big(z_\epsilon(s)\big)\big]\,ds + \epsilon^H\int_0^t \bar{b}\big(z_\epsilon(s)\big)\,d^\circ W^H(s), \tag{4}$$

where the functions $\bar{a}(x):\mathbb{R}^n \to \mathbb{R}^n$, $\bar{b}(x):\mathbb{R}^n \to \mathbb{R}^n$ and $\bar{I}(x):\mathbb{R}^n \to \mathbb{R}^n$ are measurable functions satisfying

$$\begin{aligned}
\bar{a}(x) &= \frac{1}{T}\int_0^T a(t,x)\,dt, \\
\bar{b}(x) &= \frac{1}{T}\int_0^T b(t,x)\,dt, \\
\bar{I}(x) &= \frac{1}{T}\sum_{j=1}^k I_j(x).
\end{aligned}$$

Assumption B For any $x, y \in \mathbb{R}^n$, there exist positive constants N_1 and N_2 such that

$$\big|I_j(x)\big|^2 \le N_1, \qquad \big|I_j(x) - I_j(y)\big|^2 \le N_2|x-y|^2.$$

Assumption C For all $t \in [0,T]$, $x \in \mathbb{R}^n$, the coefficients of Eq. (3) and Eq. (4) are bounded. Then there exists a positive constant M such that

$$\big|a(t,x)\big|^2 \le M, \qquad \big|b(t,x)\big|^2 \le M, \qquad \big|\bar{a}(x)\big|^2 \le M, \qquad \big|\bar{b}(x)\big|^2 \le M.$$

Now, the existence and uniqueness result for Eq. (2) is given by the following theorem.

Theorem 2.10 *Assume that Assumptions A–C are satisfied. Then, for every initial value $x_0 \in \mathbb{R}^n$, there exists a unique solution $x(t)$ to Eq.* (2) *on* $[0,T]$.

Proof The proof is a special case of the proof of Theorem 3.1 in Abouagwa et al. [38] and easy to be derived. So, we omit the proof here. □

3 Periodic averaging principle

In this section, we study the periodic averaging principle of ISDEs driven by fBm under non-Lipschitz condition.

In order to provide the periodic averaging results, we assume that the functions a and b are T-periodic in the first argument and the impulses I_j are periodic in the sense that there exist $k \in \mathbb{N}$ such that $0 \leq t_1 < t_2 < \cdots < t_k < T$, and for every $j > k$, we have $t_j = t_{j-k} + T$, $I_j = I_{j-k}$.

Following Theorem 3.6 in Mao et al. [39], we now establish our main result which is used for revealing the relationship between the processes $x_\epsilon(t)$ and $z_\epsilon(t)$.

Theorem 3.1 *Consider standard ISDE* (3) *and averaging SDE* (4) *if Assumptions* A–C *hold. Then, for* $T > 0$, *the following equality is satisfied*:

$$\lim_{\epsilon\to 0} E\big|x_\epsilon(t) - z_\epsilon(t)\big|^2 = 0. \tag{5}$$

Proof From Eqs. (3) and (4), taking expectation and employing the basic inequality $|a + b + c|^2 \leq 3|a|^2 + 3|b|^2 + 3|c|^2$, we obtain

$$\begin{aligned} E\big|x_\epsilon(t) - z_\epsilon(t)\big|^2 \leq{}& 3\epsilon^{4H} E\left|\int_0^t a\big(s, x_\epsilon(s)\big) - \bar{a}\big(z_\epsilon(s)\big)\,ds\right|^2 \\ &+ 3\epsilon^{2H} E\left|\int_0^t b\big(s, x_\epsilon(s)\big) - \bar{b}\big(z_\epsilon(s)\big)\,d^\circ W^H(s)\right|^2 \\ &+ 3\epsilon^{2H} E\left|\sum_{j=1}^{\infty} I_j\big(x_\epsilon(t_j)\big) - \int_0^t \bar{I}\big(z_\epsilon(s)\big)\,ds\right|^2 \\ ={}& \sum_{l=1}^{3} Q_l. \end{aligned}$$

Starting with the first term Q_1, we have

$$\begin{aligned} Q_1 &\leq 6\epsilon^{4H} E\left|\int_0^t a\big(s, x_\epsilon(s)\big) - a\big(s, z_\epsilon(s)\big)\,ds\right|^2 + 6\epsilon^{4H} E\left|\int_0^t a\big(s, z_\epsilon(s)\big) - \bar{a}\big(z_\epsilon(s)\big)\,ds\right|^2 \\ &= Q_{11} + Q_{12}. \end{aligned}$$

For Q_{11}, by applying the Cauchy–Schwarz inequality, Jensen's inequality, and Assumption A, one can get

$$\begin{aligned} Q_{11} &\leq 6\epsilon^{4H} E\left(t \int_0^t \big|a\big(s, x_\epsilon(s)\big) - a\big(s, z_\epsilon(s)\big)\big|^2\,ds\right) \\ &\leq 6\epsilon^{4H} \sup_{0\leq t\leq u} t\left(E\int_0^t \kappa\big(\big|x_\epsilon(s) - z_\epsilon(s)\big|^2\big)\,ds\right) \\ &\leq 6\epsilon^{4H} u \int_0^t \kappa\big(E\big|x_\epsilon(s) - z_\epsilon(s)\big|^2\big)\,ds. \end{aligned}$$

Now, to deal with Q_{12}, let m be the largest positive integer such that $mT \leq t$. Then, for every $i = 1, \ldots, m$,

$$\begin{aligned}
Q_{12} &\leq 12\epsilon^{4H} E\left|\sum_{i=1}^{m} \int_{(i-1)T}^{iT} \left[a\left(s, z_\epsilon(s)\right) - \bar{a}\left(z_\epsilon(s)\right)\right] ds\right|^2 \\
&\quad + 12\epsilon^{4H} E\left|\int_{mT}^{t} \left[a\left(s, z_\epsilon(s)\right) - \bar{a}\left(z_\epsilon(s)\right)\right] ds\right|^2 \\
&\leq 36\epsilon^{4H} E\left|\sum_{i=1}^{m} \int_{(i-1)T}^{iT} \left[a\left(s, z_\epsilon(s)\right) - a\left(s, z_\epsilon(iT)\right)\right] ds\right|^2 \\
&\quad + 36\epsilon^{4H} E\left|\sum_{i=1}^{m} \int_{(i-1)T}^{iT} \left[a\left(s, z_\epsilon(iT)\right) - \bar{a}\left(z_\epsilon(iT)\right)\right] ds\right|^2 \\
&\quad + 36\epsilon^{4H} E\left|\sum_{i=1}^{m} \int_{(i-1)T}^{iT} \left[\bar{a}\left(z_\epsilon(iT)\right) - \bar{a}\left(z_\epsilon(s)\right)\right] ds\right|^2 \\
&\quad + 12\epsilon^{4H} E\left|\int_{mT}^{t} \left[a\left(s, z_\epsilon(s)\right) - \bar{a}\left(z_\epsilon(s)\right)\right] ds\right|^2.
\end{aligned} \tag{6}$$

Note that, by the definition of $\bar{a}$, we have

$$\begin{aligned}
&E\left|\sum_{i=1}^{m} \int_{(i-1)T}^{iT} \left[a\left(s, z_\epsilon(iT)\right) - \bar{a}\left(z_\epsilon(iT)\right)\right] ds\right|^2 \\
&\quad \leq m \sum_{i=1}^{m} E\left|\int_0^T a\left(s, z_\epsilon(iT)\right) ds - T\bar{a}\left(z_\epsilon(iT)\right)\right|^2 \\
&\quad = 0,
\end{aligned}$$

thus, by the Jensen inequality and Assumptions A, C, Eq. (6) becomes

$$Q_{12} \leq 72\epsilon^{4H} mT \int_0^T \kappa\left(E\left|z_\epsilon(s) - z_\epsilon(iT)\right|^2\right) ds + 48\epsilon^{4H} uMT.$$

Then we can deduce that Q_1 has the following approximation:

$$\begin{aligned}
Q_1 &\leq 6\epsilon^{4H} u \int_0^t \kappa\left(E\left|x_\epsilon(s) - z_\epsilon(s)\right|^2\right) ds \\
&\quad + 72\epsilon^{4H} mT \int_0^T \kappa\left(E\left|z_\epsilon(s) - z_\epsilon(iT)\right|^2\right) ds + 48\epsilon^{4H} uMT \\
&:= \epsilon^H K_1 \int_0^t \kappa\left(E\left|x_\epsilon(s) - z_\epsilon(s)\right|^2\right) ds \\
&\quad + \epsilon^H K_2 \int_0^T \kappa\left(E\left|z_\epsilon(s) - z_\epsilon(iT)\right|^2\right) ds + \epsilon^H O_1.
\end{aligned} \tag{7}$$

Now, to estimate Q_2, we have

$$
\begin{aligned}
Q_2 &\leq 6\epsilon^{2H} E\left|\int_0^t b\big(s, x_\epsilon(s)\big) - b\big(s, z_\epsilon(s)\big)\, d^\circ W^H(s)\right|^2 \\
&\quad + 6\epsilon^{2H} E\left|\int_0^t b\big(s, z_\epsilon(s)\big) - \bar{b}\big(z_\epsilon(s)\big)\, d^\circ W^H(s)\right|^2 \\
&= Q_{21} + Q_{22}.
\end{aligned}
\tag{8}
$$

Thanks to Lemma 2.7 and Assumption A, we can obtain

$$
Q_{21} \leq 12\epsilon^{2H} u^{2H-1} H \int_0^t \kappa\left(E\big|x_\epsilon(s) - z_\epsilon(s)\big|^2\right) ds + 24\epsilon^{2H} u^2 C.
$$

And, similar to Eq. (6),

$$
\begin{aligned}
Q_{22} &\leq 12\epsilon^{2H} E\left|\sum_{i=1}^m \int_{(i-1)T}^{iT} \big[b\big(s, z_\epsilon(s)\big) - \bar{b}\big(z_\epsilon(s)\big)\big]\, d^\circ W^H(s)\right|^2 \\
&\quad + 12\epsilon^{2H} E\left|\int_{mT}^t \big[b\big(s, z_\epsilon(s)\big) - \bar{b}\big(z_\epsilon(s)\big)\big]\, d^\circ W^H(s)\right|^2 \\
&\leq 36\epsilon^{2H} E\left|\sum_{i=1}^m \int_{(i-1)T}^{iT} \big[b\big(s, z_\epsilon(s)\big) - b\big(s, z_\epsilon(iT)\big)\big]\, d^\circ W^H(s)\right|^2 \\
&\quad + 36\epsilon^{2H} E\left|\sum_{i=1}^m \int_{(i-1)T}^{iT} \big[b\big(s, z_\epsilon(iT)\big) - \bar{b}\big(z_\epsilon(iT)\big)\big]\, d^\circ W^H(s)\right|^2 \\
&\quad + 36\epsilon^{2H} E\left|\sum_{i=1}^m \int_{(i-1)T}^{iT} \big[\bar{b}\big(z_\epsilon(iT)\big) - \bar{b}\big(z_\epsilon(s)\big)\big]\, d^\circ W^H(s)\right|^2 \\
&\quad + 12\epsilon^{2H} E\left|\int_{mT}^t \big[b\big(s, z_\epsilon(s)\big) - \bar{b}\big(z_\epsilon(s)\big)\big]\, d^\circ W^H(s)\right|^2,
\end{aligned}
$$

employing Lemma 2.7 and Assumptions A, C implies

$$
\begin{aligned}
Q_{22} &\leq 144\epsilon^{2H} u^{2H-1} mTH \int_0^T \kappa\left(E\big|z_\epsilon(s) - z_\epsilon(iT)\big|^2\right) ds + 288\epsilon^{2H} C u^2 \\
&\quad + 72\epsilon^{2H} u^{2H-1} mTHE \int_0^T \big|b\big(s, z_\epsilon(s)\big) - \bar{b}\big(z_\epsilon(s)\big)\big|^2\, ds + 144\epsilon^{2H} C u^2 \\
&\quad + 24\epsilon^{2H} u^{2H-1} HE \int_0^T \big|b\big(s, z_\epsilon(s)\big) - \bar{b}\big(z_\epsilon(s)\big)\big|^2\, ds + 48\epsilon^{2H} C u^2 \\
&\leq 144\epsilon^{2H} u^{2H-1} mTH \int_0^T \kappa\left(E\big|z_\epsilon(s) - z_\epsilon(iT)\big|^2\right) ds + 288\epsilon^{2H} C u^2 \\
&\quad + 288\epsilon^{2H} u^{2H-1} mHT^2 M + 144\epsilon^{2H} C u^2 \\
&\quad + 96\epsilon^{2H} u^{2H-1} mHT^2 M + 48\epsilon^{2H} C u^2.
\end{aligned}
$$

Consequently, taking Q_{21} and Q_{22} into account, we conclude that

$$\begin{aligned} Q_2 &\leq 12\epsilon^{2H}u^{2H-1}H\int_0^t \kappa\left(E\left|x_\epsilon(s)-z_\epsilon(s)\right|^2\right)ds + 24\epsilon^{2H}u^2C \\ &\quad + 144\epsilon^{2H}u^{2H-1}mTH\int_0^T \kappa\left(E\left|z_\epsilon(s)-z_\epsilon(iT)\right|^2\right)ds + 288\epsilon^{2H}Cu^2 \\ &\quad + 288\epsilon^{2H}u^{2H-1}mHT^2M + 144\epsilon^{2H}Cu^2 + 96\epsilon^{2H}u^{2H-1}mHT^2M + 48\epsilon^{2H}Cu^2 \\ &:= \epsilon^H K_3\int_0^t \kappa\left(E\left|x_\epsilon(s)-z_\epsilon(s)\right|^2\right)ds \\ &\quad + \epsilon^H K_4\int_0^T \kappa\left(E\left|z_\epsilon(s)-z_\epsilon(iT)\right|^2\right)ds + \epsilon^H O_2. \end{aligned} \tag{9}$$

Arriving at the last term Q_3, we apply Assumption B to obtain

$$\begin{aligned} Q_3 &\leq 6\epsilon^{4H}k(m+1)E\sum_{j=1}^k \left|I_j\left(x_\epsilon(t_j)\right)\right|^2 \\ &\quad + 6\epsilon^{4H}k(m+1)\frac{t}{T^2}E\sum_{j=1}^k\int_0^t \left|I_j\left(z_\epsilon(s)\right)\right|^2 ds \\ &\leq 6\epsilon^{4H}k^2(m+1)N_1 + 6\epsilon^{4H}k^2(m+1)^2N_1 := \epsilon^H O_3. \end{aligned} \tag{10}$$

Now, combining (7), (9), and (10) together, we get

$$\begin{aligned} E\left|x_\epsilon(t)-z_\epsilon(t)\right|^2 &\leq \epsilon^H\tilde{O} + \epsilon^H(K_2+K_4)\int_0^T \kappa\left(E\left|z_\epsilon(s)-z_\epsilon(iT)\right|^2\right)ds \\ &\quad + \epsilon^H(K_1+K_3)\int_0^t \kappa\left(E\left|x_\epsilon(s)-z_\epsilon(s)\right|^2\right)ds, \end{aligned} \tag{11}$$

where $\tilde{O} = O_1 + O_2 + O_3$. Obviously, the function $\kappa(x)$ is nondecreasing on $\mathbb{R}_+$ and $\kappa(0) = 0$. Then, for any $t_0 > 0$, by setting $G(t) = \int_{t_0}^t \frac{ds}{\kappa(s)}$, it follows from Lemma 2.8 that

$$\begin{aligned} E\left|x_\epsilon(t)-z_\epsilon(t)\right|^2 &\leq G^{-1}\Bigg(G\left[\epsilon^H\tilde{O} + \epsilon^H(K_2+K_4)\int_0^T \kappa\left(E\left|z_\epsilon(s)-z_\epsilon(iT)\right|^2\right)ds\right] \\ &\quad + \epsilon^H(K_1+K_3)T\Bigg). \end{aligned}$$

Note that

$$\left\{\epsilon^H\tilde{O} + \epsilon^H(K_2+K_4)\int_0^T \kappa\left(E\left|z_\epsilon(s)-z_\epsilon(iT)\right|^2\right)ds\right\} \to 0,$$

as ϵ converges to zero. Recalling the condition $\int_{0^+}\frac{ds}{\kappa(s)} = \infty$, we can conclude that

$$G\left[\epsilon^H\tilde{O} + \epsilon^H(K_2+K_4)\int_0^T \kappa\left(E\left|z_\epsilon(s)-z_\epsilon(iT)\right|^2\right)ds\right] + \epsilon^H(K_1+K_3)T \to -\infty.$$

On the other hand, because the function G is strictly increasing, we obtain that G has an inverse function which is strictly increasing too, and $G^{-1}(-\infty) = 0$. Namely,

$$G^{-1}\left(G\left[\epsilon^H \tilde{O} + \epsilon^H (K_2 + K_4) \int_0^T \kappa\left(E\left|z_\epsilon(s) - z_\epsilon(iT)\right|^2\right) ds\right] + \epsilon^H (K_1 + K_3) T\right) \to 0$$

as $\epsilon \to 0$. Finally, we get

$$\begin{aligned} &\lim_{\epsilon\to 0} E\left|x_\epsilon(t) - z_\epsilon(t)\right|^2 \\ &\quad\le \lim_{\epsilon\to 0}\left(G^{-1}\left(G\left[\epsilon^H(\tilde{O}) + \epsilon^H (K_2 + K_4) \int_0^T \kappa\left(E\left|z_\epsilon(s) - z_\epsilon(iT)\right|^2\right) ds\right]\right.\right. \\ &\qquad \left.\left. + \epsilon^H (K_1 + K_3) T\right)\right) \\ &\quad = 0. \end{aligned}$$

This completes the proof. □

Remark 3.2 In Theorem 3.1, we establish the strong convergence (in the moment sense) of the processes x_ϵ and z_ϵ under non-Lipschitz condition. In other words, we have proved that, for a sufficiently small ϵ, the solutions of x_ϵ and z_ϵ are close enough.

For the sake of establishing the periodic stochastic averaging of Eq. (4) in finite time interval, we need the following auxiliary lemma.

Lemma 3.3 *Let* (4) *be averaged SDE of standard ISDE* (3). *If Assumption* C *holds, then, for* $\epsilon_1 \in (0, \epsilon_0]$, *there exist* $\epsilon \in (0, \epsilon_1]$ *and a positive constant* $D > 0$ *such that*

$$E\left|z_\epsilon(t) - z_\epsilon(iT)\right|^2 \le D$$

for all $t \in [(i-1)T, iT]$, $i = 1, 2, \dots, m$, $m \in \mathbb{N}$.

Proof By Eq. (4), taking expectation and using the simple inequality $|a+b|^2 \le 2|a|^2 + 2|b|^2$ yield

$$E\left|z_\epsilon(t) - z_\epsilon(iT)\right|^2 \le 2\epsilon^{4H} E\left|\int_{iT}^t \bar{a}\left(z_\epsilon(s)\right) ds\right|^2 + 2\epsilon^{2H} E\left|\int_{iT}^t \bar{b}\left(z_\epsilon(s)\right) d^\circ W^H(s)\right|^2.$$

Now, let $0 \le t \le u \le T$, then by the Cauchy–Schwarz inequality, Lemma 2.7, and Assumption C we get

$$E\left|\int_{iT}^t \bar{a}\left(z_\epsilon(s)\right) ds\right|^2 \le uE \int_{iT}^t \left|\bar{a}\left(z_\epsilon(s)\right)\right|^2 ds \le (m+1)T^2 M$$

and

$$\begin{aligned} E\left|\int_{iT}^t \bar{b}\left(z_\epsilon(s)\right) d^\circ W^H(s)\right|^2 &\le 2Hu^{2H-1} E \int_{iT}^t \left|\bar{b}\left(z_\epsilon(s)\right)\right|^2 ds + 4Cu^2 \\ &\le 2H(m+1)^{2H-1} T^{2H} M + 4C(m+1)^2 T^2, \end{aligned}$$

where m is the largest integer such that $mT \le t$.

Finally, one can deduce that

$$
\begin{aligned}
E\left|z_\epsilon(t)-z_\epsilon(iT)\right|^2 &\le 2\epsilon^{4H}(m+1)T^2M+4\epsilon^{2H}H(m+1)^{2H-1}T^{2H}M \\
&\quad + 8\epsilon^{2H}C(m+1)^2T^2 \\
&:= D.
\end{aligned}
$$

Hence, proved. □

Theorem 3.4 *Suppose that Assumptions A–C are fulfilled for standard ISDE (3) and for averaged SDE (4), then, for given $\beta > 0$, $\alpha \in (0,1)$, and $\epsilon_1 \in (0,\epsilon_0]$, there exist $\gamma > 0$ and $\epsilon \in (0,\epsilon_1]$ such that*

$$
E\left|x_\epsilon(t)-z_\epsilon(t)\right|^2 \le \gamma\epsilon^H \tag{12}
$$

for all $t \in [0,\beta\epsilon^{-\alpha H}]$.

Proof By Assumption A, the concave function κ satisfies

$$
\kappa(x) \le \lambda_1 x + \lambda_2,
$$

where λ_1 and λ_2 are positive constants. Applying this property for Eq. (11) yields

$$
\begin{aligned}
E\left|x_\epsilon(t)-z_\epsilon(t)\right|^2 &\le \epsilon^H K_5\lambda_1\int_0^t E\left|x_\epsilon(s)-z_\epsilon(s)\right|^2\,ds + \epsilon^H K_5\lambda_2 t \\
&\quad + \epsilon^H K_6\lambda_1\int_0^T E\left|z_\epsilon(s)-z_\epsilon(iT)\right|^2\,ds + \epsilon^H K_6\lambda_2 T + \epsilon^H\tilde{O}.
\end{aligned}
$$

Thanks to Lemma 3.3, we can obtain

$$
\begin{aligned}
E\left|x_\epsilon(t)-z_\epsilon(t)\right|^2 &\le \epsilon^H K_5\lambda_1\int_0^t E\left|x_\epsilon(s)-z_\epsilon(s)\right|^2\,ds + \epsilon^H K_5\lambda_2 t \\
&\quad + \epsilon^H K_6\lambda_1 DT + \epsilon^H K_6\lambda_2 T + \epsilon^H\tilde{O} \\
&:= \epsilon^H K\int_0^t E\left|x_\epsilon(s)-z_\epsilon(s)\right|^2\,ds + \epsilon^H O.
\end{aligned}
$$

Finally, applying Gronwall's inequality implies

$$
E\left|x_\epsilon(t)-z_\epsilon(t)\right|^2 \le \epsilon^H O e^{\epsilon^H K t}.
$$

Now, choose $\alpha \in (0,1)$ and $\beta > 0$, we can select $\epsilon_1 \in (0,\epsilon_0]$ such that, for every $\epsilon \in (0,\epsilon_1]$, $t \in [0,\beta\epsilon^{-\alpha H}] \subseteq [0,\infty)$. And let $\gamma = Oe^{K\beta}$, we conclude

$$
E\left|x_\epsilon(t)-z_\epsilon(t)\right|^2 \le \gamma\epsilon^H.
$$

Therefore, Theorem 3.4 is proved. □

Remark 3.5 *Theorem 3.4 indicates that the order of convergence of the processes x_ϵ and z_ϵ in finite time is about $\epsilon^{-\alpha H}$ for $\alpha \in (0,1)$.*

Next, we shall use the previous results to establish the convergence in probability between the solutions of Eq. (3) and Eq. (4).

Corollary 3.6 *Let Assumptions A–C hold, for arbitrary small number $\delta > 0$, there exist $\epsilon_1 \in (0,\epsilon_0]$, $\beta > 0$, and $0 < \alpha < 1$ such that, for all $\epsilon \in (0,\epsilon_1]$, we have*

$$\lim_{\epsilon\to 0} P\Bigl(\sup_{0\le t\le \beta\epsilon^{-\alpha H}} \bigl|x_\epsilon(t) - z_\epsilon(t)\bigr| > \delta\Bigr) = 0.$$

Proof By Theorem 3.4 and employing the Chebyshev–Markov inequality, for any given number $\delta > 0$, one can obtain that

$$\begin{aligned} P\Bigl(\sup_{0\le t\le \beta\epsilon^{-\alpha H}} \bigl|x_\epsilon(t) - z_\epsilon(t)\bigr| > \delta\Bigr) &\le \frac{1}{\delta^2} E\Bigl(\sup_{0\le t\le \beta\epsilon^{-\alpha H}} \bigl|x_\epsilon(t) - z_\epsilon(t)\bigr|^2\Bigr) \\ &\le \frac{\epsilon^H O e^{\epsilon^H K t}}{\delta^2}, \end{aligned}$$

letting $\epsilon \to 0$. Then the required result follows. □

4 Example

In this section, we provide an example to illustrate the foregoing averaging principle results.

Consider the following impulsive stochastic dynamical system:

$$\begin{aligned} &dx_\epsilon(t) = -\epsilon^{2H}\,dt + \epsilon^H \cos^2(t)\lambda\, d^\circ W^H(t), \quad t \ne t_j \\ &\triangle x_\epsilon(t) = \epsilon^{2H} j^3 x_\epsilon\bigl(t_j^-\bigr), \quad t = t_j, j \in \mathbb{N}, \\ &x_\epsilon(0) = x_0, \end{aligned} \tag{13}$$

where $a(t,x) = -1$, $b(t,x) = \lambda\cos^2(t)$, and $I_j(x) = j^3 x$. Let $T = 1$ and $\lambda = 3$. Then, by the definitions of $\bar{a}(\cdot)$, $\bar{b}(\cdot)$, and $\bar{I}(\cdot)$ in Sect. 2, we have

$$\begin{aligned} &\bar{a}(z_\epsilon) = \frac{1}{T}\int_0^T a(t,z_\epsilon)\,dt = -1, \\ &\bar{b}(z_\epsilon) = \frac{1}{T}\int_0^T b(t,z_\epsilon)\,dt = \lambda\int_0^1 \cos^2(t)\,dt = 3\times 0.73 = 2.19, \\ &\bar{I}(z_\epsilon) = \frac{1}{T}\sum_{j=1}^k j^3 z_\epsilon = z_\epsilon \sum_{j=1}^k j^3 = \frac{k^2(k+1)^2}{4} z_\epsilon. \end{aligned}$$

So then, the solution to averaged SDE for the impulsive dynamical system (13) can be interpreted as follows:

$$z_\epsilon(t) = x_0 + \frac{k^2(k+1)^2}{4}\epsilon^{2H}\int_0^t \bigl(z_\epsilon(s) - 1\bigr)\,ds + 2.19\epsilon^H \int_0^t d^\circ W^H(t). \tag{14}$$

It is easy to verify that the conditions of Theorem 3.1, Theorem 3.4, and Corollary 3.6 are satisfied. Then the solution of averaged SDEs (14) converges to that of standard Eq. (13) in the sense of mean square and in probability.

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