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Arbitrary-order economic production quantity model with and without deterioration: generalized point of view

Mostafijur Rahaman[1], Sankar Prasad Mondal[2], Ali Akbar Shaikh[3], Ali Ahmadian[4*], Norazak Senu[4] and Soheil Salahshour[5]

*Correspondence:
ahmadian.hosseini@gmail.com
[4]Institute for Mathematical Research, Universiti Putra Malaysia, Serdang, Malaysia
Full list of author information is available at the end of the article

Abstract

The key objective of this paper is to study and discuss the application of fractional calculus on an arbitrary-order inventory control problem. Using the concepts of fractional calculus followed by fractional derivative, we construct different possible models like generalized fractional-order economic production quantity (EPQ) model with the uniform demand and production rate and generalized fractional-order EPQ model with the uniform demand and production rate and deterioration. Also, we show that the classical EPQ model is the particular case of the corresponding generalized fractional EPQ model. This greatly facilitates the researcher a novel tactic to analyse the solution of the EPQ model in the presence of fractional index. Furthermore, this attempt also provides the solution obtained through the optimization techniques after using the real distinct poles rational approximation of the generalized Mittag-Leffler function.

Keywords: Fractional derivative; Differential equation of arbitrary-order; EPQ model; Laplace transformation; Real distinct poles rational approximation of the generalized Mittag-Leffler function; Geometric programming for optimization

1 Introduction

The journey of the concept of fractional calculus (FC) has started in the seventeenth century. Newton and Leibnitz are considered to be the first researchers among the mathematicians for working in this direction in mathematics. Through many decades the concepts regarding FC has been used to illustrate many real life problems relating to various fields of science and economy. FC explores integrals and derivatives of functions. Here the order of differentiation may be real or complex and hence in a particular case it may be of integer order. These days, global interest in FC has seemed to be exponential. Due to the different results of fractional derivatives, FC has attained much attention for modelling of the image processes, and various fields of mathematics, economics, physics and engineering [1–7].

A differential equation is understood to be a fractional differential equation (FDE) when differential operator is of a fractional order. FC simplifies the idea of a derivative of the integer order to a derivative operator of an arbitrary-order (may be real or complex). Both in the theoretic and useful outcomes had sharp out that FDEs extant outstanding outcomes in solving the different multipart problems that stand up in the field of engineering and science. It was established by several authors (see, e.g., [8–10]). It has been established that the fractional-order modelling is mostly beneficial to symbolize systems where computational effect on productions, this quality is the most important gain [11]. For the solutions of FDEs, it is complicated to acquire the analytical solutions owing their nonlocal stuff of the fractional derivative. Therefore, some systems were altered to obtain assessed solutions such as spectral methods [12], variational iteration methods [13], differential transform method [14] and Adomian decomposition method [15]. In recent years, researchers have had much attention for the field of FDEs. Recently, Hajipour et al. [16] discussed an accurate discretization of a variable-order fractional reaction–diffusion equation. The variable-order fractional description of compression deformation of amorphous glassy polymers was considered by Meng et al. [17]. The results for the nonlinear dynamical systems within the generalized fractional derivatives with Mittag-Leffler kernel were illustrated by Baleanu et al. [18]. For application purposes a new fractional analysis on the interaction of HIV with CD4+ T-cells was treated by Jajarmi and Baleanu [19]. A new aspect of the poor nutrition in the life cycle within the FC was investigated by Baleanu et al. [20]. Suboptimal control of fractional-order dynamic systems with delay argument was investigated by Jajarmi and Baleanu [21]. Recently, some interesting work regarding very realistic and physical problems was done as an application of the fractional- or arbitrary-order derivative method [22]. A fractional epidemiological model has been studied to describe computer viruses by Singh et al. [23]. Thereafter, Singh et al. [24] extended the Biswas–Milovic (BM) model which plays a vital role to describe the long distance optical communications. In addition, a rumor spreading dynamical model in a social network has been described by Singh. et al. [25]. A diabetes model and its complications with the Caputo–Frarizo fractional derivative have been investigated by Singh et al. [26]. Also, there is some work related to the application of FC on some production inventory problems which is interesting [27–31]. Also, a solution algorithm of fractional Drinfeld–Sokolov–Wilson equation in the numerical approach was given by Singh et al. [32].

1.1 Research gap between our work and related published work with our contribution

There are few papers [28–31] on the application of FC on EOQ models. On comparison, the same work on EPQ model is a little rare [27]. Moreover, the existing literature does not explain clearly the importance of the introduction of fractional concept on inventory theory. Besides that, there is no detailed explanation of the analytical or numerical optimization on the generalized EPQ model. The present study tries to overcome this deficiency.

Apart from these, in this paper we introduce the generalized EPQ model with deterioration which is totally novel. For numerical and analytical optimization, we use in a real distinct pole approximation the Mittag-Leffler function which is given by Iyiola, Asante-Asamani and Wade [33].

The contribution of the current study is:

(i) The modification of the woks by Das and Roy [27], especially in numerical result and motivation for using arbitrary order.

(ii) Arbitrary-order generalization of an EPQ model with and without deterioration when only the rate change of the inventory problem is arbitrary.

(iii) Application of a real distinct pole rational approximation to optimize the inventory problem with Mittag-Leffler function. This was not used by earlier researchers of the fractional-order inventory model.

1.2 The idea for taking the use of fractional calculus in inventory model

Firstly, though the concept of the FC is a little abstract, one of its finest physical interpretations is that it has power to remember the previous effects of the input in order to determine the current output. Again, in the real world production system, the demand varies with the environment and circumstances. One of the important issues for the marketing system is the memory effect. Generally, the selling of the products depends on the quality of the product as well as the attitude or dealing policy of the supplier to the customers. Due to the good previous experience, a consumer gains interest for buying a product. On the contrary, for the bad impression on product or supplier, the consumer's demand gradually decreases. Thus, the memory effect is an important issue for the management system and intuitionally the introduction of the FC is justified.

Secondly, our main objective in this study is to view some common EPQ models in more generalized form. In this perspective, arbitrary-order calculus is an important tool. Usually, an inventory model is described by differential equations of integer order. But one thing to remember is that the FC is not the calculus of fractional order only. Here, the order may be real, complex or in particular integer.

So, if we use the notion of fractional derivative, integration, and differential equation, then we have the following facilities.

First of all, the EPQ model described by FDE is more realistic as it illustrates the idea of memory effect of the previous experience. Secondly, introducing the fractional (arbitrary) differential equation to describe the model, we can extend the theory of EPQ. And lastly, the classical inventory model with integer-order differential equation can be described as a particular case of the fractional one.

1.3 Structure of the paper

Furthermore, the remaining structure of the paper is organized as follows. After a detailed discussion and a general overview on FC in Sect. 2, the notations, units and their descriptions are elaborated in Sect. 3. The inventory problem of EPQ type is defined in Sect. 4. Details discussion of mathematical modelling for EPQ in different scenario is illustrated in Sect. 5. Theoretical or analytical results of corresponding optimization of the problems are shown in Sect. 6. The numerical results are shown in Sect. 7. The concluding remarks are made in Sect. 8.

2 General overview on fractional calculus

2.1 Riemann–Liouville fractional derivative

Let f be a real valued continuous function. Then the left Riemann–Liouville derivative of fractional order α is defined to be

$$ {}_aD_x^{\alpha}\bigl(f(x)\bigr) = \frac{1}{\Gamma(m-\alpha)}\left(\frac{d}{dx}\right)^m \int_a^x (x-t)^{(m-\alpha-1)} f(t)\,dt, $$

where $x > 0$ and $m < \alpha < m + 1$.

And the right Riemann–Liouville derivative of fractional order α is defined to be

$$ {}_aD_x^{\alpha}\big(f(x)\big) = \frac{1}{\Gamma(m-\alpha)}\left(-\frac{d}{dx}\right)^m \int_x^a (x-t)^{(m-\alpha-1)} f(t)\,dt, $$

where $x > 0$ and $m < \alpha < m+1$.

Remark 2.1 The basic difference of the R–L derivative from the ordinary calculus is that the R–L derivative of constant term is not equal to zero.

Remark 2.2 For $m = 1$ the left and right Riemann–Liouville derivative will be of the form ${}_aD_x^{\alpha}(f(x)) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_a^x (x-t)^{-\alpha} f(t)\,dt$, where $x > 0$ and ${}_aD_x^{\alpha}(f(x)) = \frac{1}{\Gamma(1-\alpha)}(-\frac{d}{dx})\int_x^a (x-t)^{-\alpha} f(t)\,dt$, where $x > 0$.

2.2 Riemann–Liouville fractional integral

Let f be a real valued continuous function. Then the Riemann–Liouville integral of fractional order α is defined to be

$$ {}_aD_x^{-\alpha}\big(f(x)\big) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{(\alpha-1)} f(t)\,dt, \quad \text{where } x > 0 \text{ and } m < \alpha < m+1. $$

Remark 2.3 For $a = 0$ above we have the definition of the Riemann integral and for $a = -\infty$ the same is the definition of the Liouville integral.

2.3 Memory dependent derivative

Using the kernel, the derivative of any function can be written as

$$ D\big(f(x)\big) = \int_a^x K(x-t) f'(t)\,dt. $$

Remark 2.4

(1) If we take $K(x-t) = \delta(x-t)$, it gives the memory-less derivative i.e. a derivative of integer order.

(2) If we consider $K(x-t) = \frac{(x-t)^{(m-\alpha)}}{\Gamma(m-\alpha)}$, then we get the expression

$$ D_a^{\alpha}\big(f(x)\big) = \int_a^x K(x-t) f^m(t)\,dt. $$

Here f^m denotes the common derivative of mth order.

(3) The fractional derivative is not a local property. The total effects of the αth order derivative on the interval $[a,x]$ describe the variation of a system in which the instantaneous change rate depends on the past state; it is called the "memory effect".

(4) The memory strength is controlled by α. As $\alpha \to 1$, the system becomes weaker in the sense of memory and for $\alpha = 1$ it becomes totally memory-less. Lower value of α indicates a long memory of the system.

2.4 Laplace transformation of fractional derivative

The differential equation of arbitrary-order is the generalization of the differential equation. There are several ideas to find the solution of a fractional differential equation and the Laplace transformation is one of them. In this section of our study, we give a brief discussion of the Laplace transformation of a fractional derivative and its consequence.

Definition 2.1 The Laplace transform of a function $f(t)$ is given by

$$F(s) = L\{f(t);s\} = \int_0^\infty e^{-st} f(t)\,dt.$$

Definition 2.2 The inverse Laplace transform of a function $F(s)$ is given by

$$f(t) = L^{-1}(F(s)) = \int_{-\infty}^\infty e^{st} F(s)\,ds.$$

Corollary 2.1 *The Laplace transformation of the derivative of the integer order n is given by*

$$L\{f^n(t);s\} = s^n F(s) - \sum_{k=0}^{n-1} s^k f^{(n-k-1)}(0) = s^n F(s) - \sum_{k=0}^{n-1} s^{(n-k-1)} f^k(0).$$

2.5 Laplace transformation of the Riemann–Liouville derivative and integral

The Laplace transform associated with the Riemann–Liouville derivative having fractional order $p > 0$ is given by

$$L\{{}_0D_t^p f(t);s\} = s^p F(s) - \sum_{k=0}^{n-1} s^k \bigl[{}_0D_t^{p-k-1} f(0)\bigr], \quad \text{for } n-1 \le p < n.$$

The Laplace transform associated with the Riemann–Liouville integral having the order p, where $p > 0$ is

$$L\{{}_0D_t^{-p} f(t);s\} = s^{-p} F(s).$$

Corollary 2.2

(1) *For $n = 1$ the last expression takes the form*

$$L\{{}_0D_t^p f(t);s\} = s^p F(s) - {}_0D_t^{p-1} f(0), \quad \textit{where } 0 \le p < 1.$$

(2) *For $n = 2$ the last expression takes the form*

$$L\{{}_0D_t^p f(t);s\} = s^p F(s) - {}_0D_t^{p-1} f(0) - s\bigl[{}_0D_t^{p-2} f(0)\bigr], \quad \textit{where } 1 \le p < 2.$$

(3) *The m-times differentiated Mittag-Leffler function is given by*

$$E_{\alpha,\beta}^{(m)}(z) = \sum_{k=0}^\infty \frac{(k+m)!}{k!} \frac{z^k}{\Gamma(\alpha k + \alpha m + \beta)}.$$

Then $L\{t^{\alpha m+\beta-1} E_{\alpha,\beta}^{(m)}(at^\alpha);s\} = \frac{m! s^{\alpha-\beta}}{(s^\alpha - a)^{m+1}}$.

2.6 Gamma function

Let $z \in \mathbb{C}$, then the Gamma function is given by

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\,dt, \quad \text{for } \operatorname{Re}(z) > 0.$$

By splitting this integral, at a point $x \geq 0$, we obtain two incomplete gamma functions;

$$\gamma(z,x) = \int_0^x e^{-t} t^{z-1}\,dt,$$

and $\Gamma(z,x) = \int_x^\infty e^{-t} t^{z-1}\,dt$.

Remark 2.5
(1) $\gamma(z,x) + \Gamma(z,x) = \Gamma(z)$, for all $x \geq 0$ and for all $\operatorname{Re}(z) > 0$.
(2) If we consider $\Gamma(1,x) = e^{-x}$ then $\gamma(1,x) = 1 - e^{-x}$.

2.7 Beta function

Let $z, w \in \mathbb{C}$ then the Beta function is given by

$$\mathrm{B}(z,w) = \int_0^1 t^{z-1}(1-t)^{w-1}\,dt, \quad \text{for } \operatorname{Re}(z), \operatorname{Re}(w) > 0.$$

2.8 The Mittag-Leffler function

The exponential function e^z has a great importance in the study of the differential equation of integer order. This can also be written in a series form, which is given by

$$e^z = \sum_{k=0}^\infty \frac{z^k}{\Gamma(k+1)}.$$

More generally, we can consider the expression

$$E_\alpha(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k+1)}, \quad \text{which is named the Mittag-Leffler function,}$$

and the further generalization

$$E_{\alpha,\beta}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k+\beta)}, \quad \text{where } \alpha, \beta \in \mathbb{C} \text{ and } \operatorname{Re}(\alpha) > 0;$$

this is also called the Mittag-Leffler function.

Remark 2.6
(1) For the special case of $\alpha = 1$ and $\beta = 1$, we have $E_{1,1}(z) = e^z$.
(2) The Mittag-Leffler function plays a major role n the study of the functional calculus.

Corollary 2.3 *Let $z \in \mathbb{C}$, $\alpha, \beta \in \mathbb{C}$, $\operatorname{Re}(\alpha) > 0$ and $m \in \mathbb{N}$, then the m-times differentiated Mittag-Leffler function is given by*

$$E^{(m)}_{\alpha,\beta}(z) = \sum_{k=0}^\infty \frac{(k+m)!}{k!} \frac{z^k}{\Gamma(\alpha k + \alpha m + \beta)}.$$

2.9 A real distinct pole rational approximation of the generalized Mittag-Leffler function

A real distinct pole rational function is given by

$$R^*(z) = \frac{1+az}{(1-bz)(1-cz)}, \quad \text{where } a,b,c \in \mathbb{R} \text{ and } a \neq b.$$

A real distinct pole rational approximation of the generalized Mittag-Leffler function,

$$E_{\alpha,\beta}(-z) = \sum_{k=0}^{\infty} \frac{(-z)^k}{\Gamma(\alpha k+\beta)},$$

is given by

$$E_{\alpha,\beta}(-z) \approx \frac{1-az}{\Gamma(\beta)(1+bz)(1+cz)},$$

where $a = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} - b - c$, $b = \frac{\Gamma(\alpha+\beta)}{\Gamma(2\alpha+\beta)} - \frac{1}{4}$, $c = \frac{\Gamma(\beta)\Gamma(2\alpha+\beta)}{\Gamma(\alpha+\beta)(5\Gamma(2\alpha+\beta)-4\Gamma(\alpha+\beta))}$.

Special cases

(1) $E_\alpha(z) = E_{\alpha,1}(z) \approx \frac{1+az}{(1-bz)(1-cz)}$.
Here $a = \frac{1}{\alpha\Gamma(\alpha)} - b - c$, $b = \frac{\Gamma(\alpha)}{2\Gamma(2\alpha)} - \frac{1}{4}$, $c = \frac{\Gamma(2\alpha)}{\alpha\Gamma(\alpha)(5\Gamma(2\alpha)-2\Gamma(\alpha))}$.

(2) $e^z = E_{1,1}(z) \approx \frac{1+\frac{5}{12}z}{(1-\frac{1}{4}z)(1-\frac{1}{3}z)}$.

Remark 2.7 This is useful for solving scalar linear fractional differential equations.

3 Different notations, units and their description for inventory model

To describe our proposed problem, we use the following notations with certain units and description (see Tables 1 and 2).

Table 1 EPQ model without deterioration

Notations	Units	Descriptions
c_1	₴/unit	Holding cost per unit time
c_3	₴/unit	Ordering cost per unit time
K	Units	Production rate per cycle
D	Units	Demand rate per cycle
T	Year	Total time cycle
t_1	Year	Production time
$q_{\max}$	Units	Highest level of inventory
TAC	₴/Year	Total average cost
$\mathrm{TAC}_{\alpha,\beta}$	₴/Year	Total generalized average cost
α	Constant	The order of integration
β	Constant	The order of differention
Decision variable for integer-order model		
T	Year	Total time cycle
Decision variable for arbitrary-order model		
T	Year	Total time cycle
t_1	Year	Production time

Table 2 EPQ model with deterioration

Notations	Units	Descriptions
c_h	₴/unit	Holding cost per unit time
c_0	₴/unit	Ordering cost per unit time
c_p	₴/unit	Production cost per unit time
K	Units	Production rate per cycle
D	Units	Demand rate per cycle
T	Year	Total time cycle
t_1	Year	Production time
$q_{\max}$	Units	Highest level of inventory
θ_1	Constant	Rate of deterioration in $[0, t_1]$
θ_2	Constant	Rate of deterioration in $[t_1, T]$
TAC	₴/Year	Total average cost
$\mathrm{TAC}_{\alpha,\beta}$	₴/Year	Total generalized average cost
α	Constant	The order of integration
β	Constant	The order of differention
Decision variable for integer/arbitrary-order model		
T	Year	Total time cycle
t_1	Year	Production time

4 Defining the problem for inventory planning of EPQ type

4.1 Integer-order EPQ model without deterioration

The classical (integer-order) EPQ model is developed under the following assumptions:

(i) Demand is deterministic and uniform.
(ii) No shortage is allowed.
(iii) Lead time is zero.
(iv) Production rate is finite.
(v) Planning horizon is infinite.

4.2 Arbitrary-order EPQ model with deterioration

Along with all the above-mentioned assumptions, here, the additional assumption is that the EPQ model is memory sensitive i.e. the demand depends on the memory of the customer with the previous experience concerned with the behaviour of the shopkeeper or the quality of the product etc.

4.3 Integer-order EPQ model with deterioration

Along with all the assumptions, mentioned in Section 4.1, the additional assumption is that the products will be deteriorated with different rates in production time and non-productive time.

4.4 Arbitrary-order EPQ model with deterioration

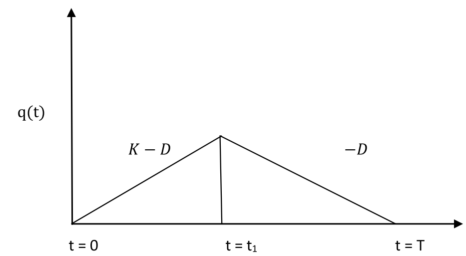
Along with all the assumptions, described in Section 4.3, the additional assumption is that the EPQ model is memory sensitive i.e. the demand depends on the memory of the customer with the previous experience concerned with the behaviour of the shopkeeper or the quality of the product.

5 Details discussion of mathematical modelling for EPQ in different scenario

5.1 Classical (integer-order) EPQ model without deterioration (Model 1)

There are two parts of this whole cycle during the first scheduling period T. The production starts at $t = 0$ and throughout the time interval $[0, t_1]$, the inventory level is increased

Figure 1 EPQ model without deterioration



gradually at the rate $K - D$ due to production rate K and to meet the demand the D. At the time $t = t_1$, the inventory reaches its highest level and here the production is stopped. Then during the time interval $[t_1, T]$, the inventory level gradually decreases as regards it being payable to happen up the customer's demands and at $t = T$ the inventory level reaches 0, as no shortage is allowed (see Fig. 1).

If $q(t)$ represents the inventory level at any time, then the corresponding differential equation is given by

$$\frac{dq(t)}{dt} = K - D, \quad \text{for } 0 \le t \le t_1, \tag{5.1}$$

$$\frac{dq(t)}{dt} = -D, \quad \text{for } t_1 \le t \le T. \tag{5.2}$$

We have the boundary conditions $q(0) = q(T) = 0$.

Now (5.1) along with the initial condition gives $q(t) = (K - D)t$, for $0 \le t \le t_1$.

Let $q(t_1) = q_{\max}$.

Now (5.2) along with the initial condition gives $q(t) = (T - t)D$, for $t_1 \le t \le T$.

So, we have

$$t_1 = \frac{q_{\max}}{K - D} \tag{5.3}$$

and

$$T - t_1 = \frac{q_{\max}}{D}. \tag{5.4}$$

So $T = t_1 + (T - t_1) = q_{\max}(\frac{1}{K-D} + \frac{1}{D})$, i.e.,

$$q_{\max} = D\left(1 - \frac{D}{K}\right)T. \tag{5.5}$$

Some relevant costs

(i) The holding cost is

$$\begin{aligned} \mathrm{HC} &= c_1\left[\int_o^{t_1} q(t)\,dt + \int_{t_1}^{T} q(t)\,dt\right] \\ &= c_1\left[(K - D)\frac{t_1^2}{2} + D\frac{(T - t_1)^2}{2}\right] \\ &= \frac{c_1}{2}\left(1 - \frac{D}{K}\right)DT^2, \quad \text{using the last three equations.} \end{aligned} \tag{5.6}$$

(ii) The total cost is

$$\mathrm{TC} = c_1 + \frac{c_1}{2}\left(1 - \frac{D}{K}\right)DT^2. \tag{5.7}$$

(iii) The total average cost is

$$\mathrm{TAC} = \frac{c_3}{T} + \frac{c_1}{2}\left(1 - \frac{D}{K}\right)\mathrm{DT}. \tag{5.8}$$

So, the classical EPQ model is

$$\begin{aligned} &\text{Minimize } \mathrm{TAC}(T) = \frac{c_3}{T} + \frac{c_1}{2}\left(1 - \frac{D}{K}\right)\mathrm{DT}. \\ &\quad \text{Subject to } T > 0. \end{aligned} \tag{5.9}$$

5.2 Generalized (arbitrary-order) EPQ model without deterioration (Model 2)

Here, we consider the memory effect with the same assumption and notations as for classical EPQ model. Then, the differential equation of fractional order α, due to memory $(0 \le \alpha \le 1)$, corresponding to the generalized EPQ model, is

$$\frac{d^\alpha q(t)}{dt^\alpha} = K - D, \quad \text{for } 0 \le t \le t_1, \tag{5.10}$$

$$\frac{d^\alpha q(t)}{dt^\alpha} = -D, \quad \text{for } t_1 \le t \le T, \tag{5.11}$$

$$\text{With the initial condition } q(0) = 0 \text{ and boundary condition } q(T) = 0. \tag{5.12}$$

Let

$$q(t_1) = q_{\max}. \tag{5.13}$$

Taking the Laplace transformation of (5.10) we get

$$s^\alpha \overline{q}(s) - s^{\alpha-0-1} q(0) = \frac{K - D}{s}.$$

Or

$$q(t) = L^{-1}\bigl(\overline{q}(s)\bigr) = (K - D)\frac{t^\alpha}{\Gamma(\alpha + 1)}. \tag{5.14}$$

Then

$$q(t_1) = q_{\max} = (K - D)\frac{t_1^\alpha}{\Gamma(\alpha + 1)}. \tag{5.15}$$

Again, taking the Laplace transformation of (5.11) we have

$$s^\alpha \overline{q}(s) - s^{\alpha-0-1} q(t_1) = \frac{-D}{s}.$$

Or

$$q(t) = L^{-1}\big(\bar{q}(s)\big) = q_{\max} - D\frac{t^{\alpha}}{\Gamma(\alpha+1)}. \tag{5.16}$$

Now, let $t_2 = T - t_1$.

Again, $q(T) = 0$ or

$$q_{\max} = \frac{DT^{\alpha}}{\Gamma(\alpha+1)}. \tag{5.17}$$

Hence, from (5.15) and (5.17) we have

$$(K-D)\frac{t_1^{\alpha}}{\Gamma(\alpha+1)} = \frac{DT^{\alpha}}{\Gamma(\alpha+1)}. \tag{5.18}$$

Then, for $0 \le t \le t_1$, we get

$$q(t) = (K-D)\frac{t^{\alpha}}{\Gamma(\alpha+1)}. \tag{5.19}$$

For $t_1 \le t \le T$, we get

$$q(t) = \frac{D}{\Gamma(\alpha+1)}\big[T^{\alpha} - t^{\alpha}\big]. \tag{5.20}$$

Now $q(t)$ is continuous at $t = t_1$.

So, $q_{\max} = q(t_1)$ or $(K-D)\frac{t_1^{\alpha}}{\Gamma(\alpha+1)} = \frac{D}{\Gamma(\alpha+1)}[T^{\alpha} - t_1^{\alpha}]$.

Or

$$Kt_1^{\alpha} = DT^{\alpha}. \tag{5.21}$$

Again, by (5.19) and (5.20) we get

$$q_{\max} = \frac{D}{\Gamma(\alpha+1)}\left(1 - \frac{D}{K}\right)T^{\alpha}. \tag{5.22}$$

Some relevant costs

(i) The holding cost $\mathrm{HC}_{\alpha,\beta}(T) = c_1 D^{-\beta} q(T)$.

Here

$$\begin{aligned} D^{-\beta}q(T) &= \frac{1}{\Gamma(\beta)}\int_0^T (T-x)^{\beta-1} q(x)\,dx \\ &= \frac{1}{\Gamma(\beta)}\left[\int_0^{t_1} (T-x)^{\beta-1} q(x)\,dx + \int_{t_1}^T (T-x)^{\beta-1} q(x)\,dx\right] \\ &= \frac{1}{\Gamma(\beta)}\left[\int_0^{t_1} (T-x)^{\beta-1}(K-D)\frac{x^{\alpha}}{\Gamma(\alpha+1)}\,dx \right. \\ &\quad \left. + \int_{t_1}^T (T-x)^{\beta-1}\frac{D}{\Gamma(\alpha+1)}\big[T^{\alpha} - x^{\alpha}\big]\,dx\right] \\ &= \frac{1}{\Gamma(\beta)}(I_1 + I_2). \end{aligned} \tag{5.23}$$

Here

$$I_1 = \int_0^{t_1} (T-x)^{\beta-1}(K-D)\frac{x^\alpha}{\Gamma(\alpha+1)}\,dx \tag{5.24}$$

and

$$I_2 = \int_{t_1}^{T} (T-x)^{\beta-1}\frac{D}{\Gamma(\alpha+1)}\left[T^\alpha - x^\alpha\right]dx. \tag{5.25}$$

So, the holding cost is

$$\mathrm{HC}_{\alpha,\beta}(T) = \frac{c_1}{\Gamma(\beta)}(I_1+I_2). \tag{5.26}$$

(ii) The total cost = the holding cost + the set up cost

$$= \frac{c_1}{\Gamma(\beta)}(I_1+I_2)+c_3.$$

(iii) The total average cost, $\mathrm{TAC}_{\alpha,\beta}(T) = \frac{c_1}{\Gamma(\beta)T}(I_1+I_2)+\frac{c_3}{T}$.
Therefore, the model will be of the form

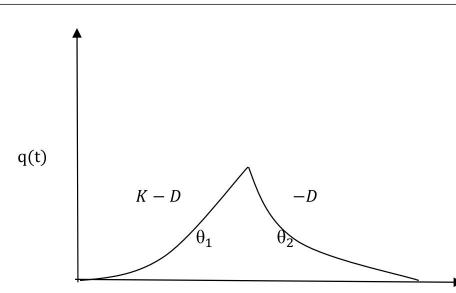
$$\begin{aligned} &\text{Minimize } \mathrm{TAC}_{\alpha,\beta}(T) = \frac{c_1}{\Gamma(\beta)T}(I_1+I_2)+\frac{c_3}{T}. \\ &\quad \text{Such that } T>0. \end{aligned} \tag{5.27}$$

This is the generalized EPQ model.

5.3 Classical (integer-order) EPQ model with deterioration (Model 3)

During the first scheduling period T, there are two parts of this whole cycle. The production starts at $t = 0$ and throughout the time interval $[0, t_1]$, the inventory level is increased at the rate $K - D$ due to the production rate K and to meet the demand we have D. Also the deterioration affects the inventory level. In the time interval $[0, t_1]$, the deterioration rate is θ_1. At the time $t = t_1$, the inventory reaches its highest level and the production is stopped. Then, throughout the time interval $[t_1, T]$, the inventory level gradually decreases as regards being payable to meet the customer's demands and for the deterioration at the rate θ_2 and at $t = T$ the inventory level reaches 0, as no shortage is allowed (see Fig. 2).

Figure 2 EPQ model with deterioration



If $q(t)$ represents the inventory level at any time, then the corresponding differential equation is given by

$$\frac{dq(t)}{dt} + \theta_1 q(t) = K - D, \quad \text{for } 0 \le t \le t_1, \tag{5.28}$$

$$\frac{dq(t)}{dt} + \theta_2 q(t) = -D, \quad \text{for } t_1 \le t \le T. \tag{5.29}$$

We have the boundary conditions

$$q(0) = q(T) = 0. \tag{5.30}$$

Also, let

$$q_{\max} = q(t_1). \tag{5.31}$$

Solving Eqs. (5.28) and (5.29) and by using the boundary conditions we have

$$q(t) = \frac{K-D}{\theta_1}\left(1 - e^{-\theta_1 t}\right), \quad \text{for } 0 \le t \le t_1. \tag{5.32}$$

$$q(t) = \frac{D}{\theta_2}\left(e^{\theta_2 (T-t)} - 1\right), \quad \text{for } t_1 \le t \le T. \tag{5.33}$$

Then,

$$q_{\max} = q(t_1) = \frac{K-D}{\theta_1}\left(1 - e^{-\theta_1 t_1}\right), \tag{5.34}$$

Also, using the continuity condition we have

$$\frac{K-D}{\theta_1}\left(1 - e^{-\theta_1 t_1}\right) = \frac{D}{\theta_2}\left(e^{\theta_2 (T-t_1)} - 1\right). \tag{5.35}$$

Some relevant costs

(i) The total holding cost is

$$\begin{aligned} \mathrm{HC} &= c_h\left[\int_0^{t_1} q(t)\,dt + \int_{t_1}^{T} q(t)\,dt\right] \\ &= c_h\left[\frac{K-D}{\theta_1} t_1 + \frac{K-D}{\theta_1^2}\left(e^{-\theta_1 t_1} - 1\right) + \frac{D}{\theta_2^2}\left\{e^{\theta_2 (T-t_1)} - \theta_2 (T - t_1) - 1\right\}\right]. \end{aligned} \tag{5.36}$$

(ii) The production cost,

$$\mathrm{PC} = c_p \int_0^{t_1} K\,dt = c_p K t_1. \tag{5.37}$$

(iii) Total cost of the system during the entire circle is given by

$$X = c_0 + \mathrm{HC} + \mathrm{PC}. \tag{5.38}$$

(iv) Total average cost of the system during the entire circle is given by

$$\mathrm{TAC} = \frac{X}{T}. \tag{5.39}$$

So, the problem can be written as

$$\begin{aligned} &\text{Minimize TAC} = \frac{X}{T}. \\ &\quad \text{Subject to } T > 0. \end{aligned} \tag{5.40}$$

5.4 Generalized (arbitrary-order) EPQ model with deterioration (Model 4)

Here, we consider the memory effect with the same assumptions and notations as the integer-order EPQ model with deterioration. Then the differential equation of fractional order α, due to memory ($0 \le \alpha \le 1$), corresponding to the generalized EPQ model, is

$$\frac{d^\alpha q(t)}{dt^\alpha} + \theta_1 q(t) = K - D, \quad \text{for } 0 \le t \le t_1, \tag{5.41}$$

$$\frac{d^\alpha q(t)}{dt^\alpha} + \theta_2 q(t) = -D, \quad \text{for } t_1 \le t \le T. \tag{5.42}$$

We have the boundary conditions

$$q(0) = q(T) = 0. \tag{5.43}$$

Also, let

$$q_{\max} = q(t_1). \tag{5.44}$$

Now, taking the Laplace transformation of (5.41) we have

$$s^\alpha \overline{q}(s) - s^{\alpha-0-1} q(0) + \theta_1 \overline{q}(s) = \frac{K-D}{s}.$$

Or

$$\overline{q}(s) = \frac{K-D}{\theta_1} \frac{\theta_1}{s(s^\alpha + \theta_1)}. \tag{5.45}$$

Therefore, using the inverse Laplace transformation of (5.45) we have

$$\begin{aligned} q(t) = L^{-1}\big(\overline{q}(s)\big) &= \frac{K-D}{\theta_1} L^{-1}\left(\frac{\theta_1}{s(s^\alpha + \theta_1)}\right) \\ &= \frac{K-D}{\theta_1}\big(1 - E_\alpha\big(-\theta_1 t^\alpha\big)\big), \quad \text{for } 0 \le t \le t_1. \end{aligned} \tag{5.46}$$

Here

$$E_\alpha\big(-\theta_1 t^\alpha\big) = \sum_{k=0}^{\infty} \frac{(-\theta_1 t^\alpha)^k}{\Gamma(\alpha k + 1)}. \tag{5.47}$$

Now, taking the Laplace transformation of (5.42) we have

$$s^{\alpha}\overline{q}(s) - s^{\alpha-0-1}q(t_1) + \theta_2\overline{q}(s) = \frac{-D}{s},$$

Or

$$\overline{q}(s) = \frac{s^{\alpha}q_{\max}}{s(s^{\alpha}+\theta_2)} - \frac{D}{\theta_2}\frac{\theta_2}{s(s^{\alpha}+\theta_2)}. \tag{5.48}$$

Therefore, using the inverse Laplace transformation of (5.48) we have

$$\begin{aligned} q(t) &= L^{-1}\bigl(\overline{q}(s)\bigr) = L^{-1}\left(\frac{s^{\alpha}q_{\max}}{s(s^{\alpha}+\theta_2)}\right) - \frac{D}{\theta_2}L^{-1}\left(\frac{\theta_2}{s(s^{\alpha}+\theta_2)}\right) \\ &= q_{\max}E_{\alpha}\bigl(-\theta_2 t^{\alpha}\bigr) - \frac{D}{\theta_2}\bigl(1-E_{\alpha}\bigl(-\theta_2 t^{\alpha}\bigr)\bigr), \quad \text{for } t_1 \le t \le T. \end{aligned} \tag{5.49}$$

Here

$$E_{\alpha}\bigl(-\theta_2 t^{\alpha}\bigr) = \sum_{k=0}^{\infty}\frac{(-\theta_2 t^{\alpha})^k}{\Gamma(\alpha k+1)}. \tag{5.50}$$

Now, let $t_2 = T - t_1$.

Again, $q(T) = 0$.

So

$$q_{\max} = \frac{D}{\theta_2}\frac{(1-E_{\alpha}(-\theta_2 T^{\alpha}))}{(E_{\alpha}(-\theta_2 T^{\alpha}))} = \frac{D}{\theta_2}\bigl\{\bigl(E_{\alpha}\bigl(-\theta_2 T^{\alpha}\bigr)\bigr)^{-1} - 1\bigr\}. \tag{5.51}$$

Then (5.49) takes the form

$$\begin{aligned} q(t) &= \frac{D}{\theta_2}\bigl\{\bigl(E_{\alpha}\bigl(-\theta_2 T^{\alpha}\bigr)\bigr)^{-1} - 1\bigr\}E_{\alpha}\bigl(-\theta_2 t^{\alpha}\bigr) - \frac{D}{\theta_2}\bigl(1-E_{\alpha}\bigl(-\theta_2 t^{\alpha}\bigr)\bigr) \\ &= \frac{D}{\theta_2}\bigl\{E_{\alpha}\bigl(\theta_2 T^{\alpha}\bigr)E_{\alpha}\bigl(-\theta_2 t^{\alpha}\bigr) - 1\bigr\}. \end{aligned} \tag{5.52}$$

Again, from (5.46) we have

$$q_{\max} = q(t_1) = \frac{K-D}{\theta_1}\bigl(1-E_{\alpha}\bigl(-\theta_1 t_1^{\alpha}\bigr)\bigr). \tag{5.53}$$

Now, from (5.51) and (5.52) we have

$$\frac{K-D}{\theta_1}\bigl(1-E_{\alpha}\bigl(-\theta_1 t_1^{\alpha}\bigr)\bigr) = \frac{D}{\theta_2}\bigl\{\bigl(E_{\alpha}\bigl(-\theta_2 T^{\alpha}\bigr)\bigr)^{-1} - 1\bigr\}. \tag{5.54}$$

Then, for $0 \le t \le t_1$, we get

$$q(t) = \frac{K-D}{\theta_1}\bigl(1-E_{\alpha}\bigl(-\theta_1 t^{\alpha}\bigr)\bigr), \tag{5.55}$$

and, for $t_1 \le t \le T$, we get

$$q(t) = \frac{D}{\theta_2}\bigl\{E_{\alpha}\bigl(\theta_2 T^{\alpha}\bigr)E_{\alpha}\bigl(-\theta_2 t^{\alpha}\bigr) - 1\bigr\}. \tag{5.56}$$

Now, $q(t)$ is continuous at $t = t_1$.

So $q_{\max} = q(t_1)$ i.e.,

$$\frac{K-D}{\theta_1}\left(1-E_\alpha\left(-\theta_1 t_1^\alpha\right)\right) = \frac{D}{\theta_2}\left\{E_\alpha\left(\theta_2 T^\alpha\right)E_\alpha\left(-\theta_2 t_1^\alpha\right)-1\right\}. \tag{5.57}$$

Some relevant costs

(i) The holding cost is

$$\mathrm{HC}_{\alpha,\beta}(T) = c_h D^{-\beta} q(T). \tag{5.58}$$

Here

$$\begin{aligned} D^{-\beta} q(T) &= \frac{1}{\Gamma(\beta)}\int_0^T (T-x)^{\beta-1} q(x)\,dx \\ &= \frac{1}{\Gamma(\beta)}\left[\int_0^{t_1}(T-x)^{\beta-1}q(x)\,dx + \int_{t_1}^T (T-x)^{\beta-1}q(x)\right]dx \\ &= \frac{1}{\Gamma(\beta)}\left[\int_0^{t_1}(T-x)^{\beta-1}\frac{K-D}{\theta_1}\left(1-E_\alpha\left(-\theta_1 x^\alpha\right)\right)dx \right. \\ &\quad \left. + \int_{t_1}^T (T-x)^{\beta-1}\frac{D}{\theta_2}\left\{E_\alpha\left(\theta_2 T^\alpha\right)E_\alpha\left(-\theta_2 x^\alpha\right)-1\right\}dx\right] \\ &= \frac{1}{\Gamma(\beta)}(I_1+I_2). \end{aligned} \tag{5.59}$$

Here

$$I_1 = \int_0^{t_1}(T-x)^{\beta-1}\frac{K-D}{\theta_1}\left(1-E_\alpha\left(-\theta_1 x^\alpha\right)\right)dx \tag{5.60}$$

and

$$I_2 = \int_{t_1}^T (T-x)^{\beta-1}\frac{D}{\theta_2}\left\{E_\alpha\left(\theta_2 T^\alpha\right)E_\alpha\left(-\theta_2 x^\alpha\right)-1\right\}dx. \tag{5.61}$$

(ii) The production cost is

$$\begin{aligned} \mathrm{PC} = c_p D^{-\beta} K &= c_p \frac{1}{\Gamma(\beta)}\int_0^{t_1}(t_1-x)^{\beta-1}K\,dx \\ &= c_p K \frac{1}{\Gamma(\beta)}\int_0^{t_1}(t_1-x)^{\beta-1}\,dx. \end{aligned} \tag{5.62}$$

(iii) The total cost of the system during the entire circle is given by

$$\mathrm{TP}_{\alpha,\beta}(T) = c_0 + \mathrm{HC} + \mathrm{PC}. \tag{5.63}$$

(iv) The total average cost of the system during the entire circle is given by

$$\mathrm{TAP}_{\alpha,\beta}(T) = \frac{\mathrm{TP}_{\alpha,\beta}(T)}{T}. \tag{5.64}$$

So, the problem can be written as

$$\begin{aligned} &\text{Minimize } \mathrm{TAP}_{\alpha,\beta}(T) = \frac{\mathrm{TP}_{\alpha,\beta}(T)}{T}. \\ &\quad \text{Subject to } T > 0. \end{aligned} \tag{5.65}$$

6 Theoretical/analytical results for optimization of the problems

6.1 Model 1

Solving a (5.9), we can show that $T^* = \sqrt{\frac{2c_3}{C_1 D(1-\frac{D}{K})}}$ is the optimal cycle time at which the optimal average cost is $\mathrm{TAC}^*(T^*) = \sqrt{2c_1c_3D(1-\frac{D}{K})}$ and $q^*_{\max} = \sqrt{\frac{2c_3D(1-\frac{D}{K})}{C_1}}$.

6.2 Model 2

Case 6.1 ($\alpha = 1$ and $\beta = 1$)

(i) The holding cost

$$\begin{aligned} \mathrm{HC}_{1,1} &= \frac{c_1}{\Gamma(1)}\left[\int_0^{t_1}(T-x)^{1-1}q(x)\,dx+\right]\int_{t_1}^{T}(T-x)^{1-1}q(x)\,dx \\ &= \frac{c_1}{\Gamma(1)}\left[\int_0^{t_1}(K-D)\frac{x}{\Gamma(1+1)}\,dx + \int_{t_1}^{T}\frac{D}{\Gamma(1+1)}[T-x]\,dx\right] \\ &= c_1\left[(K-D)\frac{t_1^2}{2} + D\frac{(T-t_1)^2}{2}\right] \\ &= \frac{c_1}{2}\left(1-\frac{D}{K}\right)DT^2, \quad \text{using } Kt_1 = \mathrm{DT}. \end{aligned}$$

(ii) The total average cost $\mathrm{TAC}_{1,1} = \frac{c_1}{2}(1-\frac{D}{K})\mathrm{DT} + \frac{c_3}{T}$.

So, the EPQ model is

$$\begin{aligned} &\text{Minimize } \mathrm{TAC}_{1,1}(T) = \frac{c_1}{2}\left(1-\frac{D}{K}\right)\mathrm{DT} + \frac{c_3}{T}. \\ &\quad \text{Such that } T > 0. \end{aligned}$$

Remark 6.1 This is the classical EPQ model. It is seen that the classical EPQ model is a particular case of the general EPQ model.

Case 6.2 (When $\beta = 1$ and α is any arbitrary number such that $0 < \alpha \le 1$)

(i) The holding cost

$$\begin{aligned} \mathrm{HC}_{\alpha,1} &= \frac{c_1}{\Gamma(1)}\left[\int_0^{t_1}(T-x)^{1-1}q(x)\,dx + \int_{t_1}^{T}(T-x)^{1-1}q(x)\,dx\right] \\ &= \frac{1}{\Gamma(1)}\left[\int_0^{t_1}(T-x)^{1-1}(K-D)\frac{x^\alpha}{\Gamma(\alpha+1)}\,dx\right. \\ &\quad \left.+ \int_{t_1}^{T}(T-x)^{1-1}\frac{D}{\Gamma(\alpha+1)}\left[T^\alpha - x^\alpha\right]dx\right] \\ &= c_1\frac{K-D}{\Gamma(\alpha+1)}\frac{t_1^{\alpha+1}}{(\alpha+1)} + c_1\frac{D}{\Gamma(\alpha+1)}\left[T^{\alpha+1} - \frac{T^{\alpha+1}}{\alpha+1} - T^\alpha t_1 + \frac{t_1^{\alpha+1}}{\alpha+1}\right] \end{aligned}$$

$$= \frac{\alpha c_1}{\Gamma(\alpha+2)} D\left[1 - \left(\frac{D}{K}\right)^{\frac{1}{\alpha}}\right] T^{\alpha+1},$$

$$\text{using } (K-D)\frac{t_1^\alpha}{\Gamma(\alpha+1)} = \frac{D}{\Gamma(\alpha+1)}\left[T^\alpha - t_1^\alpha\right] \text{ and } Kt_1^\alpha = DT^\alpha.$$

(ii) The total average cost

$$\mathrm{TAC}_{\alpha,1} = \frac{\alpha c_1}{\Gamma(\alpha+2)} D\left[1 - \left(\frac{D}{K}\right)^{\frac{1}{\alpha}}\right] T^\alpha + \frac{c_3}{T} = c_2 T^\alpha + \frac{c_3}{T},$$

$$\text{where } \frac{\alpha c_1}{\Gamma(\alpha+2)} D\left[1 - \left(\frac{D}{K}\right)^{\frac{1}{\alpha}}\right] = c_2.$$

So, the EPQ model is

$$\text{Min } \mathrm{TAC}_{\alpha,1}(T) = c_2 T^\alpha + \frac{c_3}{T}.$$

$$\text{Such that } T > 0.$$

Analytical solution using geometric programming method

The corresponding dual problem is

$$\text{Maximize } d(w) = \left(\frac{c_2}{w_1}\right)^{w_1}\left(\frac{c_3}{w_2}\right)^{w_2},$$

subject to the normalized and orthogonal conditions

$$w_1 + w_2 = 1, \qquad \alpha w_1 - w_2 = 0,$$

$$w_1, w_2 \geq 0.$$

This gives $w_1 = \frac{1}{\alpha+1}$ and $w_2 = \frac{\alpha}{\alpha+1}$.

Also, the primal-dual relationship will be

$$c_2 T^\alpha = w_1 d(w), \qquad \frac{c_3}{T} = w_2 d(w).$$

This gives $T = (\frac{c_3}{c_2\alpha})^{\frac{1}{\alpha+1}}$ and

$$\text{Maximize } d(w) = c_2^{\frac{1}{\alpha+1}} c_3^{\frac{\alpha}{\alpha+1}} \alpha^{-\frac{\alpha}{\alpha+1}}(\alpha+1).$$

So, finally the problem will be

$$\text{Minimize } \mathrm{TAC}_{\alpha,1}(T) = c_2^{\frac{1}{\alpha+1}} c_3^{\frac{\alpha}{\alpha+1}} \alpha^{-\frac{\alpha}{\alpha+1}}(\alpha+1)$$

and $T = (\frac{c_3}{c_2\alpha})^{\frac{1}{\alpha+1}}$, $q_{\max} = \frac{D}{\Gamma(\alpha+1)}(1 - \frac{D}{K})T^\alpha = \frac{D}{\Gamma(\alpha+1)}(1 - \frac{D}{K})(\frac{c_3}{c_2\alpha})^{\frac{1}{\alpha+1}}$.

Case 6.3 (When $\alpha = 1$ and β is any arbitrary number such that $0 < \beta \leq 1$) Therefore, the model will be of the form

$$\text{Min TAC}_{1,\beta}(T) = \frac{c_1}{\Gamma(\beta)T}(I_1 + I_2) + \frac{c_3}{T}.$$

Such that $T > 0$.

Here

$$\begin{aligned} I_1 &= (K-D)\int_0^{t_1} x(T-x)^{\beta-1}\,dx = (K-D)\int_0^{t_1}\bigl[T(T-x)^{\beta-1} - (T-x)^{\beta}\bigr]\,dx \\ &= (K-D)\left[\frac{T^{(\beta+1)} - (T-t_1)^{\beta}(T+\beta t_1)}{\beta(\beta+1)}\right] \end{aligned}$$

and

$$I_2 = D\int_{t_1}^{T}(T-x)^{\beta}\,dx = -\frac{D(T-t_1)^{(\beta+1)}}{(\beta+1)}.$$

Case 6.4 (For arbitrary values of α and β the analytical illustration for the model is a little tough. For the sake of simplicity we take particular values of α and β. Here in our discussion we take $\alpha = 0.5$ and $\beta = 0.5$) Then,

$$\begin{aligned} I_1 &= \frac{(K-D)}{\Gamma(1.5)}\int_0^{t_1}(T-x)^{-0.5}x^{0.5}\,dx = \frac{(K-D)}{\Gamma(1.5)}\left[T\sin^{-1}\sqrt{\frac{t_1}{T}} - \sqrt{t_1(T-t_1)}\right], \\ I_1 &= \frac{D}{\Gamma(1.5)}\int_{t_1}^{T}(T-x)^{-0.5}\bigl(T^{0.5} - x^{0.5}\bigr)\,dx \\ &= \frac{D}{\Gamma(1.5)}\left[2\sqrt{T(T-t_1)} - \frac{\pi}{2}T + T\sin^{-1}\sqrt{\frac{t_1}{T}} - \sqrt{t_1(T-t_1)}\right]. \end{aligned}$$

So the holding cost is $\text{HC}_{0.5,0.5}(T) = \frac{c_1}{\Gamma(0.5)}(I_1 + I_2)$.

Therefore the model will be of the form

$$\text{Minimize TAC}_{0.5,0.5}(T) = \frac{c_1}{\Gamma(0.5)T}(I_1 + I_2) + \frac{c_3}{T}.$$

Such that $T > 0$.

6.3 Model 3

We think that there is no need to illustrate further theoretical results after describing in detail the part of mathematical modelling.

6.4 Model 4

Case 6.5 (When $\alpha = 1$ and $\beta = 1$)

(i) The holding cost is $\text{HC}_{1,1}(T) = c_h D^{-1} q(T)$.

Here

$$\begin{aligned}
D^{-1}q(t) &= \frac{1}{\Gamma(1)}\int_0^T (T-x)^{1-1}q(x)\,dx = \left[\int_0^{t_1} q(x)\,dx + \int_{t_1}^T q(x)\right]dx \\
&= \left[\int_0^{t_1} \frac{K-D}{\theta_1}\bigl(1-E_1\bigl(-\theta_1 x^1\bigr)\bigr)\,dx \right.\\
&\quad \left. + \int_{t_1}^T \frac{D}{\theta_2}\bigl(E_1\bigl(-\theta_2 x^1\bigr) - E_1\bigl(-\theta_2 T^1\bigr)\bigr)\,dx\right] \\
&= (I_1 + I_2).
\end{aligned}$$

Here

$$\begin{aligned}
I_1 &= \int_0^{t_1} \frac{K-D}{\theta_1}\bigl(1-E_1\bigl(-\theta_1 x^1\bigr)\bigr)\,dx = \int_0^{t_1} \frac{K-D}{\theta_1}\bigl(1-e^{-\theta_1 x}\bigr)\,dx \\
&= \frac{K-D}{\theta_1}t_1 + \frac{K-D}{\theta_1^2}\bigl(e^{-\theta_1 t_1}-1\bigr)
\end{aligned}$$

and

$$\begin{aligned}
I_2 &= \int_{t_1}^T (T-x)^{1-1}\frac{D}{\theta_2}\bigl\{E_1\bigl(\theta_2 T^1\bigr)E_1\bigl(-\theta_2 x^1\bigr) - 1\bigr\}\,dx \\
&= \int_{t_1}^T \frac{D}{\theta_2}\bigl(e^{\theta_2(T-x)} - 1\bigr)\bigr)\,dx \\
&= \frac{D}{\theta_2^2}\bigl\{e^{\theta_2(T-t_1)} - \theta_2(T-t_1) - 1\bigr\}.
\end{aligned}$$

This shows that it is the holding cost for the classical EPQ model with deterioration.

(ii) The production cost is

$$\begin{aligned}
\mathrm{PC} = c_p D^{-1}K &= c_p\frac{1}{\Gamma(1)}\int_0^{t_1}(t_1-x)^{1-1}K\,dx \\
&= c_p K t_1.
\end{aligned}$$

This is the production cost of the classical EPQ model with deterioration.

(iii) The total cost of the system during the entire circle is given by

$$\mathrm{TC}_{1,1}(T) = c_0 + \mathrm{HC} + \mathrm{PC}.$$

(iv) The total average cost of the system during the entire circle is given by

$$\mathrm{TAC}_{1,1}(T) = \frac{\mathrm{TP}_{1,1}(T)}{T}.$$

So the problem can be written as

$$\text{Minimize } \mathrm{TAC}_{1,1}(T) = \frac{\mathrm{TP}_{1,1}(T)}{T}.$$

Subject to $T > 0$.

Remark 6.2 The above discussion shows that the classical EPQ model with deterioration is a particular case of the generalized fractional EPQ model with deterioration.

Case 6.6 (When α is any arbitrary number such that $0 < \alpha \leq 1$ and $\beta = 1$) Here, we use the real distinct pole approximation of the Mittag-Leffler function to avoid complexity of the function.

(i) The holding cost is

$$\begin{aligned}
&\mathrm{HC}_{\alpha,1}(T) \\
&\quad = c_h \int_0^T q(x)\,dx \\
&\quad = c_h \left[\int_0^{t_1} q(x)\,dx + \int_{t_1}^T q(x) \right] dx \\
&\quad = c_h \left[\int_0^{t_1} \frac{K-D}{\theta_1} \bigl(1 - E_\alpha\bigl(-\theta_1 x^\alpha\bigr)\bigr)\,dx + \int_{t_1}^T \frac{D}{\theta_2} \bigl\{ E_\alpha\bigl(\theta_2 T^\alpha\bigr) E_\alpha\bigl(-\theta_2 x^\alpha\bigr) - 1 \bigr\}\,dx \right] \\
&\quad = c_h \left[\int_0^{t_1} \frac{K-D}{\theta_1} \left(1 - \frac{1 - a_1(\theta_1 x^\alpha)}{(1 + a_2(\theta_1 x^\alpha))(1 + a_3(\theta_1 x^\alpha))} \right) dx \right. \\
&\qquad \left. + \int_{t_1}^T \frac{D}{\theta_2} \left\{ \frac{1 + b_1(\theta_2 T^\alpha)}{(1 - b_2(\theta_2 T^\alpha))(1 - b_3(\theta_2 T^\alpha))} \frac{1 - c_1(\theta_2 x^\alpha)}{(1 + c_2(\theta_2 x^\alpha))(1 + c_3(\theta_2 x^\alpha))} - 1 \right\} dx \right].
\end{aligned}$$

Here

$$\begin{aligned}
a_1 &= \frac{1}{\alpha \Gamma(\alpha)} - a_2 - a_3, \qquad a_2 = \frac{\Gamma(\alpha)}{2\Gamma(2\alpha)} - \frac{1}{4}, \qquad a_3 = \frac{\Gamma(2\alpha)}{\alpha \Gamma(\alpha)(5\Gamma(2\alpha) - 2\Gamma(\alpha))}; \\
b_1 &= \frac{1}{\alpha \Gamma(\alpha)} - b_2 - b_3, \qquad b_2 = \frac{\Gamma(\alpha)}{2\Gamma(2\alpha)} - \frac{1}{4}, \qquad b_3 = \frac{\Gamma(2\alpha)}{\alpha \Gamma(\alpha)(5\Gamma(2\alpha) - 2\Gamma(\alpha))}
\end{aligned}$$

and

$$c_1 = \frac{1}{\alpha \Gamma(\alpha)} - c_2 - c_3, \qquad c_2 = \frac{\Gamma(\alpha)}{2\Gamma(2\alpha)} - \frac{1}{4}, \qquad c_3 = \frac{\Gamma(2\alpha)}{\alpha \Gamma(\alpha)(5\Gamma(2\alpha) - 2\Gamma(\alpha))}.$$

For a particular value of α, say $\alpha = 0.5$ we have the following:

$$\mathrm{HC}_{0.5,1}(T) = c_h(I_1 + I_2),$$

where

$$\begin{aligned}
I_1 &= \int_0^{t_1} \frac{K-D}{\theta_1} \left(1 - \frac{1 - a_1(\theta_1 x^\alpha)}{(1 + a_2(\theta_1 x^\alpha))(1 + a_3(\theta_1 x^\alpha))} \right) dx \\
&= \frac{K-D}{\theta_1} \\
&\quad \times \left\{ t_1 + 2 \frac{(a_1 a_2 a_3 \theta_1^4 (a_2 - a_3)\sqrt{t_1} - a_2^2 \theta_1^3 (a_3 + a_1) \log(a_3 \theta_1 \sqrt{t_1} + 1) + a_3^2 \theta_1^3 (a_2 + a_1) \log(a_2 \theta_1 \sqrt{t_1} + 1))}{a_2^2 a_3^2 \theta_1^5 (a_2 - a_3)} \right\}
\end{aligned}$$

and

$$I_2 = \int_{t_1}^{T} \frac{D}{\theta_2}\left\{\frac{1+b_1(\theta_2 T^\alpha)}{(1-b_2(\theta_2 T^\alpha))(1-b_3(\theta_2 T^\alpha))}\frac{1-c_1(\theta_2 x^\alpha)}{(1+c_2(\theta_2 x^\alpha))(1+c_3(\theta_2 x^\alpha))}-1\right\}dx$$
$$= \frac{D}{\theta_2}\left[\left\{\frac{1+b_1(\theta_2\sqrt{T})}{(1-b_2(\theta_2\sqrt{T}))(1-b_3(\theta_2\sqrt{T}))}\right\}(\mathrm{J}_3-\mathrm{J}_4)-T+t_1\right].$$

Here

$$\mathrm{J}_3 = -2\frac{(c_1c_2c_3\theta_1^4(c_2-c_3)\sqrt{T}-c_2^2\theta_1^3(c_3+c_1)\log(c_3\theta_1\sqrt{T}+1)+c_3^2\theta_1^3(c_2+c_1)\log(c_2\theta_1\sqrt{T}+1))}{c_2^2c_3^2\theta_1^5(c_2-c_3)},$$
$$\mathrm{J}_4 = -2\frac{(c_1c_2c_3\theta_1^4(c_2-c_3)\sqrt{t_1}-c_2^2\theta_1^3(c_3+c_1)\log(c_3\theta_1\sqrt{t_1}+1)+c_3^2\theta_1^3(c_2+c_1)\log(c_2\theta_1\sqrt{t_1}+1))}{c_2^2c_3^2\theta_1^5(c_2-c_3)}.$$

(ii) The production cost is $\mathrm{PC}_{0.5,1} = c_p\int_0^{t_1} K\,dx = c_pKt_1$.
So the problem can be written as

$$\text{Minimize } \mathrm{TAP}_{\alpha,1}(T) = \frac{c_0+\mathrm{HC}+\mathrm{PC}}{T}.$$

Subject to $T > 0$.

Case 6.7 (When $\alpha = 1$ and β is arbitrary number such that $0 < \beta \le 1$)
(i) The holding cost is $\mathrm{HC}_{1,\beta}(T) = c_hD^{-\beta}q(T) = \frac{c_h}{\Gamma(\beta)}(I_1+I_2)$.
Here

$$I_1 = \int_0^{t_1}\frac{K-D}{\theta_1}(T-x)^{\beta-1}\left(1-e^{-\theta_1 x}\right)dx$$
$$= \frac{K-D}{\theta_1}\left[\frac{T^\beta-(T-t_1)^\beta}{\beta}-e^{-\theta_1 T}(-\theta_1)^{-\beta}\left\{\Gamma\left(\beta,\theta_1(t_1-T)\right)-\Gamma(\beta,-\theta_1 T)\right\}\right]$$

and

$$I_2 = \int_{t_1}^{T}\frac{D}{\theta_2}(T-x)^{\beta-1}\left(e^{\theta_2(T-x)}-1\right)dx$$
$$= \frac{D}{\theta_2}\left[(-\theta_2)^{-\beta}\left\{\Gamma(\beta,0)-\Gamma\left(\beta,\theta_2(t_1-T)\right)\right\}-\frac{(T-t_1)^\beta}{\beta}\right].$$

(ii) The production cost is $\mathrm{PC}_{1,\beta}(T) = c_pD^{-1}K = c_p\frac{1}{\Gamma(\beta)}\int_0^{t_1}(t_1-x)^{\beta-1}K\,dx$.
(iii) The total cost of the system during the entire circle is given by

$$\mathrm{TC}_{1,\beta}(T) = c_0+\mathrm{HC}+\mathrm{PC}.$$

So, the problem can be written as

$$\text{Minimize } \mathrm{TAC}_{1,\beta}(T) = \frac{\mathrm{TP}_{1,\beta}(T)}{T}.$$

Subject to $T > 0$.

Case 6.8 For arbitrary values of α and β the problem becomes more complicated. But for particular values of α and β the problem can be solved numerically with approach described in Case 6.6.

7 Numerical results and sensitivity analysis

Some numerical examples have been taken and sensitivity analysis on the examples has been done to justify the theoretical aspects.

7.1 Model 1

For numerical illustration, we take the following numerical values:

$$c_1 = 4, \qquad c_3 = 30, \qquad K = 2500, \qquad D = 1200.$$

Then the minimum total average cost is $\mathrm{TAC}^* = 386.9884$ which is obtained for the optimal time cycle $T^* = 96.74709$. Also then the maximum inventory level $q^*_{\max} = 96.74709$.

In Table 3, the sensitivity of the optimality with respect to the values of the parameters is shown.

From Table 3, we can make the following observations:

(i) As the values of c_1 increase, T^* gradually decreases and TAC^* gradually increases. This indicates that as the holding cost increases the total schedule time has to decrease to minimize the total average cost.

Table 3 Sensitivity table for case Model 1

Parameters	Change in values	T^*	TAC*	q^*_{amx}
c_1	3.6	0.1634301	367.1294	101.9804
	3.8	0.1590712	377.1896	99.26042
	3.9	0.1570186	382.1204	97.97959
	4	0.1550434	386.9884	96.74709
	4.1	0.1531410	391.7959	95.55997
	4.2	0.1513069	396.5451	94.41550
	4.3	0.1495372	401.2381	93.31118
c_3	27	0.1470871	367.1294	91.78235
	28	0.1497862	373.8663	93.46657
	29	0.1524375	380.4839	95.12098
	30	0.1550434	386.9884	96.74709
	31	0.1576063	393.3853	98.34633
	32	0.1601282	399.6799	99.91997
	33	0.1626109	405.8768	101.4692
K	2470	0.1559199	384.8129	96.20323
	2480	0.1556237	385.5453	96.38632
	2490	0.1553316	386/2704	96.56760
	2500	0.1550434	386.9884	96.74709
	2510	0.1547591	387.6993	96.92483
	2520	0.1544786	388.4033	97.10083
	2530	0.1542018	389.1005	97.27513
D	1170	0.1552376	386.5043	96.62608
	1180	0.1551628	386.6906	96.67264
	1190	0.1550981	386.8519	96.71298
	1200	0.1550434	386.9884	96.74709
	1210	0.1549987	387.1000	96.77500
	1220	0.1549640	387.1868	96.79669
	1230	0.1549392	387.2488	96.81219

(ii) As the values of c_3 increase, both T^* and TAC* gradually increase. This fact can be interpreted as follows: if the ordering cost increases then to minimize the total average cost the inventory procedure needs to run a long time.
(iii) As the value of K increases, T^* gradually decreases and TAC* gradually increases. That means that, for a high production rate, the total schedule time must be shorter and the total average cost will be minimized.
(iv) As the values of D increases, T^* gradually decreases and TAC* gradually increases. That means that to meet a large demand the total time must be shorter and then obviously the total average cost will be minimized.

7.2 Model 2

We consider the same numerical value as taken for Model 1. Then, for this arbitrary model we have the following cases:

Case 6.1: *When* $\alpha = 1$ *and* $\beta = 1$

In the theoretical discussion we have shown that for this case Model 2 reduces to Model 1. The same thing happens for the numerical example also, i.e., if we take the same numerical example as Model 1 with $\alpha = 1$ and $\beta = 1$ then we can illustrate Model 2 using Model 1.

Case 6.2: *When* $\alpha > 0$ *is arbitrary and* $\beta = 1$

Here we discuss the same numerical example as Model 1, i.e., $c_1 = 4$, $c_3 = 30$, $K = 2500$, $D = 1200$.

In Table 4, the sensitivity of the optimality with respect to the different values of the memory index α is shown.

From Table 4, we can make the observation that there is a critical value of the memory parameter, say = 0.3, for which the minimized total average cost becomes maximum (480.7231) and then the value of TAC* decreases above and below the table. The lower values of α indicates a large memory. So, another interpretation from the table is that for large memory the system needs more time to minimize the total average cost. That means that to reach the same minimum cost like the memory-less inventory system, the dealer has to change the policy of dealing with customers.

In Table 5, the sensitivity of the optimality with respect to the value of the parameters is shown for the fixed values of the memory parameters $\alpha = 0.5$ and $\beta = 1$.

Table 4 Results for $\alpha > 0$ with the fixed value $\beta = 1$ of Case 6.2

α	$\Gamma(\alpha+2)$	$\Gamma(\alpha+1)$	$\frac{\alpha c_1}{\Gamma(\alpha+2)} D[1-(\frac{D}{K})^{\frac{1}{\alpha}}] = c_2$	T^*	TAC*	q^*_{max}
0.1	1.04648585469	0.9135076987	458.3802	0.6801931	319.2005	631.1130
0.2	1.10180249088	0.9181687424	879.0985	0.2358365	446.3459	509.0779
0.3	1.16671190520	0.8974706963	1127.364	0.1551386	480.7231	397.5395
0.4	1.24216934450	0.8872638175	1298.952	0.1304171	476.9366	311.3620
0.5	1.32934038818	0.8862269255	1389.441	0.1230857	460.6260	247.0264
0.6	1.42962455886	0.8935153493	1421.718	0.1234093	441.9322	199.0175
0.7	1.54468584585	0.9086387329	1412.893	0.1279405	414.5195	162.8167
0.8	1.67649078776	0.9313837710	1375.371	0.1351746	409.5117	135.1363
0.9	1.82735508062	0.9617658319	1318.186	0.1443518	397.0645	113.6567
1	2.00000000000	1.0000000000	1248.000	0.1550434	386.9884	96.74709

Table 5 Sensitivity table for Case 6.2 for fix $\alpha = 0.5$ and $\beta = 1$

Parameters	Change in values	T^*	TAC*	q^*_{amx}
c_1	3.6	0.1320421	429.3816	255.8562
	3.8	0.1273674	445.1409	251.2863
	3.9	0.1251808	452.9165	249.1199
	4	0.1230857	460.6260	247.0264
	4.1	0.1210760	468.2714	245.0015
	4.2	0.1191465	475.8550	243.0414
	4.3	0.1172920	483.3786	241.1426
c_3	27	0.1147367	444.7295	238.5014
	28	0.1175525	450.1535	241.4102
	29	0.1203350	455.4500	244.2506
	30	0.1230857	460.6260	247.0264
	31	0.1258059	465.6882	249.7412
	32	0.1284971	470.6127	252.3982
	33	0.1311603	475.4950	255.0004
K	2470	0.1236897	458.3764	244.8555
	2480	0.1234851	459.1360	245.5850
	2490	0.1232838	459.8858	246.3086
	2500	0.1230857	460.6260	247.0264
	2510	0.1228907	461.3568	247.7384
	2520	0.1226988	462.0783	248.4448
	2530	0.1225099	462.7902	249.1455
D	1170	0.1239622	457.3688	247.2847
	1180	0.1236581	458.4937	247.2192
	1190	0.1233659	459.5795	247.1331
	1200	0.1230857	460.6260	247.0264
	1210	0.1220171	461.6330	246.8992
	1220	0.1225603	462.6004	246.7515
	1230	0.1223150	463.5280	246.5834

Table 6 Result for Case 6.3

β	T^*	t_1^*	TAC*	q^*_{max}
0.1	0.9578852	0.4597849	344.5089	597.7204
0.2	0.3246652	0.1558393	554.4172	202.5911
0.3	0.2031685	0.9752090	639.8628	126.1772
0.4	0.1614390	0.07749071	650.4005	100.7379
0.5	0.1446220	0.06941856	622.3119	90.24313
0.6	0.1386208	0.06653798	577.1140	86.49937
0.7	0.1383842	0.06642440	526.4847	86.35172
0.8	0.1417004	0.06801618	476.3573	88.4203
0.9	0.1474590	0.07078033	429.4978	92.0143
1.0	0.1550434	0.07442084	386.9884	96.74709

The observations in Table 5 and the interpretations regarding it are the same as that of Table 3.

Case 6.3: *When* $\alpha = 1$ *and* β *is arbitrary*

Here we fix the value of $\beta = 0.5$ and take the same example as Case 6.2. Then the minimum value of the total average cost is $\mathrm{TAC}^* = 622.3119$, which is given for the optimal time cycle $T^* = 0.1446220$. Also the maximum inventory will be $q^*_{\max} = 90.24413$, which is obtained at $t_1^* = 0.0694185$.

In Table 6, the sensitivity of the optimality with respect to the different values of the memory index β is shown.

From Table 6, we can make the observation that there is a critical value of the memory parameter, say 0.4, for which the minimized total average cost becomes maximum (650.4005) and then the value of TAC* decreases above and below the table. The lower values of β indicate a large memory. So, another interpretation from the table is that for large memory the system needs more time to minimize the total average cost. That means that to reach the same minimum cost like the memory-less inventory system, the dealer has to change the policy of dealing with customers.

In Table 7, the sensitivity of the optimality with respect to the value of the parameters is shown for the fixed values of the memory parameters $\alpha = 1$ and $\beta = 0.5$.

The observations in Table 7 and the interpretations regarding it are the same as that of Table 3.

Case 6.4: *When* α, β *are arbitrary*

In our discussion we take $\alpha = 0.5$, $\beta = 0.5$ and $c_1 = 4$, $c_3 = 40$, $K = 250$, $D = 120$.

Then, the minimum value of the total average cost, $\mathrm{TAC}^* = 146.3850$, which is given for the optimal time interval, $T^* = 1.010695$ and $t_1^* = 0.2562614$ and $q_{\max}^* = 70.78638$.

In Table 8, the sensitivity of the optimality with respect to the value of the parameters is shown.

Table 7 Sensitivity table for Case 6.3 for fixed $\alpha = 1$ and $\beta = 0.5$

Parameters	Change in values	T^*	t_1^*	TAC*	q_{max}^*
c_1	3.9	0.1470837	0.0760019	611.89	91.78025
	4	0.1446220	0.06941856	622.3119	90.24413
	4.1	0.1422608	0.06828517	632.3411	88.77972
c_3	25	0.1280698	0.06147352	585.6180	79.91558
	30	0.1446220	0.06941856	622.3119	90.24413
	35	0.1602749	0.0769319	655.1243	100.0115
K	2490	0.1449054	0.06983392	621.0949	90.08576
	2500	0.1446220	0.06941856	622.3119	90.24413
	2510	0.1443425	0.06900838	623.5168	90.40098
D	1190	0.1448451	0.06894628	621.3533	90.31963
	1200	0.1446220	0.06941856	622.3119	90.24413
	1210	0.1444108	0.06989481	623.2223	90.16430

Table 8 Sensitivity table for Case 6.4 for fixed $\alpha = 0.5$ and $\beta = 0.5$

Parameters	Change in values	T^*	TAC*	q_{max}^*
c_1	3.9	1.010935	143.7147	70.79477
	4	1.010695	146.3850	70.78638
	4.2	1.010246	151.7250	70.77065
c_3	38	1.010223	144.4057	70.76986
	40	1.010695	146.3850	70.78638
	42	1.011161	148.3634	70.80270
K	247 (non-feasible)	0.02102814	2033.175	10.09586
	250	1.010695	146.3850	70.78638
	253	1.010959	146.9759	71.57052
D	119	1.010960	145.9075	70.74572
	120	1.010695	146.3850	70.78638
	123	1.009944	147.7492	70.85533

The observations in Table 8 and the interpretations regarding it are the same as that of Table 3.

7.3 Model 3

We set an example given as follows:

$$c_h = 50, \qquad c_0 = 500, \qquad c_p = 36, \qquad K = 25{,}000,$$
$$D = 15{,}000, \qquad \theta_1 = 0.35, \qquad \theta_2 = 0.005.$$

Then the minimum cost TAC$*$ is 561,866.9, which is given for the optimal value of $T^* =$ 0.1454915 and $t_1^* = 0.08783444$. Also, then the maximum inventory level will be $q^*_{\max} =$ 81,183.

In Table 9, the sensitivity of the optimality with respect to the value of the parameters is shown.

From Table 9, we can make the following observations.

(i) When the value of c_h increases then T^* and TAC* remain the same. So, we can say the optimality is stable with respect to the change of the holding cost.

(ii) When the value of c_0 increases then both T^* and TAC* gradually increase. This fact can be interpreted as follows: if the ordering cost increases then to minimize the total average cost the inventory procedure needs to run a long time.

(iii) When the value of c_p increases TAC* gradually increases. This fact can be interpreted as follows: if the production cost increases, the total average cost increases.

Table 9 Sensitivity table for Model 3

Parameters	Change in values	T^*	t_1^*	TAC*	q^*_{max}
c_h	48	0.1454915	0.08783444	561,866.9	81,183
	50	0.1454915	0.08783444	561,866.9	81,183
	52	0.1454915	0.08783444	561,866.9	81,183
c_p	34	0.1417628	0.09164477	516,582.9	81,162.12
	36	0.1454915	0.08783444	561,866.9	81,183
	40	0.1399389	0.0844624	607,139.6	81,201.39
c_0	490	0.1440319	0.08694787	561,797.8	81,187.85
	500	0.1454915	0.08783444	561,866.9	81,183
	510	0.1469366	0.0887122	561,935.3	81,178.20
K	24,950	0.1455730	0.08805949	561,863.1	80,771.88
	25,000	0.1454915	0.08783444	561,866.9	81,183
	25,050	0.1454113	0.08761109	561,870.6	81,594.15
D	14,950	0.1456006	0.0876086	560,011.5	81,596.95
	15,000	0.1454915	0.08783444	561,866.9	81,183
	15,050	0.1453860	0.0806192	563,722.1	80,769.23
θ_1	0.345	0.1465341	0.08845998	561,818.1	83,556.77
	0.35	0.1454915	0.08783444	561,866.9	81,183
	0.355	0.1444709	0.08722204	561,915.3	78,908.90
θ_2	0.0045	0.1455589	0.08787488	561,863.7	81,182.80
	0.005	0.1454915	0.08783444	561,866.9	81,183
	0.0055	0.1454242	0.08779406	561,870.1	81,183.21

(iv) If the value of K increases then T^* gradually decreases and TAC^* gradually increases. That means that for high production rate the total time must be shorter and the total average cost will be minimized.

(v) When the value of D increases then T^* gradually decreases and TAC^* gradually increases. That means that to meet a large demand the total time must be shorter and then obviously the total average cost will be minimized.

(vi) When the values of θ_1, θ_2 increase then T^* gradually decreases and TAC^* gradually increases. That means that as the total cost gradually increases due to deterioration, for minimum cost the procedure has to run for shorter time.

7.4 Model 4

Case 6.5: *When* $\alpha = 1$ *and* $\beta = 1$

In this case Model 4 is reduces to the model 3. So if we take the same numerical data as model-3 the numerical discussion will be done for this case.

Case 6.6: *When* $\alpha > 0$ *is arbitrary and* $\beta = 1$

For example if we take $\alpha = 0.5$ and set an example given by

$$c_h = 50, \qquad c_0 = 500, \qquad c_p = 36, \qquad K = 2500,$$
$$D = 1500, \qquad \theta_1 = 0.5, \qquad \theta_2 = 0.005,$$

then the minimum cost is $\mathrm{TAC}^*_{0.5,1} = 568575.9$, which is given for the optimal value of $T^* = 9.978575.9$ and $t_1^* = 0.0004$.

In Table 10, the sensitivity of the optimality with respect to the value of the parameters is shown.

Table 10 Sensitivity table for Case 6.6 for fixed $\alpha = 0.5$ and $\beta = 1$

Parameters	Change in values	T^*	t_1^*	$\mathrm{TAC}^*_{0.5,1}$
c_h	40	9.974794	0.0004	454,871.5
	50	9.974794	0.0004	568,575.9
	60	9.974795	0.0004	682,280.2
c_p	30	9.974794	0.0004	568,575.3
	36	9.974794	0.0004	568,575.9
	40	9.974794	0.0004	568,576.3
c_0	450	9.974795	0.0004	568,570.8
	500	9.974794	0.0004	568,575.9
	550	9.974795	0.0004	568,580.8
K	2400	9.974795	0.000416666	511,600.2
	2500	9.974794	0.0004	568,575.9
	2600	9.974795	0.0003846154	625,570.2
D	1400	9.974795	0.0004	625,427.6
	1500	9.974794	0.0004	568,575.9
	1600	9.974795	0.0004	511,724.0
θ_1	0.4	77.14196	0.0004	171,729.7
	0.5	9.974794	0.0004	568,575.9
	0.6	27.81227	0.0004	99,140.86
θ_2	0.004	9.912312	0.0004	572,159.9
	0.005	9.974794	0.0004	568,575.9
	0.006	10.03836	0.0004	564,975.5

The observations in Table 10 and the interpretations regarding it are the same as that of Table 9.

8 Conclusion

In this paper, we realize that classical economic production quantity model (EPQ) may be generalized as a fractional-order EPQ model with and without deterioration. It is being perceived that holding costs and total average costs for non-fractional cases are the particular cases of generalized holding costs and generalized total average costs. That means that the classical EPQ model may be seen as a particular case of the generalized EPQ model. We have also seen that the generalized EPQ model (fractional EPQ) is not so easy to optimize analytically by any ordinary optimization method. It needs some different solution procedure. For that, we use the real distinct poles rational approximation method of the generalized Mittag-Leffler function and after simplifying we get the result of the optimization problem. In the future, we shall be looking for the analytical as well as numerical optimization method for fractional EPQ models. Hence the FC may be exploited to grow any other classical EPQ and EPQ model with different verity. Moreover, we conclude that the fractional-order inventory model mechanism has been successfully implied for a business which has been a new result. More work with practical info needs to be carried out for future features.

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Author details
[1]Department of Mathematics, Indian Institute of Engineering Science and Technology, Shibpur, Howrah, India. [2]Department of Applied Science, Maulana Abul Kalam Azad University of Technology, West Bengal, Haringhata, India. [3]Department of Mathematics, University of Burdwan, Burdwan, India. [4]Institute for Mathematical Research, Universiti Putra Malaysia, Serdang, Malaysia. [5]Faculty of Engineering and Natural Sciences, Bahcesehir University, Istanbul, Turkey.

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