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Solutions to Riemann–Liouville fractional integrodifferential equations via fractional resolvents

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Abstract

This paper is concerned with the semilinear fractional integrodifferential system with Riemann–Liouville fractional derivative. Firstly, we introduce the suitable $C_{1-\alpha}$-solution to Riemann–Liouville fractional integrodifferential equations in the new frame of fractional resolvents. Some properties of fractional resolvents are given. Then we discuss the sufficient conditions for the existence of solutions without the Lipschitz assumptions to nonlinear item. Finally, an example on fractional partial differential equations is presented to illustrate our abstract results.

MSC: 34A08; 47A10

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1 Introduction

Fractional differential equations have received much attention over the past two decades, as they are found to be important models in many physical, biological, and engineering problems. In fact, they can be regarded as alternative models to nonlinear differential equations and many physical phenomena with memory characteristics can be described by fractional differential equations; see, for instance, [1–7]. Recently, the theories of fractional differential equations with classical Caputo and Riemann–Liouville derivative have been developed and some basic properties are obtained including existence and controllability, see [8–27]. Among them, the differential equations with Caputo fractional derivative are studied extensively. By probability density functions, Wang and Zhou [13] gave a suitable concept of mild solutions to Caputo fractional evolution equations. Balachandran and Kiruthika [11], Balasubramaniam and Tamilalagan [23] proved the existence of solutions to Caputo fractional integrodifferential equations by using resolvent operators. Mallika and Baleanu et al. [26] studied the fractional neutral integrodifferential equation with nonlocal conditions by fixed point theorems and resolvent operators.

On the other hand, in the papers of Heymans and Podlubny [28], Agarwal et al. [29], Baleanu et al. [30], it was shown that Riemann–Liouville fractional differential equations are useful in physics to model viscoelasticity and have different properties from the Ca-

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puto derivative. As the Riemann–Liouville fractional derivative has a singularity at zero, the mathematical analysis to Riemann–Liouville fractional differential equations is more complicated. In this paper we will consider the following semilinear integrodifferential system with a Riemann–Liouville fractional derivative:

$$
\begin{cases}
D^\alpha x(t) = Ax(t) + f(t, x(t), \int_0^t h(t,s,x(s))\,ds), & t \in J' := (0,b], \\
\lim_{t\to 0^+} \Gamma(\alpha)t^{1-\alpha}x(t) = x_0,
\end{cases}
\tag{1.1}
$$

where $0 < \alpha < 1$, D^α is the Riemann–Liouville fractional derivative of order α, $A : D(A) \subseteq X \to X$ is the infinitesimal generator of an order-α fractional resolvent $\{S_\alpha(t), t > 0\}$ on a Banach space X, the operators $h : \Delta \times X \to X, f : J \times X \times X \to X$ are nonlinear functions, where $\Delta = \{(t,s), 0 \le s \le t \le b\}, J := [0,b]$.

Some authors have discussed the solutions to fractional differential equations with Riemann–Liouville fractional derivative [18, 31–33]. For the mild solution, there are two different types of representation that have been given. The first one was constructed in terms of a probability density function. By Laplace transformation and probability density function, Liu and Li [31] gave an appropriate concept of solutions to a semilinear differential system when A generates a C_0-semigroup. The second one was presented in terms of fractional resolvents. In [32], based on (α, k)-regularized operators, Lizama got the representation of solutions for linear fractional order differential equations. Using order-α resolvents, Li and Peng [21], Fan [33] discussed the solutions to fractional homogeneous and inhomogeneous linear differential system, respectively. As is well known, C_0-semigroup is a useful tool in the study of first order differential equations in Banach spaces. In a similar way, fractional resolvents play an important role in the theory of fractional integrodifferential equations. For a compact C_0-semigroup $T(t)$, it is continuous in the sense of operator norm for $t > 0$. Then it is a natural question to ask whether the result is valid in the case of Riemann–Liouville fractional resolvents; see Lemma 2.5. This is one motivation of this paper.

Recently some interesting results on Caputo fractional resolvents have been given in [11, 12, 26]. We note that the properties of resolvent operators for Caputo derivative and Riemann–Liouville derivative are different in essence, though neither of them has the semigroup property. For Caputo fractional resolvents $T_\alpha(t)$, $T_\alpha(0)x = x$ for every $x \in X$, but it is not valid in the case of Riemann–Liouville fractional resolvents. So another motivation of this paper is to formulate the suitable solution to problem (1.1) by Riemann–Liouville fractional resolvents in a Banach space $C_{1-\alpha}(J, X)$, which is constructed to solve the difficulty of fractional resolvents' unboundedness at $t = 0$. Then without the Lipschitz conditions, the existence of solutions to problem (1.1) is discussed.

The paper is organized as follows. In Sect. 2, we recall some concepts and facts about the fractional resolvents. Section 3 is devoted to the sufficient conditions for solutions to problem (1.1). Finally, an example is presented to illustrate the application of our results.

2 Preliminaries
We denote by $C(J, X)$ the space of X-valued continuous functions on J with the norm $\|x\| = \sup\{\|x(t)\|, t \in J\}$, $B(X)$ the space of all bounded linear operators from X to itself, $L^p(J, X)$ the space of X-valued Bochner integrable functions with the norm $\|f\|_{L^p} =$

$(\int_0^b \|f(t)\|^p \, dt)^{\frac{1}{p}}$. In order to define the solution to problem (1.1), we consider the space

$$C_{1-\alpha}(J,X) := \left\{ x(\cdot) : \cdot^{1-\alpha} x(\cdot) \in C(J,X), 0 < \alpha < 1 \right\},$$

with the norm $\|x\|_{C_{1-\alpha}} = \sup\{\|t^{1-\alpha} x(t)\|_X : t \in J\}$, where $t^{1-\alpha} x(t)|_{t=0} = \lim_{t\to 0^+} t^{1-\alpha} x(t)$. Obviously $C_{1-\alpha}(J,X)$ is a Banach space.

Now we recall some definitions and results on fractional derivative and fractional differential equations.

Definition 2.1 ([3]) The Riemann–Liouville fractional integral of a function $f \in L^1(J,X)$ of order $\alpha \in \mathbb{R}^+$ is defined by

$$I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) \, ds, \quad t > 0,$$

where $\Gamma(\cdot)$ is the gamma function.

Definition 2.2 ([3]) The Riemann–Liouville fractional order derivative of order $\alpha \in \mathbb{R}^+$ of a function $f \in L^1(J,X)$ is defined by

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} f(s) \, ds, \quad t > 0,$$

where $\alpha \in (n-1, n]$, $n \in \mathbb{N}$.

Especially for $0 < \alpha < 1$, $D^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} f(s) \, ds$, $t > 0$.

Let the symbol $*$ be the convolution $(f * g)(t) = \int_0^t f(t-s) g(s) \, ds$. For the sake of convenience, we take $g_\alpha(t) := \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ for $t > 0$ and $g_\alpha(t) = 0$ for $t \leq 0$. Then, for $0 < \alpha < 1$,

$$I_t^\alpha f(t) = (g_\alpha * f)(t), \qquad D^\alpha f(t) = \frac{d}{dt}(g_{1-\alpha} * f)(t).$$

Definition 2.3 ([21]) Let $0 < \alpha < 1$. A family $\{S_\alpha(t), t > 0\} \subseteq B(X)$ is called an order-α fractional resolvent if it satisfies the following assumptions:

(a) $S_\alpha(\cdot)x \in C(\mathbb{R}^+, X)$ and $\lim_{t\to 0^+} \Gamma(\alpha) t^{1-\alpha} S_\alpha(t)x = x$, $x \in X$;

(b) $S_\alpha(t)S_\alpha(s) = S_\alpha(s)S_\alpha(t)$, $s, t > 0$;

(c) $S_\alpha(t)I_s^\alpha S_\alpha(s) - I_t^\alpha S_\alpha(t)S_\alpha(s) = g_\alpha(t)I_s^\alpha S_\alpha(s) - g_\alpha(s)I_t^\alpha S_\alpha(t)$, $s, t > 0$.

The linear operator A defined by

$$Ax = \Gamma(2\alpha) \lim_{t\to 0^+} \frac{t^{1-\alpha} S_\alpha(t)x - \frac{1}{\Gamma(\alpha)}x}{t^\alpha}, \quad x \in D(A),$$

is the infinitesimal generator of the fractional resolvent $S_\alpha(t)$, where

$$D(A) = \left\{ x \in X : \lim_{t\to 0+} \frac{t^{1-\alpha} S_\alpha(t)x - \frac{1}{\Gamma(\alpha)}x}{t^\alpha} \text{ exists} \right\}.$$

Note that the fractional resolvent $S_\alpha(t)$ is unbounded when t is sufficiently small, but $t^{1-\alpha} S_\alpha(t)$ is bounded on $J = [0, b]$. We denote $M = \sup_{t\in J} \|t^{1-\alpha} S_\alpha(t)\|$.

Lemma 2.4 ([21]) *Let $\{S_\alpha(t), t > 0\}$ be an order-α fractional resolvent and A be its infinitesimal generator. Then*:

(a) *$S_\alpha(t)x \in D(A)$ and $AS_\alpha(t)x = S_\alpha(t)Ax$ for all $x \in D(A)$, $t > 0$.*

(b) *For all $x \in X$, $t > 0$, $S_\alpha(t)x = \frac{t^{\alpha-1}}{\Gamma(\alpha)}x + AI_t^\alpha S_\alpha(t)x$.*

(c) *For all $x \in D(A)$, $t > 0$, $S_\alpha(t)x = \frac{t^{\alpha-1}}{\Gamma(\alpha)}x + I_t^\alpha S_\alpha(t)Ax$.*

(d) *A is closed and densely defined.*

As fractional resolvents do not satisfy the property of semigroups, we need the following convergence results for resolvents in the uniform operator topology.

Lemma 2.5 *Let $\{t^{1-\alpha}S_\alpha(t), t > 0\}$ be equicontinuous and compact. Then, for every $t > 0$,*

(a) *$\lim_{h\to 0^+} \|(t+h)^{1-\alpha}S_\alpha(t+h) - \Gamma(\alpha)h^{1-\alpha}S_\alpha(h) \cdot t^{1-\alpha}S_\alpha(t)\| = 0$;*

(b) *$\lim_{h\to 0^+} \|t^{1-\alpha}S_\alpha(t) - \Gamma(\alpha)h^{1-\alpha}S_\alpha(h) \cdot (t-h)^{1-\alpha}S_\alpha(t-h)\| = 0$.*

Proof As $t^{1-\alpha}S_\alpha(t)$ is compact for $t > 0$, we have the set

$$P_t = \{t^{1-\alpha}S_\alpha(t)x : \|x\| \le 1\},$$

is precompact in X for every $t > 0$. Then we can find a finite family $\{t^{1-\alpha}S_\alpha(t)x_i : \|x_i\| \le 1\}_{i=1}^m \subset P_t$ satisfying for every x, $\|x\| \le 1$, there exists x_i, $i = 1, \ldots, m$, such that

$$\|t^{1-\alpha}S_\alpha(t)x - t^{1-\alpha}S_\alpha(t)x_i\| < \frac{\varepsilon}{3(1 + \Gamma(\alpha)M)}. \tag{2.1}$$

From Definition 2.3(a), there exists $h_1 > 0$ such that

$$\|t^{1-\alpha}S_\alpha(t)x_i - \Gamma(\alpha)h^{1-\alpha}S_\alpha(h) \cdot t^{1-\alpha}S_\alpha(t)x_i\| < \frac{\varepsilon}{3}, \tag{2.2}$$

for every $0 < h \le h_1$ and $1 \le i \le m$.

Moreover, as $t^{1-\alpha}S_\alpha(t)$ is equicontinuous for $t > 0$, we can find $h_2 > 0$ such that

$$\|(t+h)^{1-\alpha}S_\alpha(t+h)x - t^{1-\alpha}S_\alpha(t)x\| < \frac{\varepsilon}{3}, \tag{2.3}$$

for every $0 < h \le h_2$ and $\|x\| \le 1$.

Now for $0 < h \le \min\{h_1, h_2\}$ and $\|x\| \le 1$, it follows from (2.1)–(2.3) that

$$\begin{aligned}
&\|(t+h)^{1-\alpha}S_\alpha(t+h)x - \Gamma(\alpha)h^{1-\alpha}S_\alpha(h) \cdot t^{1-\alpha}S_\alpha(t)x\| \\
&\quad \le \|(t+h)^{1-\alpha}S_\alpha(t+h)x - t^{1-\alpha}S_\alpha(t)x\| \\
&\qquad + \|t^{1-\alpha}S_\alpha(t)x - t^{1-\alpha}S_\alpha(t)x_i\| \\
&\qquad + \|t^{1-\alpha}S_\alpha(t)x_i - \Gamma(\alpha)h^{1-\alpha}S_\alpha(h) \cdot t^{1-\alpha}S_\alpha(t)x_i\| \\
&\qquad + \|\Gamma(\alpha)h^{1-\alpha}S_\alpha(h) \cdot t^{1-\alpha}S_\alpha(t)x_i - \Gamma(\alpha)h^{1-\alpha}S_\alpha(h) \cdot t^{1-\alpha}S_\alpha(t)x\| \\
&\quad < \frac{\varepsilon}{3} + \frac{\varepsilon}{3(1 + \Gamma(\alpha)M)} + \frac{\varepsilon}{3} \\
&\qquad + \Gamma(\alpha))\|h^{1-\alpha}S_\alpha(h)[t^{1-\alpha}S_\alpha(t)x_i - t^{1-\alpha}S_\alpha(t)x]\| \\
&\quad \le \frac{\varepsilon}{3} + \frac{\varepsilon}{3(1 + \Gamma(\alpha)M)} + \frac{\varepsilon}{3} + \Gamma(\alpha)M\frac{\varepsilon}{3(1 + \Gamma(\alpha)M)} \\
&\quad \le \varepsilon,
\end{aligned}$$

which implies that, for every $t > 0$,

$$\lim_{h \to 0} \left\| (t + h)^{1-\alpha} S_\alpha(t + h) - \Gamma(\alpha) h^{1-\alpha} S_\alpha(h) \cdot t^{1-\alpha} S_\alpha(t) \right\| = 0.$$

(b) Let $t > 0$ and $0 < h < \min\{t, b\}$. Then, for $\|x\| \le 1$, we have

$$
\begin{aligned}
& \left\| t^{1-\alpha} S_\alpha(t) x - \Gamma(\alpha) h^{1-\alpha} S_\alpha(h) \cdot (t - h)^{1-\alpha} S_\alpha(t - h) x \right\| \\
&= \left\| t^{1-\alpha} S_\alpha(t) x - (t + h)^{1-\alpha} S_\alpha(t + h) x \right\| \\
&\quad + \left\| (t + h)^{1-\alpha} S_\alpha(t + h) x - \Gamma(\alpha) h^{1-\alpha} S_\alpha(h) \cdot t^{1-\alpha} S_\alpha(t) x \right\| \\
&\quad + \left\| \Gamma(\alpha) h^{1-\alpha} S_\alpha(h) \cdot \left[t^{1-\alpha} S_\alpha(t) x - (t - h)^{1-\alpha} S_\alpha(t - h) x \right] \right\| \\
&\le \left\| t^{1-\alpha} S_\alpha(t) x - (t + h)^{1-\alpha} S_\alpha(t + h) x \right\| \\
&\quad + \left\| (t + h)^{1-\alpha} S_\alpha(t + h) x - \Gamma(\alpha) h^{1-\alpha} S_\alpha(h) \cdot t^{1-\alpha} S_\alpha(t) x \right\| \\
&\quad + \Gamma(\alpha) M \left\| t^{1-\alpha} S_\alpha(t) x - (t - h)^{1-\alpha} S_\alpha(t - h) x \right\|,
\end{aligned}
$$

which implies the corresponding result by the conclusion of Lemma 2.5(a) and the equicontinuity of $\{ t^{1-\alpha} S_\alpha(t), t > 0 \}$. □

Definition 2.6 A function $x \in C_{1-\alpha}(J, X)$ is called a solution to problem (1.1) if it satisfies

$$x(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)} x_0 + A I_t^\alpha x(t) + I_t^\alpha f\left(t, x(t), \int_0^t h\big(t, s, x(s)\big) \, ds \right), \quad t \in J'.$$

Lemma 2.7 ([34]) *Let $f \in L^p(J, X)$ with $1 \le p < \infty$. Then*

$$\lim_{h \to 0} \int_0^b \left\| f(t + h) - f(t) \right\|^p \, dt = 0,$$

where $f(t) = 0$ for $t \ne J$.

3 Main results

In this section we shall discuss the concept of solution to problem (1.1) by fractional resolvent method and give its existence theorem without Lipschitz assumptions to nonlinear item f. Let r be a finite positive constant and set $W_r = \{ x \in C_{1-\alpha}(J, X) : \|x\|_{C_{1-\alpha}} \le r \}$. For brevity, we define the integral operator H by $(Hx)(t) = \int_0^t h(t, s, x(s)) \, ds$, $x \in C_{1-\alpha}(J, X)$. We give the following hypotheses on fractional integrodifferential system (1.1).

(H1) $\{ t^{1-\alpha} S_\alpha(t), t > 0 \}$ is equicontinuous and compact.

(H2) The function $h : \Delta \times X \to X$ satisfies the following:

    (1) For a.e. $(t, s) \in \Delta$, the function $h(t, s, \cdot) : X \to X$ is continuous and for all $x \in X$, the function $h(\cdot, \cdot, x) : \Delta \to X$ is strongly measurable;

    (2) There exists $m \in \mathbb{R}^+$ such that $\|h(t, s, x)\| \le m\|x\|$.

(H3) The function $f : J \times X \times X \to X$ satisfies the following:

    (1) $f(t, \cdot, \cdot)$ is continuous for a.e. $t \in [0, b]$ and $f(\cdot, x, y) : [0, b] \to X$ is measurable for all $x, y \in X$;

(2) For a.e. $t \in [0, b]$ and $x, y \in X$,

$$\|f(t, x, y)\| \leq \theta(t) + \rho t^{1-\alpha}(\|x\| + \|y\|),$$

where $\theta(t) \in L^p(J, X)$, $p > \frac{1}{\alpha}$ and $0 < \rho < \frac{\alpha^2}{Mb\alpha + Mb^2 m}$.

Lemma 3.1 *Let $f \in L^p(J, X)$, $p > \frac{1}{\alpha}$ and hypothesis (H1) be satisfied. Then the convolution*

$$(S_\alpha * f)(t) = \int_0^t S_\alpha(t - s)f(s)\, ds, \quad t \in J',$$

exists and defines a continuous function on J'.

Proof From Proposition 1.3.4 in [35], we know that $S_\alpha(t - \cdot)f(\cdot)$ is measurable on $(0, t)$. Moreover, we have

$$\begin{aligned}
\|(S_\alpha * f)(t)\| &= \left\| \int_0^t \left((t-s)^{1-\alpha} S_\alpha(t-s) \right) \cdot (t-s)^{\alpha-1} f(s)\, ds \right\| \\
&\leq M \int_0^t \left\| (t-s)^{\alpha-1} f(s) \right\| ds \\
&\leq M\|f\|_{L^p} \left(\int_0^t \left((t-s)^{\alpha-1} \right)^{\frac{p}{p-1}} ds \right)^{\frac{p-1}{p}} \\
&\leq M\|f\|_{L^p} \left(\left(\frac{p-1}{\alpha p - 1} \right) t^{\frac{\alpha p - 1}{p-1}} \right)^{\frac{p-1}{p}} \\
&\leq M\|f\|_{L^p} b^{\alpha - \frac{1}{p}} \left(\frac{p-1}{\alpha p - 1} \right)^{1 - \frac{1}{p}},
\end{aligned}$$

which shows that $S_\alpha * f$ exists.

Next we show that $S_\alpha * f \in C(J', X)$. Let $0 < \varepsilon < t_1 < t_2 \leq b$, then we have

$$\begin{aligned}
&\|(S_\alpha * f)(t_2) - (S_\alpha * f)(t_1)\| \\
&= \left\| \int_0^{t_2} S_\alpha(t_2 - s)f(s)\, ds - \int_0^{t_1} S_\alpha(t_1 - s)f(s)\, ds \right\| \\
&\leq \left\| \int_0^{t_1 - \varepsilon} \left[(t_2 - s)^{1-\alpha} S_\alpha(t_2 - s) - (t_1 - s)^{1-\alpha} S_\alpha(t_1 - s) \right] \cdot (t_2 - s)^{\alpha-1} f(s)\, ds \right\| \\
&\quad + \left\| \int_{t_1 - \varepsilon}^{t_1} \left[(t_2 - s)^{1-\alpha} S_\alpha(t_2 - s) - (t_1 - s)^{1-\alpha} S_\alpha(t_1 - s) \right] \cdot (t_2 - s)^{\alpha-1} f(s)\, ds \right\| \\
&\quad + \left\| \int_0^{t_1} (t_1 - s)^{1-\alpha} S_\alpha(t_1 - s) \cdot \left[(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1} \right] f(s)\, ds \right\| \\
&\quad + \left\| \int_{t_1}^{t_2} (t_2 - s)^{1-\alpha} S_\alpha(t_2 - s) \cdot (t_2 - s)^{\alpha-1} f(s)\, ds \right\| \\
&\leq \sup_{s \in [0, t_1 - \varepsilon]} \left\| (t_2 - s)^{1-\alpha} S_\alpha(t_2 - s) - (t_1 - s)^{1-\alpha} S_\alpha(t_1 - s) \right\| \cdot \|f\|_{L^p} b^{\alpha - \frac{1}{p}} \left(\frac{p-1}{\alpha p - 1} \right)^{1 - \frac{1}{p}} \\
&\quad + 2M\|f\|_{L^p} \cdot \left(\frac{p-1}{\alpha p - 1} \right)^{1 - \frac{1}{p}} \left[(t_2 - t_1 + \varepsilon)^{\frac{\alpha p - 1}{p-1}} - (t_2 - t_1)^{\frac{\alpha p - 1}{p-1}} \right]^{1 - \frac{1}{p}}
\end{aligned}$$

$$+ M\|f\|_{L^p} \left(\int_0^{t_1} \left[(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1} \right]^{\frac{p}{p-1}} \, ds \right)^{1-\frac{1}{p}}$$

$$+ M\|f\|_{L^p} \left(\frac{p-1}{\alpha p - 1} \right)^{1-\frac{1}{p}} (t_2 - t_1)^{\alpha - \frac{1}{p}}. \tag{3.1}$$

Then due to the equicontinuity of $\{t^{1-\alpha} S_\alpha(t), t > 0\}$, Lemma 2.7 and the arbitrariness of ε, we get

$$\left\| (S_\alpha * f)(t_2) - (S_\alpha * f)(t_1) \right\| \to 0, \quad \text{as } t_1 \to t_2,$$

which shows that $(S_\alpha * f)(t)$ is continuous on $(0, b]$.   □

Lemma 3.2 *Suppose that conditions* (H1)–(H3) *are satisfied. Then* $x \in C_{1-\alpha}(J, X)$ *is a solution to problem* (1.1) *if and only if* x *satisfies*

$$x(t) = S_\alpha(t)x_0 + \int_0^t S_\alpha(t - s)f\big(s, x(s), Hx(s)\big) \, ds, \quad t \in J'. \tag{3.2}$$

Proof By Lemma 2.4(b), we know that, for $t > 0$,

$$g_\alpha(t) = S_\alpha(t) - (Ag_\alpha * S_\alpha)(t).$$

Let $x(\cdot)$ be a solution to problem (1.1). Then we have

$$\begin{aligned}
g_\alpha * x &= (S_\alpha - Ag_\alpha * S_\alpha) * x \\
&= S_\alpha * x - S_\alpha * (Ag_\alpha * x) \\
&= S_\alpha * (x - Ag_\alpha * x) \\
&= S_\alpha * \big(g_\alpha x_0 + g_\alpha * f\big(\cdot, x(\cdot), Hx(\cdot)\big)\big) \\
&= g_\alpha * \big(S_\alpha x_0 + S_\alpha * f\big(\cdot, x(\cdot), Hx(\cdot)\big)\big),
\end{aligned}$$

which implies

$$x(t) = S_\alpha(t)x_0 + \int_0^t S_\alpha(t - s)f\big(s, x(s), Hx(s)\big) \, ds.$$

Conversely, suppose $x(\cdot)$ satisfies Eq. (3.2). From Lemma 3.1, we know that x is well defined on J'. For the result of $AI_t^\alpha x(t)$, by Definition 2.3(c), we have

$$\begin{aligned}
&\left(s^{1-\alpha} S_\alpha(s) - \frac{1}{\Gamma(\alpha)} \right) I_t^\alpha x(t) \\
&= \big(s^{1-\alpha} S_\alpha(s) - s^{1-\alpha} g_\alpha(s) \big) \big(I_t^\alpha S_\alpha(t)x_0 + g_\alpha * S_\alpha * f\big(\cdot, x(\cdot), Hx(\cdot)\big)(t) \big) \\
&= s^{1-\alpha} \big[S_\alpha(s) I_t^\alpha S_\alpha(t)x_0 - g_\alpha(s) I_t^\alpha S_\alpha(t)x_0 \big] \\
&\quad + s^{1-\alpha} \big[S_\alpha(s) \cdot \big(I_t^\alpha S_\alpha \big) * f\big(\cdot, x(\cdot), Hx(\cdot)\big)(t) - g_\alpha(s) \cdot \big(I_t^\alpha S_\alpha \big) * f\big(\cdot, x(\cdot), Hx(\cdot)\big)(t) \big] \\
&= s^{1-\alpha} \big[S_\alpha(t) I_s^\alpha S_\alpha(s)x_0 - g_\alpha(t) I_s^\alpha S_\alpha(s)x_0 \big]
\end{aligned}$$

$$+ s^{1-\alpha}\big[I_s^\alpha S_\alpha(s)S_\alpha(t) - I_s^\alpha S_\alpha(s)g_\alpha(t)\big] * f\big(\cdot, x(\cdot), Hx(\cdot)\big)(t)$$

$$= s^{1-\alpha}I_s^\alpha S_\alpha(s)\big[S_\alpha(t)x_0 - g_\alpha(t)x_0 + S_\alpha * f\big(\cdot, x(\cdot), Hx(\cdot)\big)(t) - g_\alpha * f\big(\cdot, x(\cdot), Hx(\cdot)\big)(t)\big]$$

$$= s^{1-\alpha}I_s^\alpha S_\alpha(s)\big[x(t) - g_\alpha(t)x_0 - I_t^\alpha f\big(t, x(t), Hx(t)\big)\big].$$

It follows that

$$AI_t^\alpha x(t)$$

$$= \lim_{s\to 0^+}\Gamma(2\alpha)\frac{(s^{1-\alpha}S_\alpha(s) - \frac{1}{\Gamma(\alpha)})I_t^\alpha x(t)}{s^\alpha}$$

$$= \lim_{s\to 0^+}\Gamma(2\alpha)s^{1-2\alpha}I_s^\alpha S_\alpha(s)\big[x(t) - g_\alpha(t)x_0 - I_t^\alpha f\big(t, x(t), Hx(t)\big)\big]. \tag{3.3}$$

Noticing that

$$\big\|\Gamma(2\alpha)s^{1-2\alpha}I_s^\alpha S_\alpha(s)x - x\big\|$$

$$= \left\|\frac{\Gamma(2\alpha)}{\Gamma(\alpha)}\int_0^s s^{1-2\alpha}(s-\tau)^{\alpha-1}S_\alpha(\tau)x\,d\tau - x\right\|$$

$$= \left\|\frac{\Gamma(2\alpha)}{\Gamma(\alpha)}\int_0^1 s^{1-\alpha}(1-\tau)^{\alpha-1}S_\alpha(s\tau)x\,d\tau - x\right\|$$

$$= \left\|\frac{\Gamma(2\alpha)}{[\Gamma(\alpha)]^2}\int_0^1 s^{1-\alpha}\Gamma(\alpha)(1-\tau)^{\alpha-1}S_\alpha(s\tau)x\,d\tau - x\right\|$$

$$= \left\|\frac{\Gamma(2\alpha)}{[\Gamma(\alpha)]^2}\int_0^1 (1-\tau)^{\alpha-1}\tau^{\alpha-1}\Gamma(\alpha)(s\tau)^{1-\alpha}S_\alpha(s\tau)x\,d\tau\right.$$

$$\left. - \frac{\Gamma(2\alpha)}{[\Gamma(\alpha)]^2}\int_0^1 (1-\tau)^{\alpha-1}\tau^{\alpha-1}x\,d\tau\right\|$$

$$\leq \frac{\Gamma(2\alpha)}{[\Gamma(\alpha)]^2}\int_0^1 (1-\tau)^{\alpha-1}\tau^{\alpha-1}\,d\tau \cdot \sup_{\tau\in[0,1]}\big\|\Gamma(\alpha)(s\tau)^{1-\alpha}S_\alpha(s\tau)x - x\big\|$$

$$\leq \sup_{\tau\in[0,1]}\big\|\Gamma(\alpha)(s\tau)^{1-\alpha}S_\alpha(s\tau)x - x\big\|.$$

By Definition 2.3(a), we get

$$\big\|\Gamma(2\alpha)s^{1-2\alpha}I_s^\alpha S_\alpha(s)x - x\big\| \to 0, \quad \text{as } s\to 0^+. \tag{3.4}$$

Combining (3.3) and (3.4), we have

$$AI_t^\alpha x(t) = x(t) - g_\alpha(t)x_0 - I_t^\alpha f\big(t, x(t), Hx(t)\big).$$

That is,

$$x(t) = g_\alpha(t)x_0 + AI_t^\alpha x(t) + I_t^\alpha f\big(t, x(t), Hx(t)\big),$$

which shows that x is a solution to problem (1.1).   □

Lemma 3.3 *Suppose that assumptions* (H1)–(H3) *are satisfied. Let* $W_r = \{x \in C_{1-\alpha}(J,X) : \|x\|_{C_{1-\alpha}} \le r\}$. *Then the mapping* $G : W_r \to C_{1-\alpha}(J,X)$ *defined by*

$$(Gx)(t) = \int_0^t S_\alpha(t-s) f(s, x(s), Hx(s)) \, ds$$

is compact.

Proof In view of the relationship between $(C_{1-\alpha}(J,X), \|\cdot\|_{C_{1-\alpha}})$ and $(C(J,X), \|\cdot\|_C)$, for the compactness of GW_r in $C_{1-\alpha}(J,X)$, it is sufficient to prove that the set

$$B = \left\{ y \in C(J,X) : y(t) = t^{1-\alpha}(Gx)(t), x \in W_r, t \in J \right\}$$

is precompact in $C(J,X)$.

Firstly, we show that $B(t) = \{y(t) : y \in B\} \subseteq X$ is precompact in X for every $t \in J$. If $t = 0$, then $B(0) = 0$ is obviously satisfied. If $t > 0$, we can define a set $B^\varepsilon(t) = \{y^\varepsilon(t), x \in W_r, t \in J'\} \subseteq X$, where

$$y^\varepsilon(t) = \varepsilon^{1-\alpha} S_\alpha(\varepsilon) \cdot \Gamma(\alpha) t^{1-\alpha} \int_0^{t-\varepsilon} S_\alpha(t-s-\varepsilon) f(s, x(s), Hx(s)) \, ds.$$

For $x \in W_r$, $s \in [0,b]$, we have

$$
\begin{aligned}
\left\| f(s, x(s), Hx(s)) \right\| &\le \theta(s) + \rho s^{1-\alpha} \left(\|x(s)\| + \left\| \int_0^s h(s, \tau, x(\tau)) \, d\tau \right\| \right) \\
&\le \theta(s) + \rho s^{1-\alpha} \left(\|x(s)\| + \int_0^s m \|x(\tau)\| \, d\tau \right) \\
&\le \theta(s) + \rho s^{1-\alpha} \|x(s)\| + \rho s^{1-\alpha} \int_0^s m \tau^{\alpha-1} \|\tau^{1-\alpha} x(\tau)\| \, d\tau \\
&\le \theta(s) + \rho r + \rho s^{1-\alpha} m \frac{s^\alpha}{\alpha} r \\
&\le \theta(s) + \rho r + \rho \frac{ms}{\alpha} r \\
&\le \theta(s) + \rho r + \rho \frac{mb}{\alpha} r.
\end{aligned}
\tag{3.5}
$$

By (3.5), for $x \in W_r$, $t \in (0,b]$, we get

$$
\begin{aligned}
&\left\| t^{1-\alpha} \int_0^{t-\varepsilon} S_\alpha(t-s-\varepsilon) f(s, x(s), Hx(s)) \, ds \right\| \\
&\quad \le b^{1-\alpha} \int_0^{t-\varepsilon} \left\| (t-s-\varepsilon)^{1-\alpha} S_\alpha(t-s-\varepsilon) \cdot (t-s-\varepsilon)^{\alpha-1} f(s, x(s), Hx(s)) \right\| \, ds \\
&\quad \le M b^{1-\alpha} \int_0^{t-\varepsilon} \left\| (t-s-\varepsilon)^{\alpha-1} f(s, x(s), Hx(s)) \right\| \, ds \\
&\quad \le M b^{1-\alpha} \int_0^{t-\varepsilon} (t-s-\varepsilon)^{\alpha-1} \left(\theta(s) + \rho r + \rho \frac{mb}{\alpha} r \right) \, ds
\end{aligned}
$$

$$\le Mb^{1-\alpha} \int_0^{t-\varepsilon} (t-s-\varepsilon)^{\alpha-1}\theta(s)\,ds + Mb^{1-\alpha} \int_0^{t-\varepsilon} (t-s-\varepsilon)^{\alpha-1}\left(\rho r + \rho\frac{mb}{\alpha}r\right)ds$$

$$\le M\left(b\frac{p-1}{\alpha p-1}\right)^{1-\frac{1}{p}} \|\theta\|_{L^p} + \frac{Mb}{\alpha}\left(\rho r + \rho\frac{mb}{\alpha}r\right)$$

$$< \infty. \tag{3.6}$$

Moreover, due to hypothesis (H1), for $\varepsilon > 0$, the operator $\varepsilon^{1-\alpha}S_\alpha(\varepsilon)$ is compact. So we know that $B^\varepsilon(t)$ is precompact in X for each $t \in J'$.

Let $t \in (0,b]$ and $\delta \in (\varepsilon,t)$. We have

$$\|y(t) - y^\varepsilon(t)\|$$

$$\le t^{1-\alpha}\Bigg[\Bigg\|\int_0^{t-\varepsilon} (t-s)^{1-\alpha}S_\alpha(t-s)\cdot(t-s)^{\alpha-1}f\big(s,x(s),Hx(s)\big)\,ds$$

$$- \varepsilon^{1-\alpha}S_\alpha(\varepsilon)\Gamma(\alpha)\int_0^{t-\varepsilon} (t-s-\varepsilon)^{1-\alpha}S_\alpha(t-s-\varepsilon)\cdot(t-s)^{\alpha-1}f\big(s,x(s),Hx(s)\big)\,ds\Bigg\|$$

$$+ \Bigg\|\varepsilon^{1-\alpha}S_\alpha(\varepsilon)\Gamma(\alpha)\int_0^{t-\varepsilon}(t-s-\varepsilon)^{1-\alpha}S_\alpha(t-s-\varepsilon)\cdot\big((t-s)^{\alpha-1}-(t-s-\varepsilon)^{\alpha-1}\big)$$

$$\times f\big(s,x(s),Hx(s)\big)\,ds\Bigg\|$$

$$+ \Bigg\|\int_{t-\varepsilon}^t (t-s)^{1-\alpha}S_\alpha(t-s)\cdot(t-s)^{\alpha-1}f\big(s,x(s),Hx(s)\big)\,ds\Bigg\|\Bigg]$$

$$\le b^{1-\alpha}\int_0^{t-\varepsilon}\Big\|\big[(t-s)^{1-\alpha}S_\alpha(t-s)-\Gamma(\alpha)\varepsilon^{1-\alpha}S_\alpha(\varepsilon)(t-s-\varepsilon)^{1-\alpha}S_\alpha(t-s-\varepsilon)\big]$$

$$\times (t-s)^{\alpha-1}f\big(s,x(s),Hx(s)\big)\Big\|\,ds$$

$$+ b^{1-\alpha}\big\|\varepsilon^{1-\alpha}S_\alpha(\varepsilon)\big\|\Gamma(\alpha)\cdot M\int_0^{t-\varepsilon}\Big\|\big[(t-s)^{\alpha-1}-(t-s-\varepsilon)^{\alpha-1}\big]f\big(s,x(s),Hx(s)\big)\Big\|\,ds$$

$$+ b^{1-\alpha}M\int_{t-\varepsilon}^t\big\|(t-s)^{\alpha-1}f\big(s,x(s),Hx(s)\big)\big\|\,ds$$

$$\le b^{1-\alpha}\int_0^{t-\delta}\big\|(t-s)^{1-\alpha}S_\alpha(t-s)-\Gamma(\alpha)\varepsilon^{1-\alpha}S_\alpha(\varepsilon)(t-s-\varepsilon)^{1-\alpha}S_\alpha(t-s-\varepsilon)\big\|$$

$$\times (t-s)^{\alpha-1}\left(\theta(s)+\rho r+\rho\frac{mb}{\alpha}r\right)ds$$

$$+ b^{1-\alpha}\int_{t-\delta}^{t-\varepsilon}\big\|(t-s)^{1-\alpha}S_\alpha(t-s)-\Gamma(\alpha)\varepsilon^{1-\alpha}S_\alpha(\varepsilon)(t-s-\varepsilon)^{1-\alpha}S_\alpha(t-s-\varepsilon)\big\|$$

$$\times (t-s)^{\alpha-1}\left(\theta(s)+\rho r+\rho\frac{mb}{\alpha}r\right)ds$$

$$+ b^{1-\alpha}M^2\Gamma(\alpha)\left(\int_0^{t-\varepsilon}\big[(t-s)^{\alpha-1}-(t-s-\varepsilon)^{\alpha-1}\big]^{\frac{p}{p-1}}\,ds\right)^{1-\frac{1}{p}}\|f\|_{L^p}$$

$$+ b^{1-\alpha}M\int_{t-\varepsilon}^t\big\|(t-s)^{\alpha-1}f\big(s,x(s),Hx(s)\big)\big\|\,ds$$

$$:= I_1 + I_2 + I_3 + I_4,$$

where

$$
\begin{aligned}
I_1 = {}& b^{1-\alpha} \int_0^{t-\delta} \left\| (t-s)^{1-\alpha} S_\alpha(t-s) - \Gamma(\alpha)\varepsilon^{1-\alpha} S_\alpha(\varepsilon)(t-s-\varepsilon)^{1-\alpha} S_\alpha(t-s-\varepsilon) \right\| \\
& \times (t-s)^{\alpha-1}\left(\theta(s) + \rho r + \rho\frac{mb}{\alpha}r \right) ds,
\end{aligned}
$$

$$
\begin{aligned}
I_2 = {}& b^{1-\alpha} \int_{t-\delta}^{t-\varepsilon} \left\| (t-s)^{1-\alpha} S_\alpha(t-s) - \Gamma(\alpha)\varepsilon^{1-\alpha} S_\alpha(\varepsilon)(t-s-\varepsilon)^{1-\alpha} S_\alpha(t-s-\varepsilon) \right\| \\
& \times (t-s)^{\alpha-1}\left(\theta(s) + \rho r + \rho\frac{mb}{\alpha}r \right) ds,
\end{aligned}
$$

$$
I_3 = b^{1-\alpha} M^2 \Gamma(\alpha) \left(\int_0^{t-\varepsilon} \left[(t-s)^{\alpha-1} - (t-s-\varepsilon)^{\alpha-1} \right]^{\frac{p}{p-1}} ds \right)^{1-\frac{1}{p}} \|f\|_{L^p},
$$

$$
I_4 = b^{1-\alpha} M \int_{t-\varepsilon}^t \left\| (t-s)^{\alpha-1} f\left(s, x(s), Hx(s) \right) \right\| ds.
$$

From Lemma 2.5, we know that $I_1 \to 0$, as $\varepsilon \to 0^+$. By the arbitrariness of ε, δ and absolute continuity of integral, we get

$$
I_2 \to 0, \qquad I_4 \to 0,
$$

as $\varepsilon, \delta \to 0^+$. The conclusion of Lemma 2.7 shows that $I_3 \to 0$, as $\varepsilon \to 0^+$. Now for $t \in J'$, we get

$$
\lim_{\varepsilon \to 0^+} \left\| y(t) - y^\varepsilon(t) \right\| = 0,
$$

which implies that $B(t) = \{ y(t) : y \in B \}$ is precompact in X as there is a family of precompact sets arbitrarily close to it.

Next, we show the equicontinuity of B on J. Similar to the computational procedure of (3.6), we can get

$$
\begin{aligned}
& \left\| \int_0^t S_\alpha(t-s) f\left(s, x(s), Hx(s) \right) ds \right\| \\
& \quad \leq M b^{\alpha-\frac{1}{p}} \left(\frac{p-1}{\alpha p - 1} \right)^{1-\frac{1}{p}} \|\theta\|_{L^p} + \frac{Mb^\alpha}{\alpha} \left(\rho r + \rho\frac{mb}{\alpha}r \right) \\
& \quad := F_r < \infty,
\end{aligned} \tag{3.7}
$$

for $t \in J$, $x \in W_r$. Let $y \in B$, $0 \leq t_1 < t_2 \leq b$. Then we have

$$
\begin{aligned}
& \left\| y(t_2) - y(t_1) \right\| \\
& \quad = \left\| t_2^{1-\alpha} \int_0^{t_2} S_\alpha(t_2-s) f\left(s, x(s), Hx(s) \right) ds - t_1^{1-\alpha} \int_0^{t_1} S_\alpha(t_1-s) f\left(s, x(s), Hx(s) \right) ds \right\| \\
& \quad \leq \left\| \left(t_2^{1-\alpha} - t_1^{1-\alpha} \right) \int_0^{t_2} S_\alpha(t_2-s) f\left(s, x(s), Hx(s) \right) ds \right\| \\
& \qquad + t_1^{1-\alpha} \left\| \int_0^{t_2} S_\alpha(t_2-s) f\left(s, x(s), Hx(s) \right) ds - \int_0^{t_1} S_\alpha(t_1-s) f\left(s, x(s), Hx(s) \right) ds \right\|
\end{aligned}
$$

$$\leq \left(t_2^{1-\alpha} - t_1^{1-\alpha}\right)F_r$$
$$+ b^{1-\alpha}\left\|\int_0^{t_2} S_\alpha(t_2 - s)f\left(s, x(s), Hx(s)\right)ds - \int_0^{t_1} S_\alpha(t_1 - s)f\left(s, x(s), Hx(s)\right)ds\right\|.$$

From (3.5), we know

$$\left\|f\left(s, x(s), Hx(s)\right)\right\| \leq \theta(s) + \rho r + \rho\frac{mb}{\alpha}r, \quad \theta \in L^p(J, X).$$

Then due to Eq. (3.1) in Lemma 3.1, we have

$$\left\|\int_0^{t_2} S_\alpha(t_2 - s)f\left(s, x(s), Hx(s)\right)ds - \int_0^{t_1} S_\alpha(t_1 - s)f\left(s, x(s), Hx(s)\right)ds\right\| \to 0,$$

as $t_1 \to t_2$, independent of $x \in W_r$. Now we can obtain

$$\lim_{t_1 \to t_2}\left\|y(t_2) - y(t_1)\right\| = 0,$$

which leads to the equicontinuity of B on J. Thus $G : W_r \to C_{1-\alpha}(J, X)$ is a compact mapping by the Ascoli–Arzela theorem. This proof is completed. □

Now we can present our main existence result to problem (1.1).

Theorem 3.4 *Assume that the hypotheses* (H1)–(H3) *are satisfied. Then the system* (1.1) *has at least one solution.*

Proof We transform the existence of solutions into a fixed point problem. For this purpose, by considering Lemma 3.2, we introduce the solution operator $\Phi : C_{1-\alpha}(J, X) \to C_{1-\alpha}(J, X)$ by

$$\Phi x(t) = S_\alpha(t)x_0 + \int_0^t S_\alpha(t - s)f\left(s, x(s), Hx(s)\right)ds.$$

It is easy to see that the fixed point of Φ is just the solution to problem (1.1). Subsequently, we shall prove that Φ has a fixed point by Schauder's fixed point theorem.

Step 1. We claim that $\Phi W_r \subseteq W_r$ in $C_{1-\alpha}(J, X)$, where

$$r \geq \frac{\alpha^2}{\alpha^2 - Mb(\alpha\rho + \rho bm)}\left[M\|x_0\| + M\left(b\frac{p-1}{\alpha p - 1}\right)^{1-\frac{1}{p}}\|\theta\|_{L^p}\right].$$

In fact, for $x \in W_r$, $t \in J$, from (3.7) we have

$$\left\|t^{1-\alpha}\Phi x(t)\right\|$$
$$\leq \left\|t^{1-\alpha}S_\alpha(t)x_0\right\| + b^{1-\alpha}\left\|\int_0^t S_\alpha(t - s)f\left(s, x(s), Hx(s)\right)ds\right\|$$
$$\leq M\|x_0\| + M\left(b\frac{p-1}{\alpha p - 1}\right)^{1-\frac{1}{p}}\|\theta\|_{L^p} + \frac{Mb}{\alpha}\left(\rho r + \rho\frac{mb}{\alpha}r\right)$$
$$\leq r.$$

Step 2. We show that Φ is continuous on $W_r \subseteq C_{1-\alpha}(J, X)$. For this purpose, we assume that $x_n \to x$ in W_r. From hypothesis (H2), (H3), for $t \in J$, we have

$$(t - s)^{\alpha-1}\big(f\big(s, x_n(s), Hx_n(s)\big) - f\big(s, x(s), Hx(s)\big)\big) \to 0, \quad \text{a.e. } s \in [0, t],$$

and from (3.5), it follows that

$$(t - s)^{\alpha-1}\big\|f\big(s, x_n(s), Hx_n(s)\big) - f\big(s, x(s), Hx(s)\big)\big\|$$
$$\leq 2(t - s)^{\alpha-1}\Big(\theta(s) + \varrho r + \rho \frac{mb}{\alpha} r\Big), \quad s \in [0, t].$$

Then, by the dominated convergence theorem, we get

$$t^{1-\alpha}\big\|(\Phi x_n)(t) - (\Phi x)(t)\big\|$$
$$\leq t^{1-\alpha} \int_0^t \big\|(t - s)^{1-\alpha} S_\alpha(t - s)\big\| \cdot (t - s)^{\alpha-1}\big\|f\big(s, x_n(s), Hx_n(s)\big) - f\big(s, x(s), Hx(s)\big)\big\| \, ds$$
$$\leq Mb^{1-\alpha} \int_0^t (t - s)^{\alpha-1}\big\|f\big(s, x_n(s), Hx_n(s)\big) - f\big(s, x(s), Hx(s)\big)\big\| \, ds$$
$$\to 0, \quad n \to \infty,$$

which implies the continuity of Φ on W_r.

Step 3. We show that the operator Φ is compact. Let

$$\Phi = \Phi_1 + \Phi_2,$$

where $\Phi_1(t) = S_\alpha(t)x_0$, $\Phi_2(t) = \int_0^t S_\alpha(t - s) f(s, x(s), Hx(s)) \, ds$. From Lemma 3.3, we have concluded that Φ_2 is compact in W_r. For the compactness of Φ_1, it is sufficient to check the set

$$V = \big\{z \in C(J, X) : z(t) = t^{1-\alpha} S_\alpha(t)x_0, x_0 \in X, t \in J\big\},$$

is precompact in $C(J, X)$. Obviously, $V(0) = \{\frac{x_0}{\Gamma(\alpha)}\}$, $V(t) = \{t^{1-\alpha} S_\alpha(t)x_0\}$, $t > 0$, is precompact in X. Suppose that $0 \leq t_1 < t_2 \leq b$. If $t_1 = 0$, in view of Definition 2.3(a), we get

$$\big\|z(t_2) - z(0)\big\| = \Big\|t_2^{1-\alpha} S_\alpha(t_2)x_0 - \frac{x_0}{\Gamma(\alpha)}\Big\| \to 0,$$

as $t_2 \to 0$. If $t_1 > 0$,

$$\big\|z(t_2) - z(t_1)\big\| \leq \big\|t_2^{1-\alpha} S_\alpha(t_2)x_0 - t_2^{1-\alpha} S_\alpha(t_2)x_0\big\| \to 0.$$

From hypothesis (H1), we know that $\|z(t_2) - z(t_1)\| \to 0$, as $t_1 \to t_2$. By the Ascoli–Arzela theorem, we see that V is precompact in $C(J, X)$. Therefore, $\Phi = \Phi_1 + \Phi_2$ is a compact operator in $C_{1-\alpha}(J, X)$.

Hence, from Schauder's fixed point theorem, there exists a fixed point x such that $\Phi x = x$, which is the solution to problem (1.1). This completes the proof. □

4 Application

Consider the following integrodifferential evolution system with Riemann–Liouville fractional derivative:

$$
\begin{cases}
D^\alpha u(t,x) = \frac{\partial^2}{\partial x^2} u(t,x) + F(t, u(t,x), \int_0^t h_1(t,s,u(s,x))\,ds), & 0 < t \le 1, 0 < x < 1, \\
u(t,0) = u(t,1) = 0, \\
\lim_{t \to 0^+} \Gamma(\alpha) t^{1-\alpha} u(t,x) = u_0(x).
\end{cases}
\tag{4.1}
$$

Take $X = L^2(0,1)$ and the operator $A : D(A) \subseteq X \to X$ defined by $Az = z''$, with

$$
D(A) = \{ z \in X : z, z' \text{ are absolutely continuous}, z'' \in X, z(0) = z(1) = 0 \}.
$$

From Pazy [36], A is the infinitesimal generator of a compact analytic semigroup $T(t)$, $t \ge 0$. It is known that A has the eigenvalues $\lambda_n = -n^2\pi^2$, $n \in \mathbb{N}$, and the corresponding eigenvectors $e_n(x) = \sqrt{2}\sin(n\pi x)$ for $n \ge 1$, $e_0 = 1$, which form an orthogonal basis for $L^2(0,1)$. Then $T(t)$ is given by

$$
T(t)z = \sum_{n=1}^\infty e^{-n^2\pi^2 t} \langle z, e_n \rangle e_n.
$$

If $u_0(x) = \sum_{n=1}^\infty c_n \sin n\pi x$, then we have

$$
T(t)u_0(x) = \sum_{n=1}^\infty e^{-n^2\pi^2 t} c_n \sin n\pi x.
$$

Moreover, from [21], we know A is the infinitesimal generator of an order-α fractional resolvent $S_\alpha(t)$ and

$$
S_\alpha(t)u_0(x) = \sum_{n=1}^\infty t^{\alpha-1} E_{\alpha,\alpha}\left(-n^2\pi^2 t^\alpha\right) c_n \sin n\pi x.
$$

Employing the method in [13, 31], by Laplace transformation and probability density functions, we can have

$$
t^{1-\alpha} S_\alpha(t)u_0(x) = \alpha \int_0^\infty \theta \xi_\alpha(\theta) T\left(t^\alpha \theta\right) u_0(x)\,d\theta,
\tag{4.2}
$$

for any $u_0 \in X$, where

$$
\xi_\alpha(\theta) = \frac{1}{\alpha} \theta^{-1-\frac{1}{\alpha}} \varpi_\alpha\left(\theta^{-\frac{1}{\alpha}}\right),
$$

$$
\varpi_\alpha(\theta) = \frac{1}{\pi} \sum_{n=1}^\infty (-1)^{n-1} \theta^{-n\alpha-1} \frac{\Gamma(n\alpha+1)}{n!} \sin(n\pi\alpha), \quad \theta \in (0,\infty).
$$

Equation (4.2) shows

$$
t^{1-\alpha} S_\alpha(t) = \alpha \int_0^\infty \theta \xi_\alpha(\theta) T\left(t^\alpha \theta\right)\,d\theta.
$$

From Lemma 2.9 of [13], it follows that the family of operators $\{t^{1-\alpha}S_\alpha(t) : t > 0\}$ is equicontinuous, compact and $\|t^{1-\alpha}S_\alpha(t)\| \leq \frac{\alpha M'}{\Gamma(1+\alpha)} := M$, where $M' = \sup\{\|T(t)\| : 0 \leq t \leq 1\}$. Then hypothesis (H1) is satisfied.

Now, we define a continuous function $f : [0, b] \times X \times X \to X$ by

$$f(t, u, h)(x) = F\big(t, u(t, x), h(t, x)\big), \quad 0 < t \leq 1, 0 < x < 1,$$
$$h(t, x) = \int_0^t h_1\big(t, s, u(s, x)\big)\, ds.$$

We take

$$F\left(t, u(t, x), \int_0^t h_1\big(t, s, u(s, x)\big)\, ds\right)$$
$$= e^{-t}\cos\big(u(t, x)\big) + \rho t^{1-\alpha}\left(u(t, x) + \int_0^t \cos(ts)u(s, x)\, ds\right),$$

where $0 < \rho < \frac{\alpha^2}{M\alpha+M}$. So the functions f, h satisfy hypotheses (H2) and (H3). Let $u(t)x = u(t, x)$, for $t, x \in (0, 1)$. Then the differential system (4.1) can be presented in the abstract form (1.1) and all the conditions of Theorem 3.4 are satisfied. Hence there exists a function $u \in C_{1-\alpha}(J, L^2(0, 1))$ which is a solution of (4.1).

5 Conclusions

By using fractional resolvents, this paper introduces the solution to semilinear Riemann–Liouville fractional integrodifferential equations and discuss its existence results. There are two points worth of attention in the study. One is that the Riemann–Liouville fractional resolvent $S_\alpha(t)$ is not bounded at $t = 0$, which is essentially different from the case for Caputo fractional resolvents, and the other is $S_\alpha(t)$ does not have the property of semigroups, which means that the compactness of $S_\alpha(t)$ (or $t^{1-\alpha}S_\alpha(t)$) cannot conclude the equicontinuity of $S_\alpha(t)$ (or $t^{1-\alpha}S_\alpha(t)$). As the existence of solutions is the basis of qualitative study to differential equations, we can continue to discuss the controllability and stability of the solution using the similar approach. On the other hand, some new general fractional derivatives are introduced and studied, such as some types of extended Riemann–Liouville fractional derivative and the fractional derivative without singular kernel of exponential function [37–41]. Especially, the Hilfer fractional derivative is often used as a generalized Riemann–Liouville fractional derivative, which includes Riemann–Liouville and Caputo derivatives; see [4, 9, 42]. To the best of our knowledge, most of the existence and controllability results on the Hilfer fractional differential system are studied under the frame that A generates a strongly continuous semigroup and the solution is given by semigroup and probability density functions. It is still an open problem how to define fractional resolvents to Hilfer and other general fractional equations and it is worth discussing later.

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