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Existence and uniqueness of solutions for system of Hilfer–Hadamard sequential fractional differential equations with two point boundary conditions

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Abstract

In this paper, we study existence and uniqueness of solutions for a system of Hilfer–Hadamard sequential fractional differential equations via standard fixed point theorems. The existence is proved by using the Leray–Schauder alternative, while the existence and uniqueness by the Banach contraction mapping principle. Illustrative examples are also discussed.

MSC: 34A08; 34B10

Keywords: Hilfer–Hadamard fractional differential equations; Boundary conditions; Fixed point

1 Introduction

Fractional differential equations have been applied in many fields such as physics, chemistry, biology, engineering, and so on. Fractional differential equations have several kinds of fractional derivatives, such as Riemann–Liouville fractional derivative, Caputo fractional derivative, Grunwald–Letnikov fractional derivative, Hadamard fractional derivative, etc. The reader interested in the subject of fractional calculus is referred to the books by Kilbas *et al.* [1], Podlubny [2], Samko *et al.* [3], Miller and Ross [4], and Diethelm [5]. A generalization of derivatives of both Riemann–Liouville and Caputo was given by Hilfer in [6] when he studied fractional time evolution in physical phenomena. He named it *generalized fractional derivative of order* $\alpha \in (0,1)$ *and type* $\beta \in [0,1]$ which can be reduced to the Riemann–Liouville and Caputo fractional derivatives when $\beta = 0$ and $\beta = 1$, respectively. Many authors call it *the Hilfer fractional derivative*. Such derivative interpolates between the Riemann–Liouville and Caputo derivatives. For other current definitions of fractional derivatives, see [7–11].

Fractional-order boundary value problems have been extensively studied by many researchers. In particular, coupled systems of fractional-order differential equations have attracted special attention in view of their occurrence in the mathematical modeling of physical phenomena like chaos synchronization [12], anomalous diffusion [13], ecological effects [14], disease models [15], etc. Additionally, fixed point theory can be used to develop the existence theory for the coupled systems of fractional differential equations. For some recent theoretical results on coupled systems of fractional-order differential equations, for example, see [16–30].

Alsaedi *et al.* [23] studied the existence of solutions for a Riemann–Liouville coupled system of nonlinear fractional integro-differential equations given by

$$\begin{cases} D^{\alpha}u(t) = f(t,u(t),v(t),(\phi_1 u)(t),(\psi_1 v)(t)), & t \in [0,T], \\ D^{\beta}v(t) = g(t,u(t),v(t),(\phi_2 u)(t),(\psi_2 v)(t)), & 1 < \alpha,\beta \leq 2, \end{cases}$$

subject to the coupled Riemann–Liouville integro-differential boundary conditions

$$\begin{cases} D^{\alpha-2}u(0^+) = 0, \qquad D^{\alpha-1}u(0^+) = \nu I^{\alpha-1}v(\eta), & 0 < \eta < T, \\ D^{\beta-2}v(0^+) = 0, \qquad D^{\beta-1}v(0^+) = \mu I^{\beta-1}u(\sigma), & 0 < \sigma < T, \end{cases}$$

where $D^{(\cdot)}$, $I^{(\cdot)}$ denote the Riemann–Liouville derivatives and integral of fractional order $(\cdot)$, respectively, $f,g : [0,T] \times \mathbb{R}^4 \to \mathbb{R}$ are given continuous functions, ν, μ are real constants, and

$$(\phi_1 u)(t) = \int_0^t \gamma_1(t,s)u(s)\,ds, \qquad (\phi_2 u)(t) = \int_0^t \gamma_2(t,s)u(s)\,ds,$$
$$(\psi_1 v)(t) = \int_0^t \delta_1(t,s)v(s)\,ds, \qquad (\psi_2 v)(t) = \int_0^t \delta_2(t,s)v(s)\,ds,$$

with γ_i and δ_i $(i = 1,2)$ are continuous functions on $[0,T] \times [0,T]$.

Alsulami *et al.* [24] studied a new system of coupled Caputo type fractional differential equations

$$\begin{cases} {}^cD^{\alpha}u(t) = f(t,u(t),v(t)), & t \in [0,T], 1 < \alpha \leq 2, \\ {}^cD^{\beta}v(t) = g(t,u(t),v(t)), & t \in [0,T], 1 < \beta \leq 2, \end{cases}$$

subject to the following non-separated coupled boundary conditions:

$$\begin{cases} u(0) = \lambda_1 v(T), & u'(0) = \lambda_2 v'(T), \\ v(0) = \mu_1 u(T), & v'(0) = \mu_2 u'(T), \end{cases}$$

where ${}^cD^{\alpha}$, ${}^cD^{\beta}$ denote the Caputo fractional derivatives of order α and β, respectively, $f,g : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are appropriately chosen functions and λ_i, μ_i, $i = 1,2$, are real constants with $\lambda_i\mu_i \neq 1$, $i = 1,2$.

Ahmad *et al.* [25] studied the existence and uniqueness of solutions for the following boundary value problem of nonlinear Caputo sequential fractional differential equations:

$$\begin{cases} ({}^cD^{\alpha} + k_1 {}^cD^{\alpha-1})u(t) = f(t,u(t),v(t)), & 1 < \alpha \leq 2, t \in (0,T), \\ ({}^cD^{\beta} + k_2 {}^cD^{\beta-1})v(t) = g(t,u(t),v(t)), & 1 < \beta \leq 2, t \in (0,T), \end{cases}$$

supplemented with coupled boundary conditions

$$\begin{cases} u(0) = a_1 v(T), & u'(0) = a_2 v'(T), \\ v(0) = b_1 u(T), & v'(0) = b_2 u'(T), \end{cases}$$

where ${}^cD^\alpha$, ${}^cD^\beta$ denote the Caputo fractional derivatives of order α and β, respectively, $k_1, k_2 \in \mathbb{R}_+$, $T > 0$, $f, g : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions, and a_1, a_2, b_1, and b_2 are real constants with $a_1 b_1 \neq 1$ and $a_2 b_2 e^{-(k_1 T + k_2 T)} \neq 1$.

Aljoudi *et al.* [29] studied a coupled system of Hadamard type sequential fractional differential equations with coupled strip conditions given by

$$\begin{cases} (D^q + kD^{q-1})u(t) = f(t, u(t), v(t), D^\alpha v(t)), & k > 0, 1 < q \le 2, 0 < \alpha < 1, \\ (D^p + kD^{p-1})v(t) = g(t, u(t), v(t), D^\delta u(t)), & 1 < p \le 2, 0 < \delta < 1, \\ u(1) = 0, \quad u(e) = I^\gamma v(\eta) = \frac{1}{\Gamma(\gamma)} \int_1^\eta (\log \frac{\eta}{s})^{\gamma-1} \frac{v(s)}{s}\, ds, & \gamma > 0, 1 < \eta < e, \\ v(1) = 0, \quad v(e) = I^\beta v(\zeta) = \frac{1}{\Gamma(\beta)} \int_1^\zeta (\log \frac{\zeta}{s})^{\beta-1} \frac{u(s)}{s}\, ds, & \beta > 0, 1 < \zeta < e, \end{cases}$$

where $D^{(\cdot)}$ and $I^{(\cdot)}$ denote the Hadamard fractional derivative and the Hadamard fractional integral, respectively, and $f, g : [1, e] \times \mathbb{R}^3 \to \mathbb{R}$ are given continuous functions.

Motivated by the research going on in this direction, in this paper, we study existence and uniqueness of solutions for a new class of systems of Hilfer–Hadamard sequential fractional differential equations

$$\begin{cases} ({}_HD_{1^+}^{\alpha_1,\beta_1} + k_1 {}_HD_{1^+}^{\alpha_1-1,\beta_1})u(t) = f(t, u(t), v(t)), & 1 < \alpha_1 \le 2, t \in [1, e], \\ ({}_HD_{1^+}^{\alpha_2,\beta_2} + k_2 {}_HD_{1^+}^{\alpha_2-1,\beta_2})v(t) = g(t, u(t), v(t)), & 1 < \alpha_2 \le 2, t \in [1, e], \end{cases} \tag{1}$$

with two-point boundary conditions

$$\begin{cases} u(1) = 0, & u(e) = A_1, \\ v(1) = 0, & v(e) = A_2, \end{cases} \tag{2}$$

where ${}_HD^{\alpha_i,\beta_i}$ is the Hilfer–Hadamard fractional derivative of order $\alpha_i \in (1, 2]$ and type $\beta_i \in [0, 1]$ for $i \in \{1, 2\}$, $k_1, k_2, A_1, A_2 \in \mathbb{R}_+$ and $f, g : [1, e] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions.

To the best of our knowledge, this is the first paper dealing with a system containing Hilfer–Hadamard fractional derivative of order $\alpha_i \in (1, 2]$, $i = 1, 2$. For some recent results on coupled systems of Hilfer–Hadamard fractional derivatives of order $\alpha_i \in (0, 1]$, $i = 1, 2$, we refer to [31, 32], and the references cited therein.

The paper is organized as follows. In Sect. 2, we present some preliminary concepts of fractional calculus. Section 3 contains the main results. The first result, Theorem 3.2, is proved by using the Leray–Schauder alternative and the second result of existence and uniqueness, Theorem 3.3, by the Banach contraction mapping principle. Finally, Sect. 4 provides some examples for the illustration of the main results. We emphasize that our results are new and contribute significantly to the topic addressed in this paper.

2 Preliminaries

In this section, some basic definitions, lemmas, and theorems are mentioned.

Definition 2.1 (Hadamard fractional integral [1]) The Hadamard fractional integral of order $\alpha \in \mathbb{R}_{++}$ for a function $f : [a, \infty) \to \mathbb{R}$ is defined as follows:

$$ {}_H I_{a^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{\tau} \right)^{\alpha-1} \frac{f(\tau)}{\tau}\, d\tau \quad (t > a) \tag{3} $$

provided the integral exists, where $\log(\cdot) = \log_e(\cdot)$.

Definition 2.2 (Hadamard fractional derivative [1]) The Hadamard fractional derivative of order $\alpha > 0$, applied to the function $f : [a, \infty) \to \mathbb{R}$, is defined as follows:

$$ {}_H D_{a^+}^{\alpha} f(t) = \delta^n \left({}_H I_{a^+}^{n-\alpha} f(t) \right), \quad n - 1 < \alpha < n, n = [\alpha] + 1, \tag{4} $$

where $\delta^n = (t \frac{d}{dt})^n$ and $[\alpha]$ denotes the integer part of the real number α.

Definition 2.3 (Hilfer–Hadamard fractional derivative [6, 33]) Let $0 < \alpha < 1$ and $0 \leq \beta \leq 1$, $f \in L^1(a, b)$. The Hilfer–Hadamard fractional derivative of order α and type β of f is defined as follows:

$$ \begin{aligned} \left({}_H D_{a^+}^{\alpha,\beta} f \right)(t) &= \left({}_H I_{a^+}^{\beta(1-\alpha)} \delta\, {}_H I_{a^+}^{(1-\alpha)(1-\beta)} f \right)(t) \\ &= \left({}_H I_{a^+}^{\beta(1-\alpha)} \delta\, {}_H I_{a^+}^{1-\gamma} f \right)(t); \quad \gamma = \alpha + \beta - \alpha\beta \\ &= \left({}_H I_{a^+}^{\beta(1-\alpha)}\, {}_H D_{a^+}^{\gamma} f \right)(t), \end{aligned} $$

where ${}_H I_{a^+}^{(\cdot)}$ and ${}_H D_{a^+}^{(\cdot)}$ are the Hadamard fractional integral and derivative defined by (3) and (4), respectively.

The Hilfer–Hadamard fractional derivative may be viewed as interpolating the Hadamard fractional derivative. Indeed, for $\beta = 0$, this derivative reduces to the Hadamard fractional derivative.

Definition 2.4 (Hilfer–Hadamard fractional derivative [34]) Let $n - 1 < \alpha < n$ and $0 \leq \beta \leq 1$, $f \in L^1(a, b)$. The Hilfer–Hadamard fractional derivative of order α and type β of f is defined as follows:

$$ \begin{aligned} \left({}_H D_{a^+}^{\alpha,\beta} f \right)(t) &= \left({}_H I_{a^+}^{\beta(n-\alpha)} \delta^n\, {}_H I_{a^+}^{(n-\alpha)(1-\beta)} f \right)(t) \\ &= \left({}_H I_{a^+}^{\beta(n-\alpha)} \delta^n\, {}_H I_{a^+}^{n-\gamma} f \right)(t); \quad \gamma = \alpha + n\beta - \alpha\beta \\ &= \left({}_H I_{a^+}^{\beta(n-\alpha)}\, {}_H D_{a^+}^{\gamma} f \right)(t), \end{aligned} $$

where ${}_H I_{a^+}^{(\cdot)}$ and ${}_H D_{a^+}^{(\cdot)}$ are the Hadamard fractional integral and derivative defined by (3) and (4), respectively.

We recommend some lemmas and theorems of the Hadamard fractional integral and derivative by Kilbas *et al.* [1].

Theorem 2.5 ([1, 35]) *Let $\alpha > 0$, $n = [\alpha] + 1$, and $0 < a < b < \infty$. If $f \in L^1(a,b)$ and $({}_HI_{a+}^{n-\alpha}f)(t) \in AC_\delta^n[a,b]$, then*

$$\left({}_HI_{a+}^{\alpha}{}_HD_{a+}^{\alpha}f\right)(t) = f(t) - \sum_{j=0}^{n-1} \frac{(\delta^{(n-j-1)}({}_HI_{a+}^{n-\alpha}f))(a)}{\Gamma(\alpha-j)} \left(\log\frac{t}{a}\right)^{\alpha-j-1},$$

where $f(t) \in AC_\delta^n = \{f : [a,b] \to \mathbb{R} : \delta^{(n-1)}f(t) \in AC[a,b], \delta = t\frac{d}{dt}\}$.

Theorem 2.6 ([33]) *Let $\alpha > 0$, $0 \le \beta \le 1$, $\gamma = \alpha + n\beta - \alpha\beta$, $n-1 < \gamma \le n$, $n = [\alpha] + 1$, and $0 < a < b < \infty$. If $f \in L^1(a,b)$ and $({}_HI_{a+}^{n-\gamma}f)(t) \in AC_\delta^n[a,b]$, then*

$${}_HI_{a+}^{\alpha}\left({}_HD_{a+}^{\alpha,\beta}f\right)(t) = {}_HI_{a+}^{\gamma}\left({}_HD_{a+}^{\gamma}f\right)(t) = f(t) - \sum_{j=0}^{n-1} \frac{(\delta^{(n-j-1)}({}_HI_{a+}^{n-\gamma}f))(a)}{\Gamma(\gamma-j)} \left(\log\frac{t}{a}\right)^{\gamma-j-1}.$$

From this theorem, we notice that if $\beta = 0$ the formulae reduce to the formulae in Theorem 2.5.

We will use the following well-known fixed point theorems on Banach space for proving the existence and uniqueness of Hilfer–Hadamard fractional differential systems.

Theorem 2.7 (Leray–Schauder alternative [36]) *Let $T : E \to E$ be a completely continuous operator (i.e., a continuous map T restricted to any bounded set in E is compact). Let $\varepsilon(T) = \{x \in E : x = \lambda T(x), 0 \le \lambda \le 1\}$. Then either the set $\varepsilon(T)$ is unbounded or T has at least one fixed point.*

Theorem 2.8 (Banach fixed point theorem [37]) *Let X be a Banach space, $D \subset X$ be closed, and $F : D \to D$ be a strict contraction, i.e., $\|Fx - Fy\| \le k\|x - y\|$ for some $k \in (0,1)$ and all $x, y \in D$. Then F has a fixed point in D.*

3 Existence and uniqueness results

In this section, we prove existence and uniqueness of solutions for a system of Hilfer–Hadamard sequential fractional differential equations with boundary conditions (1) and (2). The following lemma concerns a linear variant of system (1) and (2).

Lemma 3.1 *Let $h_1, h_2 \in C([1,e],\mathbb{R})$. Then $u, v \in C([1,e],\mathbb{R})$ are solutions of the system of fractional differential equations*

$$\begin{cases} ({}_HD_{1^+}^{\alpha_1,\beta_1} + k_1\,{}_HD_{1^+}^{\alpha_1-1,\beta_1})u(t) = h_1(t), & 1 < \alpha_1 \le 2, t \in [1,e], \\ ({}_HD_{1^+}^{\alpha_2,\beta_2} + k_2\,{}_HD_{1^+}^{\alpha_2-1,\beta_2})v(t) = h_2(t), & 1 < \alpha_2 \le 2, t \in [1,e], \end{cases} \tag{5}$$

supplemented with the boundary conditions (2) if and only if

$$\begin{aligned} u(t) = {} & A_1(\log t)^{\gamma_1-1} + k_1\left[(\log t)^{\gamma_1-1}\int_1^e \frac{u(s)}{s}\,ds - \int_1^t \frac{u(s)}{s}\,ds\right] \\ & + \frac{1}{\Gamma(\alpha_1)}\left[\int_1^t \left(\log\frac{t}{s}\right)^{\alpha_1-1}\frac{h_1(s)}{s}\,ds - (\log t)^{\gamma_1-1}\int_1^e \left(\log\frac{e}{s}\right)^{\alpha_1-1}\frac{h_1(s)}{s}\,ds\right] \end{aligned} \tag{6}$$

and

$$v(t) = A_2(\log t)^{\gamma_2-1} + k_2\left[(\log t)^{\gamma_2-1}\int_1^e \frac{v(s)}{s}\,ds - \int_1^t \frac{v(s)}{s}\,ds\right] + \frac{1}{\Gamma(\alpha_2)}\left[\int_1^t \left(\log\frac{t}{s}\right)^{\alpha_2-1}\frac{h_2(s)}{s}\,ds - (\log t)^{\gamma_2-1}\int_1^e \left(\log\frac{e}{s}\right)^{\alpha_2-1}\frac{h_2(s)}{s}\,ds\right]. \tag{7}$$

Proof From the first equation of (5), we have

$$_HD_{1^+}^{\alpha_1,\beta_1}u(t) + k_1\,{_HD_{1^+}^{\alpha_1-1,\beta_1}}u(t) = h_1(t). \tag{8}$$

Taking the Hadamard fractional integral of order α_1 to both sides of (8), we get

$$_HI_{1^+}^{\alpha_1}\,{_HD_{1^+}^{\alpha_1,\beta_1}}u(t) + k_1\,{_HI_{1^+}^{\alpha_1}}\,{_HD_{1^+}^{\alpha_1-1,\beta_1}}u(t) = {_HI_{1^+}^{\alpha_1}}h_1(t).$$

By Theorem 2.6, one has

$$u(t) - \frac{\delta({_HI_{1^+}^{2-\gamma_1}}u)(1)}{\Gamma(\gamma_1)}(\log t)^{\gamma_1-1} - \frac{({_HI_{1^+}^{2-\gamma_1}}u)(1)}{\Gamma(\gamma_1-1)}(\log t)^{\gamma_1-2} + k_1\,{_HI_{1^+}^{\alpha_1}}\,{_HD_{1^+}^{\alpha_1-1,\beta_1}}u(t) = {_HI_{1^+}^{\alpha_1}}h_1(t). \tag{9}$$

From equation (9), by Definition 2.4, we get

$$u(t) - \frac{\delta({_HI_{1^+}^{2-\gamma_1}}u)(1)}{\Gamma(\gamma_1)}(\log t)^{\gamma_1-1} - \frac{({_HI_{1^+}^{2-\gamma_1}}u)(1)}{\Gamma(\gamma_1-1)}(\log t)^{\gamma_1-2} + k_1\,{_HI_{1^+}}u(t) = {_HI_{1^+}^{\alpha_1}}h_1(t). \tag{10}$$

Equation (10) can be written as follows:

$$u(t) = c_0(\log t)^{\gamma_1-1} + c_1(\log t)^{\gamma_1-2} - k_1\int_1^t \frac{u(s)}{s}\,ds + \frac{1}{\Gamma(\alpha_1)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha_1-1}\frac{h_1(s)}{s}\,ds. \tag{11}$$

In a similar way, one can obtain

$$v(t) = d_0(\log t)^{\gamma_2-1} + d_1(\log t)^{\gamma_2-2} - k_2\int_1^t \frac{v(s)}{s}\,ds + \frac{1}{\Gamma(\alpha_2)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha_2-1}\frac{h_2(s)}{s}\,ds, \tag{12}$$

where c_0, c_1, d_0, and d_1 are arbitrary constants. Now, boundary conditions (2) together with (11), (12) yield

$$u(1) = c_0(\log 1)^{\gamma_1-1} + \frac{c_1}{(\log t)^{2-\gamma_1}} - k_1\int_1^1 \frac{u(s)}{s}\,ds + \frac{1}{\Gamma(\alpha_1)}\int_1^1\left(\log\frac{1}{s}\right)^{\alpha_1-1}\frac{h_1(s)}{s}\,ds = 0, \tag{13}$$

$$
\begin{aligned}
v(1) &= d_0(\log 1)^{\gamma_2-1} + \frac{d_1}{(\log t)^{2-\gamma_2}} \\
&\quad - k_2\int_1^1 \frac{v(s)}{s}\,ds + \frac{1}{\Gamma(\alpha_2)}\int_1^1\left(\log\frac{1}{s}\right)^{\alpha_2-1}\frac{h_2(s)}{s}\,ds \\
&= 0,
\end{aligned}
$$

from which we have $c_1 = 0$ and $d_1 = 0$. Equations (13) can be written as

$$
u(t) = c_0(\log t)^{\gamma_1-1} - k_1\int_1^t \frac{u(s)}{s}\,ds + \frac{1}{\Gamma(\alpha_1)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha_1-1}\frac{h_1(s)}{s}\,ds \tag{14}
$$

and

$$
v(t) = d_0(\log t)^{\gamma_2-1} - k_2\int_1^t \frac{v(s)}{s}\,ds + \frac{1}{\Gamma(\alpha_2)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha_2-1}\frac{h_2(s)}{s}\,ds. \tag{15}
$$

Next, boundary conditions (2) together with (14), (15) yield

$$
\begin{aligned}
u(e) &= c_0(\log e)^{\gamma_1-1} - k_1\int_1^e \frac{u(s)}{s}\,ds + \frac{1}{\Gamma(\alpha_1)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha_1-1}\frac{h_1(s)}{s}\,ds = A_1, \\
v(e) &= d_0(\log e)^{\gamma_2-1} - k_2\int_1^e \frac{v(s)}{s}\,ds + \frac{1}{\Gamma(\alpha_2)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha_2-1}\frac{h_2(s)}{s}\,ds = A_2,
\end{aligned}
$$

from which we have

$$
\begin{aligned}
c_0 &= A_1 + k_1\int_1^e \frac{u(s)}{s}\,ds - \frac{1}{\Gamma(\alpha_1)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha_1-1}\frac{h_1(s)}{s}\,ds, \\
d_0 &= A_2 + k_2\int_1^e \frac{v(s)}{s}\,ds - \frac{1}{\Gamma(\alpha_2)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha_2-1}\frac{h_2(s)}{s}\,ds.
\end{aligned}
$$

Substituting the values of c_0, c_1, d_0, and d_1 in (11) and (12), we get integral equations (6) and (7). The converse follows by direct computation. This completes the proof. □

Let us introduce the Banach space $X = C([1,e])$ endowed with the norm defined by $\|u\| := \max_{t\in[1,e]}|u(t)|$. Thus, the product space $X \times X$ equipped with the norm $\|(u,v)\| = \|u\| + \|v\|$ is a Banach space. In view of Lemma 3.1, we define an operator $\mathcal{T} : X \times X \to X \times X$ by

$$
\mathcal{T}(u,v)(t) = \bigl(\mathcal{T}_1(u,v)(t), \mathcal{T}_2(u,v)(t)\bigr), \tag{16}
$$

where

$$
\begin{aligned}
\mathcal{T}_1(u,v)(t) &= A_1(\log t)^{\gamma_1-1} + k_1\left[(\log t)^{\gamma_1-1}\int_1^e \frac{u(s)}{s}\,ds - \int_1^t \frac{u(s)}{s}\,ds\right] \\
&\quad + \frac{1}{\Gamma(\alpha_1)}\left[\int_1^t\left(\log\frac{t}{s}\right)^{\alpha_1-1}\frac{f(s,u(s),v(s))}{s}\,ds\right. \\
&\quad \left. - (\log t)^{\gamma_1-1}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha_1-1}\frac{f(s,u(s),v(s))}{s}\,ds\right]
\end{aligned} \tag{17}
$$

and

$$\begin{aligned}\mathcal{T}_2(u,v)(t) &= A_2(\log t)^{\gamma_2-1} + k_2\left[(\log t)^{\gamma_2-1}\int_1^e \frac{v(s)}{s}\,ds - \int_1^t \frac{v(s)}{s}\,ds\right] \\ &\quad + \frac{1}{\Gamma(\alpha_2)}\left[\int_1^t \left(\log\frac{t}{s}\right)^{\alpha_2-1}\frac{g(s,u(s),v(s))}{s}\,ds\right. \\ &\quad \left. - (\log t)^{\gamma_2-1}\int_1^e \left(\log\frac{e}{s}\right)^{\alpha_2-1}\frac{g(s,u(s),v(s))}{s}\,ds\right]. \end{aligned} \tag{18}$$

We need the following hypotheses in the sequel:

(H_1) Assume that there exist real constants $m_i, n_i \geq 0$ $(i = 1,2)$ and $m_0 > 0$, $n_0 > 0$ such that, for all $t \in [1,e]$, $x_i \in \mathbb{R}$, $i = 1,2$,

$$\begin{aligned} \left|f(t,x_1,x_2)\right| &\leq m_0 + m_1|x_1| + m_2|x_2|, \\ \left|g(t,x_1,x_2)\right| &\leq n_0 + n_1|x_1| + n_2|x_2|. \end{aligned}$$

(H_2) There exist positive constants $L, \bar{L}$, such that, for all $t \in [1,e]$, $u_i, v_i \in \mathbb{R}$, $i = 1,2$,

$$\begin{aligned} \left|f(t,u_1,u_2) - f(t,v_1,v_2)\right| &\leq L\big(|u_1 - v_1| + |u_2 - v_2|\big), \\ \left|g(t,u_1,u_2) - g(t,v_1,v_2)\right| &\leq \bar{L}\big(|u_1 - v_1| + |u_2 - v_2|\big). \end{aligned}$$

3.1 Existence result via Leray–Schauder alternative

In the first theorem, we prove an existence result based on the Leray–Schauder alternative.

Theorem 3.2 *Assume that* (H_1) *holds. In addition it is assumed that* $\max\{Q_1, Q_2\} < 1$, *where*

$$Q_1 := 2\left(k_1 + \frac{m_1}{\Gamma(\alpha_1+1)} + \frac{n_1}{\Gamma(\alpha_2+1)}\right), \qquad Q_2 := 2\left(k_2 + \frac{m_2}{\Gamma(\alpha_1+1)} + \frac{n_2}{\Gamma(\alpha_2+1)}\right).$$

Then system (1)–(2) *has at least one solution on* $[1,e]$.

Proof We will use the Leray–Schauder alternative to prove that $\mathcal{T}$, defined by (16), has a fixed point. We divide the proof into two steps.

Step I: We show that the operator $\mathcal{T} : X \times X \to X \times X$, defined by (16), is completely continuous.

First we show that $\mathcal{T}$ is continuous. Let $\{(u_n, v_n)\}$ be a sequence such that $(u_n, v_n) \to (u, v)$ in $X \times X$. Then, for each $t \in [1,e]$, we have

$$\begin{aligned} &\left|\mathcal{T}_1(u_n,v_n)(t) - \mathcal{T}_1(u,v)(t)\right| \\ &\quad \leq k_1\left[\left|(\log t)^{\gamma_1-1}\right|\left|\int_1^e \frac{(u_n(s)-u(s))}{s}\,ds\right| + \left|\int_1^t \frac{(u_n(s)-u(s))}{s}\,ds\right|\right] \\ &\qquad + \frac{1}{\Gamma(\alpha_1)}\left[\left|\int_1^t \left(\log\frac{t}{s}\right)^{\alpha_1-1}\frac{(f(s,u_n(s),v_n(s)) - f(s,u(s),v(s)))}{s}\,ds\right|\right. \\ &\qquad \left. + \left|(\log t)^{\gamma_1-1}\right|\left|\int_1^e \left(\log\frac{e}{s}\right)^{\alpha_1-1}\frac{(f(s,u_n(s),v_n(s)) - f(s,u(s),v(s)))}{s}\,ds\right|\right] \end{aligned}$$

$$\leq k_1\left[\int_1^e \frac{|u_n(s)-u(s)|}{s}\,ds + \int_1^t \frac{|u_n(s)-u(s)|}{s}\,ds\right] + \frac{1}{\Gamma(\alpha_1)}\left[\int_1^t \left(\log\frac{t}{s}\right)^{\alpha_1-1}\frac{|f(s,u_n(s),v_n(s))-f(s,u(s),v(s))|}{s}\,ds + \int_1^e \left(\log\frac{e}{s}\right)^{\alpha_1-1}\frac{|f(s,u_n(s),v_n(s))-f(s,u(s),v(s))|}{s}\,ds\right].$$

Since f is continuous, we get

$$\left|f\bigl(s,u_n(s),v_n(s)\bigr)-f\bigl(s,u(s),v(s)\bigr)\right| \to 0 \quad \text{as } (u_n,v_n)\to(u,v).$$

Then

$$\left\|\mathcal{T}_1(u_n,v_n)-\mathcal{T}_1(u,v)\right\| \to 0 \quad \text{as } (u_n,v_n)\to(u,v). \tag{19}$$

In the same way, we obtain

$$\left\|\mathcal{T}_2(u_n,v_n)-\mathcal{T}_2(u,v)\right\| \to 0 \quad \text{as } (u_n,v_n)\to(u,v). \tag{20}$$

It follows from (19) and (20) that $\|\mathcal{T}(u_n,v_n)-\mathcal{T}(u,v)\|\to 0$ as $(u_n,v_n)\to(u,v)$. Hence $\mathcal{T}$ is continuous.

Now we show that $\mathcal{T}$ is compact. Let $\Omega\subset X\times X$ be bounded. Then there exist positive constants L_1 and L_2 such that $|f(t,u(t),v(t))|\leq L_1$, $|g(t,u(t),v(t))|\leq L_2$, $\forall(u,v)\in\Omega$. Let $(u,v)\in\Omega$. Then there exists M such that $\|(u,v)\|=\|u\|+\|v\|\leq M$, $\forall(u,v)\in\Omega$. We have

$$\begin{aligned}
&\left|\mathcal{T}_1(u,v)(t)\right| \\
&\quad\leq A_1 + k_1\left[\int_1^e \frac{|u(s)|}{s}\,ds + \int_1^t \frac{|u(s)|}{s}\,ds\right] \\
&\qquad + \frac{1}{\Gamma(\alpha_1)}\left[\int_1^t \left(\log\frac{t}{s}\right)^{\alpha_1-1}\frac{|f(s,u(s),v(s))|}{s}\,ds + \int_1^e \left(\log\frac{e}{s}\right)^{\alpha_1-1}\frac{|f(s,u(s),v(s))|}{s}\,ds\right] \\
&\quad\leq A_1 + k_1\left[\int_1^e \frac{\max_{s\in[1,e]}|u(s)|}{s}\,ds + \int_1^t \frac{\max_{s\in[1,e]}|u(s)|}{s}\,ds\right] \\
&\qquad + \frac{L_1}{\Gamma(\alpha_1)}\left[\int_1^t \left(\log\frac{t}{s}\right)^{\alpha_1-1}\frac{ds}{s} + \int_1^e \left(\log\frac{e}{s}\right)^{\alpha_1-1}\frac{ds}{s}\right] \\
&\quad\leq A_1 + k_1\|u\|\bigl[1+(\log e)\bigr] + \frac{L_1}{\Gamma(\alpha_1+1)}\bigl[(\log e)^{\alpha_1}+1\bigr],
\end{aligned}$$

which, on taking the norm for $t\in[1,e]$, yields

$$\left\|\mathcal{T}_1(u_n,v_n)\right\| \leq A_1 + 2\left[k_1\|u\| + \frac{L_1}{\Gamma(\alpha_1+1)}\right].$$

In the same way, we obtain

$$\left\|\mathcal{T}_2(u_n,v_n)\right\| \leq A_2 + 2\left[k_2\|v\| + \frac{L_2}{\Gamma(\alpha_2+1)}\right].$$

It follows that

$$\begin{aligned}\|\mathcal{T}(u,v)\| &\leq A_1 + A_2 + 2\left[k_1\|u\| + k_2\|v\| + \frac{L_1}{\Gamma(\alpha_1+1)} + \frac{L_2}{\Gamma(\alpha_2+1)}\right] \\ &\leq A_1 + A_2 + 2\left[M(k_1+k_2) + \frac{L_1}{\Gamma(\alpha_1+1)} + \frac{L_2}{\Gamma(\alpha_2+1)}\right].\end{aligned}$$

This mean that there is $P = A_1 + A_2 + 2[M(k_1 + k_2) + \frac{L_1}{\Gamma(\alpha_1+1)} + \frac{L_2}{\Gamma(\alpha_2+1)}]$ such that $\|\mathcal{T}(u,v)\| \leq P$. Hence $\mathcal{T}$ is uniformly bounded.

Finally we show that $\mathcal{T}$ is equicontinuous. Let $t, t_0 \in [1,e]$ with $t_0 < t$. Then we have

$$\begin{aligned}&\left|\mathcal{T}_1(u,v)(t) - \mathcal{T}_1(u,v)(t_0)\right| \\ &\quad\leq A_1\left[(\log t)^{\gamma_1-1} - (\log t_0)^{\gamma_1-1}\right] \\ &\qquad + k_1\left[\left((\log t)^{\gamma_1-1} - (\log t_0)^{\gamma_1-1}\right)\int_1^e \frac{|u(s)|}{s}\,ds + \int_{t_0}^t \frac{|u(s)|}{s}\,ds\right] \\ &\qquad + \frac{1}{\Gamma(\alpha_1)}\left[\int_1^{t_0}\left(\left(\log\frac{t}{s}\right)^{\alpha_1-1} - \left(\log\frac{t_0}{s}\right)^{\alpha_1-1}\right)\frac{|f(s,u(s),v(s))|}{s}\,ds\right. \\ &\qquad + \int_{t_0}^t\left(\log\frac{t}{s}\right)^{\alpha_1-1}\frac{|f(s,u(s),v(s))|}{s}\,ds \\ &\qquad \left. + \left((\log t)^{\gamma_1-1} - (\log t_0)^{\gamma_1-1}\right)\int_1^e\left(\log\frac{e}{s}\right)^{\alpha_1-1}\frac{|f(s,u(s),v(s))|}{s}\,ds\right] \\ &\quad\leq A_1\left[(\log t)^{\gamma_1-1} - (\log t_0)^{\gamma_1-1}\right] \\ &\qquad + k_1\left[\|u\|\left((\log t)^{\gamma_1-1} - (\log t_0)^{\gamma_1-1}\right) + \|u\|(\log t - \log t_0)\right] \\ &\qquad + \frac{L_1}{\Gamma(\alpha_1)}\left[\int_1^{t_0}\left(\log\frac{t}{s}\right)^{\alpha_1-1}\frac{ds}{s} - \int_1^{t_0}\left(\log\frac{t_0}{s}\right)^{\alpha_1-1}\frac{ds}{s} + \int_{t_0}^t\left(\log\frac{t}{s}\right)^{\alpha_1-1}\frac{ds}{s}\right. \\ &\qquad \left. + \left((\log t)^{\gamma_1-1} - (\log t_0)^{\gamma_1-1}\right)\int_1^e\left(\log\frac{e}{s}\right)^{\alpha_1-1}\frac{ds}{s}\right] \\ &\quad\leq A_1\left[(\log t)^{\gamma_1-1} - (\log t_0)^{\gamma_1-1}\right] \\ &\qquad + k_1M\left[\left((\log t)^{\gamma_1-1} - (\log t_0)^{\gamma_1-1}\right) + (\log t - \log t_0)\right] \\ &\qquad + \frac{L_1}{\Gamma(\alpha_1+1)}\left[\left((\log t)^{\gamma_1-1} - (\log t_0)^{\gamma_1-1}\right) + \left((\log t)^{\alpha_1} - (\log t_0)^{\alpha_1}\right)\right]\end{aligned} \tag{21}$$

and

$$\begin{aligned}&\left|\mathcal{T}_2(u,v)(t) - \mathcal{T}_2(u,v)(t_0)\right| \\ &\quad\leq A_2\left[(\log t)^{\gamma_2-1} - (\log t_0)^{\gamma_2-1}\right] \\ &\qquad + k_2\left[\left((\log t)^{\gamma_2-1} - (\log t_0)^{\gamma_2-1}\right)\int_1^e \frac{|v(s)|}{s}\,ds + \int_{t_0}^t \frac{|v(s)|}{s}\,ds\right] \\ &\qquad + \frac{1}{\Gamma(\alpha_2)}\left[\int_1^{t_0}\left(\left(\log\frac{t}{s}\right)^{\alpha_2-1} - \left(\log\frac{t_0}{s}\right)^{\alpha_2-1}\right)\frac{|g(s,u(s),v(s))|}{s}\,ds\right. \\ &\qquad + \int_{t_0}^t\left(\log\frac{t}{s}\right)^{\alpha_2-1}\frac{|g(s,u(s),v(s))|}{s}\,ds\end{aligned}$$

$$
\begin{aligned}
&+ \left((\log t)^{\gamma_2-1} - (\log t_0)^{\gamma_2-1}\right) \int_1^e \left(\log \frac{e}{s}\right)^{\alpha_2-1} \frac{|g(s,u(s),v(s))|}{s}\, ds\Bigg] \\
&\le A_2\left[(\log t)^{\gamma_2-1} - (\log t_0)^{\gamma_2-1}\right] \\
&\quad + k_2\left[\|v\|\left((\log t)^{\gamma_2-1} - (\log t_0)^{\gamma_2-1}\right) + \|v\|(\log t - \log t_0)\right] \\
&\quad + \frac{L_2}{\Gamma(\alpha_2)}\Bigg[\int_1^{t_0} \left(\log \frac{t}{s}\right)^{\alpha_2-1} \frac{ds}{s} - \int_1^{t_0} \left(\log \frac{t_0}{s}\right)^{\alpha_2-1} \frac{ds}{s} + \int_{t_0}^{t} \left(\log \frac{t}{s}\right)^{\alpha_2-1} \frac{ds}{s} \\
&\quad + \left((\log t)^{\gamma_2-1} - (\log t_0)^{\gamma_2-1}\right) \int_1^e \left(\log \frac{e}{s}\right)^{\alpha_2-1} \frac{ds}{s}\Bigg] \\
&\le A_2\left[(\log t)^{\gamma_2-1} - (\log t_0)^{\gamma_2-1}\right] \\
&\quad + k_2 M\left[\left((\log t)^{\gamma_2-1} - (\log t_0)^{\gamma_2-1}\right) + (\log t - \log t_0)\right] \\
&\quad + \frac{L_2}{\Gamma(\alpha_2+1)}\left[\left((\log t)^{\gamma_2-1} - (\log t_0)^{\gamma_2-1}\right) + \left((\log t)^{\alpha_2} - (\log t_0)^{\alpha_2}\right)\right].
\end{aligned}
\tag{22}
$$

Take $t \to t_0$, from (21) and (22), we have

$$
\left|\mathcal{T}_1(u,v)(t) - \mathcal{T}_1(u,v)(t_0)\right| \to 0 \quad \text{and} \quad \left|\mathcal{T}_2(u,v)(t) - \mathcal{T}_2(u,v)(t_0)\right| \to 0 \quad \text{as } t \to t_0.
$$

Hence $\mathcal{T}$ is equicontinuous. By Arzelá–Ascoli theorem, we get that $\mathcal{T}(\Omega)$ is compact, that is, $\mathcal{T}$ is compact on Ω. Therefore $\mathcal{T}$ is completely continuous.

Step II: We show that the set $\varepsilon = \{(u,v) \in X \times X \mid (u,v) = \lambda\mathcal{T}(u,v), 0 \le \lambda \le 1\}$ is bounded.

Let $(u,v) \in \varepsilon$, then $(u,v) = \lambda\mathcal{T}(u,v)$. For any $t \in [1,e]$, we have $u(t) = \lambda\mathcal{T}_1(u,v)(t)$, $v(t) = \lambda\mathcal{T}_2(u,v)(t)$. Then, in view of assumption (H_1), we obtain

$$
\begin{aligned}
\left|u(t)\right| &\le \left|\mathcal{T}_1(u,v)(t)\right| \\
&\le A_1 + k_1\left[\int_1^e \frac{|u(s)|}{s}\, ds + \int_1^t \frac{|u(s)|}{s}\, ds\right] \\
&\quad + \frac{1}{\Gamma(\alpha_1)}\Bigg[\int_1^t \left(\log \frac{t}{s}\right)^{\alpha_1-1} \frac{|f(s,u(s),v(s))|}{s}\, ds \\
&\quad + \int_1^e \left(\log \frac{e}{s}\right)^{\alpha_1-1} \frac{|f(s,u(s),v(s))|}{s}\, ds\Bigg] \\
&\le A_1 + k_1\left[\|u\| \int_1^e \frac{ds}{s} + \|u\| \int_1^t \frac{ds}{s}\right] \\
&\quad + \frac{(m_0 + m_1\|u\| + m_2\|v\|)}{\Gamma(\alpha_1)}\left[\int_1^t \left(\log \frac{t}{s}\right)^{\alpha_1-1} \frac{ds}{s} + \int_1^e \left(\log \frac{e}{s}\right)^{\alpha_1-1} \frac{ds}{s}\right] \\
&\le A_1 + k_1\|u\|\left[1 + (\log e)\right] + \frac{(m_0 + m_1\|u\| + m_2\|v\|)}{\Gamma(\alpha_1+1)}\left[(\log e)^{\alpha_1} + 1\right],
\end{aligned}
$$

which, on taking maximum for $t \in [1,e]$, yields

$$
\|u\| \le A_1 + 2k_1\|u\| + 2\left(\frac{m_0 + m_1\|u\| + m_2\|v\|}{\Gamma(\alpha_1+1)}\right).
\tag{23}
$$

In a similar manner, one can obtain

$$\|v\| \leq A_2 + 2k_2\|v\| + 2\left(\frac{n_0 + n_1\|u\| + n_2\|v\|}{\Gamma(\alpha_2+1)}\right). \tag{24}$$

From (23) and (24), we have

$$\begin{aligned}
\|(u,v)\| &= \|u\| + \|v\| \\
&\leq A_1 + A_2 + \frac{2m_0}{\Gamma(\alpha_1+1)} + \frac{2n_0}{\Gamma(\alpha_2+1)} \\
&\quad + 2\left(k_1 + \frac{m_1}{\Gamma(\alpha_1+1)} + \frac{n_1}{\Gamma(\alpha_2+1)}\right)\|u\| \\
&\quad + 2\left(k_2 + \frac{m_2}{\Gamma(\alpha_1+1)} + \frac{n_2}{\Gamma(\alpha_2+1)}\right)\|v\| \\
&\leq A_1 + A_2 + \frac{2m_0}{\Gamma(\alpha_1+1)} + \frac{2n_0}{\Gamma(\alpha_2+1)} + \max\{Q_1,Q_2\}\|(u,v)\|,
\end{aligned}$$

and consequently,

$$\|(u,v)\| \leq \frac{A_1 + A_2 + \frac{2m_0}{\Gamma(\alpha_1+1)} + \frac{2n_0}{\Gamma(\alpha_2+1)}}{1 - \max\{Q_1,Q_2\}}.$$

Therefore the set ε is bounded. By Theorem 2.7, we get that the operator $\mathcal{T}$ has at least one fixed point. Therefore, problem (1)–(2) has at least one solution on $[1,e]$. □

3.2 Existence and uniqueness result via the Banach fixed point theorem

Next, we prove an existence and uniqueness result based on the Banach contraction mapping principle.

Theorem 3.3 *Assume that* (H_2) *holds. Then system* (1)–(2) *has a unique solution on* $[1,e]$ *provided that*

$$\mu := 2\left(k_1 + k_2 + \frac{L}{\Gamma(\alpha_1+1)} + \frac{\bar{L}}{\Gamma(\alpha_2+1)}\right) < 1. \tag{25}$$

Proof We will use the Banach fixed point theorem to prove that $\mathcal{T}$, defined by (16), has a unique fixed point. Fixing $N_1 = \max_{t\in[1,e]} |f(t,0,0)| < \infty$, $N_2 = \max_{t\in[1,e]} |g(t,0,0)| < \infty$ and using assumption (H_2), we obtain

$$\begin{aligned}
\big|f\big(t,u(t),v(t)\big)\big| &= \big|f\big(t,u(t),v(t)\big) - f(t,0,0) + f(t,0,0)\big| \leq L\big(\|u\| + \|v\|\big) + N_1, \\
\big|g\big(t,u(t),v(t)\big)\big| &= \big|g\big(t,u(t),v(t)\big) - g(t,0,0) + g(t,0,0)\big| \leq \bar{L}\big(\|u\| + \|v\|\big) + N_2.
\end{aligned} \tag{26}$$

We choose

$$r \geq \frac{A_1 + A_2 + 2(\frac{N_1}{\Gamma(\alpha_1+1)} + \frac{N_2}{\Gamma(\alpha_2+1)})}{1 - 2(k_1 + k_2 + \frac{L}{\Gamma(\alpha_1+1)} + \frac{\bar{L}}{\Gamma(\alpha_2+1)})}.$$

We divide the proof into two steps.

Step I : First we show that $\mathcal{T}(B_r) \subset B_r$, where $B_r = \{(u,v) \in X \times X : \|(u,v)\| \leq r\}$.

Let $(u,v) \in B_r$. Then, using (26), we obtain

$$
\begin{aligned}
\left|\mathcal{T}_1(u,v)(t)\right| &\leq A_1 + k_1\left[\int_1^e \frac{|u(s)|}{s}\,ds + \int_1^t \frac{|u(s)|}{s}\,ds\right] \\
&\quad + \frac{1}{\Gamma(\alpha_1)}\left[\int_1^t \left(\log\frac{t}{s}\right)^{\alpha_1-1}\frac{|f(s,u(s),v(s))|}{s}\,ds\right. \\
&\quad \left. + \int_1^e \left(\log\frac{e}{s}\right)^{\alpha_1-1}\frac{|f(s,u(s),v(s))|}{s}\,ds\right] \\
&\leq A_1 + k_1\left[\int_1^e \frac{\max_{s\in[1,e]}|u(s)|}{s}\,ds + \int_1^t \frac{\max_{s\in[1,e]}|u(s)|}{s}\,ds\right] \\
&\quad + \frac{L(\|u\| + \|v\|) + N_1}{\Gamma(\alpha_1)}\left[\int_1^t \left(\log\frac{t}{s}\right)^{\alpha_1-1}\frac{ds}{s} + \int_1^e \left(\log\frac{e}{s}\right)^{\alpha_1-1}\frac{ds}{s}\right] \\
&\leq A_1 + 2k_1 r + \frac{2}{\Gamma(\alpha_1+1)}(Lr + N_1),
\end{aligned}
$$

which, on taking the norm for $t \in [1,e]$, yields

$$
\left\|\mathcal{T}_1(u,v)\right\| \leq A_1 + 2k_1 r + \frac{2}{\Gamma(\alpha_1+1)}(Lr + N_1).
$$

In the same way, one has

$$
\left\|\mathcal{T}_2(u,v)\right\| \leq A_2 + 2k_2 r + \frac{2}{\Gamma(\alpha_2+1)}(\bar{L}r + N_2).
$$

Then we have

$$
\begin{aligned}
\left\|\mathcal{T}(u,v)\right\| &\leq A_1 + A_2 + 2(k_1+k_2)r \\
&\quad + 2\left(\frac{L}{\Gamma(\alpha_1+1)} + \frac{\bar{L}}{\Gamma(\alpha_2+1)}\right)r + 2\left(\frac{N_1}{\Gamma(\alpha_1+1)} + \frac{N_2}{\Gamma(\alpha_2+1)}\right) \\
&\leq r.
\end{aligned}
$$

Thus $\|\mathcal{T}(u,v)\| \leq r$, that is, $\mathcal{T}(u,v) \in B_r$. Hence $\mathcal{T}(B_r) \subset B_r$.

Step II : We show that the operator $\mathcal{T}$ is a contraction.

Let $(u_2,v_2),(u_1,v_1) \in X \times X$. Then, for any $t \in [1,e]$, we have

$$
\begin{aligned}
&\left|\mathcal{T}_1(u_2,v_2)(t) - \mathcal{T}_1(u_1,v_1)(t)\right| \\
&\quad\leq k_1\left[\int_1^e \frac{|u_2(s)-u_1(s)|}{s}\,ds + \int_1^t \frac{|u_2(s)-u_1(s)|}{s}\,ds\right] \\
&\qquad + \frac{1}{\Gamma(\alpha_1)}\left[\int_1^t \left(\log\frac{t}{s}\right)^{\alpha_1-1}\frac{|f(s,u_2(s),v_2(s)) - f(s,u_1(s),v_1(s))|}{s}\,ds\right. \\
&\qquad \left. + \int_1^e \left(\log\frac{e}{s}\right)^{\alpha_1-1}\frac{|f(s,u_2(s),v_2(s)) - f(s,u_1(s),v_1(s))|}{s}\,ds\right] \\
&\quad\leq 2k_1\|u_2-u_1\| + \frac{2L}{\Gamma(\alpha_1+1)}\big(\|u_2-u_1\| + \|v_2-v_1\|\big) \\
&\quad\leq 2k_1\big(\|u_2-u_1\| + \|v_2-v_1\|\big) + \frac{2L}{\Gamma(\alpha_1+1)}\big(\|u_2-u_1\| + \|v_2-v_1\|\big),
\end{aligned}
$$

which, on taking the norm for $t \in [1,e]$, yields

$$\left\| \mathcal{T}_1(u_2,v_2) - \mathcal{T}_1(u_1,v_1) \right\| \leq \left(2k_1 + \frac{2L}{\Gamma(\alpha_1+1)} \right) \left(\|u_2-u_1\| + \|v_2-v_1\| \right). \tag{27}$$

Similarly,

$$\left\| \mathcal{T}_2(u_2,v_2) - \mathcal{T}_2(u_1,v_1) \right\| \leq \left(2k_2 + \frac{2\bar{L}}{\Gamma(\alpha_1+1)} \right) \left(\|u_2-u_1\| + \|v_2-v_1\| \right). \tag{28}$$

It follows from (27) and (28) that $\|\mathcal{T}(u_2,v_2) - \mathcal{T}(u_1,v_1)\| \leq \mu(\|u_2-u_1\| + \|v_2-v_1\|)$, which, in view of (25), shows that the operator $\mathcal{T}$ is a contraction. From Steps I and II, by Theorem 2.8, we get that the operator $\mathcal{T}$ has a unique fixed point. Therefore system (1)–(2) has a unique solution on $[1,e]$. □

4 Examples

In this section, we give two examples to illustrate our main results.

Example 4.1 Consider the following system:

$$\begin{cases} ({}_HD^{\frac{3}{2},\frac{1}{2}} + \frac{1}{6}{}_HD^{\frac{1}{2},\frac{1}{2}})u(t) = \frac{|u(t)|}{(t+3)^4(1+|u(t)|)} + \frac{|v(t)|}{90(1+|v(t)|)} + \frac{1}{16}, & t \in [1,e], \\ ({}_HD^{\frac{3}{2},\frac{1}{2}} + \frac{1}{8}{}_HD^{\frac{1}{2},\frac{1}{2}})v(t) = \frac{\sin(\pi u(t))}{80\pi} + \frac{1}{15\sqrt{t+8}} + \frac{|v(t)|}{100(1+|v(t)|)}, & t \in [1,e], \\ u(1) = 0, \qquad u(e) = \frac{1}{2}, \qquad v(1) = 0, \qquad v(e) = \frac{1}{4}. \end{cases} \tag{29}$$

Here $\alpha_1 = \frac{3}{2}$, $\alpha_2 = \frac{3}{2}$, $\beta_1 = \frac{1}{2}$, $\beta_2 = \frac{1}{2}$, $A_1 = \frac{1}{2}$, $A_2 = \frac{1}{4}$, $k_1 = \frac{1}{6}$, $k_2 = \frac{1}{8}$.

We see that (H_1) holds, because

$$\left| f(t,u,v) \right| \leq \frac{1}{16} + \frac{1}{256}|u| + \frac{1}{90}|v| \quad \text{and} \quad \left| g(t,u,v) \right| \leq \frac{1}{45} + \frac{1}{80}|u| + \frac{1}{100}|v|,$$

with

$$m_0 = \frac{1}{16}, \qquad m_1 = \frac{1}{256}, \qquad m_2 = \frac{1}{90}, \qquad n_0 = \frac{1}{45}, \qquad n_1 = \frac{1}{80}, \qquad n_2 = \frac{1}{100}.$$

In addition, $Q_1 \approx 0.3580 < 1$, $Q_2 \approx 0.2818 < 1$, and $\max\{Q_1,Q_2\} \approx 0.6420$. Thus, the hypotheses of Theorem 3.2 are satisfied. Therefore, by Theorem 3.2, system (29) has at least one solution on $[1,e]$.

Example 4.2 Consider the following Hilfer–Hadamard system:

$$\begin{cases} ({}_HD^{\frac{5}{4},\frac{1}{2}} + \frac{1}{7}{}_HD^{\frac{1}{4},\frac{1}{2}})u(t) = (1+\log t)(\frac{|u(t)|}{100+|u(t)|}) + \frac{|v(t)|}{(8+t)^3(1+|v(t)|)} + \frac{1}{\sqrt{t+15}}, & t \in [1,e], \\ ({}_HD^{\frac{3}{2},\frac{1}{2}} + \frac{1}{9}{}_HD^{\frac{1}{2},\frac{1}{2}})v(t) = \frac{\sin(u(t))}{(7+t)^3} + \frac{7}{49+t^2} + \frac{|v(t)|}{\sqrt{99+t^2}(4+|v(t)|)}, & t \in [1,e], \\ u(1) = 0, \qquad u(e) = \frac{1}{3}, \qquad v(1) = 0, \qquad v(e) = \frac{1}{5}. \end{cases} \tag{30}$$

Here $\alpha_1 = \frac{5}{4}$, $\alpha_2 = \frac{3}{2}$, $\beta_1 = \frac{1}{2}$, $\beta_2 = \frac{1}{2}$, $A_1 = \frac{1}{3}$, $A_2 = \frac{1}{5}$, $k_1 = \frac{1}{7}$, $k_2 = \frac{1}{9}$.

Note that (H_2) holds, because

$$\left| f(t,u_1,u_2) - f(t,v_1,v_2) \right| \leq \frac{1}{50}\left(|u_1-v_1| + |u_2-v_2| \right)$$

and

$$\left|g(t,u_1,u_2)-g(t,v_1,v_2)\right| \leq \frac{1}{40}\left(|u_1-v_1|+|u_2-v_2|\right),$$

with $L=\frac{1}{50}$, $\bar{L}=\frac{1}{40}$. In addition,

$$\mu := 2\left(k_1+k_2+\frac{L}{\Gamma(\alpha_1+1)}+\frac{\bar{L}}{\Gamma(\alpha_2+1)}\right) \approx 0.580854 < 1.$$

Thus, all the conditions of Theorem 3.3 are satisfied. Therefore, by Theorem 3.3, system (30) has a unique solution on $[1,e]$.

5 Conclusion

In this paper, we studied existence and uniqueness of solutions for a system of Hilfer–Hadamard sequential fractional differential equations with two-point boundary conditions. The existence result is proved by using the Leray–Schauder alternative while the Banach contraction mapping principle is used to obtain the existence and uniqueness result. Examples illustrating the obtained results are also presented. Our results on a system of Hilfer–Hadamard fractional derivatives are new in the given configuration. We emphasize that we used Hilfer-Hadamard derivative of order $1<\alpha_i\leq 2$, $i=1,2$. In the context of sequential fractional differential equations with two-point boundary conditions, the present paper significantly contributes to the existing literature on the topic. The problems studied in this paper can be extended to cover other kinds of boundary conditions.

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