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Fractional operators with generalized Mittag-Leffler k-function

Shahid Mubeen[1*] and Rana Safdar Ali[1]

*Correspondence:
smjhanda@gmail.com
[1]Department of Mathematics, University of Sargodha, Sargodha, Pakistan

Abstract

In this paper, our main aim is to deal with two integral transforms involving the Gauss hypergeometric functions as their kernels. We prove some composition formulas for such generalized fractional integrals with Mittag-Leffler k-function. The results are established in terms of the generalized Wright hypergeometric function. The Euler integral k-transformation for Mittag-Leffler k-functions has also been developed.

Keywords: Fractional integral operators; Generalized Mittag-Leffler k-function; Generalized Wright function

1 Introduction

Mittag-Leffler functions are important in studying solutions of fractional differential equations, and they are associated with a wide range of problems in many areas of mathematics and physics. The importance and great considerations of Mittag-Leffler functions have led many researchers in the theory of special functions to exploring possible generalizations and applications. Many more extensions or unifications for these functions are found in a large number of papers [1–5]. A useful generalization of the Mittag-Leffler function, the so-called Mittag-Leffler k-function has been introduced and studied in [6]. Many mathematicians discussed and obtained new results [7–13], seen as theoretical developments to the fractional operators. These considerations have led various researchers in the field of special functions for exploring possible extensions of and applications to the Mittag-Leffler function. Recently, fractional calculus gained more attention due to its wide variety of applications in various fields [14–18]. In the literature of fractional calculus, it is distinctly observed that the fractional integral operators and fractional integral formulas containing special functions occupied an influential place in computational and applied mathematics [19–21]. The fractional calculus of various types of special functions is used in many research papers [22–25]. For more details about the recent works in the field of dynamic system theory, stochastic systems, nonequilibrium statistical mechanics, and quantum mechanics, we refer the interesting readers to [26–32]. Throughout this paper, we denote by $\mathbb{C}$, $\mathbb{N}$, $\mathbb{R}^+$, and $\mathbb{R}$ the sets of complex numbers, natural numbers, positive real numbers, and real numbers, respectively.

The Gauss hypergeometric function is defined as follows [33]: for all $d, e, f \in \mathbb{C}$, $f \neq 0, -1, -2, \dots$, and $|z| < 1$,

$$ {}_2F_1(d, e; f; z) = \sum_{n=0}^{\infty} \frac{(d)_n (e)_n}{(f)_n} \frac{z^n}{n!}, \tag{1} $$

where $(d)_n$, $(e)_n$, and $(f)_n$ are the Pochhammer symbols.

The Pochhammer symbols are defined as [34]

$$ (l)_n = \begin{cases} 1 & \text{for } n = 0, l \neq 0, \\ l(l+1)(l+2)\cdots(l+n-1) & \text{for } n \geq 1, \end{cases} \tag{2} $$

where $l \in \mathbb{C}$ and $n \in \mathbb{N}$.

The gamma function [34] for $\Re(u) > 0$ is defined as

$$ \Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} \, dt. \tag{3} $$

The beta function [34] is defined as

$$ \beta(l, h) = \int_0^1 t^{l-1} (1-t)^{h-1} \, dt, \quad \Re(l) > 0, \Re(h) > 0. \tag{4} $$

The beta k-function [33] is defined as

$$ \beta_k(l, h) = \frac{1}{k} \int_0^1 s^{\frac{l}{k}-1} (1-s)^{\frac{h}{k}-1} \, ds, \quad \Re(l) > 0, \Re(h) > 0. \tag{5} $$

The generalized fractional integration operators are defined for $u > 0$, $d, e, f \in \mathbb{C}$, and $\Re(d) > 0$ as follows (see [35–37]):

$$ \left(I_{0,u}^{a,b,c} h\right)(u) = \frac{u^{-a-b}}{\Gamma(a)} \int_0^u (u-t)^{a-1} {}_2F_1\left(a+b, -c; a; 1 - \frac{t}{u}\right) h(t) \, dt \tag{6} $$

and

$$ \left(I_{z,\infty}^{d,e,f} h\right)(z) = \frac{1}{\Gamma(d)} \int_z^{\infty} (t-z)^{d-1} t^{-d-e} {}_2F_1\left(d+e, -f; d; 1 - \frac{z}{t}\right) h(t) \, dt, \tag{7} $$

where Γ is the gamma function [38], and ${}_2F_1$ is the hypergeometric series defined by Rainville [39].

The Mittag-Leffler function $E_\alpha(z)$is defined by [40, 41]

$$ E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)} \tag{8} $$

for $z \in \mathbb{C}$ and $\alpha \geq 0$. The Mittag-Leffler function $E_\alpha(z)$ has been extended in a number of ways and, together with its extensions, applied in various research areas such as engineering and (in particular) statistics. The Mittag-Leffler functions and related distributions were given in [32].

The generalization of $E_\alpha(z)$, also known as the Wiman function [42], is given by

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \tag{9}$$

for $\alpha, \beta \in \mathbb{C}$ with $\Re(\alpha) > 0$, $\Re(\beta) > 0$.

In 1971, Prabhakar [43] proposed the more general function

$$E^{\delta}_{\nu,\rho}(z) = \sum_{n=0}^{\infty} \frac{(\delta)_n z^n}{\Gamma(\nu n + \rho) n!}. \tag{10}$$

A useful generalization of the Mittag-Leffler, the so-called Mittag-Leffler k-function has been introduced and studied in [2, 6]. The generalized Mittag-Leffler k-function [44] is defined as

$$E^{\delta}_{k,\nu,\rho}(t) = \sum_{n=0}^{\infty} \frac{(\delta)_{n,k} t^n}{\Gamma_k(\nu n + \rho) n!}, \tag{11}$$

for $k \in \mathbb{R}^+$, $\nu, \rho, \delta, t \in \mathbb{C}$ with $\Re(\nu) > 0$, $\Re(\rho) > 0$.

The integral form of the generalized gamma k-function is given by [45]

$$\Gamma_k(z) = \int_0^{\infty} t^{z-1} e^{\frac{-t^k}{k}}\, dt \tag{12}$$

for $k \in \mathbb{R}^+$ and $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$. By inspection we conclude the following relations:

$$\Gamma_k(z + k) = k\Gamma_k(z), \tag{13}$$

$$\Gamma_k(\gamma) = (k)^{\frac{\gamma}{k}-1} \Gamma\left(\frac{\gamma}{k}\right). \tag{14}$$

If k approaches one, then the generalized Mittag-Leffler k-function reduces to the generalized Mittag-Leffler function.

The generalized hypergeometric function is defined as [46]

$$_pF_q(d_1, \dots, d_p, e_1, \dots, e_q; t) = \sum_{n=0}^{\infty} \frac{(d_1)_n \cdots (d_p)_n}{(e_1)_n \cdots (e_q)_n} \frac{t^n}{n!}, \tag{15}$$

where $d_i, e_j \in \mathbb{C}$, $e_j \neq 0, -1, \dots$ $(i = 1, 2, \dots, p; j = 1, 2, \dots, q)$.

The generalized Wright hypergeometric function is defined as [47]

$$_l\psi_h(t) = {}_l\Psi_h \left[\begin{array}{c|c} (c_i, p_i)_{1,l} & \\ & t \\ (d_j, q_j)_{1,h} & \end{array} \right] \equiv \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{l} \Gamma(c_i + p_i n) t^n}{\prod_{j=1}^{h} \Gamma(d_j + q_j n) n!}, \tag{16}$$

where $t \in \mathbb{C}$, $c_i, d_j \in \mathbb{C}$, and $p_i, q_j \in \mathbb{R}$ $(i = 1, 2, \dots, l; j = 1, 2, \dots, h)$.

The following identity of Gauss hypergeometric function holds:

$$_2F_1(e, f; d; 1) = \frac{\Gamma(d)\Gamma(d - e - f)}{\Gamma(d - e)\Gamma(d - f)}, \quad \Re(d - e - f) > 0. \tag{17}$$

The hypergeometric k-function [48] is defined as

$$
{}_2F_{1,k}\big((\alpha',k),(\beta',k);(\eta',k);t\big)=\sum_{m=0}^{\infty}\frac{(\alpha')_{m,k}(\beta')_{m,k}}{(\eta')_{m,k}}\frac{t^m}{m!},\quad k>0, \tag{18}
$$

where

$$
\alpha',\beta',\eta'\in\mathbb{C},\qquad \eta'\neq 0,-1,-2,-3,\ldots,\qquad |t|<1.
$$

2 Preliminary lemmas

In this section, we derive the fundamental results of left- and right-sided generalized k-fractional integration with power k-function. The following lemmas proved in [35] are needed to prove our main results.

Lemma 1 ([35]) *For $a,b,c,\rho\in\mathbb{C}$ with*

$$
\Re(a)>0\quad\text{and}\quad\Re(\rho+c-b)>0,
$$
$$
\big(I_{0,u}^{a,b,c}t^{\rho-1}\big)(u)=u^{\rho-b-1}\frac{\Gamma(\rho)\Gamma(\rho+c-b)}{\Gamma(\rho-b)\Gamma(\rho+a+c)}.
$$

Lemma 2 ([35]) *For $a,b,c\in\mathbb{C}$ with*

$$
\Re(a)>0\quad\text{and}\quad\Re(\rho)<1+\min\big[\Re(b),\Re(c)\big],
$$

we have

$$
\big(I_{u,\infty}^{a,b,c}t^{\rho-1}\big)(u)=u^{\rho-b-1}\frac{\Gamma(b-\rho+1)\Gamma(c-\rho+1)}{\Gamma(1-\rho)\Gamma(a+b+c-\rho+1)}.
$$

Theorem 1 *Let $\alpha',\beta',\eta'\in\mathbb{C}$, $k\in\mathbb{R}^+$ with $\Re(\alpha')>0$ and $\Re(\sigma')>\max[0,\Re(\beta'-\eta')]$. Then*

$$
\big(I_{0,y}^{\alpha',\beta',\eta'}s^{\frac{\sigma'}{k}-1}\big)_k(y)=y^{\frac{\sigma'-\beta'}{k}-1}\frac{\Gamma_k(\sigma')\Gamma_k(\sigma'+\eta'-\beta')}{\Gamma_k(\sigma'-\beta')\Gamma_k(\sigma'+\alpha'+\eta')}.
$$

Proof Consider the left-sided generalized k-fractional integral operator

$$
\begin{aligned}
\big(I_{0,y}^{\alpha',\beta',\eta'}g\big)_k(y)={}&\frac{y^{\frac{-\alpha'-\beta'}{k}}}{k\Gamma_k(\alpha')}\int_0^y(y-s)^{\frac{\alpha'}{k}-1}\\
&\times{}_2F_{1,k}\bigg(\big(\alpha'+\beta',k\big),\big(-\eta',k\big);\big(\alpha',k\big);1-\frac{s}{y}\bigg)g(s)\,ds.
\end{aligned} \tag{19}
$$

Using the power k-function in equation (19), we have

$$
\begin{aligned}
\big(I_{0,y}^{\alpha',\beta',\eta'}s^{\frac{\sigma'}{k}-1}\big)_k(y)={}&\frac{y^{\frac{-\alpha'-\beta'}{k}}}{k\Gamma_k(\alpha')}\int_0^y(y-s)^{\frac{\alpha'}{k}-1}\\
&\times{}_2F_{1,k}\bigg(\big(\alpha'+\beta',k\big),\big(-\eta',k\big);\big(\alpha',k\big);1-\frac{s}{y}\bigg)s^{\frac{\sigma'}{k}-1}\,ds.
\end{aligned} \tag{20}
$$

Using equation (18) in equation (20), we get

$$\big(I_{0,y}^{\alpha',\beta',\eta'} s^{\frac{\sigma'}{k}-1}\big)_k(y) = \frac{y^{\frac{-\alpha'-\beta'}{k}}}{k\Gamma_k(\alpha')} \int_0^y (y-s)^{\frac{\alpha'}{k}-1} \times \sum_{m=0}^{\infty} \frac{(\alpha'+\beta')_{m,k}(-\eta')_{m,k}}{(\alpha')_{m,k} m!} \left(1-\frac{s}{y}\right)^m s^{\frac{\sigma'}{k}-1}\,ds. \tag{21}$$

By putting

$$\begin{aligned} s = vy &\implies ds = y\,dv, \\ s = 0 &\implies v = 0, \\ s = y &\implies v = 1 \end{aligned}$$

in equation (21), we obtain

$$\begin{aligned}
&\big(I_{0,y}^{\alpha',\beta',\eta'} s^{\frac{\sigma'}{k}-1}\big)_k(y) \\
&= \frac{y^{\frac{-\alpha'-\beta'}{k}}}{k\Gamma(\alpha')} \int_0^1 (y-vy)^{\frac{\alpha'}{k}-1}(1-v)^m (vy)^{\frac{\sigma'}{k}-1} \sum_{m=0}^{\infty} \frac{(\alpha'+\beta')_{m,k}(-\eta')_{m,k}}{(\alpha')_{m,k} m!} y\,dv \\
&= \frac{y^{\frac{-\alpha'-\beta'}{k}}}{k\Gamma(\alpha')} \sum_{m=0}^{\infty} \frac{(\alpha'+\beta')_{m,k}(-\eta')_{m,k}}{(\alpha')_{m,k} m!} \int_0^1 y^{\frac{\alpha'}{k}-1}(1-v)^{\frac{\alpha'}{k}-1}(1-v)^m v^{\frac{\sigma'}{k}-1} y^{\frac{\sigma'}{k}-1} y\,dv \\
&= \frac{y^{\frac{-\alpha'-\beta'}{k}}}{k\Gamma(\alpha')} \sum_{m=0}^{\infty} \frac{(\alpha'+\beta')_{m,k}(-\eta')_{m,k}}{(\alpha')_{m,k} m!} \int_0^1 y^{\frac{\alpha'}{k}-1+\frac{\sigma'}{k}-1+1}(1-v)^{\frac{\alpha'}{k}+m-1} v^{\frac{\sigma'}{k}-1}\,dv \\
&= \frac{y^{-\frac{\alpha'}{k}-\frac{\beta'}{k}+\frac{\alpha'}{k}+\frac{\sigma'}{k}-1}}{k\Gamma(\alpha')} \sum_{m=0}^{\infty} \frac{(\alpha'+\beta')_{m,k}(-\eta')_{m,k}}{(\alpha')_{m,k} m!} \int_0^1 (1-v)^{\frac{\alpha'}{k}+m-1} v^{\frac{\sigma'}{k}-1}\,dv \\
&= \frac{y^{\frac{\sigma'-\beta'}{k}-1}}{\Gamma_k(\alpha')} \sum_{m=0}^{\infty} \frac{(\alpha'+\beta')_{m,k}(-\eta')_{m,k}}{(\alpha')_{m,k} m!} \frac{1}{k} \int_0^1 (1-v)^{\frac{\alpha'+mk}{k}-1} v^{\frac{\sigma'}{k}-1}\,dv.
\end{aligned} \tag{22}$$

Since

$$\beta_k(l,h) = \frac{\Gamma_k(l)\Gamma_k(h)}{\Gamma_k(l+h)}, \tag{23}$$

by equations (5) and (22) we have

$$= \frac{y^{\frac{\sigma'-\beta'}{k}-1}}{\Gamma_k(\alpha')} \sum_{m=0}^{\infty} \frac{(\alpha'+\beta')_{m,k}(-\eta')_{m,k}}{(\alpha')_{m,k} m!} \frac{\Gamma_k(\alpha'+mk)\Gamma_k(\sigma')}{\Gamma_k(\alpha'+\sigma'+mk)}. \tag{24}$$

Since

$$\Gamma_k(t+mk) = (t)_{m,k}\Gamma_k(t), \tag{25}$$

from equation (24) we get

$$\begin{aligned}\left(I_{0,y}^{\alpha',\beta',\eta'}s^{\frac{\sigma'}{k}-1}\right)_k(y) &= \frac{y^{\frac{\sigma'-\beta'}{k}-1}}{\Gamma_k(\alpha')}\sum_{m=0}^{\infty}\frac{(\alpha'+\beta')_{m,k}(-\eta')_{m,k}}{(\alpha')_{m,k}m!}\frac{(\alpha')_{m,k}\Gamma_k(\alpha')\Gamma_k(\sigma')}{(\alpha'+\sigma')_{m,k}\Gamma_k(\alpha'+\sigma')}\\ &= y^{\frac{\sigma'-\beta'}{k}-1}\frac{\Gamma_k(\sigma')}{\Gamma_k(\alpha'+\sigma')}\sum_{m=0}^{\infty}\frac{(\alpha'+\beta')_{m,k}(-\eta')_{m,k}}{(\alpha'+\sigma')_{m,k}m!}.\end{aligned} \tag{26}$$

Using equation (18), from equation (26) we have

$$\left(I_{0,y}^{\alpha',\beta',\eta'}s^{\sigma'-1}\right)_k(y) = y^{\frac{\sigma'-\beta'}{k}-1}\frac{\Gamma_k(\sigma')}{\Gamma_k(\alpha'+\sigma')}{}_2F_{1,k}\big((\alpha'+\beta',k),(-\eta',k);(\alpha'+\sigma',k);1\big).$$

We can also write

$$\begin{aligned}\left(I_{0,y}^{\alpha',\beta',\eta'}s^{\frac{\sigma'}{k}-1}\right)_k(y) &= y^{\frac{\sigma'-\beta'}{k}-1}\frac{\Gamma_k(\sigma')}{\Gamma_k(\alpha'+\sigma')}\\ &\quad\times {}_2F_{1,k}\big((\alpha'+\beta',k),(-\eta',k);(\alpha'+\sigma',k);1\big).\end{aligned} \tag{27}$$

Since

$${}_2F_{1,k}\big[(\alpha',k),(\beta',k)(\eta',k);1\big] = \frac{\Gamma_k(\eta')\Gamma_k(\eta'-\beta'-\alpha')}{\Gamma_k(\eta'-\alpha')\Gamma_k(\eta'-\beta')}, \tag{28}$$

from equation (27) we obtain

$$\begin{aligned}\left(I_{0,y}^{\alpha',\beta',\eta'}s^{\frac{\sigma'}{k}-1}\right)_k(y) &= y^{\frac{\sigma'-\beta'}{k}-1}\frac{\Gamma_k(\sigma')}{\Gamma_k(\alpha'+\sigma')}\frac{\Gamma_k(\alpha'+\sigma')\Gamma_k(\alpha'+\sigma'-\alpha'-\beta'+\eta')}{\Gamma_k(\alpha'+\sigma'-\alpha'-\beta')\Gamma_k(\alpha'+\sigma'+\eta')},\\ \left(I_{0,y}^{\alpha',\beta',\eta'}s^{\frac{\sigma'}{k}-1}\right)_k(y) &= y^{\frac{\sigma'-\beta'}{k}-1}\frac{\Gamma_k(\sigma')\Gamma_k(\sigma'+\eta'-\beta')}{\Gamma_k(\sigma'-\beta')\Gamma_k(\sigma'+\alpha'+\eta')}.\end{aligned}$$

□

Theorem 2 *Let $\alpha',\beta',\eta' \in \mathbb{C}$. Then*

$$\left(I_{y,\infty}^{\alpha',\beta',\eta'}s^{\frac{\sigma'}{k}-1}\right)_k(y) = y^{\frac{\sigma'-\beta'}{k}-1}\frac{\Gamma_k(\beta'-\sigma'+k)\Gamma_k(-\sigma'+k+\eta')}{\Gamma_k(-\sigma'+k)\Gamma_k(\alpha'-\sigma'+\beta'+k+\eta')}. \tag{29}$$

Proof Consider the right-sided generalized k-fractional integral operator

$$\begin{aligned}\left(I_{y,\infty}^{\alpha',\beta',\eta'}g\right)_k(y) &= \frac{1}{k\Gamma_k(\alpha')}\int_y^{\infty}(s-y)^{\frac{\alpha'}{k}-1}s^{\frac{-\alpha'-\beta'}{k}}\\ &\quad\times {}_2F_{1,k}\left((\alpha'+\beta',k),(-\eta',k);(\alpha',k);1-\frac{y}{s}\right)g(s)\,ds.\end{aligned} \tag{30}$$

Using the power k-function in (30), we have

$$\begin{aligned}\left(I_{y,\infty}^{\alpha',\beta',\eta'}s^{\frac{\sigma'}{k}-1}\right)_k(y) &= \frac{1}{k\Gamma_k(\alpha')}\int_y^{\infty}(s-y)^{\frac{\alpha'}{k}-1}s^{\frac{-\alpha'-\beta'}{k}}\\ &\quad\times {}_2F_{1,k}\left((\alpha'+\beta',k),(-\eta',k);(\alpha',k);1-\frac{y}{s}\right)s^{\frac{\sigma'}{k}-1}\,ds.\end{aligned} \tag{31}$$

Using equation (18) in equation (31), we get

$$\begin{aligned}\left(I_{y,\infty}^{\alpha',\beta',\eta'} s^{\frac{\sigma'}{k}-1}\right)_k(y) &= \frac{1}{k\Gamma_k(\alpha')}\int_y^\infty (s-y)^{\frac{\alpha'}{k}-1} s^{\frac{-\alpha'-\beta'}{k}} \\ &\quad \times \sum_{m=0}^\infty \frac{(\alpha'+\beta')_{m,k}(-\eta')_{m,k}}{(\alpha')_{m,k}m!}\left(1-\frac{y}{s}\right)^m s^{\frac{\sigma'}{k}-1}\,ds. \end{aligned}\tag{32}$$

Putting

$$\begin{aligned} &s=\frac{y}{v} \quad\Longrightarrow\quad ds=-\frac{y}{v^2}\,dv,\\ &s=y \quad\Longrightarrow\quad v=1,\\ &s=\infty \quad\Longrightarrow\quad v=0 \end{aligned}$$

in equation (32), we obtain

$$\begin{aligned} &\left(I_{y,\infty}^{\alpha',\beta',\eta'} s^{\frac{\sigma'}{k}-1}\right)_k(y)\\ &\quad= \frac{1}{k\Gamma_k(\alpha')}\int_1^0 \left(\frac{y}{v}-y\right)^{\frac{\alpha'}{k}-1}\left(\frac{y}{v}\right)^{\frac{-\alpha'-\beta'}{k}}\sum_{m=0}^\infty \frac{(\alpha'+\beta')_{m,k}(-\eta')_{m,k}}{(\alpha')_{m,k}m!}\\ &\qquad\times(1-v)^m\left(\frac{y}{v}\right)^{\frac{\sigma'}{k}-1}\left(-\frac{y}{v^2}\right)dv\\ &\quad= \frac{1}{k\Gamma_k(\alpha')}\sum_{m=0}^\infty \frac{(\alpha'+\beta')_{m,k}(-\eta')_{m,k}}{(\alpha')_{m,k}m!}\int_0^1 y^{\frac{\alpha'}{k}-1}\left(\frac{1-v}{v}\right)^{\frac{\alpha'}{k}-1} y^{\frac{-\alpha'-\beta'}{k}} v^{\frac{\alpha'+\beta'}{k}}\\ &\qquad\times(1-v)^m y^{\frac{\sigma'}{k}-1} v^{1-\frac{\sigma'}{k}} y v^{-2}\,dv,\\ &\left(I_{y,\infty}^{\alpha',\beta',\eta'} s^{\frac{\sigma'}{k}-1}\right)_k(y)\\ &\quad= \frac{1}{k\Gamma_k(\alpha')}\sum_{m=0}^\infty \frac{(\alpha'+\beta')_{m,k}(-\eta')_{m,k}}{(\alpha')_{m,k}m!}\int_0^1 y^{\frac{\alpha'}{k}-1-\frac{\alpha'}{k}-\frac{\beta'}{k}+\frac{\sigma'}{k}-1+1}(1-v)^{\frac{\alpha'}{k}+m-1}\\ &\qquad\times v^{1-\frac{\alpha'}{k}+\frac{\alpha'}{k}+\frac{\beta'}{k}+1-\frac{\sigma'}{k}-2}\,dv\\ &\quad= \frac{y^{\frac{\sigma'}{k}-\frac{\beta'}{k}-1}}{k\Gamma_k(\alpha')}\sum_{m=0}^\infty \frac{(\alpha'+\beta')_{m,k}(-\eta')_{m,k}}{(\alpha')_{m,k}m!}\int_0^1 (1-v)^{\frac{\alpha'}{k}+m-1} v^{\frac{\beta'-\sigma'}{k}}\,dv\\ &\quad= \frac{y^{\frac{\sigma'-\beta'}{k}-1}}{\Gamma_k(\alpha')}\sum_{m=0}^\infty \frac{(\alpha'+\beta')_{m,k}(-\eta')_{m,k}}{(\alpha')_{m,k}m!}\frac{1}{k}\int_0^1 (1-v)^{\frac{\alpha'+mk}{k}-1} v^{\frac{\beta'-\sigma'}{k}+1-1}\,dv. \end{aligned}\tag{33}$$

Using equation (5) and equation (23) in equation (33), we get

$$\begin{aligned}\left(I_{y,\infty}^{\alpha',\beta',\eta'} s^{\frac{\sigma'}{k}-1}\right)_k(y) &= \frac{y^{\frac{\sigma'-\beta'}{k}-1}}{\Gamma_k(\alpha')}\\ &\quad\times\sum_{m=0}^\infty \frac{(\alpha'+\beta')_{m,k}(-\eta')_{m,k}}{(\alpha')_{m,k}m!}\frac{\Gamma_k(\alpha'+mk)\Gamma_k(\beta'-\sigma'+k)}{\Gamma_k(\alpha'+\beta'-\sigma'+mk+k)}. \end{aligned}\tag{34}$$

Using equation (25) in equation (34), we obtain

$$\begin{aligned}
&\left(I_{y,\infty}^{\alpha',\beta',\eta'} s^{\frac{\sigma'}{k}-1}\right)_k(y) \\
&= \frac{y^{\frac{\sigma'-\beta'}{k}-1}}{\Gamma_k(\alpha')} \sum_{m=0}^{\infty} \frac{(\alpha'+\beta')_{m,k}(-\eta')_{m,k}}{(\alpha')_{m,k} m!} \frac{(\alpha')_{m,k}\Gamma_k(\alpha')\Gamma_k(\beta'-\sigma'+k)}{(\alpha'-\sigma'+\beta'+k)_{m,k}\Gamma_k(\alpha'-\sigma'+\beta'+k)} \\
&= y^{\frac{\sigma'-\beta'}{k}-1} \frac{\Gamma_k(\beta'-\sigma'+k)}{\Gamma_k(\alpha'-\sigma'+\beta'+k)} \sum_{m=0}^{\infty} \frac{(\alpha'+\beta')_{m,k}(-\eta')_{m,k}}{(\alpha'-\sigma'+\beta'+k)_{m,k} m!}.
\end{aligned} \tag{35}$$

Using equation (18) in equation (35), we have

$$\begin{aligned}
\left(I_{y,\infty}^{\alpha',\beta',\eta'} s^{\frac{\sigma'}{k}-1}\right)_k(y) &= y^{\frac{\sigma'-\beta'}{k}-1} \frac{\Gamma_k(\beta'-\sigma'+k)}{\Gamma_k(\alpha'-\sigma'+\beta'+k)} \\
&\quad \times {}_2F_{1,k}\left((\alpha'+\beta',k),(-\eta',k);(\alpha'-\sigma'+\beta'+k,k);1\right).
\end{aligned} \tag{36}$$

Using equation (28) in equation (36), we get

$$\begin{aligned}
\left(I_{y,\infty}^{\alpha',\beta',\eta'} s^{\frac{\sigma'}{k}-1}\right)_k(y) &= y^{\frac{\sigma'-\beta'}{k}-1} \frac{\Gamma_k(\beta'-\sigma'+k)}{\Gamma_k(\alpha'-\sigma'+\beta'+k)} \\
&\quad \times \frac{\Gamma_k(\alpha'-\sigma'+\beta'+k)\Gamma_k(\alpha'-\sigma'+\beta'+k-\alpha'-\beta'+\eta')}{\Gamma_k(\alpha'-\sigma'+\beta'+k-\alpha'-\beta')\Gamma_k(\alpha'-\sigma'+\beta'+k+\eta')}, \\
\left(I_{y,\infty}^{\alpha',\beta',\eta'} s^{\frac{\sigma'}{k}-1}\right)_k(y) &= y^{\frac{\sigma'-\beta'}{k}-1} \frac{\Gamma_k(\beta'-\sigma'+k)\Gamma_k(-\sigma'+k+\eta')}{\Gamma_k(-\sigma'+k)\Gamma_k(\alpha'-\sigma'+\beta'+k+\eta')}.
\end{aligned}$$

□

3 Generalized fractional integrals in terms of Wright functions

In this section, we solve the composition of the Mittag-Leffler with power function to generalized left- and right-sided fractional integral operators and also discuss k-calculus.

Theorem 3 *For $a, b, c, \rho, \delta \in \mathbb{C}$ with*

$$\Re(a) > 0 \quad \text{and} \quad \Re(\rho + c - b) > 0, \qquad \nu > 0, \qquad \lambda > 0, \qquad w \in \mathbb{R},$$

we have

$$\begin{aligned}
\left(I_{0,u}^{a,b,c} t^{\rho-1} E_{\nu,\rho}^{\delta}\left(wt^{\lambda}\right)\right)(u) &= \frac{u^{-b-1+\rho}}{\Gamma(\delta)} \\
&\quad \times {}_3\Psi_3\left[\left.\begin{matrix}(c-b+\rho,\lambda),(\rho,\lambda),(\delta,1) \\ (a+c+\rho,\lambda),(\rho-b,\lambda),(\rho,\nu)\end{matrix}\right| wu^{\lambda}\right].
\end{aligned}$$

Proof Using the power function and (10) in (6), we have

$$\begin{aligned}
\left(I_{0,u}^{a,b,c} t^{\rho-1} E_{\nu,\rho}^{\delta}\left(wt^{\lambda}\right)\right)(u) &= \frac{u^{-a-b}}{\Gamma(a)} \int_0^u (u-t)^{a-1} \\
&\quad \times {}_2F_1\left(a+b,-c;a;1-\frac{t}{u}\right) t^{\rho-1} \sum_{n=0}^{\infty} \frac{(\delta)_n}{\Gamma(\nu n+\rho) n!} \left(wt^{\lambda}\right)^n dt \quad (37) \\
&= \sum_{n=0}^{\infty} \frac{w^n (\delta)_n}{\Gamma(\nu n+\rho) n!} \left(I_{0,u}^{a,b,c} t^{\rho+\lambda n-1}\right)(u). \quad (38)
\end{aligned}$$

Since for $n = 0, 1, 2, \dots$, $\Re(\rho + \lambda n) \geq \Re(\rho + c - b) > 0$, using Lemma 1 with ρ replaced by $\rho + \lambda n$ in equation (38), we obtain

$$\begin{aligned}&\left(I_{0,u}^{a,b,c} t^{\rho-1} E_{\nu,\rho}^{\delta}\left(wt^{\lambda}\right)\right)(u)\\&\quad = \frac{u^{\rho-b-1}}{\Gamma(\delta)} \sum_{n=0}^{\infty} \frac{\Gamma(\delta+n)\Gamma(\rho+\lambda n)\Gamma(c-b+\rho+\lambda n)}{\Gamma(-b+\rho+\lambda n)\Gamma(a+c+\rho+\lambda n)\Gamma(\nu n+\rho)n!}\left(wu^{\lambda}\right)^{n}.\end{aligned} \tag{39}$$

Using (16) in (39), we get

$$\begin{aligned}\left(I_{0,u}^{a,b,c} t^{\rho-1} E_{\nu,\rho}^{\delta}\left(wt^{\lambda}\right)\right)(u) &= \frac{u^{-b-1+\rho}}{\Gamma(\delta)}\\&\quad \times {}_3\Psi_3\left[\begin{matrix}(c-b+\rho,\lambda),(\rho,\lambda),(\delta,1)\\(a+c+\rho,\lambda),(\rho-b,\lambda),(\rho,\nu)\end{matrix}\middle| wu^{\lambda}\right].\end{aligned}$$ □

Theorem 4 *For $a, b, c, \rho, \delta \in \mathbb{C}$ with*

$$\begin{aligned}&\Re(a) > 0 \quad \textit{and} \quad \Re(a+\rho) > \max\left[-\Re(b), -\Re(c)\right], \qquad \Re(b) \neq \Re(c),\\&\nu > 0, \qquad \lambda > 0, \qquad w \in \mathbb{R},\end{aligned}$$

we have

$$\begin{aligned}\left(I_{u,\infty}^{a,b,c} t^{\rho-1} E_{\nu,\rho}^{\delta}\left(wt^{-\lambda}\right)\right)(u) &= \frac{u^{\rho-b-1}}{\Gamma(\delta)}\\&\quad \times {}_3\Psi_3\left[\begin{matrix}(b-\rho+1,\lambda),(1+c-\rho,\lambda),(\delta,1)\\(1-\rho,\lambda),(a+b-\rho+c+1,\lambda),(\rho,\nu)\end{matrix}\middle| wu^{-\lambda}\right].\end{aligned}$$

Proof Using the power function and (10) in (7), we have

$$\begin{aligned}\left(I_{u,\infty}^{a,b,c} t^{\rho-1} E_{\nu,\rho}^{\delta}\left(wt^{-\lambda}\right)\right)(u) &= \frac{1}{\Gamma(a)} \int_u^{\infty} (t-u)^{a-1} t^{-a-b}\\&\quad \times {}_2F_1\left(a+b, -c; a; 1-\frac{u}{t}\right) t^{\rho-1} \sum_{n=0}^{\infty} \frac{(\delta)_n}{\Gamma(\nu n+\rho)n!}\left(wt^{-\lambda}\right)^n dt\\&= \sum_{n=0}^{\infty} \frac{w^n(\delta)_n}{\Gamma(\nu n+\rho)n!}\left(I_{u,\infty}^{a,b,c} t^{\rho-\lambda n-1}\right)(u).\end{aligned} \tag{40}$$

Since for $n = 0, 1, 2, \dots$, $\Re(\rho-\lambda n-1) \leq \Re(\rho+a-1) > 1+\max[-\Re(b), -\Re(c)]$, using Lemma 2 with ρ replaced by $\rho - \lambda n$, we reduce equation (40) to

$$\begin{aligned}&\left(I_{u,\infty}^{a,b,c} t^{\rho-1} E_{\nu,\rho}^{\delta}\left(wt^{-\lambda}\right)\right)(u)\\&\quad = \frac{u^{\rho-b-1}}{\Gamma(\delta)} \sum_{n=0}^{\infty} \frac{\Gamma(\delta+n)\Gamma(1-a+a+b+\lambda n-\rho)\Gamma(1-a-b+c+a+b+\lambda n-\rho)}{\Gamma(1-a-b+a+b+\lambda n-\rho)\Gamma(1+c+a+b+\lambda n-\rho)\Gamma(\nu n+\rho)n!}\\&\qquad \times \left(wu^{-\lambda}\right)^n.\end{aligned} \tag{41}$$

Using (16) in (41), we get

$$\left(I_{u,\infty}^{a,b,c} t^{\rho-1} E_{\nu,\rho}^{\delta}\left(wt^{-\lambda}\right)\right)(u) = \frac{u^{\rho-b-1}}{\Gamma(\delta)} \times {}_3\Psi_3\left[\begin{matrix}(b-\rho+1,\lambda),(1+c-\rho,\lambda),(\delta,1)\\(1-\rho,\lambda),(a+b-\rho+c+1,\lambda),(\rho,\nu)\end{matrix}\middle| wu^{-\lambda}\right]. \qquad \square$$

Theorem 5 *For $a,b,c,\rho,\delta \in \mathbb{C}$ with*

$$\Re(a) > 0 \quad \text{and} \quad \Re(\rho + c - b) > 0, \qquad \nu > 0, \qquad \lambda > 0, \qquad w \in \mathbb{R},$$

we have

$$\left(I_{0,u}^{a,b,c} t^{\frac{\rho}{k}-1} E_{k,\nu,\rho}^{\delta}\left(wt^{\frac{\lambda}{k}}\right)\right)(u) = \frac{k^{1-\frac{\rho}{k}} u^{-b-1+\frac{\rho}{k}}}{\Gamma(\frac{\delta}{k})} \times {}_3\Psi_3\left[\begin{matrix}(c-b+\frac{\rho}{k},\frac{\lambda}{k}),(\frac{\rho}{k},\frac{\lambda}{k}),(\frac{\delta}{k},1)\\(a+c+\frac{\rho}{k},\frac{\lambda}{k}),(\frac{\rho}{k}-b,\frac{\lambda}{k}),(\frac{\rho}{k},\frac{\nu}{k})\end{matrix}\middle| k^{1-\frac{\nu}{k}} wu^{\frac{\lambda}{k}}\right].$$

Proof Using the power k-function and (11) in (6), we have

$$\begin{aligned}&\left(I_{0,u}^{a,b,c} t^{\frac{\rho}{k}-1} E_{k,\nu,\rho}^{\delta}\left(wt^{\frac{\lambda}{k}}\right)\right)(u)\\ &\quad= \frac{u^{-a-b}}{\Gamma(a)} \int_0^u (u-t)^{a-1} {}_2F_1\left(a+b,-c;a;1-\frac{t}{u}\right) t^{\frac{\rho}{k}-1} \sum_{n=0}^{\infty} \frac{(\delta)_{n,k}}{\Gamma_k(\nu n+\rho) n!}\left(wt^{\frac{\lambda}{k}}\right)^n dt \qquad (42)\\ &\quad= \sum_{n=0}^{\infty} \frac{w^n (\delta)_{n,k}}{\Gamma_k(\nu n+\rho) n!}\left(I_{0,u}^{a,b,c} t^{\frac{\rho+\lambda n}{k}-1}\right)_k(u). \qquad (43)\end{aligned}$$

Since for $n = 0, 1, 2, \ldots$, $\Re(\rho + \lambda n) \geq \Re(\rho + c - b) > 0$, using Lemma 1 with ρ replaced by by $\frac{\rho+\lambda n}{k}$, we reduce equation (43) to

$$\begin{aligned}&\left(I_{0,u}^{a,b,c} t^{\frac{\rho}{k}-1} E_{k,\nu,\rho}^{\delta}\left(wt^{\lambda}\right)\right)(u)\\ &\quad= \frac{k^{1-\frac{\rho}{k}} u^{-b-1+\frac{\rho}{k}}}{\Gamma(\frac{\delta}{k})} \sum_{n=0}^{\infty} \frac{\Gamma(\frac{\delta+nk}{k})\Gamma(\frac{\rho+\lambda n}{k})\Gamma(c-b+\frac{\rho+\lambda n}{k})}{\Gamma(-b+\frac{\rho+\lambda n}{k})\Gamma(a+c+\frac{\rho+\lambda n}{k})\Gamma(\frac{\nu n+\rho}{k}) n!}\left(wk^{1-\frac{\nu}{k}} t^{\frac{\lambda}{k}}\right)^n. \qquad (44)\end{aligned}$$

Using (16) in (44), we get

$$\left(I_{0,u}^{a,b,c} t^{\frac{\rho}{k}-1} E_{k,\nu,\rho}^{\delta}\left(wt^{\frac{\lambda}{k}}\right)\right)(u) = \frac{k^{1-\frac{\rho}{k}} u^{-b-1+\frac{\rho}{k}}}{\Gamma(\frac{\delta}{k})} \times {}_3\Psi_3\left[\begin{matrix}(c-b+\frac{\rho}{k},\frac{\lambda}{k}),(\frac{\rho}{k},\frac{\lambda}{k}),(\frac{\delta}{k},1)\\(a+c+\frac{\rho}{k},\frac{\lambda}{k}),(\frac{\rho}{k}-b,\frac{\lambda}{k}),(\frac{\rho}{k},\frac{\nu}{k})\end{matrix}\middle| k^{1-\frac{\nu}{k}} wu^{\frac{\lambda}{k}}\right]. \qquad \square$$

Remark 1 If we replace k by one, then we get the result of [3].

Theorem 6 *For $a,b,c,\rho,\delta \in \mathbb{C}$ with*

$$\Re(a) > 0 \quad \text{and} \quad \Re(a+\rho) > \max\left[-\Re(b), -\Re(c)\right], \qquad \Re(b) \neq \Re(c),$$

$$\nu > 0, \qquad \lambda > 0, \qquad w \in \mathbb{R},$$

we have

$$\begin{aligned}&\left(I^{a,b,c}_{u,\infty} t^{\frac{\rho}{k}-1} E^{\delta}_{k,\nu,\rho}\left(wt^{\frac{-\lambda}{k}}\right)\right)(u) \\ &\quad = \frac{k^{1-\frac{\rho}{k}} u^{\frac{\rho-a-b}{k}+a-1}}{\Gamma(\frac{\delta}{k})} {}_3\Psi_3\left[\begin{matrix}(1+b-\frac{\rho}{k},\frac{\lambda}{k}),(1+c-\frac{\rho}{k},\frac{\lambda}{k}),(\frac{\delta}{k},1)\\(1-\frac{\rho}{k},\frac{\lambda}{k}),(1+a+b+c-\frac{\rho}{k},\frac{\lambda}{k}),(\frac{\rho}{k},\frac{\nu}{k})\end{matrix}\middle| k^{1-\frac{\nu}{k}} wu^{\frac{-\lambda}{k}}\right].\end{aligned} \tag{45}$$

Proof Using the power k-function and (11) in (7), we have

$$\begin{aligned}&\left(I^{a,b,c}_{u,\infty} t^{\frac{\rho}{k}-1} E^{\delta}_{k,\nu,\rho}\left(wt^{\frac{-\lambda}{k}}\right)\right)(u) \\ &\quad = \frac{1}{\Gamma(a)}\int_u^\infty (t-u)^{a-1} t^{-a-b} {}_2F_1\left(a+b,-c;a;1-\frac{t}{u}\right) t^{\frac{\rho}{k}-1}\sum_{n=0}^{\infty}\frac{(\delta)_{n,k}}{\Gamma_k(\nu n+\rho)n!} \\ &\qquad \times \left(wt^{\frac{-\lambda}{k}}\right)^n dt \qquad (46)\\ &\quad = \sum_{n=0}^{\infty}\frac{w^n(\delta)_{n,k}}{\Gamma_k(\nu n+\rho)n!}\left(I^{a,b,c}_{u,\infty} t^{\frac{\rho-\lambda n}{k}-1}\right)(u). \qquad (47)\end{aligned}$$

Since for $n = 0, 1, 2, \ldots, \Re(\rho-\lambda n-1) \leq \Re(\rho+a-1) > 1+\max[-\Re(b), -\Re(c)]$, using Lemma 2 with ρ replaced by $\frac{\rho-\lambda n}{k}$, we reduce equation (47) to

$$\begin{aligned}&\left(I^{a,b,c}_{u,\infty} t^{\frac{\rho}{k}-1} E^{\delta}_{k,\nu,\rho}\left(wt^{\frac{-\lambda}{k}}\right)\right)(u) \\ &\quad = \frac{k^{1-\frac{\rho+\nu n}{k}} u^{\frac{\rho}{k}-b-1}}{\Gamma(\frac{\delta}{k})}\sum_{n=0}^{\infty}\frac{\Gamma(\frac{\delta+nk}{k})\Gamma(1+b-\frac{\rho-\lambda n}{k})\Gamma(1+c-\frac{\rho-\lambda n}{k})}{\Gamma(1-\frac{\rho-\lambda n}{k})\Gamma(1+a+b+c-\frac{\rho-\lambda n}{k})\Gamma(\frac{\nu n+\rho}{k})n!}\left(kwu^{\frac{-\lambda}{k}}\right)^n.\end{aligned} \tag{48}$$

Using (16) in (48), we get

$$\begin{aligned}&\left(I^{a,b,c}_{u,\infty} t^{\rho-1} E^{\delta}_{\nu,\rho}\left(wt^{\frac{-\lambda}{k}}\right)\right)(u) \\ &\quad = \frac{k^{1-\frac{\rho}{k}} u^{\frac{\rho-a-b}{k}+a-1}}{\Gamma(\frac{\delta}{k})} {}_3\Psi_3\left[\begin{matrix}(1+b-\frac{\rho}{k},\frac{\lambda}{k}),(1+c-\frac{\rho}{k},\frac{\lambda}{k}),(\frac{\delta}{k},1)\\(1-\frac{\rho}{k},\frac{\lambda}{k}),(1+a+b+c-\frac{\rho}{k},\frac{\lambda}{k}),(\frac{\rho}{k},\frac{\nu}{k})\end{matrix}\middle| k^{1-\frac{\nu}{k}} wu^{\frac{-\lambda}{k}}\right]. \qquad \square\end{aligned}$$

Remark 2 If we replace k by one, then we get the result of [4].

4 Euler transform for Mittag-Leffler function

In this section, we investigate the Euler integral transformation for the Mittag-Leffler k-function. We also derive the Euler k-transformation of the Mittag-Leffler k-function.

Theorem 7 *The Euler integral operator for the generalized Mittag-Leffler function is*

$$\left(I_0^1 t^{a-1}(1-t)^{b-1} E^{\delta}_{\nu,\rho}\left(wt^{\lambda}\right)\right) dt = \frac{\Gamma(b)}{\Gamma(\delta)} {}_2\Psi_2\left[\begin{matrix}(\delta,1),(a,\lambda)\\(\rho,\nu),(a+b,\lambda)\end{matrix}\middle| w\right].$$

Proof

$$\begin{aligned}\left(I_0^1 t^{a-1}(1-t)^{b-1}E^{\delta}_{\nu,\rho}\left(wt^{\lambda}\right)\right)dt &= \sum_{n=0}^{\infty}\frac{(\delta)_n}{\Gamma(\nu n+\rho)n!}(w)^n\int_0^1 t^{a+\lambda n-1}(1-t)^{b-1}\,dt\\ &= \frac{\Gamma(b)}{\Gamma(\delta)}\sum_{n=0}^{\infty}\frac{\Gamma(\delta+n)\Gamma(a+\lambda n)}{\Gamma(a+b+\lambda n)\Gamma(\nu n+\rho)n!}(w)^n\\ &= \frac{\Gamma(b)}{\Gamma(\delta)}\,{}_2\Psi_2\left[\left.\begin{matrix}(\delta,1),(a,\lambda)\\(\rho,\nu),(a+b,\lambda)\end{matrix}\right| w\right].\end{aligned}$$

□

Theorem 8 *The Euler integral operator for the generalized Mittag Leffler k-function is*

$$\left(I_0^1 t^{a-1}(1-t)^{b-1}E^{\delta}_{k,\nu,\rho}\left(wt^{\frac{\lambda}{k}}\right)\right)dt = \frac{\Gamma(b)k^{1-\frac{\rho}{k}}}{\Gamma(\delta)}\,{}_2\Psi_2\left[\left.\begin{matrix}(\frac{\delta}{k},1),(a,\frac{\lambda}{k})\\(\frac{\rho}{k},\frac{\nu}{k}),(a+b,\frac{\lambda}{k})\end{matrix}\right| k^{1-\frac{\nu}{k}}w\right].$$

Proof

$$\begin{aligned}\left(I_0^1 t^{a-1}(1-t)^{b-1}E^{\delta}_{k,\nu,\rho}\left(wt^{\frac{\lambda}{k}}\right)\right)dt &= \sum_{n=0}^{\infty}\frac{(\delta)_{n,k}}{\Gamma_k(\nu n+\rho)n!}(w)^n\int_0^1 t^{a+\frac{\lambda n}{k}-1}(1-t)^{b-1}\,dt\\ &= \frac{\Gamma(b)k^{1-\frac{\rho}{k}}}{\Gamma(\delta)}\sum_{n=0}^{\infty}\frac{\Gamma(\frac{\delta}{k}+n)\Gamma(a+\frac{\lambda n}{k})}{\Gamma(a+b+\frac{\lambda n}{k})\Gamma(\frac{\nu n}{k}+\frac{\rho}{k})n!}\left(k^{1-\frac{\nu}{k}}w\right)^n\\ &= \frac{\Gamma(b)k^{1-\frac{\rho}{k}}}{\Gamma(\delta)}\,{}_2\Psi_2\left[\left.\begin{matrix}(\frac{\delta}{k},1),(a,\frac{\lambda}{k})\\(\frac{\rho}{k},\frac{\nu}{k}),(a+b,\frac{\lambda}{k})\end{matrix}\right| k^{1-\frac{\nu}{k}}w\right].\end{aligned}$$

□

Theorem 9 *Let $a,c,\rho,\nu,\lambda\in\mathbb{C}$, $w\in\mathbb{R}$, and $k\in\mathbb{R}^+$. Then the Euler k-transformation for the generalized Mittag Leffler k-function is*

$$\left(\frac{1}{k}I_0^1 t^{\frac{a}{k}-1}(1-t)^{\frac{b}{k}-1}E^{\delta}_{k,\nu,\rho}\left(wt^{\frac{\lambda}{k}}\right)\right)dt = \frac{\Gamma_k(b)}{\Gamma_k(\delta)}\,{}_2\Psi_2^k\left[\left.\begin{matrix}(\delta,k),(a,\lambda)\\(a+b,\lambda),(\rho,\nu)\end{matrix}\right| w\right].$$

Proof

$$\begin{aligned}\left(I_{0,k}^1 t^{\frac{a}{k}-1}(1-t)^{\frac{b}{k}-1}E^{\delta}_{k,\nu,\rho}\left(wt^{\frac{\lambda}{k}}\right)\right)dt &= \sum_{n=0}^{\infty}\frac{(\delta)_{n,k}}{\Gamma_k(\nu n+\rho)n!}\frac{1}{k}\int_0^1 t^{\frac{a}{k}-1}(1-t)^{\frac{b}{k}-1}\left(wt^{\frac{\lambda}{k}}\right)^n\,dt.\\ &= \sum_{n=0}^{\infty}\frac{(\delta)_{n,k}w^n}{\Gamma(\nu n+\rho)n!}\frac{1}{k}\int_0^1 (t)^{\frac{a+\lambda n}{k}-1}(1-t)^{\frac{b}{k}-1}\,dt.\\ &= \frac{\Gamma_k(b)}{\Gamma_k(\delta)}\sum_{n=0}^{\infty}\frac{\Gamma_k(\delta+nk)\Gamma_k(a+\lambda n)}{\Gamma_k(a+b+\lambda n)\Gamma_k(\nu n+\rho)n!}(w)^n.\\ &= \frac{\Gamma_k(b)}{\Gamma_k(\delta)}\,{}_2\Psi_2^k\left[\left.\begin{matrix}(a,\lambda),(\delta,k)\\(a+b,\lambda),(\rho,\nu)\end{matrix}\right| w\right].\end{aligned}$$

□

5 Conclusion

In this paper, we have discussed two integral transforms involving the Gauss hypergeometric functions as their kernels. We have proved some composition formulae for these generalized fractional integrals with the Mittag-Leffler k-function. The results have been established in terms of the generalized Wright hypergeometric function. We have also developed the Euler integral k-transformation for the Mittag-Leffler k-function. Furthermore, if we take $k = 1$, then we find out the classical results.

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