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# Convergence of solutions for functional integro-differential equations with nonlinear boundary conditions

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**Abstract**

This paper is concerned with the convergence of solutions for a class of functional integro-differential equations with nonlinear boundary conditions. New comparison principles are obtained. By using the comparison principles and quasilinearization method, we present two monotone iterative sequences uniformly and monotonically converging to the unique solution with rate of order 2. Meanwhile, an example is given to demonstrate applications of the result reported.

**MSC:** 34B15

**Keywords:** Functional integro-differential equations; Monotone iterative; Quasilinearization; Nonlinear boundary conditions; Coupled lower and upper solutions

## 1 Introduction

Integro-differential equations are widely used in many fields such as control theory, biology, and mechanics, and the qualitative theory of integro-differential equations creates an important branch of nonlinear analysis; see, for instance, the monographs [2, 6, 7] and the papers [5, 10–12, 16, 24, 26, 31, 33, 34]. For the results of existence of solutions and existence of extremal solutions for such equations under different boundary conditions, we refer the reader to the monographs by Guo et al. [13] and Lakshmikantham and Rama Mohana Rao [23], the related literature for integro-differential equations [1, 4, 8, 15, 27, 30], and for functional integro-differential equations [3, 14, 17, 20, 21, 28, 35, 36], and the references cited therein.

Recently, various classes of differential/difference equations with nonlinear boundary conditions have attracted extensive attention of researchers. For instance, Franco et al. [9] discussed the existence conditions of solutions for first-order differential equations with nonlinear boundary conditions; Jankowski [19] obtained the existence conditions of first-order advanced differential equations with nonlinear boundary conditions; Mahdavi [29] investigated the nonlinear boundary value problems involving abstract Volterra operators; Wang et al. [36] presented the existence conditions of extreme solutions for first-order functional difference equations with nonlinear boundary conditions; Wang et al. [37] proved uniform convergence approximate solutions for second-order functional differential equations with periodic boundary conditions; Wang [38] and Wang and Tian [39] established the existence conditions of extreme solutions for causal differential equations and impulsive differential equations with causal operators, respectively. However, we noticed that the previous studies mostly focused on the existence of solutions and extremal solutions as well as the uniform convergence approximate solutions via the method of upper and lower solutions coupled with the monotone iterative technique; see [22, 31]. There are few results of rapid convergence for integro-differential equations with nonlinear boundary conditions. From the perspective of application, the convergence rate of the solution is both important and meaningful. In [18], by using the quasilinearization method [25], Jankowski obtained the quadratic approximation of solutions for differential equations with nonlinear boundary conditions. In [32], Sun et al. presented quadratic approximation of solutions for boundary value problems with nonlocal boundary conditions. Inspired by [18, 25, 32], in this paper, we consider the following functional integro-differential equation with nonlinear boundary conditions:

$$\begin{cases} x'(t) = f(t, x(t), x(\theta(t)), (Sx)(t)), & t \in J, \\ 0 = g(x(0), x(T)), \end{cases} \tag{1.1}$$

where $f \in C(J \times R^3, R)$, $g \in C(R \times R, R)$, $J = [0, T]$, $\theta \in C(J, R_+)$, $\theta(t) \le t$, $(Sx)(t) = \int_0^t k(t,s)x(s)\,ds$, $k \in C(D, R_+)$, $D = \{(t,s) \in J \times J : 0 \le s \le t, t \in J\}$, $k_0 = \max_{(t,s)\in D} k(t,s)$.

The aim of this paper is to investigate the problem of the convergence of solutions for Eq. (1.1). By employing the comparison principles and the quasilinearization method, we obtain two monotone sequences of iterates converging uniformly and quadratically to the unique solution of the problem. Meanwhile, an example is given to demonstrate applications of the result established. Equation (1.1) contains many special types. In addition, the nonlinear boundary conditions of Eq. (1.1) contain a lot of special types. For instance, Eq. (1.1) can be reduced to initial value problems for $g(x(0), x(T)) = x(0) - c$, that is, $x(0) = c$; Eq. (1.1) reduces to anti-periodic boundary value problems for $g(x(0), x(T)) = x(0) + x(T)$, that is, $x(0) = -x(T)$.

## 2 Preliminaries

We introduce the following definitions and lemmas which are used throughout this paper.

**Definition 2.1** We say that a function $\alpha \in C^1(J, R)$ is a lower solution of Eq. (1.1) if

$$\begin{cases} \alpha'(t) \le f(t, \alpha(t), \alpha(\theta(t)), (S\alpha)(t)), & t \in J, \\ g(\alpha(0), \alpha(T)) \le 0. \end{cases}$$

**Definition 2.2** We say that a function $\beta \in C^1(J, R)$ is an upper solution of Eq. (1.1) if

$$\begin{cases} \beta'(t) \ge f(t, \beta(t), \beta(\theta(t)), (S\beta)(t)), & t \in J, \\ g(\beta(0), \beta(T)) \ge 0. \end{cases}$$

**Definition 2.3** We say that the functions $\alpha, \beta \in C^1(J,R)$ are coupled quasisolutions of Eq. (1.1) if

$$\begin{cases} \alpha'(t) = f(t, \alpha(t), \alpha(\theta(t)), (S\alpha)(t)), & t \in J, \\ g(\alpha(0), \beta(T)) = 0, \end{cases}$$

or

$$\begin{cases} \beta'(t) = f(t, \beta(t), \beta(\theta(t)), (S\beta)(t)), & t \in J, \\ g(\beta(0), \alpha(T)) = 0. \end{cases}$$

**Definition 2.4** We say that the functions $\alpha, \beta \in C^1(J,R)$ are coupled lower solution and upper solution of Eq. (1.1) if

$$\begin{cases} \alpha'(t) \le f(t, \alpha(t), \alpha(\theta(t)), (S\alpha)(t)), & t \in J, \\ g(\alpha(0), \beta(T)) \le 0, \end{cases} \tag{2.1}$$

and

$$\begin{cases} \beta'(t) \ge f(t, \beta(t), \beta(\theta(t)), (S\beta)(t)), & t \in J, \\ g(\beta(0), \alpha(T)) \ge 0, \end{cases} \tag{2.2}$$

respectively.

**Lemma 2.1** *Assume that the following condition holds.*
$(H_{2.1})$ *There exist integrable functions $h_i(t) < 0$, $i = 1,2,3$ such that*

$$\int_0^T \bigl\{h_1(t) + h_2(t) + k_0 T h_3(t)\bigr\}\,dt \ge -1. \tag{2.3}$$

*If there exists a function $p \in C^1(J,R)$ such that*

$$\begin{cases} p'(t) \le h_1(t)p(t) + h_2(t)p(\theta(t)) + h_3(t)(Sp)(t), & t \in J, \\ p(0) \le 0, \end{cases} \tag{2.4}$$

*then $p(t) \le 0$.*

*Proof* Suppose, to the contrary, that there exists a $t^* \in (0,T]$ such that $p(t^*) > 0$. Let $t_* \in [0,t^*]$ be such that $p(t_*) = \inf p(t) = -b$, $b \ge 0$. By virtue of (2.4), we have

$$p'(t) \le -b\bigl[h_1(t) + h_2(t) + k_0 T h_3(t)\bigr].$$

Integrating both sides of the above inequality, we get

$$p\bigl(t^*\bigr) - p(t_*) \le -b \int_{t_*}^{t^*} \bigl\{h_1(t) + h_2(t) + k_0 T h_3(t)\bigr\}\,dt,$$

and so

$$0 < p(t^*) \le p(t_*) - b \int_{t_*}^{t^*} \{h_1(t) + h_2(t) + k_0 T h_3(t)\} \, dt.$$

Furthermore, we obtain

$$b < -b \int_0^T \{h_1(t) + h_2(t) + k_0 T h_3(t)\} \, dt, \tag{2.5}$$

which contradicts (2.3), and thus $p(t) \le 0$. This completes the proof. □

**Lemma 2.2** *Assume that condition* $(H_{2.1})$ *is satisfied. If there exist functions* $p \in C^1(J, R)$ *and* $q \in C^1(J, R)$ *such that*

$$\begin{cases} p'(t) \le h_1(t)p(t) + h_2(t)p(\theta(t)) + h_3(t)(Sp)(t), & p(0) \le Aq(T), t \in J, \\ q'(t) \le h_1(t)q(t) + h_2(t)q(\theta(t)) + h_3(t)(Sq)(t), & q(0) \le Ap(T), t \in J, \end{cases} \tag{2.6}$$

*where* $0 < A \le 1$ *is a constant, then* $p(t) \le 0$ *and* $q(t) \le 0$, $t \in J$.

*Proof* Without loss of generality, we prove one case of $p(t) \le 0$. Suppose that the conclusion is not true. We consider the following two cases where $p(0) \le 0$ and $p(0) > 0$, respectively.

*Case 1*. Let $p(0) \le 0$. As in the proof of Lemma 2.1, we arrive at (2.5), which contradicts (2.3).

*Case 2*. Let $p(0) > 0$. There are two cases where for all $t \in J$, $p(t) > 0$ and there exist $\bar{t}$, $\underline{t}$ such that $p(\underline{t}) \le 0$, $p(\bar{t}) > 0$, respectively.

*Case 2.1*. When $p(t) > 0$, $t \in J$, if $q(0) \le 0$, Lemma 2.1 implies that $q(T) \le 0$, then we have $p(0) \le Aq(T) \le 0$, which is a contradiction.

If $q(0) > 0$, by $0 < p(0) \le Aq(T)$, then $q(T) > 0$. Hence, there are two cases where for all $t \in J$, $q(t) > 0$ and there exist $\tilde{t}$, $\hat{t}$ such that $q(\tilde{t}) \le 0$, $q(\hat{t}) > 0$, respectively.

*Case 2.1.1*. First, when $q(t) > 0$ for all $t \in J$, condition (2.6) implies that $q'(t) < 0$, and so $q$ is decreasing. The inequalities $p(t) > 0$ and (2.6) yield $p'(t) < 0$, and hence $p$ is decreasing and $A^2 q(0) > A^2 q(T) > Ap(0) > Ap(T) > q(0)$, which is a contradiction.

*Case 2.1.2*. Then there exist $\tilde{t}$, $\hat{t}$ such that $q(\tilde{t}) \le 0$, $q(\hat{t}) > 0$, and we obtain $q(\tilde{t}_*) = \inf q(t) = -b$, where $\tilde{t}_* \in (0, T)$, $b \ge 0$. Condition (2.6) implies that

$$q'(t) \le -b[h_1(t) + h_2(t) + k_0 T h_3(t)].$$

Integrating the above inequality from $\tilde{t}_*$ to $T$, we deduce that

$$0 < q(T) \le q(\tilde{t}_*) - b \int_{\tilde{t}_*}^{T} \{h_1(t) + h_2(t) + k_0 T h_3(t)\} \, dt,$$

and thus

$$b < -b \int_0^T \{h_1(t) + h_2(t) + k_0 T h_3(t)\} \, dt,$$

which is a contradiction.

*Case 2.2.* Next, we consider the following case, there exist $\bar{t}$, $\underline{t}$ such that $p(\underline{t}) \le 0$, $p(\bar{t}) > 0$. If $q(0) \le 0$, Lemma 2.1 yields $q(t) \le 0$, then $p(0) \le Aq(T) \le 0$, it is a contradiction.

If $q(0) > 0$ and $0 < p(0) \le Aq(T)$, then $q(T) > 0$, and so $q(\bar{t}_*) = \inf q(t) = -b$, where $\bar{t}_* \in (0,T)$, $b \ge 0$. Similarly to the proof of Case 2.1.2, we can get a contradiction. Therefore, we conclude that $p(t) \le 0$.

Similarly, the case of $q(t) \le 0$ can be proved. The proof is complete. □

*Remark* 2.1 When $p(T) = q(T)$, Lemma 2.2 is also valid.

**Lemma 2.3** *Assume that condition* $(H_{2.1})$ *is satisfied and*

$$f\bigl(t,\bar{\gamma},\bar{\gamma}(\theta),T\bar{\gamma}\bigr) - f\bigl(t,\gamma,\gamma(\theta),T\gamma\bigr) \ge h_1(t)[\bar{\gamma}-\gamma] + h_2(t)\bigl[\bar{\gamma}(\theta)-\gamma(\theta)\bigr] + h_3(t)\bigl[(S\bar{\gamma})(t)-(S\gamma)(t)\bigr], \tag{2.7}$$

*where* $\gamma(t) \le \bar{\gamma}(t)$, $\gamma(\theta) \le \bar{\gamma}(\theta)$, $(S\gamma)(t) \le (S\bar{\gamma})(t)$, $t \in J$, $h_i(t) < 0$, $i = 1,2,3$.

$(H_{2.2})$ *There exist two constants* $0 < N_2 < N_1$ *such that*

$$g(\bar{\alpha},\beta) - g(\alpha,\bar{\beta}) \le N_1[\bar{\alpha}-\alpha] - N_2[\bar{\beta}-\beta], \tag{2.8}$$

*where* $\alpha(0) \le \bar{\alpha}(0)$, $\beta(T) \le \bar{\beta}(T)$.

*Moreover, let* $u, v \in C^1(J,R)$ *be coupled lower and upper solutions of problem* (1.1). *If* $y, z \in C^1(J,R)$ *and*

$$\begin{cases} y'(t) = f(t,u(t),u(\theta(t)),(Su)(t)) + h_1(t)[y(t)-u(t)] + h_2(t)[y(\theta)-u(\theta)] \\ \qquad + h_3(t)[(Sy)(t)-(Su)(t)], \quad t \in J, \\ 0 = g(u(0),v(T)) + N_1[y(0)-u(0)] + N_2[z(T)-v(T)], \end{cases} \tag{2.9}$$

$$\begin{cases} z'(t) = f(t,v(t),v(\theta(t)),(Sv)(t)) + h_1(t)[z(t)-v(t)] + h_2(t)[z(\theta)-v(\theta)] \\ \qquad + h_3(t)[(Sz)(t)-(Sv)(t)], \quad t \in J, \\ 0 = g(v(0),u(T)) + N_1[z(0)-v(0)] + N_2[y(T)-u(T)], \end{cases} \tag{2.10}$$

*then* $u(t) \le y(t) \le z(t) \le v(t)$, *and* $y$, $z$ *are coupled lower and upper solutions of* (1.1), *respectively.*

*Proof* First, we need to prove that the inequalities $u(t) \le y(t)$ and $z(t) \le v(t)$ hold. Set

$$p(t) = u(t) - y(t) \quad \text{and} \quad q(t) = z(t) - v(t).$$

This yields

$$\begin{aligned} p'(t) &\le f\bigl(t,u(t),u(\theta(t)),(Su)(t)\bigr) - f\bigl(t,u(t),u(\theta(t)),(Su)(t)\bigr) \\ &\quad - h_1(t)\bigl[y(t)-u(t)\bigr] - h_2(t)\bigl[y(\theta)-u(\theta)\bigr] - h_3(t)\bigl[(Sy)(t)-(Su)(t)\bigr] \\ &= h_1(t)p(t) + h_2(t)p(\theta) + h_3(t)(Sp)(t). \end{aligned}$$

Condition (2.9) implies that $0 = g(u(0),v(T)) - N_1p(0) + N_2q(T)$. By virtue of (2.1), we obtain $p(0) \le (N_2/N_1)q(T)$.

Similarly, we can conclude that

$$\begin{cases} q'(t) \le h_1(t)q(t) + h_2(t)q(\theta) + h_3(t)(Sq)(t), & t \in J, \\ q(0) \le \frac{N_2}{N_1} p(T). \end{cases}$$

It follows from Lemma 2.2 that $p(t) \le 0$, $q(t) \le 0$, that is, $u(t) \le y(t)$, $z(t) \le v(t)$.

Now, we prove that $y(t) \le z(t)$. Letting $m(t) = y(t) - z(t)$, we have

$$\begin{aligned} m'(t) \le{}& f\bigl(t, u(t), u\bigl(\theta(t)\bigr), (Su)(t)\bigr) - f\bigl(t, v(t), v\bigl(\theta(t)\bigr), (Sv)(t)\bigr) \\ &+ h_1(t)\bigl[y(t) - u(t)\bigr] + h_2(t)\bigl[y(\theta) - u(\theta)\bigr] + h_3(t)\bigl[(Sy)(t) - (Su)(t)\bigr] \\ &- h_1(t)\bigl[z(t) - v(t)\bigr] - h_2(t)\bigl[z(\theta) - v(\theta)\bigr] - h_3(t)\bigl[(Sz)(t) - (Sv)(t)\bigr] \\ \le{}& -h_1(t)\bigl[v(t) - u(t)\bigr] - h_2(t)\bigl[v(\theta) - u(\theta)\bigr] - h_3(t)\bigl[(Sv)(t) - (Su)(t)\bigr] \\ &+ h_1(t)\bigl[v(t) - u(t) + m(t)\bigr] + h_2(t)\bigl[v(\theta) - u(\theta) + m(\theta)\bigr] \\ &+ h_3(t)\bigl[(Sv)(t) - (Su)(t) + (Sm)(t)\bigr] \\ ={}& h_1(t)m(t) + h_2(t)m(\theta) + h_3(t)(Sm)(t). \end{aligned}$$

In view of (2.9) and (2.10), we arrive at

$$\begin{aligned} 0 ={}& g\bigl(u(0), v(T)\bigr) - g\bigl(v(0), u(T)\bigr) + N_1\bigl[y(0) - u(0)\bigr] + N_2\bigl[z(T) - v(T)\bigr] \\ &- N_1\bigl[z(0) - v(0)\bigr] - N_2\bigl[y(T) - u(T)\bigr] \\ \ge{}& -N_1\bigl[v(0) - u(0)\bigr] + N_2\bigl[v(T) - u(T)\bigr] + N_1\bigl[v(0) - u(0) + m(0)\bigr] \\ &+ N_2\bigl[m(T) - v(T) + u(T)\bigr] \\ ={}& N_1 m(0) - N_2 m(T), \end{aligned}$$

which finally gives $m(0) \le (N_2/N_1)m(T)$. It follows now from Lemma 2.2 that $m(t) \le 0$.

Now, we need to prove that $y$ and $z$ are coupled lower and upper solutions of Eq. (1.1). In fact, by (2.7) and (2.8), we get

$$\begin{aligned} y'(t) ={}& f\bigl(t, u(t), u\bigl(\theta(t)\bigr), (Su)(t)\bigr) - f\bigl(t, y(t), y\bigl(\theta(t)\bigr), (Sy)(t)\bigr) \\ &+ f\bigl(t, y(t), y\bigl(\theta(t)\bigr), (Sy)(t)\bigr) + h_1(t)\bigl[y(t) - u(t)\bigr] \\ &+ h_2(t)\bigl[y(\theta) - u(\theta)\bigr] + h_3(t)\bigl[(Sy)(t) - (Su)(t)\bigr] \\ \le{}& f\bigl(t, y(t), y\bigl(\theta(t)\bigr), (Sy)(t)\bigr) \end{aligned}$$

and

$$\begin{aligned} g\bigl(y(0), z(T)\bigr) - g\bigl(u(0), v(T)\bigr) &\le N_1\bigl[y(0) - u(0)\bigr] - N_2\bigl[v(T) - z(T)\bigr] \\ &\le g\bigl(u(0), v(T)\bigr) + N_1\bigl[y(0) - u(0)\bigr] - N_2\bigl[v(T) - z(T)\bigr] \\ &= 0. \end{aligned}$$

Similarly, we deduce that $z'(t) \ge f(t, z(t), z(\theta(t)), (Sz)(t))$ and $g(z(0), y(T)) \ge 0$. This proves that $y$ and $z$ are coupled lower and upper solutions of Eq. (1.1). □

## 3 Main result

In this section, the quadratic convergence of successive approximation sequences is proved by the quasilinearization method.

**Theorem 3.1** *Set* $\Omega = \{(t,u) \in J \times R : y_0(t) \le u(t) \le z_0(t)\}$, $\Omega_1 = [y_0(0), z_0(0)]$, *and* $\Omega_2 = [y_0(T), z_0(T)]$. *Assume that the following conditions hold.*

($A_{3.1}$) $y_0$, $z_0$ *are coupled lower and upper solutions of Eq.* (1.1), *and* $y_0(t) \le z_0(t)$ *on* $J$;

($A_{3.2}$) $f_x \in C(\Omega, R)$, $g_x, g_y \in C(\Omega_1 \times \Omega_2, R)$, $f_x < 0$, $f_y < 0$, $f_z < 0$, $0 < g_y < g_x < 1$;

($A_{3.3}$) $f_{xx}, f_{xy}, f_{yy} \in C(\Omega, R)$, $g_{xx}, g_{xy}, g_{yy} \in C(\Omega_1 \times \Omega_2, R)$, $f_{xx} \ge 0$, $f_{xy} \ge 0$, $f_{xz} \ge 0$, $f_{yy} \ge 0$, $f_{zz} \ge 0$, $f_{yz} \ge 0$, $g_{xx} \le 0$, $g_{xy} \le 0$, $g_{yy} \le 0$.

*If*

$$\begin{aligned} &\int_0^T \bigl\{f_x\bigl(t, y_0, y_0(\theta), (Sy_0)(t)\bigr) + f_y\bigl(t, y_0, y_0(\theta), (Sy_0)(t)\bigr) + k_0 T f_z\bigl(t, y_0, y_0(\theta), (Sy_0)(t)\bigr)\bigr\}\, dt \\ &\quad \ge -1, \end{aligned} \tag{3.1}$$

*then there exist the monotone sequences* $\{y_n(t)\}$ *and* $\{z_n(t)\}$ *converging uniformly to the unique solution* $x$ *of Eq.* (1.1) *and the convergence is quadratic, that is,*

$$\begin{aligned} &\max_{t\in J} \bigl|x(t) - y_{n+1}(t)\bigr| \le d_1 \max_{t\in J} \bigl|x(t) - y_n(t)\bigr|^2 + d_2 \max_{t\in J} \bigl|x(t) - z_n(t)\bigr|^2, \\ &\max_{t\in J} \bigl|x(t) - z_{n+1}(t)\bigr| \le d_3 \max_{t\in J} \bigl|x(t) - y_n(t)\bigr|^2 + d_4 \max_{t\in J} \bigl|x(t) - z_n(t)\bigr|^2, \end{aligned}$$

*where the coefficients* $d_1$, $d_2$, $d_3$, *and* $d_4$ *are nonnegative constants.*

*Proof* Consider the following problems:

$$\begin{cases} y'_{n+1}(t) = f(t, y_n(t), y_n(\theta(t)), (Sy_n)(t)) + f_x(t, y_n(t), y_n(\theta(t)), (Sy_n)(t))[y_{n+1}(t) \\ \qquad - y_n(t)] + f_y(t, y_n(t), y_n(\theta(t)), (Sy_n)(t))[y_{n+1}(\theta) - y_n(\theta)] \\ \qquad + f_z(t, y_n(t), y_n(\theta(t)), (Sy_n)(t))[(Sy_{n+1})(t) - (Sy_n)(t)], \quad t \in J, \\ 0 = g(y_n(0), z_n(T)) + g_x(y_n(0), y_n(T))[y_{n+1}(0) - y_n(0)] \\ \quad + g_y(y_n(0), z_n(T))[z_{n+1}(T) - z_n(T)], \end{cases} \tag{3.2}$$

$$\begin{cases} z'_{n+1}(t) = f(t, z_n(t), z_n(\theta(t)), (Sz_n)(t)) + f_x(t, y_n(t), y_n(\theta(t)), (Sy_n)(t))[z_{n+1}(t) \\ \qquad - z_n(t)] + f_y(t, y_n(t), y_n(\theta(t)), (Sy_n)(t))[z_{n+1}(\theta) - z_n(\theta)] \\ \qquad + f_z(t, y_n(t), y_n(\theta(t)), (Sy_n)(t))[(Sz_{n+1})(t) - (Sz_n)(t)], \quad t \in J, \\ 0 = g(z_n(0), y_n(T)) + g_x(y_n(0), y_n(T))[z_{n+1}(0) - z_n(0)] \\ \quad + g_y(y_n(0), z_n(T))[y_{n+1}(T) - y_n(T)], \end{cases} \tag{3.3}$$

in which $n = 0, 1, \ldots$. By the mean value theorem, we conclude that

$$\begin{aligned} g(\bar\alpha, \beta) - g(\alpha, \bar\beta) &= g(\bar\alpha, \beta) - g(\alpha, \beta) + g(\alpha, \beta) - g(\alpha, \bar\beta) \\ &\le g_x(\delta_1, \beta)[\bar\alpha - \alpha] - g_y(\alpha, \delta_2)[\bar\beta - \beta], \end{aligned}$$

where $\alpha(0) \le \delta_1 \le \bar{\alpha}(0)$, $\beta(T) \le \delta_2 \le \bar{\beta}(T)$. Note that

$$\begin{aligned}
&f\bigl(t,\bar{\gamma}(t),\bar{\gamma}(\theta),(S\bar{\gamma})(t)\bigr)-f\bigl(t,\gamma(t),\gamma(\theta),(S\gamma)(t)\bigr)\\
&\quad=f_x\bigl(t,\delta_3,\bar{\gamma}(\theta),(S\bar{\gamma})(t)\bigr)\bigl[\bar{\gamma}(t)-\gamma(t)\bigr]+f_y\bigl(t,\gamma(t),\delta_4,(S\bar{\gamma})(t)\bigr)\bigl[\bar{\gamma}(\theta)-\gamma(\theta)\bigr]\\
&\qquad+f_z\bigl(t,\gamma(t),\gamma(\theta),\delta_5\bigr)\bigl[(S\bar{\gamma})(t)-(S\gamma)(t)\bigr]\\
&\quad\ge f_x\bigl(t,\gamma(t),\gamma(\theta),(S\gamma)(t)\bigr)\bigl[\bar{\gamma}(t)-\gamma(t)\bigr]+f_y\bigl(t,\gamma(t),\gamma(\theta),(S\gamma)(t)\bigr)\bigl[\bar{\gamma}(\theta)-\gamma(\theta)\bigr]\\
&\qquad+f_z\bigl(t,\gamma(t),\gamma(\theta),(S\gamma)(t)\bigr)\bigl[(S\bar{\gamma})(t)-(S\gamma)(t)\bigr],
\end{aligned}$$

where $\gamma(t) \le \delta_3 \le \bar{\gamma}(t)$, $\gamma(\theta) \le \delta_4 \le \bar{\gamma}(\theta)$, $(S\gamma)(t) \le \delta_5 \le (S\bar{\gamma})(t)$, and

$$\begin{aligned}
&\int_0^T \bigl\{f_x\bigl(t,z_n,z_n(\theta),(Sz_n)(t)\bigr)+f_y\bigl(t,z_n,z_n(\theta),(Sz_n)(t)\bigr)+k_0Tf_z\bigl(t,z_n,z_n(\theta),(Sz_n)(t)\bigr)\bigr\}\,dt\\
&\quad\ge\int_0^T \bigl\{f_x\bigl(t,y_n,y_n(\theta),(Sy_n)(t)\bigr)+f_y\bigl(t,y_n,y_n(\theta),(Sy_n)(t)\bigr)\\
&\qquad+k_0Tf_z\bigl(t,y_n,y_n(\theta),(Sy_n)(t)\bigr)\bigr\}\,dt\\
&\quad\ge\int_0^T \bigl\{f_x\bigl(t,y_0,y_0(\theta),(Sy_0)(t)\bigr)+f_y\bigl(t,y_0,y_0(\theta),(Sy_0)(t)\bigr)\\
&\qquad+k_0Tf_z\bigl(t,y_0,y_0(\theta),(Sy_0)(t)\bigr)\bigr\}\,dt\\
&\quad\ge-1.
\end{aligned}$$

Using Lemma 2.3 and mathematical induction, we can deduce that

$$y_0(t)\le y_1(t)\le\cdots\le y_n(t)\le z_n(t)\le\cdots\le z_1(t)\le z_0(t),\quad n=0,1,\ldots,t\in J.$$

Thus, the sequences $\{y_n\}$ and $\{z_n\}$ are uniformly bounded and equicontinuous on $J$.

$$\begin{aligned}
\bigl|y_n(t)-y_n(s)\bigr|&=\biggl|y_n(0)+\int_0^t \bigl\{f\bigl(\varphi,y_n(\varphi),y_n\bigl(\theta(\varphi)\bigr),(Sy_n)(\varphi)\bigr)\\
&\quad+f_x\bigl(\varphi,y_n(\varphi),y_n\bigl(\theta(\varphi)\bigr),(Sy_n)(\varphi)\bigr)\bigl[y_{n+1}(\varphi)\\
&\quad-y_n(\varphi)\bigr]+f_y\bigl(\varphi,y_n(\varphi),y_n\bigl(\theta(\varphi)\bigr),(Sy_n)(\varphi)\bigr)\bigl[y_{n+1}(\theta)-y_n(\theta)\bigr]\\
&\quad+f_z\bigl(\varphi,y_n(\varphi),y_n\bigl(\theta(\varphi)\bigr),(Sy_n)(\varphi)\bigr)\bigl[(Sy_{n+1})(\varphi)-(Sy_n)(\varphi)\bigr]\bigr\}\,d\varphi\\
&\quad-\biggl\{y_n(0)+\int_0^s \bigl\{f\bigl(\varphi,y_n(\varphi),y_n\bigl(\theta(\varphi)\bigr),(Sy_n)(\varphi)\bigr)\\
&\quad+f_x\bigl(\varphi,y_n(\varphi),y_n\bigl(\theta(\varphi)\bigr),(Sy_n)(\varphi)\bigr)\bigl[y_{n+1}(\varphi)\\
&\quad-y_n(\varphi)\bigr]+f_y\bigl(\varphi,y_n(\varphi),y_n\bigl(\theta(\varphi)\bigr),(Sy_n)(\varphi)\bigr)\bigl[y_{n+1}(\theta)-y_n(\theta)\bigr]\\
&\quad+f_z\bigl(\varphi,y_n(\varphi),y_n\bigl(\theta(\varphi)\bigr),(Sy_n)(\varphi)\bigr)\bigl[(Sy_{n+1})(\varphi)-(Sy_n)(\varphi)\bigr]\bigr\}\,d\varphi\biggr\}\biggr|\\
&=\biggl|\int_s^t \bigl\{f\bigl(\varphi,y_n(\varphi),y_n\bigl(\theta(\varphi)\bigr),(Sy_n)(\varphi)\bigr)\\
&\quad+f_x\bigl(\varphi,y_n(\varphi),y_n\bigl(\theta(\varphi)\bigr),(Sy_n)(\varphi)\bigr)\bigl[y_{n+1}(\varphi)
\end{aligned}$$

$$
\begin{aligned}
&- y_n(\varphi)\big] + f_y\big(\varphi, y_n(\varphi), y_n\big(\theta(\varphi)\big), (Sy_n)(\varphi)\big)\big[y_{n+1}(\theta) - y_n(\theta)\big] \\
&+ f_z\big(\varphi, y_n(\varphi), y_n\big(\theta(\varphi)\big), (Sy_n)(\varphi)\big)\big[(Sy_{n+1})(\varphi) - (Sy_n)(\varphi)\big]\big\}\, d\varphi\bigg| \\
\le{}& \int_s^t \big|f\big(\varphi, y_n(\varphi), y_n\big(\theta(\varphi)\big), (Sy_n)(\varphi)\big) \\
&+ f_x\big(\varphi, y_n(\varphi), y_n\big(\theta(\varphi)\big), (Sy_n)(\varphi)\big)\big[y_{n+1}(\varphi) \\
&- y_n(\varphi)\big] + f_y\big(\varphi, y_n(\varphi), y_n\big(\theta(\varphi)\big), (Sy_n)(\varphi)\big)\big[y_{n+1}(\theta) - y_n(\theta)\big] \\
&+ f_z\big(\varphi, y_n(\varphi), y_n\big(\theta(\varphi)\big), (Sy_n)(\varphi)\big)\big[(Sy_{n+1})(\varphi) - (Sy_n)(\varphi)\big]\big|\, d\varphi \\
\le{}& M|t-s|.
\end{aligned}
$$

By virtue of Arzelà–Ascoli theorem, there exist the subsequences $\{y_{n_k}\}$ and $\{z_{n_k}\}$ converging uniformly on $J$ to some continuous functions $y$ and $z$, respectively, and

$$
\begin{cases}
y'_{n_{k+1}}(t) = f(t, y_{n_k}(t), y_{n_k}(\theta(t)), (Sy_{n_k})(t)) + f_x(t, y_{n_k}(t), y_{n_k}(\theta(t)), (Sy_{n_k})(t))[y_{n_{k+1}}(t) \\
\qquad - y_{n_k}(t)] + f_y(t, y_{n_k}(t), y_{n_k}(\theta(t)), (Sy_{n_k})(t))[y_{n_{k+1}}(\theta) - y_{n_k}(\theta)] \\
\qquad + f_z(t, y_{n_k}(t), y_{n_k}(\theta(t)), (Sy_{n_k})(t))[(Sy_{n_{k+1}})(t) - (Sy_{n_k})(t)], \quad t \in J, \\
0 = g(y_{n_k}(0), z_{n_k}(T)) + g_x(y_{n_k}(0), y_{n_k}(T))[y_{n_{k+1}}(0) - y_{n_k}(0)] \\
\quad + g_y(y_{n_k}(0), z_{n_k}(T))[z_{n_{k+1}}(T) - z_{n_k}(T)],
\end{cases}
$$

$$
\begin{cases}
z'_{n_{k+1}}(t) = f(t, z_{n_k}(t), z_{n_k}(\theta(t)), (Sz_{n_k})(t)) + f_x(t, y_{n_k}(t), y_{n_k}(\theta(t)), (Sy_{n_k})(t))[z_{n_{k+1}}(t) \\
\qquad - z_{n_k}(t)] + f_y(t, y_{n_k}(t), y_{n_k}(\theta(t)), (Sy_{n_k})(t))[z_{n_{k+1}}(\theta) - z_{n_k}(\theta)] \\
\qquad + f_z(t, y_{n_k}(t), y_{n_k}(\theta(t)), (Sy_{n_k})(t))[(Sz_{n_{k+1}})(t) - (Sz_{n_k})(t)], \quad t \in J, \\
0 = g(z_{n_k}(0), y_{n_k}(T)) + g_x(y_{n_k}(0), y_{n_k}(T))[z_{n_{k+1}}(0) - z_{n_k}(0)] \\
\quad + g_y(y_{n_k}(0), z_{n_k}(T))[y_{n_{k+1}}(T) - y_{n_k}(T)],
\end{cases}
$$

when $n_k \to \infty$, $y$ and $z$ satisfy the equations

$$
\begin{cases}
y'(t) = f(t, y(t), y(\theta(t)), (Sy)(t)), \quad t \in J, \\
0 = g(y(0), z(T)),
\end{cases}
$$

and

$$
\begin{cases}
z'(t) = f(t, z(t), z(\theta(t)), (Sz)(t)), \quad t \in J, \\
0 = g(z(0), y(T)).
\end{cases}
$$

Thus, $y, z \in C^1(J, R)$ are coupled solutions of Eq. (1.1).

Now, we prove that $y = z$ is a unique solution of Eq. (1.1). Clearly, $y(t) \le z(t)$. Let $p(t) = z(t) - y(t)$. Then

$$
\begin{aligned}
p'(t) &= f\big(t, z(t), z\big(\theta(t)\big), (Sz)(t)\big) - f\big(t, y(t), y\big(\theta(t)\big), (Sy)(t)\big) \\
&= f_x\big(t, \xi_1, z\big(\theta(t)\big), (Sz)(t)\big)p(t) + f_y\big(t, y(t), \xi_2, (Sz)(t)\big)p(\theta) \\
&\quad + f_z\big(t, y(t), y\big(\theta(t)\big), \xi_3\big)(Sp)(t),
\end{aligned}
$$

where $y(t) \le \xi_1 \le z(t)$, $y(\theta(t)) \le \xi_2 \le z(\theta(t))$, $(Sy)(t) \le \xi_3 \le (Sz)(t)$, and

$$\begin{aligned} g\big(z(0), y(T)\big) - g\big(y(0), z(T)\big) &\le N_1\big[z(0) - y(0)\big] - N_2\big[z(T) - y(T)\big] \\ &= N_1 p(0) - N_2 p(T). \end{aligned}$$

In view of (2.1), we get $p(0) \le (N_2/N_1)p(T)$. An application of Lemma 2.2 yields $p(t) \le 0$, that is, $z(t) \le y(t)$. Hence, we have $y(t) = z(t)$.

Let $x \in [y_0, z_0]$ be any solution of Eq. (1.1). It is not difficult to prove that $y_n(t) \le x(t) \le z_n(t)$. Letting $n \to \infty$, then $y(t) = z(t) = x(t)$, it means that $\{y_{n_k}\}$ and $\{z_{n_k}\}$ converge to the unique solution $x$ of Eq. (1.1).

Finally, we prove the quadratic convergence of $\{y_n\}$ and $\{z_n\}$ to $x$. Let $p_{n+1}(t) = x(t) - y_{n+1}(t) \ge 0$ and $q_{n+1}(t) = z_{n+1}(t) - x(t) \ge 0$. Then

$$\begin{aligned} p'_{n+1}(t) = {} & f_x\big(t, \rho_1, x\big(\theta(t)\big), (Sx)(t)\big)p_n(t) + f_y\big(t, y_n(t), \rho_2, (Sx)(t)\big)p_n(\theta) \\ & + f_z\big(t, y_n(t), y_n\big(\theta(t)\big), \rho_3\big)(Sp_n)(t) \\ & - f_x\big(t, y_n(t), y_n\big(\theta(t)\big), (Sy_n)(t)\big)p_n(t) \\ & - f_y\big(t, y_n(t), y_n\big(\theta(t)\big), (Sy_n)(t)\big)p_n(\theta) \\ & - f_z\big(t, y_n(t), y_n\big(\theta(t)\big), (Sy_n)(t)\big)(Sp_n)(t) \\ & + f_x\big(t, y_n(t), y_n\big(\theta(t)\big), (Sy_n)(t)\big)p_{n+1}(t) \\ & + f_y\big(t, y_n(t), y_n\big(\theta(t)\big), (Sy_n)(t)\big)p_{n+1}(\theta) \\ & + f_z\big(t, y_n(t), y_n\big(\theta(t)\big), (Sy_n)(t)\big)(Sp_{n+1})(t) \\ \le {} & f_{xx}\big(t, \rho_4, x\big(\theta(t)\big), (Sx)(t)\big)p_n^2(t) \\ & + f_{xy}\big(t, y_n(t), \rho_5, (Sx)(t)\big)p_n(t)p_n(\theta) \\ & + f_{xz}\big(t, y_n(t), y_n\big(\theta(t)\big), \rho_6\big)p_n(t)(Sp_n)(t) \\ & + f_{yy}\big(t, y_n(t), \rho_7, (Sx)(t)\big)p_n^2(\theta) \\ & + f_{yz}\big(t, y_n(t), y_n\big(\theta(t)\big), \rho_8\big)p_n(\theta)(Sp_n)(t) \\ & + f_{zz}\big(t, y_n(t), y_n\big(\theta(t)\big), (Sy_n)(t)\big)(Sp_n)^2(t) \\ & + f_x\big(t, y_n(t), y_n\big(\theta(t)\big), (Sy_n)(t)\big)p_{n+1}(t) \\ & + f_y\big(t, y_n(t), y_n\big(\theta(t)\big), (Sy_n)(t)\big)p_{n+1}(\theta) \\ & + f_z\big(t, y_n(t), y_n\big(\theta(t)\big), (Sy_n)(t)\big)(Sp_{n+1})(t) \\ \le {} & f_{xx}\big(t, \rho_4, x\big(\theta(t)\big), (Sx)(t)\big)p_n^2(t) + \frac{1}{2}f_{xy}\big(t, y_n(t), \rho_5, (Sx)(t)\big)\big[p_n^2(t) \\ & + p_n^2(\theta)\big] + \frac{1}{2}f_{xz}\big(t, y_n(t), y_n\big(\theta(t)\big), \rho_6\big)\big[p_n^2(t) + (Sp_n)^2(t)\big] \\ & + f_{yy}\big(t, y_n(t), \rho_7, (Sx)(t)\big)p_n^2(\theta) + \frac{1}{2}f_{yz}\big(t, y_n(t), y_n\big(\theta(t)\big), \rho_8\big)\big[p_n^2(\theta) \\ & + (Sp_n)^2(t)\big] + f_{zz}\big(t, y_n(t), y_n\big(\theta(t)\big), (Sy_n)(t)\big)(Sp_n)^2(t) \\ \le {} & \bigg\{f_{xx}\big(t, \rho_4, x\big(\theta(t)\big), (Sx)(t)\big) + \frac{1}{2}f_{xy}\big(t, y_n(t), \rho_5, (Sx)(t)\big) \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}f_{xz}\bigl(t,y_n(t),y_n\bigl(\theta(t)\bigr),\rho_6\bigr)\biggr\}p_n^2(t)+\biggl\{f_{yy}\bigl(t,y_n(t),\rho_7,(Sx)(t)\bigr)\\
&+\frac{1}{2}f_{xy}\bigl(t,y_n(t),\rho_5,(Sx)(t)\bigr)+\frac{1}{2}f_{yz}\bigl(t,y_n(t),y_n\bigl(\theta(t)\bigr),\rho_8\bigr)\biggr\}p_n^2(\theta)\\
&+\biggl\{f_{zz}\bigl(t,y_n(t),y_n\bigl(\theta(t)\bigr),(Sy_n)(t)\bigr)+\frac{1}{2}f_{xz}\bigl(t,y_n(t),y_n\bigl(\theta(t)\bigr),\rho_6\bigr)\\
&+\frac{1}{2}f_{yz}\bigl(t,y_n(t),y_n\bigl(\theta(t)\bigr),\rho_8\bigr)\biggr\}(Sp_n)^2(t)\\
\le{}& D_0p_n^2(t)+D_1p_n^2(\theta)+D_2(Sp_n)^2(t),
\end{aligned}
$$

where $y_n(t)\le\rho_4\le\rho_1\le x(t)$, $y_n(\theta)\le\rho_5\le x(\theta)$, $(Sy_n)(t)\le\rho_3,\rho_6,\rho_8\le(Sx)(t)$, $y_n(\theta)\le\rho_7\le\rho_2\le x(\theta)$. Hence, we conclude that

$$
\begin{aligned}
p_{n+1}(t)&\le p_{n+1}(0)+\int_0^t\bigl\{D_0p_n^2(s)+D_1p_n^2(\theta)+D_2(Sp_n)^2(s)\bigr\}\,ds\\
&\le p_{n+1}(0)+\int_0^T\bigl\{D_0p_n^2(s)+D_1p_n^2(\theta)+D_2(Sp_n)^2(s)\bigr\}\,ds\\
&\le p_{n+1}(0)+C_0\max_{t\in J}p_n^2(t),
\end{aligned}
\tag{3.4}
$$

where $C_0=T[D_0+D_1+D_2T^2k_0^2]$. Moreover, we obtain

$$
\begin{aligned}
0={}&g\bigl(x(0),x(T)\bigr)-g\bigl(y_n(0),z_n(T)\bigr)-g_x\bigl(y_n(0),y_n(T)\bigr)\bigl[-p_{n+1}(0)+p_n(0)\bigr]\\
&-g_y\bigl(y_n(0),z_n(T)\bigr)\bigl[q_{n+1}(T)-q_n(T)\bigr]\\
={}&g_x\bigl(\delta_1,x(T)\bigr)p_n(0)-g_y\bigl(y_n(0),\delta_2\bigr)q_n(T)-g_x\bigl(y_n(0),y_n(T)\bigr)\bigl[-p_{n+1}(0)\\
&+p_n(0)\bigr]-g_y\bigl(y_n(0),z_n(T)\bigr)\bigl[q_{n+1}(T)-q_n(T)\bigr]
\end{aligned}
$$

and

$$
\begin{aligned}
&g_x\bigl(y_n(0),y_n(T)\bigr)p_{n+1}(0)\\
&\quad=\bigl[g_y\bigl(y_n(0),\delta_2\bigr)-g_y\bigl(y_n(0),z_n(T)\bigr)\bigr]q_n(T)\\
&\qquad+\bigl[g_x\bigl(y_n(0),y_n(T)\bigr)-g_x\bigl(\delta_1,x(T)\bigr)\bigr]p_n(0)+g_y\bigl(y_n(0),z_n(T)\bigr)q_{n+1}(T)\\
&\quad=-g_{yy}\bigl(y_n(0),\delta_3\bigr)q_n^2(T)-g_{xx}\bigl(\delta_4,x(T)\bigr)p_n^2(0)-g_{xy}\bigl(y_n(0),\delta_5\bigr)p_n(T)p_n(0)\\
&\qquad+g_y\bigl(y_n(0),z_n(T)\bigr)q_{n+1}(T),
\end{aligned}
$$

where $y_n(0)\le\delta_4\le\delta_1\le x(0)$, $x(T)\le\delta_2\le\delta_3\le z_n(T)$. Therefore, we deduce that

$$
\begin{aligned}
p_{n+1}(0)&\le B_0q_n^2(T)+B_1p_n^2(0)+B_2p_n^2(T)+g_y\bigl(y_n(0),z_n(T)\bigr)q_{n+1}(T)\\
&\le B_0\max_{t\in J}q_n^2(t)+B_1\max_{t\in J}p_n^2(t)+B_2\max_{t\in J}p_n^2(t)+\frac{g_y(y_n(0),z_n(T))}{g_x(y_n(0),y_n(T))}q_{n+1}(T)\\
&\le C_1\max_{t\in J}p_n^2(t)+C_2\max_{t\in J}q_n^2(t)+C_1^0q_{n+1}(T),
\end{aligned}
\tag{3.5}
$$

where

$$B_0 = -\frac{g_{yy}(y_n(0),\delta_3)}{g_x(y_n(0),y_n(T))}, \qquad B_1 = -\frac{g_{xx}(\delta_4,x(T))}{g_x(y_n(0),y_n(T))},$$
$$B_2 = -\frac{1}{2}\frac{g_{xy}(y_n(0),\delta_5)}{g_x(y_n(0),y_n(T))}, \qquad C_1 = B_1 + B_2, \qquad C_2 = B_0, \qquad C_1^0 = \frac{g_y(y_n(0),z_n(T))}{g_x(y_n(0),y_n(T))}.$$

In a similar way, we can arrive at

$$\begin{aligned}
q'_{n+1}(t) \le{}& f_{xx}\bigl(t,\xi_4,z_n\bigl(\theta(t)\bigr),(Sz_n)(t)\bigr)q_n(t)\bigl(q_n(t)+p_n(t)\bigr)\\
&+f_{xy}\bigl(t,y_n(t),\xi_5,(Sz_n)(t)\bigr)q_n(t)\bigl(q_n(\theta)+p_n(\theta)\bigr)\\
&+f_{xz}\bigl(t,y_n(t),y_n\bigl(\theta(t)\bigr),\xi_6\bigr)q_n(t)\bigl[(Sq_n)(t)+(Sp_n)(t)\bigr]\\
&+f_{yx}\bigl(t,\xi_7,\xi_2,(Sz_n)(t)\bigr)q_n(\theta)p_n(t)\\
&+f_{yy}\bigl(t,y_n(t),\xi_8,(Sz_n)(t)\bigr)q_n(\theta)\bigl[p_n(\theta)+q_n(\theta)\bigr]\\
&+f_{yz}\bigl(t,y_n(t),y_n\bigl(\theta(t)\bigr),\xi_9\bigr)q_n(\theta)\bigl[(Sp_n)(t)+(Sq_n)(t)\bigr]\\
&+f_{zx}\bigl(t,\xi_{10},x\bigl(\theta(t)\bigr),\xi_3\bigr)p_n(t)(Sq_n)(t)\\
&+f_{zy}\bigl(t,y_n(t),\xi_{11},\xi_3\bigr)p_n(\theta)(Sq_n)(t)\\
&+f_{zz}\bigl(t,y_n(t),y_n\bigl(\theta(t)\bigr),\xi_{12}\bigr)\bigl[(Sp_n)(t)+(Sq_n)(t)\bigr](Sq_n)(t)\\
\le{}& \biggl\{\frac{3}{2}f_{xx}\bigl(t,\xi_4,z_n\bigl(\theta(t)\bigr),(Sz_n)(t)\bigr)+f_{xy}\bigl(t,y_n(t),\xi_5,(Sz_n)(t)\bigr)\\
&+f_{xz}\bigl(t,y_n(t),y_n\bigl(\theta(t)\bigr),\xi_6\bigr)\biggr\}q_n^2(t)+\frac{1}{2}\bigl\{f_{xx}\bigl(t,\xi_4,z_n\bigl(\theta(t)\bigr),(Sz_n)(t)\bigr)\\
&+f_{yx}\bigl(t,\xi_7,\xi_2,(Sz_n)(t)\bigr)+f_{zx}\bigl(t,\xi_{10},x\bigl(\theta(t)\bigr),\xi_3\bigr)\bigr\}p_n^2(t)\\
&+\biggl\{\frac{1}{2}f_{xy}\bigl(t,y_n(t),\xi_5,(Sz_n)(t)\bigr)+\frac{1}{2}f_{yx}\bigl(t,\xi_7,\xi_2,(Sz_n)(t)\bigr)\\
&+\frac{3}{2}f_{yy}\bigl(t,y_n(t),\xi_8,(Sz_n)(t)\bigr)+f_{yx}\bigl(t,\xi_7,\xi_2,(Sz_n)(t)\bigr)\biggr\}q_n^2(\theta)\\
&+\frac{1}{2}\bigl\{f_{xy}\bigl(t,y_n(t),\xi_5,(Sz_n)(t)\bigr)+f_{yy}\bigl(t,y_n(t),\xi_8,(Sz_n)(t)\bigr)\\
&+f_{zy}\bigl(t,y_n(t),\xi_{11},\xi_3\bigr)\bigr\}p_n^2(\theta)+\frac{1}{2}\bigl\{f_{xz}\bigl(t,y_n(t),y_n\bigl(\theta(t)\bigr),\xi_6\bigr)\\
&+f_{yz}\bigl(t,y_n(t),y_n\bigl(\theta(t)\bigr),\xi_9\bigr)+f_{zx}\bigl(t,\xi_{10},x\bigl(\theta(t)\bigr),\xi_3\bigr)\\
&+f_{zy}\bigl(t,y_n(t),\xi_{11},\xi_3\bigr)+3f_{zz}\bigl(t,y_n(t),y_n\bigl(\theta(t)\bigr),\xi_{12}\bigr)\bigr\}(Sq_n)^2(t)\\
&+\frac{1}{2}\bigl\{f_{xz}\bigl(t,y_n(t),y_n\bigl(\theta(t)\bigr),\xi_6\bigr)+f_{yz}\bigl(t,y_n(t),y_n\bigl(\theta(t)\bigr),\xi_9\bigr)\\
&+f_{zz}\bigl(t,y_n(t),y_n\bigl(\theta(t)\bigr),\xi_{12}\bigr)\bigr\}(Sp_n)^2(t)\\
\le{}& D_3q_n^2(t)+D_4p_n^2(t)+D_5q_n^2(\theta)+D_6p_n^2(\theta)+D_7(Sq_n)^2(t)+D_8(Sp_n)^2(t)
\end{aligned}$$

and

$$\begin{aligned} q_{n+1}(t) &\le q_{n+1}(0) + \int_0^T \bigl\{D_3 q_n^2(s) + D_4 p_n^2(s) + D_5 q_n^2(\theta) + D_6 p_n^2(\theta) \\ &\quad + D_7 (Sq_n)^2(s) + D_8 (Sp_n)^2(s)\bigr\}\, ds \\ &\le q_{n+1}(0) + \max_{t\in J} p_n^2(t) T\bigl[D_4 + D_6 + D_8 T^2 k_0^2\bigr] + \max_{t\in J} q_n^2(t) T\bigl[D_3 \\ &\quad + D_5 + D_7 T^2 k_0^2\bigr] \\ &= q_{n+1}(0) + C_3 \max_{t\in J} p_n^2(t) + C_4 \max_{t\in J} q_n^2(t), \end{aligned} \tag{3.6}$$

where $C_3 = T[D_4 + D_6 + D_8 T^2 k_0^2]$, $C_4 = T[D_3 + D_5 + D_7 T^2 k_0^2]$, $y_n(t) \le \xi_4 \le \xi_1$, $y_n(\theta) \le \xi_5 \le z_n(\theta)$, $x(t) \le \xi_1 \le z_n(t)$, $(Sy_n)(t) \le \xi_6, \xi_9 \le (Sz_n)(t)$, $x(\theta) \le \xi_2 \le z_n(\theta)$, $y_n(t) \le \xi_7, \xi_{10} \le x(t)$, $(Sx)(t) \le \xi_3 \le (Sz_n)(t)$, $y_n(\theta) \le \xi_8 \le \xi_2$, $y_n(\theta) \le \xi_{11} \le x(\theta)$, $(Sy_n)(t) \le \xi_{12} \le \xi_3$. Meanwhile, we have

$$\begin{aligned} 0 &= -g\bigl(x(0), x(T)\bigr) + g\bigl(z_n(0), y_n(T)\bigr) + g_x\bigl(y_n(0), y_n(T)\bigr)\bigl[q_{n+1}(0) - q_n(0)\bigr] \\ &\quad + g_y\bigl(y_n(0), z_n(T)\bigr)\bigl[-p_{n+1}(T) + p_n(T)\bigr] \\ &= g_x\bigl(\alpha_1, y_n(T)\bigr) q_n(0) - g_y\bigl(x(0), \alpha_2\bigr) p_n(T) \\ &\quad + g_x\bigl(y_n(0), y_n(T)\bigr)\bigl[q_{n+1}(0) - q_n(0)\bigr] + g_y\bigl(y_n(0), z_n(T)\bigr)\bigl[-p_{n+1}(T) + p_n(T)\bigr] \end{aligned}$$

and

$$\begin{aligned} &g_x\bigl(y_n(0), y_n(T)\bigr) q_{n+1}(0) \\ &\quad \le -g_{xx}\bigl(\alpha_3, y_n(T)\bigr) q_n(0)\bigl[q_n(0) + p_n(0)\bigr] \\ &\qquad - g_{yy}\bigl(y(0), \alpha_5\bigr) p_n(T)\bigl[q_n(T) + p_n(T)\bigr] + g_{yx}\bigl(\alpha_4, \alpha_2(T)\bigr) p_n(T) p_n(0) \\ &\qquad + g_y\bigl(y_n(0), z_n(T)\bigr) p_{n+1}(T) \\ &\quad \le -g_{xx}\bigl(\alpha_3, y_n(T)\bigr) q_n(0)\bigl[q_n(0) + p_n(0)\bigr] - g_{yy}\bigl(y(0), \alpha_5\bigr) p_n(T)\bigl[q_n(T) + p_n(T)\bigr] \\ &\qquad + g_y\bigl(y_n(0), z_n(T)\bigr) p_{n+1}(T). \end{aligned}$$

Hence, we conclude that

$$\begin{aligned} q_{n+1}(0) &\le B_3 q_n^2(T) + B_4 q_n^2(0) + B_5 p_n^2(0) + B_6 p_n^2(T) + g_y\bigl(y_n(0), z_n(T)\bigr) p_{n+1}(T) \\ &\le B_3 \max_{t\in J} q_n^2(t) + B_4 \max_{t\in J} q_n^2(t) + B_5 \max_{t\in J} p_n^2(t) + B_6 \max_{t\in J} p_n^2(t) \\ &\quad + \frac{g_y(y_n(0), z_n(T))}{g_x(y_n(0), y_n(T))} p_{n+1}(T) \\ &\le C_5 \max_{t\in J} p_n^2(t) + C_6 \max_{t\in J} q_n^2(t) + C_2^0 p_{n+1}(T), \end{aligned} \tag{3.7}$$

where

$$
\begin{aligned}
&B_3 = -\frac{1}{2}\frac{g_{yy}(y(0),\alpha_5)}{g_x(y_n(0),y_n(T))}, \qquad B_4 = -\frac{3}{2}\frac{g_{xx}(\alpha_3,y_n(T))}{g_x(y_n(0),y_n(T))},\\
&B_5 = -\frac{1}{2}\frac{g_{xx}(\alpha_3,y_n(T))}{g_x(y_n(0),y_n(T))}, \qquad B_6 = -\frac{3}{2}\frac{g_{yy}(y(0),\alpha_5)}{g_x(y_n(0),y_n(T))},\\
&C_5 = B_5 + B_6, \qquad C_6 = B_3 + B_4, \qquad C_2^0 = \frac{g_y(y_n(0),z_n(T))}{g_x(y_n(0),y_n(T))}.
\end{aligned}
$$

It follows from (3.4)–(3.7) that

$$
\begin{aligned}
&p_{n+1}(0) \le C_1^1 p_n^2(t) + C_2^1 q_n^2(t),\\
&q_{n+1}(0) \le C_3^1 p_n^2(t) + C_4^1 q_n^2(t),
\end{aligned}
\tag{3.8}
$$

where

$$
\begin{aligned}
&C_1^1 = \frac{1}{(1-C_2^0)(1-C_1^0)}\bigl[C_1^0 C_2^0 C_0 + C_1^0 C_5 + C_1^0 C_3 + C_1\bigr],\\
&C_2^1 = \frac{1}{(1-C_2^0)(1-C_1^0)}\bigl[C_1^0 C_2 + C_1^0 C_4 + C_6\bigr],\\
&C_3^1 = C_5 + C_2^1 C_2^0 + C_0 C_2^0, \qquad C_4^1 = C_6 + C_2^1 C_2^0.
\end{aligned}
$$

By virtue of (3.4), (3.6), and (3.8), we see that

$$
\begin{aligned}
&\max_{t\in J}\bigl|x(t)-y_{n+1}(t)\bigr| \le d_1 \max_{t\in J}\bigl|x(t)-y_n(t)\bigr|^2 + d_2 \max_{t\in J}\bigl|x(t)-z_n(t)\bigr|^2,\\
&\max_{t\in J}\bigl|x(t)-z_{n+1}(t)\bigr| \le d_3 \max_{t\in J}\bigl|x(t)-y_n(t)\bigr|^2 + d_4 \max_{t\in J}\bigl|x(t)-z_n(t)\bigr|^2,
\end{aligned}
$$

where $d_1 = C_1^1 + C_0$, $d_2 = C_2^1$, $d_3 = C_3^1 + C_3$, $d_4 = C_4^1 + C_4$. This completes the proof. □

## 4 Example

To illustrate the validity of the theoretical result obtained in the previous section, we give the following example.

*Example* 4.1 Consider the boundary value problem

$$
\begin{cases}
x'(t) = \frac{1}{10}x^2(t) - \frac{1}{4}x(\frac{t}{2}) - \frac{1}{80}\int_0^t x(s)\,ds - \frac{1}{10}x(t), \quad t\in[0,1],\\
\frac{1}{2}x(0) - \frac{1}{12}x^2(1) + \frac{1}{12}x(1) + \frac{1}{4} = 0,
\end{cases}
\tag{4.1}
$$

where

$$
\begin{aligned}
&f\bigl(t,x(t),x(\theta(t)),(Sx)(t)\bigr) = \frac{1}{10}x^2(t) - \frac{1}{4}x\left(\frac{t}{2}\right) - \frac{1}{80}\int_0^t x(s)\,ds - \frac{1}{10}x(t), \quad t\in[0,1],\\
&g\bigl(x(0),x(1)\bigr) = \frac{1}{2}x(0) - \frac{1}{12}x^2(1) + \frac{1}{12}x(1) + \frac{1}{4}.
\end{aligned}
$$

Letting $y_0(t) = -1$, $z_0(t) = 0$, $t \in [0,1]$, then $y_0(t) < z_0(t)$ and

$$f\bigl(t, y_0(t), y_0\bigl(\theta(t)\bigr), (Sy_0)(t)\bigr) \geq y_0'(t), \quad t \in [0,1], \qquad g\bigl(y_0(0), z_0(1)\bigr) = -\frac{1}{4},$$
$$f\bigl(t, z_0(t), z_0\bigl(\theta(t)\bigr), (Sz_0)(t)\bigr) = 0 = z_0'(t), \quad t \in [0,1], \qquad g\bigl(z_0(0), y_0(1)\bigr) = \frac{1}{12}.$$

Thus, $y_0$ and $z_0$ are coupled lower and upper solutions of Eq. (4.1), and $y_0(t) \leq x(t) \leq z_0(t)$, that is, $-1 \leq x(t) \leq 0$. Moreover, by Eq. (4.1), we have

$$\int_0^1 \bigl\{ f_x\bigl(t, y_0, y_0(\theta), (Sy_0)(t)\bigr) + f_y\bigl(t, y_0, y_0(\theta), (Sy_0)(t)\bigr) + f_z\bigl(t, y_0, y_0(\theta), (Sy_0)(t)\bigr) k_0 T \bigr\}\, dt$$
$$= -\frac{9}{16} \geq -1.$$

It is not difficult to verify that all conditions of Theorem 3.1 are satisfied. Therefore, there exist the monotone sequences $\{y_n(t)\}$ and $\{z_n(t)\}$ converging uniformly to the unique solution $x$ of equation (4.1) and the convergence is quadratic.

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**Authors' contributions**
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**References**

1. Aftabizadeh, A.R., Leela, S.: Existence results for boundary value problems of integro-differential equations. In: Proc. Colloquia on Qualitative Theory of Differential Equations at Bolyai Institute, pp. 23–35 (1984)
2. Burton, T.A.: Volterra Integral and Differential Equations. Academic Press, New York (1983)
3. Cabada, A., Pouso, R.L., Liz, E.: A generalization of the method of upper and lower solutions for discontinuous first order problems with nonlinear boundary conditions. Appl. Math. Comput. **114**, 135–148 (2000)
4. Chen, L.J., Sun, J.T.: Nonlinear boundary value problem for first order impulsive integro-differential equations of mixed type. J. Math. Anal. Appl. **325**, 830–842 (2007)
5. Corduneanu, C.: Some control problems for abstract Volterra functional-differential equations. In: Proc. Int. Symp. MTNS-89, vol. III, pp. 331–338. Birkhäuser, Basel (1990)
6. Corduneanu, C.: Equations with Abstract Volterra Operators as Modelling Tools in Science and Engineering. Pitman Research Notes in Mathematics Series. Longman, London (1994)
7. Cushing, J.M.: Integrodifferential Equations and Delay Models in Population Dynamics. Springer, New York (1977)

8. Dhakne, M.B., Kendre, S.D.: On a nonlinear Volterra integrodifferential equation in Banach spaces. Math. Inequal. Appl. **9**, 725–738 (2006)
9. Franco, D., Nieto, J.J., O'Regan, D.: Existence of solutions for first order ordinary differential equations with nonlinear boundary conditions. Appl. Math. Comput. **153**, 793–802 (2004)
10. Grace, S.R., Graef, J.R., Tunç, E.: On the asymptotic behavior of solutions of certain integro-differential equations. J. Appl. Anal. Comput. **9**, 1305–1318 (2019)
11. Graef, J.R., Grace, S.R., Tunç, E.: On the oscillation of certain integral equations. Publ. Math. (Debr.) **90**, 195–204 (2017)
12. Graef, J.R., Grace, S.R., Tunç, E.: Asymptotic behavior of solutions of certain integro-differential equations. Panam. Math. J. **29**, 45–60 (2019)
13. Guo, D.J., Lakshmikantham, V., Liu, X.Z.: Nonlinear Integral Equations in Abstract Spaces. Kluwer Academic, Dordrecht (1996)
14. Han, J.F., Liu, Y.L., Zhao, J.: Integral boundary value problems for first order nonlinear impulsive functional integro-differential equations. Appl. Math. Comput. **218**, 5002–5009 (2012)
15. He, Z.M., He, X.M.: Monotone iterative technique for impulsive integro-differential equations with periodic boundary conditions. Comput. Math. Appl. **48**, 73–84 (2004)
16. Hristova, S., Tunç, C.: Stability of nonlinear Volterra integro-differential equations with Caputo fractional derivative and bounded delays. Electron. J. Differ. Equ. **2019**, Article ID 30 (2019)
17. Jankowski, T.: Existence of solutions of boundary value problems for differential equations with delayed argument. J. Comput. Appl. Math. **156**, 239–252 (2003)
18. Jankowski, T.: Quadratic approximation of solutions for differential equations with nonlinear boundary conditions. Comput. Math. Appl. **47**, 1619–1626 (2004)
19. Jankowski, T.: Advanced differential equations with nonlinear boundary conditions. J. Math. Anal. Appl. **304**, 490–503 (2005)
20. Jankowski, T.: First-order impulsive functional differential equations. Commun. Appl. Anal. **11**, 359–378 (2007)
21. Kwapisz, M.: Remarks on the existence and uniqueness of solutions of Volterra functional equations in $L^p$-spaces. J. Integral Equ. Appl. **3**, 383–397 (1991)
22. Ladde, G.S., Lakshmikantham, V., Vatsala, A.S.: Monotone Iterative Techniques for Nonlinear Differential Equations. Pitman, London (1985)
23. Lakshmikantham, V., Rama Mohana Rao, M.: Theory of Integro-Differential Equations. Gordon & Breach, New York (1995)
24. Lakshmikantham, V., Shahzad, N., Nieto, J.J.: Methods of generalized quasilinearization for periodic boundary value problems. Nonlinear Anal. **27**, 143–151 (1996)
25. Lakshmikantham, V., Vatsala, A.S.: Generalized Quasilinearization for Nonlinear Problems. Kluwer Academic, Dordrecht (1998)
26. Li, T., Pintus, N., Viglialoro, G.: Properties of solutions to porous medium problems with different sources and boundary conditions. Z. Angew. Math. Phys. **70**, Article ID 86 (2019)
27. Liu, X.Z., Guo, D.J.: Method of upper and lower solutions for second-order impulsive integro-differential equations in a Banach space. Comput. Math. Appl. **38**, 213–223 (1999)
28. Liu, Z.H., Liang, J.T.: A class of boundary value problems for first-order impulsive integro-differential equations with deviating arguments. J. Comput. Appl. Math. **237**, 477–486 (2013)
29. Mahdavi, M.: Nonlinear boundary value problems involving abstract Volterra operators. Libertas Math. **13**, 17–26 (1993)
30. Meehan, M., O'Regan, D.: Existence theory for nonlinear Volterra integrodifferential and integral equations. Nonlinear Anal. **31**, 317–341 (1998)
31. Nieto, J.J.: An abstract monotone iterative technique. Nonlinear Anal. **28**, 1923–1933 (1997)
32. Sun, L., Zhou, M.R., Wang, G.W.: Quadratic approximation of solutions for boundary value problems with nonlocal boundary conditions. Comput. Math. Appl. **61**, 1547–1558 (2011)
33. Tunç, C., Tunç, O.: New qualitative criteria for solutions of Volterra integro-differential equations. Arab J. Basic Appl. Sci. **25**, 158–165 (2018)
34. Tunç, C., Tunç, O.: A note on the qualitative analysis of Volterra integro-differential equations. J. Taibah Univ. Sci. **13**, 490–496 (2019)
35. Wang, G.T.: Boundary value problems for systems of nonlinear integro-differential equations with deviating arguments. J. Comput. Appl. Math. **234**, 1356–1363 (2010)
36. Wang, P.G., Tian, S.H., Wu, Y.H.: Monotone iterative method for first-order functional difference equations with nonlinear boundary value conditions. Appl. Math. Comput. **203**, 266–272 (2008)
37. Wang, W.B., Shen, J.H., Nieto, J.J.: Periodic boundary value problems for second order functional differential equations. J. Appl. Math. Comput. **36**, 173–186 (2011)
38. Wang, W.L.: Monotone iterative technique for causal differential equations with upper and lower solutions in the reversed order. Bound. Value Probl. **2016**, Article ID 140 (2016)
39. Wang, W.L., Tian, J.F.: Nonlinear boundary value problems for impulsive differential equations with causal operators. Differ. Equ. Appl. **9**, 161–170 (2017)