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Nonoscillatory solutions to fourth-order neutral dynamic equations on time scales

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Abstract

In this paper, we present some sufficient conditions and necessary conditions for the existence of nonoscillatory solutions to a class of fourth-order nonlinear neutral dynamic equations on time scales by employing Banach spaces and Krasnoselskii's fixed point theorem. Two examples are given to illustrate the applications of the results.

MSC: 34N05; 34C10; 39A13

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1 Introduction

In this paper, we consider the existence of nonoscillatory solutions to fourth-order nonlinear neutral dynamic equations of the form

$$\bigl(r_1(t)\bigl(r_2(t)\bigl(r_3(t)\bigl(x(t)+p(t)x\bigl(g(t)\bigr)\bigr)^{\Delta}\bigr)^{\Delta}\bigr)^{\Delta}\bigr)^{\Delta}+f\bigl(t,x\bigl(h(t)\bigr)\bigr)=0 \tag{1}$$

on a time scale $\mathbb{T}$ satisfying $\sup\mathbb{T}=\infty$, where $t\in[t_0,\infty)_{\mathbb{T}}$ with $t_0\in\mathbb{T}$.

The oscillation and nonoscillation of nonlinear differential and difference equations have been developed rapidly in the recent decades. Afterwards, the theory of time scale united the differential and difference ones, and since then many researchers have investigated the oscillation and nonoscillation criteria of nonlinear dynamic equations on time scales; see, for instance, the papers [1–19] and the references cited therein. The majority of the scholars obtained the sufficient conditions to ensure that the solutions of the equations oscillate or tend to zero by using the generalized Riccati transformation and integral averaging technique. The correlative research has made a great achievement. However, we note that there is not much research into the field of existence of nonoscillatory solutions to dynamic equations on time scales. Generally, a functional space and a fixed point theorem would be employed to analyze it. We refer the reader to [20–25] for details of the theory of time scale and to [4–16, 19] for the studies on the existence of nonoscillatory solutions to nonlinear neutral dynamic equations on time scales.

Definition 1.1 A solution x to (1) is defined to be eventually positive (or eventually negative) if there exists $c \in \mathbb{T}$ such that $x(t) > 0$ (or $x(t) < 0$) on $[c, \infty)_{\mathbb{T}}$. If a solution is either eventually positive or eventually negative, then we say it is nonoscillatory.

For a class of nth-order nonlinear neutral dynamic equations as follows:

$$R_n\bigl(t, x(t)\bigr) + f\bigl(t, x\bigl(h(t)\bigr)\bigr) = 0, \tag{2}$$

where

$$R_k\bigl(t, x(t)\bigr) = \begin{cases} z(t) = x(t) + p(t)x(g(t)), & k = 0, \\ r_{n-k}(t)R_{k-1}^{\Delta}(t, x(t)), & 1 \le k \le n-1, \\ R_{n-1}^{\Delta}(t, x(t)), & k = n, \end{cases}$$

some scholars had been concerned with the existence of nonoscillatory solutions to (2), successively. Without loss of generality, only the eventually positive solutions were discussed. Zhu and Wang [19] considered (2) for $n = 1$. The authors introduced a Banach space

$$\mathrm{BC}[T_0, \infty)_{\mathbb{T}} = \Bigl\{x \in \mathrm{C}\bigl([T_0, \infty)_{\mathbb{T}}, \mathbb{R}\bigr) : \sup_{t \in [T_0, \infty)_{\mathbb{T}}} \bigl|x(t)\bigr| < \infty\Bigr\}$$

with the norm $\|x\| = \sup_{t \in [T_0, \infty)_{\mathbb{T}}} |x(t)|$, where $\mathrm{C}([T_0, \infty)_{\mathbb{T}}, \mathbb{R})$ denotes all continuous functions mapping $[T_0, \infty)_{\mathbb{T}}$ into $\mathbb{R}$, and established the existence of nonoscillatory solutions to (2) by Krasnoselskii's fixed point theorem. Note that there exist only two cases for every eventually positive solution x to (2): $\lim_{t \to \infty} x(t) = a > 0$ or $\lim_{t \to \infty} x(t) = 0$.

Gao and Wang [5] and Deng and Wang [4] investigated (2) for $n = 2$ under different conditions $\int_{t_0}^{\infty} 1/r_1(t)\Delta t < \infty$ and $\int_{t_0}^{\infty} 1/r_1(t)\Delta t = \infty$, respectively. Gao and Wang [5] introduced the same Banach space as in [19] and concluded that all the eventually positive solutions to (2) converge to a positive constant or zero. Deng and Wang [4] defined an improved Banach space

$$\mathrm{BC}_{\lambda}[T_0, \infty)_{\mathbb{T}} = \Biggl\{x \in \mathrm{C}\bigl([T_0, \infty)_{\mathbb{T}}, \mathbb{R}\bigr) : \sup_{t \in [T_0, \infty)_{\mathbb{T}}} \biggl|\frac{x(t)}{R^{2\lambda}(t)}\biggr| < \infty\Biggr\} \tag{3}$$

with the norm $\|x\|_{\lambda} = \sup_{t \in [T_0, \infty)_{\mathbb{T}}} |x(t)/R^{2\lambda}(t)|$, where $\lambda = 0, 1$ and $R(t) = 1 + \int_{t_0}^{t} 1/r_1(s)\Delta s$, and presented four cases for the eventually positive solution x to (2): (i) $x \in A(0, 0)$, (ii) $x \in A(b, 0)$ for some positive constant b, (iii) $x \in A(\infty, b)$ for some positive constant b, (iv) $\limsup_{t \to \infty} x(t) = \infty$ and $\lim_{t \to \infty} x(t)/R(t) = 0$, where

$$A(\alpha, \beta) = \Biggl\{x : \lim_{t \to \infty} x(t) = \alpha \text{ and } \lim_{t \to \infty} \frac{x(t)}{R(t)} = \beta\Biggr\}.$$

Therefore, it is not difficult to see that the existences of nonoscillatory solutions to (2) are greatly different for diverse cases of convergence and divergence of the integrals $\int_{t_0}^{\infty} 1/r_i(t)\Delta t$, $i = 1, 2, \ldots, n-1$.

For $n = 3$, Qiu [12] studied (2) under the assumption $\int_{t_0}^{\infty} 1/r_1(t)\Delta t = \int_{t_0}^{\infty} 1/r_2(t)\Delta t = \infty$ and summarized five cases for the eventually positive solutions to (2). To obtain the existence and asymptotic behavior of nonoscillatory solutions to (2) for other cases of $\int_{t_0}^{\infty} 1/r_i(t)\Delta t$, $i = 1,2$, Qiu et al. [16] were concerned with (2) under $\int_{t_0}^{\infty} 1/r_1(t)\Delta t = \infty$ and $\int_{t_0}^{\infty} 1/r_2(t)\Delta t < \infty$, while the case that $\int_{t_0}^{\infty} 1/r_1(t)\Delta t < \infty$ and $\int_{t_0}^{\infty} 1/r_2(t)\Delta t = \infty$ was considered in Qiu et al. [14]. There are four cases for the eventually positive solutions to (2). However, the existence and asymptotic behavior of the solutions are evidently different between [14] and [16].

Furthermore, for $n \geq 3$, Qiu and Wang [15] investigated (2) under the condition $\int_{t_0}^{\infty} 1/r_i(t)\Delta t < \infty$, $i = 1,2,\dots,n-1$, and deduced that every eventually positive solution converges to a positive constant or zero, which complements and unites the results in [5, 19]. Qiu et al. [13] continued to study (2) with $\int_{t_0}^{\infty} 1/r_1(t)\Delta t = \infty$ and $\int_{t_0}^{\infty} 1/r_i(t)\Delta t < \infty$, $i = 2,3,\dots,n-1$. Four cases for the eventually positive solutions have also been presented, and the results are consistent with those in [16] when $n = 3$.

In this paper, we consider the existence of nonoscillatory solutions to (1), which is (2) for $n = 4$, under the following conditions:

(C1) $r_1, r_2, r_3 \in \mathrm{C}_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}}, (0,\infty))$ and there exist two positive constants M_1 and M_3 such that

$$\int_{t_0}^{\infty} \frac{\Delta t}{r_i(t)} = M_i < \infty, \quad i = 1,3 \quad \text{and} \quad \int_{t_0}^{\infty} \frac{\Delta t}{r_2(t)} = \infty;$$

(C2) $p \in \mathrm{C}_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}}, \mathbb{R})$ and $-1 < \lim_{t\to\infty} p(t) = p_0 < 1$;

(C3) $g, h \in \mathrm{C}_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}}, \mathbb{T})$, $g(t) \leq t$, and $\lim_{t\to\infty} g(t) = \lim_{t\to\infty} h(t) = \infty$; if $p_0 \in (-1,0]$, then there exists a sequence $\{c_k\}_{k\geq 0}$ such that $\lim_{k\to\infty} c_k = \infty$ and $g(c_{k+1}) = c_k$;

(C4) $f \in \mathrm{C}([t_0,\infty)_{\mathbb{T}} \times \mathbb{R}, \mathbb{R})$, $f(t,x)$ is nondecreasing in x, and $xf(t,x) > 0$ for $x \neq 0$;

(C5) if

$$\int_{t_0}^{\infty} \int_{t_0}^{u_3} \frac{\Delta u_2 \Delta u_3}{r_2(u_2) r_3(u_3)} = \infty, \tag{4}$$

then we define

$$R(t) = 1 + \int_{t_0}^{t} \int_{t_0}^{u_3} \frac{\Delta u_2 \Delta u_3}{r_2(u_2) r_3(u_3)},$$

where $\lim_{t\to\infty} R(g(t))/R(t) = \eta \in (0,1]$ is satisfied.

In view of (C1), it is clear that the results in the references are not available for (1). The conclusions in this paper can bring a deeper understanding of the existence and asymptotic behavior of nonoscillatory solutions to (2). In addition, we provide two significant examples to illustrate our results.

2 Auxiliary results

Firstly, we state Krasnoselskii's fixed point theorem (see [3]) as follows, which will be used in the sequel.

Lemma 2.1 *Suppose that X is a Banach space and Ω is a bounded, convex, and closed subset of X. If there exist two operators $U, V : \Omega \to X$ such that $Ux + Vy \in \Omega$ for all $x, y \in \Omega$, U is a contraction mapping, and V is completely continuous, then $U + V$ has a fixed point in Ω.*

Then, we have another lemma to show the relationship between the functions z and x. The proof is omitted since it is similar to those in [4, Lemma 2.3], [5, Theorem 1], [15, Lemma 2.3], and [19, Theorem 7].

Lemma 2.2 *Suppose that x is an eventually positive solution to* (1) *and*

$$\lim_{t\to\infty}\frac{z(t)}{R^{\lambda}(t)}=a,\qquad \lambda=0,1,$$

where $\lambda = 1$ only if (4) *holds. If a is finite, then we have*

$$\lim_{t\to\infty}\frac{x(t)}{R^{\lambda}(t)}=\frac{a}{1+p_0\eta^{\lambda}},$$

or $\limsup_{t\to\infty} x(t)/R^{\lambda}(t)=\infty$.

Finally, we need to divide all the eventually positive solutions to (1) into four groups for the sake of simplicity. In addition, define

$$A(\alpha)=\left\{x\in S:\lim_{t\to\infty}x(t)=\infty \text{ and } \lim_{t\to\infty}\frac{x(t)}{R(t)}=\alpha\right\},$$

where S is the set of all eventually positive solutions to (1).

Theorem 2.3 *Suppose that x is an eventually positive solution to* (1). *Then the solution x belongs to one of the following four cases*:

(A1) $\lim_{t\to\infty} x(t)=0$;
(A2) $\lim_{t\to\infty} x(t)=b$, *where b is a positive constant*;
(A3) $x\in A(b)$, *where b is a positive constant*;
(A4) $\limsup_{t\to\infty} x(t)=\infty$ *and* $\lim_{t\to\infty} x(t)/R(t)=0$.

Proof In view of (C2) and (C3), for any eventually positive solution x to (1), there always exist $t_1\in[t_0,\infty)_{\mathbb{T}}$ and p_1 with $|p_0|<p_1<1$ such that $x(t)>0$, $x(g(t))>0$, $x(h(t))>0$, and $|p(t)|\le p_1$, $t\in[t_1,\infty)_{\mathbb{T}}$. From (1) and (C4), for $t\in[t_1,\infty)_{\mathbb{T}}$, we have

$$R_4\bigl(t,x(t)\bigr)=R_3^{\Delta}\bigl(t,x(t)\bigr)=-f\bigl(t,x\bigl(h(t)\bigr)\bigr)<0,$$

which means that R_3 is strictly decreasing on $[t_1,\infty)_{\mathbb{T}}$. Hence, it follows that

$$R_2^{\Delta}\bigl(t,x(t)\bigr)\le\frac{r_1(t_1)R_2^{\Delta}(t_1,x(t_1))}{r_1(t)},\qquad t\in[t_1,\infty)_{\mathbb{T}}.\tag{5}$$

From (5) we deduce that there are two cases for R_2. Assume that there exists $T\in[t_1,\infty)_{\mathbb{T}}$ satisfying $R_2^{\Delta}(T,x(T))\le 0$, then it is clear that R_2^{Δ} is eventually negative. If not, then we get $R_2^{\Delta}(t,x(t))>0$, $t\in[t_1,\infty)_{\mathbb{T}}$. That is, R_2 is always eventually monotonic. Substituting s for t in (5) and integrating (5) from t_1 to t, where $t\in[\sigma(t_1),\infty)_{\mathbb{T}}$, we obtain

$$R_2\bigl(t,x(t)\bigr)-R_2\bigl(t_1,x(t_1)\bigr)\le r_1(t_1)R_2^{\Delta}\bigl(t_1,x(t_1)\bigr)\int_{t_1}^{t}\frac{\Delta s}{r_1(s)}$$

$$< r_1(t_1)\left|R_2^{\Delta}\left(t_1, x(t_1)\right)\right| \cdot M_1,$$

which implies that R_2 is upper bounded. Therefore, R_2 (or $r_2R_1^{\Delta}$) is eventually monotonic and upper bounded. There are two cases to be discussed.

Case 1. $r_2R_1^{\Delta}$ is eventually decreasing. It follows that

$$-\infty \le \lim_{t\to\infty} r_2(t)R_1^{\Delta}\left(t, x(t)\right) = L_1 < \infty.$$

(a) If $-\infty \le L_1 < 0$, then there exist $t_2 \in [t_1, \infty)_{\mathbb{T}}$ and a constant $a_1 < 0$ such that $r_2(t)R_1^{\Delta}(t, x(t)) \le a_1$ or

$$R_1^{\Delta}\left(t, x(t)\right) \le \frac{a_1}{r_2(t)}, \quad t \in [t_2, \infty)_{\mathbb{T}}. \tag{6}$$

Letting t be replaced by s in (6) and integrating (6) from t_2 to t, where $t \in [\sigma(t_2), \infty)_{\mathbb{T}}$, we derive

$$r_3(t)z^{\Delta}(t) = R_1\left(t, x(t)\right) \le R_1\left(t_2, x(t_2)\right) + a_1 \int_{t_2}^{t} \frac{\Delta s}{r_2(s)} \to -\infty, \quad t \to \infty,$$

which means that z^{Δ} is eventually negative, and thus z is eventually strictly decreasing. It is easy to see that z is eventually positive or eventually negative, but we can claim that z is eventually positive in terms of (C3). Assume not; then we have $-1 < p_0 \le 0$. Moreover, there exists $t_3 \in [t_2, \infty)_{\mathbb{T}}$ such that $x(t) < -p(t)x(g(t)) \le p_1x(g(t))$ for $t \in [t_3, \infty)_{\mathbb{T}}$. Choose a positive integer N satisfying $c_k \in [t_3, \infty)_{\mathbb{T}}$ for all $k \ge N$. For any $k > N$, we always have

$$x(c_k) < p_1x(c_{k-1}) < p_1^2x(c_{k-2}) < \cdots < p_1^{k-N}x(c_N),$$

which implies that $\lim_{k\to\infty} x(c_k) = \lim_{k\to\infty} z(c_k) = 0$. It causes a contradiction, since z is eventually strictly decreasing and eventually negative. Therefore, we deduce

$$0 \le \lim_{t\to\infty} z(t) = L_0 < \infty.$$

By virtue of Lemma 2.2, it follows that case (A1) or (A2) holds.

(b) If $0 < L_1 < \infty$, then there exist $t_4 \in [t_1, \infty)_{\mathbb{T}}$ and a constant $a_2 > 0$ such that $r_2(t)R_1^{\Delta}(t, x(t)) \ge a_2$ or

$$R_1^{\Delta}\left(t, x(t)\right) \ge \frac{a_2}{r_2(t)}, \quad t \in [t_4, \infty)_{\mathbb{T}}.$$

Similarly, we have

$$r_3(t)z^{\Delta}(t) = R_1\left(t, x(t)\right) \ge R_1\left(t_4, x(t_4)\right) + a_2 \int_{t_4}^{t} \frac{\Delta s}{r_2(s)} \to \infty, \quad t \to \infty.$$

It means that z^{Δ} is eventually positive and thus z is eventually strictly increasing. Similar to the proof in (a), we derive

$$0 \le \lim_{t\to\infty} z(t) = L_0 \le \infty. \tag{7}$$

If $L_0 = \infty$, then we deduce

$$\lim_{t\to\infty} r_2(t)R_1^{\Delta}\bigl(t,x(t)\bigr) = \lim_{t\to\infty}\frac{z(t)}{R(t)} = L_1$$

according to L'Hôpital's rule (see [24, Theorem 1.120]). By Lemma 2.2 we know that one of cases (A1)–(A3) holds.

(c) If $L_1 = 0$, since $r_2R_1^{\Delta}$ is eventually strictly decreasing, then $r_2R_1^{\Delta}$ and R_1^{Δ} are both eventually positive, and so R_1 (or r_3z^{Δ}) is eventually strictly increasing. It follows that r_3z^{Δ} and z^{Δ} are both eventually positive or eventually negative, and z is always eventually monotonic, which implies that z is eventually positive or eventually negative. Similarly, we also obtain (7). In view of (C3), we see that one of cases (A1), (A2), and (A4) holds.

Case 2. $r_2R_1^{\Delta}$ is eventually increasing. It follows that

$$-\infty < \lim_{t\to\infty} r_2(t)R_1^{\Delta}\bigl(t,x(t)\bigr) = L_1 < \infty.$$

(a) If $-\infty < L_1 < 0$, then there also exist $t_5 \in [t_1,\infty)_{\mathbb{T}}$ and a constant $a_3 < 0$ such that $r_2(t)R_1^{\Delta}(t,x(t)) \le a_3$ for $t \in [t_5,\infty)_{\mathbb{T}}$. Similar to the proof in Case 1, we get one of cases (A1) or (A2) holds.

(b) If $0 < L_1 < \infty$, similarly, then one of cases (A1)–(A3) holds.

(c) If $L_1 = 0$, since $r_2R_1^{\Delta}$ is eventually strictly increasing, then $r_2R_1^{\Delta}$ and R_1^{Δ} are both eventually negative. Hence, R_1 (or r_3z^{Δ}) is eventually strictly decreasing. Similarly, we always have (7), and one of cases (A1), (A2), and (A4) holds.

The proof is complete. □

3 Sufficient conditions

In this section, we firstly present some sufficient conditions for the existence of each type of eventually positive solutions to (1).

Theorem 3.1 *If there exists some constant $K > 0$ such that*

$$\int_{t_0}^{\infty}\int_{t_0}^{u_3}\int_{t_0}^{u_2}\int_{t_0}^{u_1}\frac{f(u_0,K)}{r_1(u_1)r_2(u_2)r_3(u_3)}\Delta u_0\Delta u_1\Delta u_2\Delta u_3 < \infty, \tag{8}$$

then (1) *has an eventually positive solution x with $\lim_{t\to\infty} x(t) = b$, where b is a positive constant.*

Proof Suppose that there exists some constant $K > 0$ satisfying (8). For $0 \le p_0 < 1$, there are two cases $p_0 > 0$ and $p_0 = 0$. If $p_0 > 0$, then we choose a constant p_1 with $p_0 < p_1 < (1 + 4p_0)/5 < 1$, and thus there exists $T_0 \in [t_0,\infty)_{\mathbb{T}}$ such that for $t \in [T_0,\infty)_{\mathbb{T}}$, we have $p(t) > 0$, $(5p_1 - 1)/4 \le p(t) \le p_1 < 1$, and

$$\int_{T_0}^{\infty}\int_{T_0}^{u_3}\int_{T_0}^{u_2}\int_{T_0}^{u_1}\frac{f(u_0,K)}{r_1(u_1)r_2(u_2)r_3(u_3)}\Delta u_0\Delta u_1\Delta u_2\Delta u_3 \le \frac{(1-p_1)K}{8}.$$

If $p_0 = 0$, then we take p_1 satisfying that $|p(t)| \le p_1 \le 1/13$ for $t \in [T_0,\infty)_{\mathbb{T}}$.

Choose $T_1 \in (T_0,\infty)_{\mathbb{T}}$ such that $g(t) \ge T_0$ and $h(t) \ge T_0$ for $t \in [T_1,\infty)_{\mathbb{T}}$. Let $\Omega_1 = \{x \in \mathrm{BC}_0[T_0,\infty)_{\mathbb{T}} : K/2 \le x(t) \le K\}$, where $\mathrm{BC}_0[T_0,\infty)_{\mathbb{T}}$ is defined as (3) when $\lambda = 0$, and

$U_1, V_1 : \Omega_1 \to \mathrm{BC}_0[T_0,\infty)_{\mathbb{T}}$ as follows:

$$(U_1x)(t) = \begin{cases} (U_1x)(T_1), & t \in [T_0,T_1)_{\mathbb{T}}, \\ 3Kp_1/4 - p(t)x(g(t)), & t \in [T_1,\infty)_{\mathbb{T}}, \end{cases}$$

$$(V_1x)(t) = \begin{cases} (V_1x)(T_1), & t \in [T_0,T_1)_{\mathbb{T}}, \\ 3K/4 + \int_t^\infty \int_{T_1}^{u_3} \int_{T_1}^{u_2} \int_{T_1}^{u_1} \frac{f(u_0,x(h(u_0)))}{r_1(u_1)r_2(u_2)r_3(u_3)} \Delta u_0 \Delta u_1 \Delta u_2 \Delta u_3, & t \in [T_1,\infty)_{\mathbb{T}}. \end{cases}$$

Similar to the proofs in [4, Theorem 2.5], [5, Theorem 2], [12, Theorem 3.1], [15, Theorem 3.1], and [19, Theorem 8], we omit the explanation that U_1 and V_1 satisfy the conditions in Lemma 2.1. Then there exists $x \in \Omega_1$ such that $(U_1 + V_1)x = x$, which means that, for $t \in [T_1,\infty)_{\mathbb{T}}$, we obtain

$$\begin{aligned} x(t) = {}& \frac{3(1+p_1)K}{4} - p(t)x\bigl(g(t)\bigr) \\ & + \int_t^\infty \int_{T_1}^{u_3} \int_{T_1}^{u_2} \int_{T_1}^{u_1} \frac{f(u_0,x(h(u_0)))}{r_1(u_1)r_2(u_2)r_3(u_3)} \Delta u_0 \Delta u_1 \Delta u_2 \Delta u_3. \end{aligned}$$

Letting $t \to \infty$, from (C4) and Lemma 2.2, we deduce

$$\lim_{t\to\infty} z(t) = \frac{3(1+p_1)K}{4} \quad \text{and} \quad \lim_{t\to\infty} x(t) = \frac{3(1+p_1)K}{4(1+p_0)} > 0.$$

For $-1 < p_0 < 0$, change p_1 to satisfy $-p_0 < p_1 < (1-4p_0)/5 < 1$ and $(5p_1-1)/4 \le -p(t) \le p_1 < 1$ for $t \in [T_0,\infty)_{\mathbb{T}}$. Let

$$(\overline{U}_1x)(t) = \begin{cases} (\overline{U}_1x)(T_1), & t \in [T_0,T_1)_{\mathbb{T}}, \\ -3Kp_1/4 - p(t)x(g(t)), & t \in [T_1,\infty)_{\mathbb{T}}. \end{cases}$$

Similarly, there also exists $x \in \Omega_1$ such that $(\overline{U}_1 + V_1)x = x$, and we obtain

$$\lim_{t\to\infty} z(t) = \frac{3(1-p_1)K}{4} \quad \text{and} \quad \lim_{t\to\infty} x(t) = \frac{3(1-p_1)K}{4(1+p_0)} > 0.$$

This completes the proof. □

Theorem 3.2 *Assume that (4) holds. If there exists some constant $K > 0$ such that*

$$\int_{t_0}^\infty \int_{t_0}^{u_1} \frac{f(u_0,KR(h(u_0)))}{r_1(u_1)} \Delta u_0 \Delta u_1 < \infty, \tag{9}$$

then (1) has an eventually positive solution $x \in A(b)$, where b is a positive constant.

Proof Suppose that there exists some constant $K > 0$ such that (9) holds. Proceed as in the proof of Theorem 3.1, except that, for $p_0 > 0$, take $T_0 \in [t_0,\infty)_{\mathbb{T}}$ satisfying $p(t) > 0$, $(5p_1-1)/4 \le p(t) \le p_1 < 1$, $p(t)R(g(t))/R(t) \ge (5p_1-1)\eta/4$ for $t \in [T_0,\infty)_{\mathbb{T}}$, and

$$\int_{T_0}^\infty \int_{T_0}^{u_1} \frac{f(u_0,KR(h(u_0)))}{r_1(u_1)} \Delta u_0 \Delta u_1 \le \frac{(1-p_1\eta)K}{8}.$$

Let $\Omega_2 = \{x \in \mathrm{BC}_1[T_0,\infty)_{\mathbb{T}} : KR(t)/2 \le x(t) \le KR(t)\}$, where $\mathrm{BC}_1[T_0,\infty)_{\mathbb{T}}$ is defined as (3) when $\lambda = 1$, and $U_2, V_2 : \Omega_2 \to \mathrm{BC}_1[T_0,\infty)_{\mathbb{T}}$ as follows:

$$(U_2x)(t) = \begin{cases} 3Kp_1\eta R(t)/4 - p(T_1)x(g(T_1))R(t)/R(T_1), & t \in [T_0,T_1)_{\mathbb{T}}, \\ 3Kp_1\eta R(t)/4 - p(t)x(g(t)), & t \in [T_1,\infty)_{\mathbb{T}}, \end{cases}$$

$$(V_2x)(t) = \begin{cases} 3KR(t)/4, & t \in [T_0,T_1)_{\mathbb{T}}, \\ 3KR(t)/4 \\ \quad + \int_{T_1}^{t}\int_{T_1}^{u_3}\int_{u_2}^{\infty}\int_{T_1}^{u_1} \frac{f(u_0,x(h(u_0)))}{r_1(u_1)r_2(u_2)r_3(u_3)}\Delta u_0\Delta u_1\Delta u_2\Delta u_3, & t \in [T_1,\infty)_{\mathbb{T}}, \end{cases}$$

where T_1 is defined as in Theorem 3.1. Similarly, there exists $x \in \Omega_2$ such that $(U_2 + V_2)x = x$. For $t \in [T_1,\infty)_{\mathbb{T}}$, it follows that

$$\begin{aligned} x(t) = {}& \frac{3(1+p_1\eta)KR(t)}{4} - p(t)x\bigl(g(t)\bigr) \\ & + \int_{T_1}^{t}\int_{T_1}^{u_3}\int_{u_2}^{\infty}\int_{T_1}^{u_1} \frac{f(u_0,x(h(u_0)))}{r_1(u_1)r_2(u_2)r_3(u_3)}\Delta u_0\Delta u_1\Delta u_2\Delta u_3. \end{aligned}$$

Letting $t \to \infty$, we derive

$$\lim_{t\to\infty}\frac{z(t)}{R(t)} = \frac{3(1+p_1\eta)K}{4} \quad \text{and} \quad \lim_{t\to\infty}\frac{x(t)}{R(t)} = \frac{3(1+p_1\eta)K}{4(1+p_0\eta)} > 0.$$

For $-1 < p_0 < 0$, similar to the proof in Theorem 3.1, we have

$$\lim_{t\to\infty}\frac{z(t)}{R(t)} = \frac{3(1-p_1\eta)K}{4} \quad \text{and} \quad \lim_{t\to\infty}\frac{x(t)}{R(t)} = \frac{3(1-p_1\eta)K}{4(1+p_0\eta)} > 0.$$

Moreover, it is clear that $\lim_{t\to\infty} x(t) = \infty$. The proof is complete. □

Theorem 3.3 *Assume that* (4) *holds. If there exists a positive constant M satisfying that $|p(t)R(t)| \le M$ for $t \in [t_0,\infty)_{\mathbb{T}}$,*

$$\int_{t_0}^{\infty}\int_{t_0}^{u_1} \frac{f(u_0,R(h(u_0)))}{r_1(u_1)}\Delta u_0\Delta u_1 < \infty, \tag{10}$$

and

$$\int_{t_0}^{\infty}\int_{t_0}^{u_3}\int_{u_2}^{\infty}\int_{t_0}^{u_1} \frac{f(u_0,M+3/4)}{r_1(u_1)r_2(u_2)r_3(u_3)}\Delta u_0\Delta u_1\Delta u_2\Delta u_3 = \infty, \tag{11}$$

then (1) *has an eventually positive solution $x \in A(0)$.*

Proof Suppose that there exists a constant $M > 0$ such that $|p(t)R(t)| \le M$ for $t \in [t_0,\infty)_{\mathbb{T}}$, and both of (10) and (11) hold. It is easy to see that $p_0 = 0$. There exist $T_0 \in [t_0,\infty)_{\mathbb{T}}$ and p_1 with $0 < p_1 < 1$ such that, for $t \in [T_0,\infty)_{\mathbb{T}}$, we have $|p(t)| \le p_1 < 1$, $2M + 3/2 \le R(t)/4$, and

$$\int_{T_0}^{\infty}\int_{T_0}^{u_1} \frac{f(u_0,R(h(u_0)))}{r_1(u_1)}\Delta u_0\Delta u_1 \le \frac{1-p_1}{8}.$$

Let $\Omega_3 = \{x \in \mathrm{BC}_1[T_0,\infty)_{\mathbb{T}} : M + 3/4 \le x(t) \le R(t)\}$ and $U_3, V_3 : \Omega_3 \to \mathrm{BC}_1[T_0,\infty)_{\mathbb{T}}$ as follows:

$$(U_3x)(t) = \begin{cases} M + 3/4 - p(T_1)x(g(T_1))R(t)/R(T_1), & t \in [T_0, T_1)_{\mathbb{T}}, \\ M + 3/4 - p(t)x(g(t)), & t \in [T_1,\infty)_{\mathbb{T}}, \end{cases}$$

$$(V_3x)(t) = \begin{cases} M + 3/4, & t \in [T_0, T_1)_{\mathbb{T}}, \\ M + 3/4 + \int_{T_1}^{t} \int_{T_1}^{u_3} \int_{u_2}^{\infty} \int_{T_1}^{u_1} \frac{f(u_0,x(h(u_0)))}{r_1(u_1)r_2(u_2)r_3(u_3)} \Delta u_0 \Delta u_1 \Delta u_2 \Delta u_3, & t \in [T_1,\infty)_{\mathbb{T}}, \end{cases}$$

where T_1 is defined as in Theorem 3.1. Similarly, there exists $x \in \Omega_3$ such that, for $t \in [T_1,\infty)_{\mathbb{T}}$, we have

$$x(t) = 2M + \frac{3}{2} - p(t)x\bigl(g(t)\bigr) + \int_{T_1}^{t} \int_{T_1}^{u_3} \int_{u_2}^{\infty} \int_{T_1}^{u_1} \frac{f(u_0,x(h(u_0)))}{r_1(u_1)r_2(u_2)r_3(u_3)} \Delta u_0 \Delta u_1 \Delta u_2 \Delta u_3.$$

Letting $t \to \infty$, it is not difficult to see that

$$\lim_{t\to\infty} z(t) = \infty \quad \text{and} \quad \lim_{t\to\infty} \frac{z(t)}{R(t)} = 0,$$

which implies that

$$\lim_{t\to\infty} x(t) = \infty \quad \text{and} \quad \lim_{t\to\infty} \frac{x(t)}{R(t)} = 0$$

since $|p(t)x(g(t))| \le |p(t)R(t)| \le M$ for $t \in [T_1,\infty)_{\mathbb{T}}$. This completes the proof. □

Remark 3.4 It is not easy to find a sufficient condition for the existence of nonoscillatory solutions tending to zero to (1) since their asymptotic behaviors are more complex than those of other solutions. However, we refer the reader to [4, Theorem 2.8 and Remark 2.9], [5, Theorem 3], [12, Theorems 3.5 and 3.6], [15, Theorems 3.2 and 3.3], and [19, Theorems 9 and 10], where some instructive results are presented.

4 Necessary conditions

Some necessary conditions for the existence of eventually positive solutions to (1) are provided in this section, where an additional assumption is needed as follows:

$$\int_{t_0}^{\infty} \int_{t_0}^{u_3} \int_{t_0}^{u_2} \frac{\Delta u_1 \Delta u_2 \Delta u_3}{r_1(u_1)r_2(u_2)r_3(u_3)} < \infty. \tag{12}$$

Theorem 4.1 *Assume that* (12) *holds and*

$$\int_{t_0}^{\infty} \int_{t_0}^{u_3} \frac{\Delta u_2 \Delta u_3}{r_2(u_2)r_3(u_3)} < \infty. \tag{13}$$

If (1) *has an eventually positive solution* x *with* $\lim_{t\to\infty} x(t) = b$, *where* b *is a positive constant, then there exists some constant* $K > 0$ *satisfying* (8).

Proof Suppose that (12) and (13) hold, and (1) has an eventually positive solution x with $\lim_{t\to\infty} x(t) = b > 0$. It follows that $\lim_{t\to\infty} z(t) = (1+p_0)b$, and there exists $t_1 \in [t_0,\infty)_{\mathbb{T}}$ such that $x(t) > 0$, $x(g(t)) > 0$, and $x(h(t)) \geq b/2$ for $t \in [t_1,\infty)_{\mathbb{T}}$. Letting t be replaced by u_0 in (1) and integrating (1) from t_1 to u_1, where $u_1 \in [\sigma(t_1),\infty)_{\mathbb{T}}$, we have

$$R_3\bigl(u_1,x(u_1)\bigr) = R_3\bigl(t_1,x(t_1)\bigr) - \int_{t_1}^{u_1} f\bigl(u_0,x\bigl(h(u_0)\bigr)\bigr)\Delta u_0$$

and thus

$$R_2^{\Delta}\bigl(u_1,x(u_1)\bigr) = \frac{R_3(t_1,x(t_1))}{r_1(u_1)} - \int_{t_1}^{u_1} \frac{f(u_0,x(h(u_0)))}{r_1(u_1)}\Delta u_0. \tag{14}$$

Integrating (14) with respect to u_1 from t_1 to u_2, where $u_2 \in [\sigma(t_1),\infty)_{\mathbb{T}}$, we have

$$\begin{aligned} R_2\bigl(u_2,x(u_2)\bigr) &= R_2\bigl(t_1,x(t_1)\bigr) + R_3\bigl(t_1,x(t_1)\bigr)\int_{t_1}^{u_2} \frac{\Delta u_1}{r_1(u_1)} \\ &\quad - \int_{t_1}^{u_2}\int_{t_1}^{u_1} \frac{f(u_0,x(h(u_0)))}{r_1(u_1)}\Delta u_0\Delta u_1. \end{aligned} \tag{15}$$

By analogy, we deduce

$$\begin{aligned} z(t) &= z(t_1) + R_1\bigl(t_1,x(t_1)\bigr)\int_{t_1}^{t} \frac{\Delta u_3}{r_3(u_3)} + R_2\bigl(t_1,x(t_1)\bigr)\int_{t_1}^{t}\int_{t_1}^{u_3} \frac{\Delta u_2\Delta u_3}{r_2(u_2)r_3(u_3)} \\ &\quad + R_3\bigl(t_1,x(t_1)\bigr)\int_{t_1}^{t}\int_{t_1}^{u_3}\int_{t_1}^{u_2} \frac{\Delta u_1\Delta u_2\Delta u_3}{r_1(u_1)r_2(u_2)r_3(u_3)} \\ &\quad - \int_{t_1}^{t}\int_{t_1}^{u_3}\int_{t_1}^{u_2}\int_{t_1}^{u_1} \frac{f(u_0,x(h(u_0)))}{r_1(u_1)r_2(u_2)r_3(u_3)}\Delta u_0\Delta u_1\Delta u_2\Delta u_3. \end{aligned} \tag{16}$$

Letting $t\to\infty$, in terms of (C1), (C4), (12), (13), and the fact that $x(h(t)) \geq b/2$ for $t \in [t_1,\infty)_{\mathbb{T}}$, we conclude

$$\int_{t_1}^{\infty}\int_{t_1}^{u_3}\int_{t_1}^{u_2}\int_{t_1}^{u_1} \frac{f(u_0,b/2)}{r_1(u_1)r_2(u_2)r_3(u_3)}\Delta u_0\Delta u_1\Delta u_2\Delta u_3 < \infty,$$

which means that (8) holds. The proof is complete. □

Here is a lemma to present a sufficient condition to ensure that (4) is satisfied, and then Theorems 4.3 and 4.4 follow.

Lemma 4.2 *Assume that* (12) *holds. If* (1) *has an eventually positive solution x with* $\lim_{t\to\infty} x(t) = \infty$, *then* (4) *holds.*

Proof If (1) has an eventually positive solution x with $\lim_{t\to\infty} x(t) = \infty$, then we claim that $\lim_{t\to\infty} z(t) = \infty$. Assume not; by Theorem 2.3 we have $0 \leq \lim_{t\to\infty} z(t) < \infty$, which implies that $0 \leq \lim_{t\to\infty} x(t) < \infty$ in view of Lemma 2.2. Hence, we deduce $\lim_{t\to\infty} z(t) = \infty$. Similar to the proof in Theorem 4.1, we arrive at (16). Letting $t\to\infty$, from (C1) and (12) we obtain (4) and complete the proof. □

Theorem 4.3 *Assume that* (12) *holds. If* (1) *has an eventually positive solution* $x \in A(b)$, *where* b *is a positive constant, then there exists some constant* $K > 0$ *satisfying* (9).

Proof Suppose that (12) holds and (1) has an eventually positive solution $x \in A(b)$, where b is a positive constant, then (4) holds according to Lemma 4.2. Define R as in (C5), by Lemma 2.2 and Theorem 2.3 we obtain

$$\lim_{t\to\infty} z(t) = \infty \quad \text{and} \quad \lim_{t\to\infty} R_2\bigl(t, x(t)\bigr) = \lim_{t\to\infty} \frac{z(t)}{R(t)} = (1 + p_0\eta)b.$$

Then there exists $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $x(t) > 0$, $x(g(t)) > 0$, and $x(h(t)) \geq bR(h(t))/2$ for $t \in [t_1, \infty)_{\mathbb{T}}$. Proceeding as the proof in Theorem 4.1, we get (15). Letting $u_2 \to \infty$, by (C1) and (C4) we conclude

$$\int_{t_1}^{\infty} \int_{t_1}^{u_1} \frac{f(u_0, bR(h(u_0))/2)}{r_1(u_1)} \Delta u_0 \Delta u_1 < \infty,$$

which implies that (9) holds. The proof is complete. □

Theorem 4.4 *Assume that* (12) *holds. If* (1) *has an eventually positive solution* $x \in A(0)$, *then we have*

$$\int_{t_0}^{\infty} \int_{t_0}^{u_1} \frac{f(u_0, 3/4)}{r_1(u_1)} \Delta u_0 \Delta u_1 < \infty \tag{17}$$

and

$$\int_{t_0}^{\infty} \int_{t_0}^{u_3} \int_{u_2}^{\infty} \int_{t_0}^{u_1} \frac{f(u_0, R(h(u_0)))}{r_1(u_1) r_2(u_2) r_3(u_3)} \Delta u_0 \Delta u_1 \Delta u_2 \Delta u_3 = \infty. \tag{18}$$

Proof Suppose that (12) holds and (1) has an eventually positive solution $x \in A(0)$. It follows that (4) holds, and we define R as in (C5), then we obtain

$$\lim_{t\to\infty} z(t) = \infty \quad \text{and} \quad \lim_{t\to\infty} R_2\bigl(t, x(t)\bigr) = \lim_{t\to\infty} \frac{z(t)}{R(t)} = 0.$$

There exists $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $3/4 \leq x(t) \leq R(t)$, $3/4 \leq x(g(t)) \leq R(g(t))$, and $3/4 \leq x(h(t)) \leq R(h(t))$ for $t \in [t_1, \infty)_{\mathbb{T}}$. Similar to the proof in Theorem 4.1, we get (14). Integrating (14) with respect to u_1 from t_2 to t, where $t_2 \in [\sigma(t_1), \infty)_{\mathbb{T}}$ and $t \in [\sigma(t_2), \infty)_{\mathbb{T}}$, we have

$$\begin{aligned} R_2\bigl(t, x(t)\bigr) &= R_2\bigl(t_2, x(t_2)\bigr) + R_3\bigl(t_1, x(t_1)\bigr) \int_{t_2}^{t} \frac{\Delta u_1}{r_1(u_1)} \\ &\quad - \int_{t_2}^{t} \int_{t_1}^{u_1} \frac{f(u_0, x(h(u_0)))}{r_1(u_1)} \Delta u_0 \Delta u_1. \end{aligned}$$

Letting $t \to \infty$, we obtain

$$R_2\bigl(t_2, x(t_2)\bigr) = -R_3\bigl(t_1, x(t_1)\bigr) \int_{t_2}^{\infty} \frac{\Delta u_1}{r_1(u_1)}$$

$$+\int_{t_2}^{\infty}\int_{t_1}^{u_1}\frac{f(u_0,x(h(u_0)))}{r_1(u_1)}\Delta u_0\Delta u_1. \tag{19}$$

According to (C1) and (C4), it follows that

$$\int_{t_2}^{\infty}\int_{t_1}^{u_1}\frac{f(u_0,3/4)}{r_1(u_1)}\Delta u_0\Delta u_1<\infty,$$

which implies that (17) holds. Substituting u_2 for t_2 in (19) and integrating (19) with respect to u_2 from t_1 to u_3, where $u_3\in[\sigma(t_1),\infty)_{\mathbb{T}}$, we have

$$\begin{aligned}R_1\bigl(u_3,x(u_3)\bigr)&=R_1\bigl(t_1,x(t_1)\bigr)-R_3\bigl(t_1,x(t_1)\bigr)\int_{t_1}^{u_3}\int_{u_2}^{\infty}\frac{\Delta u_1\Delta u_2}{r_1(u_1)r_2(u_2)}\\&\quad+\int_{t_1}^{u_3}\int_{u_2}^{\infty}\int_{t_1}^{u_1}\frac{f(u_0,x(h(u_0)))}{r_1(u_1)r_2(u_2)}\Delta u_0\Delta u_1\Delta u_2.\end{aligned}$$

Similarly, it follows that

$$\begin{aligned}z(t)&=z(t_1)+R_1\bigl(t_1,x(t_1)\bigr)\int_{t_1}^{t}\frac{\Delta u_3}{r_3(u_3)}\\&\quad-R_3\bigl(t_1,x(t_1)\bigr)\int_{t_1}^{t}\int_{t_1}^{u_3}\int_{u_2}^{\infty}\frac{\Delta u_1\Delta u_2\Delta u_3}{r_1(u_1)r_2(u_2)r_3(u_3)}\\&\quad+\int_{t_1}^{t}\int_{t_1}^{u_3}\int_{u_2}^{\infty}\int_{t_1}^{u_1}\frac{f(u_0,x(h(u_0)))}{r_1(u_1)r_2(u_2)r_3(u_3)}\Delta u_0\Delta u_1\Delta u_2\Delta u_3.\end{aligned}$$

Letting $t\to\infty$, we conclude that (18) holds and complete the proof. □

Now, we can present some necessary and sufficient conditions for the existence of eventually positive solutions to (1). In fact, according to Theorems 3.1 and 4.1, we obtain the following corollary.

Corollary 4.5 *Assume that* (12) *and* (13) *hold. Then* (1) *has an eventually positive solution* x *with* $\lim_{t\to\infty}x(t)=b>0$ *if and only if there exists some constant* $K>0$ *satisfying* (8).

Similarly, we get another corollary in terms of Theorems 3.2 and 4.3.

Corollary 4.6 *Assume that* (4) *holds. Then* (1) *has an eventually positive solution* $x\in A(b)$ *if and only if there exists some constant* $K>0$ *satisfying* (9), *where* b *is a positive constant.*

5 Examples

To illustrate the applications of the conclusions in this paper, two interesting examples are presented as follows.

Example 5.1 Let $\mathbb{T}=\bigcup_{n=0}^{\infty}[3^n,2\cdot3^n]$. For $t\in[3,\infty)_{\mathbb{T}}$, consider

$$\left(t^3\left(\frac{1}{t^\alpha}\left(t^\beta\left(x(t)-\frac{t+\cos t}{3t}x\left(\frac{t}{3}\right)\right)^{\Delta}\right)^{\Delta}\right)^{\Delta}\right)^{\Delta}+t^2x(3t)=0. \tag{20}$$

Here, $r_1(t) = t^3$, $r_2(t) = 1/t^\alpha$, $r_3(t) = t^\beta$, $p(t) = -(t + \cos t)/(3t)$, $g(t) = t/3$, $h(t) = 3t$, $f(t,x) = t^2 x$, $t_0 = 3$, and $p_0 = -1/3$, where $\alpha \geq 0$ and $\beta > \alpha + 3$. It is clear that conditions (C1)–(C4) are satisfied. Moreover, we have

$$\begin{aligned}
&\int_3^\infty \int_3^{u_3} \int_3^{u_2} \int_3^{u_1} \frac{f(u_0,1)}{r_1(u_1)r_2(u_2)r_3(u_3)} \Delta u_0 \Delta u_1 \Delta u_2 \Delta u_3 \\
&\quad = \int_3^\infty \int_3^{u_3} \int_3^{u_2} \int_3^{u_1} \frac{u_0^2 u_2^\alpha}{u_1^3 u_3^\beta} \Delta u_0 \Delta u_1 \Delta u_2 \Delta u_3 \\
&\quad < \frac{1}{3} \int_3^\infty \int_3^{u_3} \int_3^{u_2} \frac{u_2^\alpha}{u_3^\beta} \Delta u_1 \Delta u_2 \Delta u_3 < \frac{1}{3} \int_3^\infty \int_3^{u_3} \frac{u_2^{\alpha+1}}{u_3^\beta} \Delta u_2 \Delta u_3 \\
&\quad < \frac{1}{3(\alpha+2)} \int_3^\infty \frac{\Delta u_3}{u_3^{\beta-\alpha-2}} < \infty.
\end{aligned}$$

According to Theorem 3.1, we conclude that (20) has an eventually positive solution x with $\lim_{t\to\infty} x(t) = b > 0$.

Example 5.2 Let $\mathbb{T} = [1,\infty)_\mathbb{R}$. For $t \in [2,\infty)_\mathbb{T}$, consider

$$\left(t^\lambda \left(\frac{1}{t}\left(t^2\left(x(t) + \frac{1}{2t}x(t-1)\right)'\right)'\right)'\right)' + \frac{x(t)}{t^2} = 0. \tag{21}$$

Here, we have $r_1(t) = t^\lambda$, $r_2(t) = 1/t$, $r_3(t) = t^2$, $p(t) = 1/(2t)$, $g(t) = t-1$, $h(t) = t$, $f(t,x) = x/t^2$, $t_0 = 2$, and $p_0 = 0$, where $1 < \lambda \leq 2$. It is obvious that conditions (C1)–(C4) and (4) are satisfied. From (C5), we have

$$R(t) = 1 + \int_2^t \int_2^{u_3} \frac{du_2\, du_3}{r_2(u_2)r_3(u_3)} = 1 + \int_2^t \int_2^{u_3} \frac{u_2}{u_3^2}\, du_2\, du_3 = \frac{t}{2} - 1 + \frac{2}{t},$$

which satisfies that $\eta = \lim_{t\to\infty} R(g(t))/R(t) = 1$. Moreover, we deduce

$$\begin{aligned}
&\left|p(t)R(t)\right| = \frac{1}{2t} \cdot \left(\frac{t}{2} - 1 + \frac{2}{t}\right) \leq M = \frac{1}{4}, \quad t \in [2,\infty)_\mathbb{T}, \\
&\int_2^\infty \int_2^{u_1} \frac{f(u_0, R(h(u_0)))}{r_1(u_1)}\, du_0\, du_1 \\
&\quad = \int_2^\infty \int_2^{u_1} \frac{u_0/2 - 1 + 2/u_0}{u_0^2 u_1^\lambda}\, du_0\, du_1 \\
&\quad < \frac{1}{2} \int_2^\infty \int_2^{u_1} \frac{1}{u_0 u_1^\lambda}\, du_0\, du_1 < \frac{1}{2} \int_2^\infty \frac{\ln u_1}{u_1^\lambda}\, du_1 < \infty,
\end{aligned}$$

and

$$\begin{aligned}
&\int_2^\infty \int_2^{u_3} \int_{u_2}^\infty \int_2^{u_1} \frac{f(u_0, M + 3/4)}{r_1(u_1)r_2(u_2)r_3(u_3)}\, du_0\, du_1\, du_2\, du_3 \\
&\quad = \int_2^\infty \int_2^{u_3} \int_{u_2}^\infty \int_2^{u_1} \frac{u_2}{u_0^2 u_1^\lambda u_3^2}\, du_0\, du_1\, du_2\, du_3 = \infty.
\end{aligned}$$

By virtue of Theorems 3.2 and 3.3, we conclude that (21) has two eventually positive solutions $x_1 \in A(b)$ for some positive constant b and $x_2 \in A(0)$.

On the other hand, if we take $r_3(t) = t^4$ and $f(t,x) = t \cdot x$, but other functions remain unchanged in (21), then (C1)–(C4) are still satisfied. In addition, we derive

$$\int_2^\infty \int_2^{u_3} \int_2^{u_2} \frac{u_2}{u_1^\lambda u_3^4}\,du_1\,du_2\,du_3 < \infty \quad \text{and} \quad \int_2^\infty \int_2^{u_3} \frac{u_2}{u_3^4}\,du_2\,du_3 < \infty,$$

which means that both of (12) and (13) hold. However, for all $K > 0$, we always have

$$\begin{aligned} &\int_2^\infty \int_2^{u_3} \int_2^{u_2} \int_2^{u_1} \frac{f(u_0,K)}{r_1(u_1)r_2(u_2)r_3(u_3)}\,du_0\,du_1\,du_2\,du_3 \\ &\quad = K \int_2^\infty \int_2^{u_3} \int_2^{u_2} \int_2^{u_1} \frac{u_0 u_2}{u_1^\lambda u_3^4}\,du_0\,du_1\,du_2\,du_3 = \infty, \end{aligned}$$

which implies that (8) is not satisfied. Therefore, we deduce that (21) has no eventually positive solution tending to a positive constant in terms of Theorem 4.1 or Corollary 4.5.

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