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Dynamics of swine influenza model with optimal control

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Abstract

Transmission dynamics of swine influenza pandemic is analysed through a deterministic model. Qualitative analysis of the model includes global asymptotic stability of disease-free and endemic equilibria under a certain condition based on the reproduction number. Sensitivity analysis to ponder the effect of model parameters on the reproduction number is performed and control strategies are designed. It is also verified that the obtained numerical results are in good agreement with the analytical ones.

Keywords: Dynamical system; Swine flu; Epidemic model; Lyapunov function; Sensitivity analysis; Optimal control

1 Introduction

Flu, as a pig infection, was first diagnosed at the time of Spanish influenza pandemic of 1918–1919. First of all, Koen depicted the flu ailment in those families for which this disease appeared in their swine herds [1]. The first seclusion of influenza virus from pigs happened in 1930 [2] and several years later from humans [3]. Swine influenza virus was first separated from a human in 1974, affirming the hypothesis that swine flu infections could contaminate people [4].

In the twentieth century, the recognition of influenza subtypes made it possible to diagnose transmission to humans accurately. The recorded number of confirmed transmissions was 50. Some of its strains passed from human to human. The 2009 H1N1 virus was not zoonotic swine flu, as it was not transported from pigs to people, but from one person to another.

The standard methods of the transmission of influenza viruses in people are through the dissemination of large droplets and coughing of an infected person [5]. There is additional potential for transmission through contact with fomites that are tainted with respiratory or gastrointestinal material. Since numerous patients suffering from swine influenza infection have had loose bowels, the potential for fecal viral shedding and resulting fecal-oral transmission is considered and studied [6].

It is estimated that 2009 pandemic influenza A H1N1 caused more than 18,300 deaths across 74 countries of the world [7]. In the same year, the world also experienced epidemic

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of H1N1 influenza termed as swine influenza that led to more than 16,455 mortalities globally [8]. In Canada, this type of epidemic appeared twice. First, it occurred during spring 2009 and then in October 2009. It lasted for about three months [9]. In 2014, 937 cases of swine flu had been reported in India and 218 deaths had been observed due to this disease. In 2015, the recorded cases and casualties had performed better than the past numbers. There is a huge aggregate of affirmed cases, and these cases are greater than 33,000, which involves approximately 2000 passed away cases [10].

Isolation (of people dreaded to have been presented to a transmittable illness) is one of the most seasoned general well-being control strategy of transferable maladies in given population. During flare-ups of a transmittable infection in human populations, essential general well-being control measures, eminent isolation (of people associated with being presented to the sickness), and detachment (of people with clinical side effects of the sickness) are commonly actualized to control or moderate the malady trouble (estimated in terms of the number of new cases, hospitalization, grimness, mortality). These measures have been viably utilized in the control of various developing and reappearing human and creature illnesses such as sickness, plague, cholera, typhus, yellow fever, smallpox, diphtheria, tuberculosis, measles, ebola, serious intense respiratory disorder, cow-like tuberculosis, rinderpest, foot-and-mouth, psittacosis, and all the more as of the late 2009 swine flu pandemic [11–13].

Optimal control hypothesis is an incredible scientific apparatus to settle on choice including complex dynamical frameworks. It has additionally been connected to irresistible infection issues previously, and is a decent strategy for deciding how to control a sickness best, for modeling what level of the population ought to be immunized as time advances in a given pestilence model to limit both the quantity of contaminated individuals and the expense of actualizing the immunization technique. Various examinations have utilized the uses of ideal control hypothesis in epidemiological models. Lenhart and Workman [14] introduced many examples of optimal control applied to different problems, including an infectious disease problem. Sweilam et al. [15] studied an optimal control problem for the fractional tuberculosis (TB) infection models. Different displaying studies have additionally been made to think about the job of optimal control using epidemic models [16, 17].

For the comprehension of the transmission dynamics of influenza, there exist numerous mathematical models. Brian et al. [18] presented the literature about influenza models and discussed how these models are helpful to study the dynamics of swine flu. Martcheva [19] discussed the evolutionary model of influenza A with drift and shift. Rahman and Zou [20] analyzed two strains influenza model with inoculation for strain 1 being executed. Transmission dynamics of H1N1 influenza has been discussed by different authors through mathematical models [21–25]. However, to the authors' knowledge, no such model of swine flu has been developed in which the impact of quarantine individuals is discussed and optimal control strategies have been designed on the basis of model sensitive parameters.

The version used in this work incorporates an extension presented in Imran et al. [25] that accounts for the quarantine class. The potential highlights of the proposed model are summarized as follows:

- A novel epidemiological model is designed by considering the effect of quarantines.

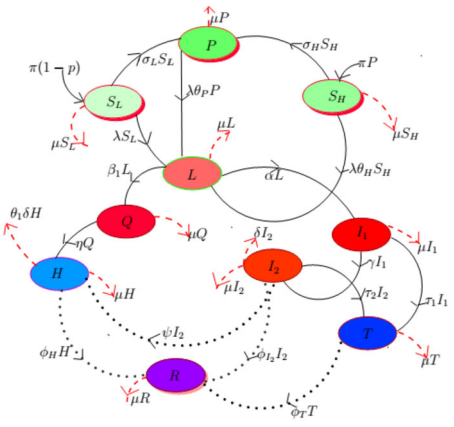

Figure 1 Schematic flow diagram of influenza model

- The effect of model parameters on the dynamics of swine flu is analyzed using sensitivity analysis.
- Control strategies have been designed on the basis of sensitive parameters.

The objective of this study is to analyse new features of the model under consideration. In the first phase, qualitative analysis has been completed including the sensitivity analysis. Secondly, to project the potential impact of pandemic influenza, optimal control strategies have been designed. The remaining part of the paper is arranged as follows: In Sect. 2, model formulation, with parameter description including parameter values, is presented. In Sect. 3, we discuss the model analysis, equilibria, and their stabilities. In Sect. 4, sensitivity analysis is performed. Section 5 is devoted to the formulation of optimal control problem. Optimal control existence is given in Sect. 6. In Sect. 7, analytical results are verified through numerical simulations. In Sect. 8, concluding remarks are presented.

2 Model formulation

The whole population is divided into ten heterogeneous compartments, namely low-risk susceptibles, high-risk susceptibles, vaccinated, exposed, quarantined, symptomatic individuals at early stage, symptomatic people at later stage, hospitalized, treated, and recovered denoted by $S_L(t)$, $S_H(t)$, $P(t)$, $L(t)$, $Q_(t)$, $I_1(t)$, $I_2(t)$, $H(t)$, $T(t)$, and $R(t)$ at any time t, respectively. Thus the total human population can be written as

$$N(t) = S_L(t) + S_H(t) + P(t) + L(t) + Q(t) + I_1(t) + I_2(t) + H(t) + T(t) + R(t).$$

Pregnant ladies, juveniles, health care workers, elderly, and other immune-compromised individuals are considered as high-risk susceptibles and the remaining population has been thought to be at low risk to get infected by swine flu. Flow diagram of model (1) is shown in Fig. 1.

We have the accompanying system of nonlinear ODEs to portray our problem:

$$\frac{dS_L}{dt} = \pi - \pi p - \lambda S_L - \sigma_L S_L - \mu S_L,$$
$$\frac{dS_H}{dt} = \pi p - \theta_H \lambda S_H - \sigma_H S_H - \mu S_H,$$
$$\frac{dP}{dt} = \sigma_L S_L + \sigma_H S_H - \theta_P \lambda P - \mu P,$$

$$\begin{aligned}
&\frac{dL}{dt} = \lambda(S_L + \theta_H S_H + \theta_P P) - (\alpha + \beta_1 + \mu)L,\\
&\frac{dQ}{dt} = \beta_1 L - (\mu + \eta)Q,\\
&\frac{dI_1}{dt} = \alpha L - (\tau_1 + \gamma + \mu)I_1,\\
&\frac{dI_2}{dt} = \gamma I_1 - (\tau_2 + \psi + \phi_{I_2} + \mu + \delta)I_2,\\
&\frac{dH}{dt} = \psi I_2 + \eta Q - (\phi_H + \mu + \theta_1\delta)H,\\
&\frac{dT}{dt} = \tau_1 I_1 + \tau_2 I_2 - (\phi_T + \mu)T,\\
&\frac{dR}{dt} = \phi_{I_2} I_2 + \phi_H H + \phi_T T - \mu R,
\end{aligned} \tag{1}$$

where "λ", the infection rate, is given by

$$\lambda = \beta \frac{\eta_1 L + \eta Q + I_1 + \eta_2 I_2 + \eta_3 H + \eta_4 T}{N}. \tag{2}$$

In (2), β is the compelling contact rate, and the refinement parameters $0 \leq \eta_i < 1$ ($i \in \{1, 2, 3, 4\}$) describe the supposed reduction of infectiousness of exposed, symptomatic individuals at later stage, hospitalized, and treated individuals, respectively, in relation to the symptomatically-infected (infectious) individuals in the I_1 class. Similarly, $0 \leq \eta < 1$ represents the assumed depreciation of virulence of quarantined individuals in connection to people in the I_1 class.

Susceptible individuals who are at low risk are reduced by infection (at the rate λ), vaccination (at a rate σ_L), and natural death (at a rate μ). Similarly high-risk susceptible individuals are reduced by infection (at the rate $\theta_H\lambda$), vaccination (at a rate σ_H), and natural death (at a rate μ). The parameter $\theta_H > 1$ represents the supposition that the people which are at high risk are more sensitive to get tainted as compared to the people which are at low risk. We assume that people in all epidemiological compartments are expected to suffer same natural death rate μ, whereas disease-induced death rates of individuals in the I_2 class and the hospitalized class are δ and $\theta_1\delta$ ($0 < \theta_1 < 1$), respectively.

Vaccinated individuals can become tainted at a decreased rate $\theta_P\lambda$, where $1 - \theta_P$ ($0 < \theta_P < 1$) is the adequacy of the antiviral in averting infection. Latently-infected individuals (E) are produced at the rate λ and reduced by the formation of clinical indications of the malady (at a rate α) and quarantined (at a rate β_1). Latent individuals move to the symptomatic initial level of infection I_1, and afterward with the rate γ they move to I_2, which is the later class of infectious. People in classes I_1 and I_2 respectively get the medication at rates τ_1 and τ_2. Infected individuals I_2 are hospitalized at the rate ψ and quarantined individuals at the rate η. Recovery rates of I_2, hospitalized, and treated individuals are ϕ_{I_2}, ϕ_H, and ϕ_T, respectively. We consider that recovered individuals are permanently immune against re-infection with $H1N1$. Parameters along with numerical values are described in Table 1.

Table 1 Description of the model parameters and nominal values

Parameters	Explanation	Value	Ref
π	Human birth rate	$1{,}119{,}583\frac{1}{80*365}$	[25]
$\frac{1}{\mu}$	Average lifespan of humans	80*365	[25]
p	Part of high-risk susceptible individuals	0.4	[25]
β	Effectual contact rate for spreading H1N1 influenza	0.9	[25]
σ_L	Cure rate of low-risk susceptible individualsby using antiviral drugs	0.3	[26]
σ_H	The rate at which high-risk susceptible individuals get cured by using antiviral drugs	0.5	[26]
α	Rate at which latent individuals become infected	1/1.9	[26]
τ_1	Medication rate of individuals at the early stage of infection	1/5	[26]
τ_2	Medication rate of individuals at the later stage of infection	1/3	[26]
ϕ_{I_2}	Cure rate of symptomatic infectious individuals at the later stage	1/5	[26]
ϕ_T	Cure rate of treated individuals	1/3	[26]
η_1	Refinement parameter (see text)	0.1	[26]
η_2	Refinement parameter (see text)	1/2	[26]
η_3	Refinement parameter (see text)	1.2	[26]
η_4	Refinement parameter (see text)	1	[26]
θ_H	Refinement parameter for infection rate of high risk	1.2	[26]
$1-\theta_P$	Drug efficacy against infection	0.5	[26]
ψ	Hospitalized rate of individuals in the I_2 class	0.5	[26]
γ	Progression rate from I_1 to I_2 classes	0.06	[26]
δ	It denotes the rate at which the people in class I_2 die	1/100	[26]
$\theta_1\delta$	It denotes the rate at which the people in class H die	1/100	[26]

3 Analysis of the model

3.1 Basic properties

All the mentioned parameters in model (1) are nonnegative because it deals with human population. It can be easily proved that the closed set $D = \{(S_L, S_H, P, L, Q, I_1, I_2, H, T, R) \in R_+^{10} : N \leq \frac{\pi}{\mu}\}$ is positively-invariant and attracting with respect to model (1).

3.2 Disease-free equilibrium and its stability

The disease-free equilibrium (DFE) of system (1) is given by

$$\begin{aligned}\varepsilon_0 &= \left(S_L^*, S_H^*, P^*, L^*, Q^*, I_1^*, I_2^*, H^*, T^*, R^*\right)\\ &= \left(S_L^*, S_H^*, \frac{\sigma_L S_L^* + \sigma_H S_H^*}{\mu}, 0, 0, 0, 0, 0, 0, 0\right),\end{aligned} \tag{3}$$

with $S_L^* = \frac{\pi(1-P)}{\sigma_L+\mu}$ and $S_H^* = \frac{\pi P}{\sigma_H+\mu}$.

Following the notation given in [27], the nonnegative matrix F consisting of the new infection terms and the matrix V of the progression terms involved in model (1) are given, respectively, by

$$F = \begin{bmatrix} \beta\eta_1\Omega & \beta\eta\Omega & \beta\Omega & \beta\eta_2\Omega & \beta\eta_3\Omega & \beta\eta_4\Omega \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$V = \begin{bmatrix} K_1 & 0 & 0 & 0 & 0 & 0 \\ -\beta_1 & K & 0 & 0 & 0 & 0 \\ -\alpha & 0 & K_2 & 0 & 0 & 0 \\ 0 & 0 & -\gamma & K_3 & 0 & 0 \\ 0 & -\eta & 0 & -\psi & K_4 & 0 \\ 0 & 0 & -\tau_1 & -\tau_2 & 0 & K_5 \end{bmatrix},$$

where $K_1 = \alpha + \beta_1 + \mu$, $K = \mu + \eta$, $K_2 = \tau_1 + \gamma + \mu$, $K_3 = \tau_2 + \psi + \phi_{I_2} + \mu + \delta$, $K_4 = \phi_H + \mu + \theta_1\delta$, $K_5 = \phi_T + \mu$, and $\Omega = \frac{S_L^{*+\theta_H} S_H^{*+\theta_P} P^*}{N^*}$.

It pursues the control reproduction number, signified by $R_C = \rho(FV^{-1})$, which is given as follows:

$$\begin{aligned} R_C &= \rho\left(FV^{-1}\right) \\ &= \frac{\Omega\beta}{K_1}\left(\eta_1 + \frac{\alpha}{K_2} + \frac{\beta_1\eta}{K} + \alpha\gamma\frac{\eta_2}{K_2K_3} + \frac{\eta_4}{K_2K_3K_5}(\alpha\gamma\tau_2 + \alpha\tau_1K_3)\right. \\ &\quad \left. + \frac{1}{K}\frac{\eta_3}{K_2K_3K_4}(K\alpha\gamma\psi + \eta\beta_1K_2K_3)\right), \end{aligned}$$

where ρ denotes the overwhelming eigenvalue in the absolute value of FV^{-1}. Utilizing Theorem 2 in [27], the accompanying outcome is set up as follows.

Lemma 3.1 *The DFE of model* (1), *given by* (3), *is locally-asymptotically stable* (*LAS*) *if* $R_C < 1$ *and unstable if* $R_C > 1$.

The control reproduction number R_C represents the average number of new cases generated by a primary infectious individual in a population where some susceptible individuals receive antiviral prophylaxis. Lemma (3.1) shows that, for R_C <1, the H1N1 pandemic can be removed from the population if the basin of attraction of DFE(ε_0) contains the initial sub-populations. Global stability of the DFE is proved in the following theorem to ensure that illness can be destroyed totally if the control reproduction number is less than one.

Theorem 3.2 *The DFE,* ε_0, *of model* (1) *is GAS in D if* $R_C \leq R^* = \frac{\Omega}{\theta_H}$.

Proof Consider the Lyapunov function

$$G(t) = g_1L + gQ + g_2I_1 + g_3I_2 + g_4H + g_5T,$$

where

$$\begin{aligned} g_1 &= \eta_1K_2K_3K_4K_5 + \alpha K_3k_4K_5 + \alpha\gamma\eta_2K_4K_5 + \alpha\gamma\tau_2\eta_4K_4 + \alpha\tau_1\eta_4K_3K_4 \\ &\quad + \alpha\gamma\psi\eta_3K_5 + \frac{\beta_1\eta K_2K_3K_4K_5}{K} + \eta_3\eta\beta_1\frac{K_5K_2K_3}{K}, \\ g &= \frac{\eta\eta_3K_1K_2K_3K_5}{K} + \eta\frac{K_1K_2K_3K_4K_5}{K}, \\ g_2 &= \gamma\tau_2\eta_4K_1K_4 + \gamma\psi\eta_3K_1K_5 + \gamma\eta_2K_1K_4K_5 + \tau_1\eta_4K_1K_3K_4 + K_1K_3K_4K_5, \end{aligned}$$

$$g_3 = \eta_2 K_1 K_2 K_4 K_5 + \psi \eta_3 K_1 K_2 K_5 + \tau_2 \eta_4 K_1 K_2 K_4,$$

$$g_4 = \eta_3 K_1 K_2 K_3 K_5,$$

$$g_5 = \eta_4 K_1 K_2 K_3 K_4.$$

The time derivative of $G(t)$ is given by

$$\begin{aligned}
G'(t) &= g_1 L' + gQ' + g_2 I_1' + g_3 I_2' + g_4 H' + g_5 T' \\
&= g_1\big(\lambda(S_L + \theta_H S_H + \theta_P P) - K_1 L\big) + g(\beta_1 L - KQ) + g_2(\alpha L - K_2 I_1) + g_3(\gamma I_1 - K_3 I_2) \\
&\quad + g_4(\psi I_2 + \eta Q - K_4 H) + g_5(\tau_1 I_1 + \tau_2 I_2 - K_5 T) \\
&= \bigg(\eta_1 K_2 K_3 K_4 K_5 + \alpha K_3 K_4 K_5 + \alpha\gamma\eta_2 K_4 K_5 + \alpha\gamma\tau_2 K_4 \eta_4 + \alpha\tau_1 K_3 K_4 \eta_4 + \alpha\gamma\psi\eta_3 K_5 \\
&\quad + \frac{\beta_1 \eta K_2 K_3 K_4 k_5}{K} + \eta_3 \eta \beta_1 \frac{K_5}{K} K_2 K_3\bigg)\lambda(S_L + \theta_H S_H + \theta_P P) \\
&\quad - K_1 K_2 K_3 K_4 K_5(\eta_1 L + \eta Q + I_1 + \eta_2 I_2 + \eta_3 H + \eta_4 T) \\
&\le g_1 \lambda \theta_H N - K_1 K_2 K_3 K_4 K_5 \frac{\lambda N}{\beta} \\
&\le K_1 K_2 K_3 K_4 K_5 \frac{\lambda N}{\beta}\left(\frac{R_C}{R^*} - 1\right).
\end{aligned}$$

Thus $G'(t) \le 0$ if $R_C \le R^*$ and $G'(t) = 0$ if and only if $L = Q = I_1 = I_2 = H = T = 0$. Moreover, the greatest compact invariant set in $\{(S_L, S_H, P, L, Q, I_1, I_2, H, T, R) \in D : G' = 0\}$ is the singleton set $\{\varepsilon_0\}$. According to LaSalle's invariance principle [28], every solution to system (1) converges to ε_0, as $t \to \infty$. Hence the DFE is globally asymptotically stable. □

3.3 Endemic equilibrium and its stability

In this section, the existence of endemic equilibrium (EE) for system (1) (that is, equilibria where the infected classes are taken nonzero) and its stability are established. Let

$$E_1 = \big(S_L^{**}, S_H^{**}, P^{**}, L^{**}, Q^{**}, I_1^{**}, I_2^{**}, H^{**}, T^{**}, R^{**}\big)$$

be EE of model (1). Further, suppose that

$$\lambda^{**} = \frac{\beta(\eta_1 L^{**} + \eta Q^{**} + I_1^{**} + \eta_2 I_2^{**} + \eta_3 H^{**} + \eta_4 T^{**})}{N^{**}}$$

denotes the infection force at the steady-state. By simplifying the model at steady-state, we have

$$\begin{aligned}
S_L^{**} &= \frac{\pi - pip}{\lambda^{**} + \sigma_L + \mu}, \\
S_H^{**} &= \frac{\pi p}{\theta_H \lambda^{**} + \sigma_H + \mu}, \\
P^{**} &= \pi \frac{((1-p)\theta_H \sigma_L + p\sigma_H)\lambda^{**} + (1-p)\sigma_L(\sigma_H + \mu) + p\sigma_H(\sigma_L + \mu)}{Q}, \\
L^{**} &= \frac{\lambda^{**}(\pi \theta_P \theta_H \lambda^{**2} + m_1 \lambda^{**} + m_2)}{K_1 Q},
\end{aligned}$$

$$
\begin{aligned}
Q^{**} &= \frac{\lambda^{**}\beta_1(\pi\theta_P\theta_H\lambda^{**2} + m_1\lambda^{**} + m_2)}{K_1KQ}, \\
I_1^{**} &= \frac{\alpha\lambda^{**}(\pi\theta_P\theta_H\lambda^{**2} + m_1\lambda^{**} + m_2)}{K_1K_2Q}, \\
I_2^{**} &= \frac{\gamma\alpha\lambda^{**}(\pi\theta_P\theta_H\lambda^{**2} + m_1\lambda^{**} + m_2)}{K_1K_2K_3Q}, \\
H^{**} &= \frac{\alpha\gamma\psi\lambda^{**}(\pi\theta_P\theta_H\lambda^{**2} + m_1\lambda^{**} + m_2)}{K_1k_2K_3K_4Q} + \frac{\lambda^{**}\eta\beta_1(\pi\theta_P\theta_H\lambda^{**2} + m_1\lambda^{**} + m_2)}{K_1KK_4Q}, \\
T^{**} &= \frac{\tau_1\alpha\lambda^{**}(\pi\theta_P\theta_H\lambda^{**2} + m_1\lambda^{**} + m_2)}{K_1K_2K_5Q} + \frac{\tau_2\gamma\alpha\lambda^{**}(\pi\theta_P\theta_H\lambda^{**2} + m_1\lambda^{**} + m_2)}{K_1K_2K_3K_5Q},
\end{aligned} \tag{4}
$$

where

$$
\begin{aligned}
Q &= \big(\lambda^{**} + \sigma_L + \mu\big)\big(\theta_H\lambda^{**} + \sigma_H + \mu\big)\big(\theta_P\lambda^{**} + \mu\big), \\
m_1 &= \pi\big\{(1-p)\big[\mu\theta_H + (\sigma_H + \mu)\theta_P\big] + p\theta_H\big[\mu + (\sigma_L + \mu)\theta_P\big] + \theta_P A\big\}, \\
m_2 &= \pi\big\{(1-p)(\sigma_H + \mu)\mu + p\theta_H(\sigma_L + \mu)\mu + \theta_P B\big\}, \\
A &= (1-p)\theta_H\sigma_L + p\sigma_H, \\
B &= (1-p)\sigma_L(\sigma_H + \mu) + p\sigma_H(\sigma_L + \mu).
\end{aligned}
$$

Using (4) in the expression of λ^{**}, we get

$$
a_0\lambda^{**3} + b_0\lambda^{**2} + c_0\lambda^{**} + d_0 = 0, \tag{5}
$$

where

$$
\begin{aligned}
a_0 &= A_1A_8, \\
b_0 &= \pi\theta_H\theta_P\left(1 - \frac{R_C}{\Omega}\right) + \pi p\theta_P(1-\theta_H) + A_2A_8, \\
c_0 &= \pi(1-p)\big(\theta_P(\sigma_H + \mu) + \mu\theta_H\big)\left\{1 - \frac{R_C}{\Omega}\right\} + \pi p\big(\theta_P(\mu + \sigma_L + \mu)\big)\left\{1 - \frac{\theta_H R_C}{\Omega}\right\} \\
&\quad + \pi A_7\left(1 - \frac{\theta_P R_C}{\Omega}\right) + A_3A_8, \\
d_0 &= \pi(1-p)(\sigma_H + \mu)\left\{\mu\left(1 - \frac{R_C}{\Omega}\right) + \sigma_L\left(1 - \theta_P\frac{R_C}{\Omega}\right)\right\} \\
&\quad + \pi P(\sigma_L + \mu)\left\{\mu\left(1 - \theta_H\frac{R_C}{\Omega}\right) + \sigma_H\left(1 - \theta_P\frac{R_C}{\Omega}\right)\right\}, \\
A_1 &= \pi\theta_H\theta_P, \\
A_2 &= \pi\big\{(1-p)\big[\mu\theta_H + (\sigma_H + \mu)\theta_P\big] + p\theta_H\big[\mu + (\sigma_L + \mu)\theta_P\big] + \theta_P A\big\}, \\
A_3 &= \pi\big\{(1-p)(\sigma_H + \mu)\mu + p\theta_H(\sigma_L + \mu)\mu + \theta_P B\big\}, \\
A_7 &= (1-p)\theta_H\sigma_L + p\sigma_H, \\
A_8 &= \frac{1}{K_1} + \frac{\beta_1}{K_1K} + \frac{\alpha}{K_1K_2} + \frac{\gamma\alpha}{K_1K_2K_3} + \frac{\alpha\gamma\psi}{K_1K_2K_3K_4} + \frac{\eta\beta_1}{K_1KK_4} + \frac{\tau_1\alpha}{K_1K_2K_5}
\end{aligned}
$$

$$+\frac{\tau_2\gamma\alpha}{K_1K_2K_3K_5}+\frac{\gamma\alpha\phi_{I_2}}{\mu K_1K_2K_3}+\frac{\phi_H\alpha\gamma\psi}{\mu K_1K_2K_3K_4}+\frac{\phi_H\eta\beta_1}{\mu K_1KK_4}+\frac{\phi_T\tau_1\alpha}{\mu K_1K_2K_5}\frac{\phi_T\tau_2\gamma\alpha}{\mu K_1K_2K_3K_5}.$$

It can be easily verified that coefficients of Eq. (5) are positive when $\frac{R_C}{\Omega}<\frac{1}{\theta_H}$, then by Descarte's rule of sign, there is no positive root. When $\frac{R_C}{\Omega}>\frac{1}{\theta_P}$, all the coefficients of Eq. (5) are positive other than d_0, thus in this case, the sign changes only once. Hence, we conclude the above discussion as follows.

Theorem 3.3 *If $\frac{R_C}{\Omega}>\frac{1}{\theta_P}$, then there exists one and only one EE for system (1). However, in the case of $\frac{R_C}{\Omega}<\frac{1}{\theta_H}$, no EE exists.*

Now, the global stability of EE calculated for model (1) is given for the exceptional situation where the disease-induced fatality is negligible. It is noted that the setting $\delta=0$ implies $N\to\frac{\pi}{\mu}$ as $t\to\infty$. Using $N=\frac{\pi}{\mu}$ gives $\lambda=\beta_2(\eta_1L+\eta Q+I_1+\eta_2I_2+\eta_3H+\eta_4T)$, where $\beta_2=\beta\frac{\mu}{\pi}$. Consider the accompanying change of variables: $\frac{S_L}{S_L^*}=x_1$, $\frac{S_H}{S_H^*}=x_2$, $\frac{P}{P^*}=x_3$, $\frac{L}{L^*}=x_4$, $\frac{Q}{Q^*}=x_5$, $\frac{I_1}{I_1^*}=x_6$, $\frac{I_2}{I_2^*}=x_7$, $\frac{H}{H^*}=x_8$, $\frac{T}{T^*}=x_9$, $\frac{R}{R^*}=x_{10}$.

The Lyapunov function for the sub-framework comprising the initial nine equations of (1) is as follows:

$$\begin{aligned}\mathtt{L}(\mathtt{t})&=a_1(x_1-1-\log x_1)+a_2(x_2-1-\log x_2)+a_3(x_3-1-\log x_3)\\&\quad+a_4(x_4-1-\log x_4)+a_5(x_5-1-\log x_5)+a_6(x_6-1-\log x_6)\\&\quad+a_7(x_7-1-\log x_7)+a_8(x_8-1-\log x_8)+a_9(x_9-1-\log x_9),\end{aligned}$$

where a_i $(i=1,2,\dots,9)$ are constants and their values are found later. Now, differentiating $\mathtt{L}$ w.r.t. *time* along the solutions of (1), we have

$$\begin{aligned}\mathtt{L}'&=a_1\frac{\pi}{S_L^*}(1-p)\left(2-\frac{1}{x_1}-x_1\right)+a_1\beta_2\eta_1L^*(x_1+x_4-x_1x_4-1)\\&\quad+a_1\beta_2I_1^*(x_1+x_6-x_1x_6-1)+a_1\beta_2\eta_2I_2^*(x_1+x_7-x_1x_7-1)\\&\quad+a_1\beta_2\eta Q^*(x_1+x_5-x_1x_5-1)+a_1\beta_2\eta_3H^*(x_1+x_8-x_1x_8-1)\\&\quad+a_1\beta_2\eta_4T^*(x_1+x_9-x_1x_9-1)+a_2\frac{\pi p}{S_H^*}\left(2-\frac{1}{x_2}-x_2\right)\\&\quad+a_2\beta_2\eta_1L^*\theta_H(x_2+x_4-x_2x_4-1)\\&\quad+a_2\beta_2I_1^*\theta_H(x_2+x_6-x_2x_6-1)+a_2\beta_2\eta_2I_2^*\theta_H(x_2+x_7-x_2x_7-1)\\&\quad+a_2\beta_2\eta Q^*\theta_H(x_2+x_5-x_2x_5-1)+a_2\beta_2\eta_3H^*\theta_H(x_2+x_8-x_2x_8-1)\\&\quad+a_2\beta_2\eta_4T^*\theta_H(x_2+x_9-x_2x_9-1)+a_3\sigma_H\frac{S_H^*}{P^*}\left(x_2-x_3-\frac{x_2}{x_3}+1\right)\\&\quad+a_3\sigma_L\frac{S_L^*}{P^*}\left(x_1-x_3-\frac{x_1}{x_3}+1\right)\\&\quad+a_3\beta_2\eta_1L^*\theta_P(x_3+x_4-x_3x_4-1)+a_3\beta_2I_1^*\theta_P(x_3+x_6-x_3x_6-1)\\&\quad+a_3\beta_2\eta_2I_2^*\theta_P(x_3+x_7-x_3x_7-1)+a_3\beta_2\eta Q^*\theta_P(x_3+x_5-x_3x_5-1)\\&\quad+a_3\beta_2\eta_3H^*\theta_P(x_3+x_8-x_3x_8-1)+a_3\beta_2\eta_4T^*\theta_P(x_3+x_9-x_3x_9-1)\end{aligned}$$

$$
\begin{aligned}
&+ a_4\beta_2\eta_1 S_L^*(x_1x_4 - x_4 - x_1 + 1) + a_4\beta_2\eta_1\theta_P P^*(x_3x_4 - x_4 - x_3 + 1)\\
&+ a_4\beta_2\eta_1\theta_H S_H^*(x_2x_4 - x_4 - x_2 + 1) + a_4\beta_2 I_1^*\frac{S_L^*}{L^*}\left(x_1x_6 - x_4 - \frac{x_1}{x_4}x_6 + 1\right)\\
&+ a_4\beta_2\theta_H I_1^*\frac{S_H^*}{L^*}\left(x_2x_6 - x_4 - \frac{x_2}{x_4}x_6 + 1\right) + a_4\beta_2\eta_2 I_2^*\frac{S_L^*}{L^*}\left(x_1x_7 - x_4 - \frac{x_1}{x_4}x_7 + 1\right)\\
&+ a_4\beta_2\theta_P\frac{I_1^*}{L^*}P^*\left(x_3x_6 - x_4 - \frac{x_3}{x_4}x_6 + 1\right) + a_4\eta\beta_2\frac{S_L^*}{L^*}Q^*\left(x_1x_5 - x_4 - \frac{x_1}{x_4}x_5 + 1\right)\\
&+ a_4\beta_2\eta_3 S_L^*\frac{H^*}{L^*}\left(x_1x_8 - x_4 - \frac{x_1}{x_4}x_8 + 1\right) + a_4\beta_2\eta_4\frac{S_L^*}{L^*}T^*\left(x_1x_9 - x_4 - \frac{x_1}{x_4}x_9 + 1\right)\\
&+ a_4\eta\beta_2\theta_H\frac{S_H^*}{L^*}Q^*\left(x_2x_5 - x_4 - \frac{x_2}{x_4}x_5 + 1\right)\\
&+ a_4\beta_2\eta_3\theta_H S_H^*\frac{H^*}{L^*}\left(x_2x_8 - x_4 - \frac{x_2}{x_4}x_8 + 1\right)\\
&+ a_4\beta_2\eta_4\theta_H\frac{S_H^*}{L^*}T^*\left(x_2x_9 - x_4 - \frac{x_2}{x_4}x_9 + 1\right)\\
&+ a_4\beta_2\eta_2\theta_H I_2^*\frac{S_H^*}{L^*}\left(x_2x_7 - x_4 - \frac{x_2}{x_4}x_7 + 1\right)\\
&+ a_4\eta\beta_2\frac{\theta_P}{L^*}P^*Q^*\left(x_3x_5 - x_4 - \frac{x_3}{x_4}x_5 + 1\right)\\
&+ a_4\beta_2\eta_3\theta_P\frac{H^*}{L^*}P^*\left(x_3x_8 - x_4 - \frac{x_3}{x_4}x_8 + 1\right)\\
&+ a_4\beta_2\eta_4\frac{\theta_P}{L^*}P^*T^*\left(x_3x_9 - x_4 - \frac{x_3}{x_4}x_9 + 1\right)\\
&+ a_4\beta_2\eta_2\theta_P\frac{I_2^*}{L^*}P^*\left(x_3x_7 - x_4 - \frac{x_3}{x_4}x_7 + 1\right)\\
&+ a_5\beta_1\frac{L^*}{Q^*}\left[x_4 - x_5 - \frac{x_4}{x_5} + 1\right] + a_6\frac{\alpha}{I_1^*}L^*\left[x_4 - x_6 - \frac{x_4}{x_6} + 1\right]\\
&+ a_7\gamma\frac{I_1^*}{I_2^*}\left[x_6 - x_7 - \frac{x_6}{x_7} + 1\right]\\
&+ a_8\left[\frac{\eta}{H^*}Q^*\left(x_5 - x_8 - \frac{x_5}{x_8} + 1\right) + \psi\frac{I_2^*}{H^*}\left(x_7 - x_8 - \frac{x_7}{x_8} + 1\right)\right]\\
&+ a_9\left[\tau_1\frac{I_1^*}{T^*}\left(x_6 - x_9 - \frac{x_6}{x_9} + 1\right) + \tau_2\frac{I_2^*}{T^*}\left(x_7 - x_9 - \frac{x_7}{x_9} + 1\right)\right].
\end{aligned}
$$

In order to find the values of a_i $(i = 1, 2, \ldots, 9)$, putting the coefficients of $x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_1x_4, x_1x_5, x_1x_6, x_1x_7, x_1x_8, x_1x_9, x_2x_4, x_2x_5, x_2x_6, x_2x_7, x_2x_8, x_2x_9, x_3x_4, x_3x_5, x_3x_6, x_3x_7, x_3x_8, x_3x_9$ equal to zero, we get

$$
\begin{aligned}
&a_1 = a_4\frac{S_L^*}{L^*}, \qquad a_2 = a_4\frac{S_H^*}{L^*}, \qquad a_3 = a_4\frac{P^*}{L^*},\\
&a_5 = \frac{Q^*}{\beta_1 L^*}\left(a_1\beta_2\eta Q^* + a_2\beta_2\eta Q^*\theta_H + a_3\beta_2\eta Q^*\theta_P + a_8\frac{\eta}{H^*}Q^*\right),\\
&a_6 = \frac{I_1^*}{\alpha L^*}\left(a_1\beta_2 I_1^* + a_2\beta_2 I_1^*\theta_H + a_3\beta_2 I_1^*\theta_P + a_7\gamma\frac{I_1^*}{I_2^*} + a_9\tau_1\frac{I_1^*}{T^*}\right),
\end{aligned}
$$

$$a_7 = \frac{I_2^*}{\gamma I_1^*}\left(a_1\beta_2\eta_2 I_2^* + a_2\beta_2\eta_2 I_2^*\theta_H + a_3\beta_2\eta_2 I_2^*\theta_P + a_8\psi\frac{I_2^*}{H^*} + a_9\tau_2\frac{I_2^*}{T^*}\right),$$

$$a_8 = \frac{1}{(\frac{\eta}{H^*}Q^* + \psi\frac{I_2^*}{H^*})}\left(a_1\beta_2\eta_3 H^* + a_2\beta_2\eta_3 H^*\theta_H + a_3\beta_2\eta_3 H^*\theta_P\right),$$

$$a_9 = \frac{1}{(\tau_1\frac{I_1^*}{T^*} + \tau_2\frac{I_2^*}{T^*})}\left(a_1\beta_2\eta_4 T^* + a_2\beta_2\eta_4 T^*\theta_H + a_3\beta_2\eta_4 T^*\theta_P\right).$$

Now

$$\begin{aligned}
\mathtt{L}' = {} & a_1\frac{\pi}{S_L^*}(1-p)\left(1-\frac{1}{x_1}\right) + a_1(\mu+\sigma_L) + a_2\frac{\pi p}{S_H^*}\left(1-\frac{1}{x_2}\right) + a_2(\sigma_H+\mu) \\
& + a_3\sigma_H\frac{S_H^*}{P^*}\left(1-\frac{x_2}{x_3}\right) + a_3\sigma_L\frac{S_L^*}{P^*}\left(1-\frac{x_1}{x_3}\right) + a_3\mu + a_4\beta_2\eta_1 S_L^* + a_4\beta_2\eta_1\theta_P P^* \\
& + a_4\beta_2\eta_1\theta_H S_H^* + a_4\beta_2 I_1^*\frac{S_L^*}{L^*}\left(1-\frac{x_1}{x_4}x_6\right) + a_4\beta_2\theta_H I_1^*\frac{S_H^*}{L^*}\left(1-\frac{x_2}{x_4}x_6\right) \\
& + a_4\beta_2\eta_2 I_2^*\frac{S_L^*}{L^*}\left(1-\frac{x_1}{x_4}x_7\right) + a_4\beta_2\theta_P\frac{I_1^*}{L^*}P^*\left(1-\frac{x_3}{x_4}x_6\right) + a_4\eta\beta_2\frac{S_L^*}{L^*}Q^*\left(1-\frac{x_1}{x_4}x_5\right) \\
& + a_4\beta_2\eta_3 S_L^*\frac{H^*}{L^*}\left(1-\frac{x_1}{x_4}x_8\right) + a_4\beta_2\eta_4\frac{S_L^*}{L^*}T^*\left(1-\frac{x_1}{x_4}x_9\right) \\
& + a_4\eta\beta_2\theta_H\frac{S_H^*}{L^*}Q^*\left(1-\frac{x_2}{x_4}x_5\right) + a_4\beta_2\eta_3\theta_H S_H^*\frac{H^*}{L^*}\left(1-\frac{x_2}{x_4}x_8\right) \\
& + a_4\beta_2\eta_4\theta_H\frac{S_H^*}{L^*}T^*\left(1-\frac{x_2}{x_4}x_9\right) + a_4\beta_2\eta_2\theta_H I_2^*\frac{S_H^*}{L^*}\left(1-\frac{x_2}{x_4}x_7\right) \\
& + a_4\eta\beta_2\frac{\theta_P}{L^*}P^*Q^*\left(1-\frac{x_3}{x_4}x_5\right) \\
& + a_4\beta_2\eta_3\theta_P\frac{H^*}{L^*}P^*\left(1-\frac{x_3}{x_4}x_8\right) + a_4\beta_2\eta_4\frac{\theta_P}{L^*}P^*T^*\left(1-\frac{x_3}{x_4}x_9\right) \\
& + a_4\beta_2\eta_2\theta_P\frac{I_2^*}{L^*}P^*\left(1-\frac{x_3}{x_4}x_7\right) \\
& + a_5\beta_1\frac{L^*}{Q^*}\left[1-\frac{x_4}{x_5}\right] + a_6\frac{\alpha}{I_1^*}L^*\left[1-\frac{x_4}{x_6}\right] + a_7\gamma\frac{I_1^*}{I_2^*}\left[1-\frac{x_6}{x_7}\right] \\
& + a_8\left[\frac{\eta}{H^*}Q^*\left(1-\frac{x_5}{x_8}\right) + \psi\frac{I_2^*}{H^*}\left(1-\frac{x_7}{x_8}\right)\right] \\
& + a_9\left[\tau_1\frac{I_1^*}{T^*}\left(1-\frac{x_6}{x_9}\right) + \tau_2\frac{I_2^*}{T^*}\left(1-\frac{x_7}{x_9}\right)\right] \\
= {} & P(x_1, x_2, \ldots, x_9).
\end{aligned}$$

Let us construct the following function to show that $\mathtt{L}' \leq 0$ in D, which is a positively invariant region:

$$\mathcal{U}(x_1, x_2, \ldots, x_9) = \sum_{k=1}^{15} U_k(x_1, x_2, \ldots, x_9),$$

where

$$
\begin{aligned}
U_1 &= d_1\left(3-\frac{1}{x_1}-\frac{x_1}{x_4}x_6-\frac{x_4}{x_6}\right),\\
U_2 &= d_2\left(4-\frac{1}{x_2}-\frac{x_2}{x_3}-\frac{x_3}{x_4}x_6-\frac{x_4}{x_6}\right),\\
U_3 &= d_3\left(4-\frac{x_1}{x_3}-\frac{1}{x_1}-\frac{x_3}{x_4}x_5-\frac{x_4}{x_5}\right),\\
U_4 &= d_4\left(4-\frac{x_1}{x_4}x_9-\frac{1}{x_1}-\frac{x_4}{x_6}-\frac{x_6}{x_9}\right),\\
U_5 &= d_5\left(4-\frac{x_1}{x_4}x_7-\frac{1}{x_1}-\frac{x_4}{x_6}-\frac{x_6}{x_7}\right),\\
U_6 &= d_6\left(3-\frac{x_2}{x_4}x_6-\frac{1}{x_2}-\frac{x_4}{x_6}\right),\\
U_7 &= d_7\left(3-\frac{x_1}{x_4}x_5-\frac{1}{x_1}-\frac{x_4}{x_5}\right),\\
U_8 &= d_8\left(4-\frac{x_1}{x_4}x_8-\frac{1}{x_1}-\frac{x_5}{x_8}-\frac{x_4}{x_5}\right),\\
U_9 &= d_9\left(3-\frac{x_2}{x_4}x_5-\frac{1}{x_2}-\frac{x_4}{x_5}\right),\\
U_{10} &= d_{10}\left(5-\frac{x_2}{x_4}x_8-\frac{x_7}{x_8}-\frac{1}{x_2}-\frac{x_6}{x_7}-\frac{x_4}{x_6}\right),\\
U_{11} &= d_{11}\left(6-\frac{x_3}{x_4}x_8-\frac{x_2}{x_3}-\frac{1}{x_2}-\frac{x_7}{x_8}-\frac{x_6}{x_7}-\frac{x_4}{x_6}\right),\\
U_{12} &= d_{12}\left(6-\frac{x_3}{x_4}x_9-\frac{x_2}{x_3}-\frac{1}{x_2}-\frac{x_7}{x_9}-\frac{x_6}{x_7}-\frac{x_4}{x_6}\right),\\
U_{13} &= d_{13}\left(5-\frac{x_3}{x_4}x_7-\frac{x_2}{x_3}-\frac{1}{x_2}-\frac{x_6}{x_7}-\frac{x_4}{x_6}\right),\\
U_{14} &= d_{14}\left(5-\frac{x_2}{x_4}x_9-\frac{1}{x_2}-\frac{x_7}{x_9}-\frac{x_6}{x_7}-\frac{x_4}{x_6}\right),\\
U_{15} &= d_{15}\left(4-\frac{x_2}{x_4}x_7-\frac{1}{x_2}-\frac{x_6}{x_7}-\frac{x_4}{x_6}\right).
\end{aligned}
$$

Comparison of the same terms between $P(x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8,x_9)$ and

$$\sum_{k=1}^{15} U_k(x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8,x_9)$$

yields the following equations:

$$
\begin{aligned}
&d_1 = a_4\beta_2 I_1^*\frac{S_L^*}{L^*}, \qquad d_2 = a_4\beta_2\theta_P\frac{I_1^*}{L^*}P^*, \qquad d_3 = a_4\eta\beta_2\frac{\theta_P}{L^*}P^*Q^*,\\
&d_4 = a_4\beta_2\eta_4\frac{S_L^*}{L^*}T^*, \qquad d_5 = a_4\beta_2\eta_2 I_2^*\frac{S_L^*}{L^*},\\
&d_6 = a_4\beta_2\theta_H I_1^*\frac{S_H^*}{L^*}, \qquad d_7 = a_4\eta\beta_2\frac{S_L^*}{L^*}Q^*, \qquad d_8 = a_4\beta_2\eta_3 S_L^*\frac{H^*}{L^*},
\end{aligned}
$$

$$d_9 = a_4\eta\beta_2\theta_H\frac{S_H^*}{L^*}Q^*, \qquad d_{10} = a_4\beta_2\eta_3\theta_H S_H^*\frac{H^*}{L^*},$$
$$d_{11} = a_4\beta_2\eta_3\theta_P\frac{H^*}{L^*}P^*, \qquad d_{12} = a_4\beta_2\eta_4\frac{\theta_P}{L^*}P^*T^*, \qquad d_{13} = a_4\beta_2\eta_2\theta_P\frac{I_2^*}{L^*}P^*,$$
$$d_{14} = a_4\beta_2\eta_4\theta_H\frac{S_H^*}{L^*}T^*, \qquad d_{15} = a_4\beta_2\eta_2\theta_H I_2^*\frac{S_H^*}{L^*}.$$

Thus

$$\begin{aligned}
\mathtt{L}' = {} & a_4\beta_2 I_1^*\frac{S_L^*}{L^*}\left(3-\frac{1}{x_1}-\frac{x_1}{x_4}x_6-\frac{x_4}{x_6}\right)+a_4\beta_2\theta_P\frac{I_1^*}{L^*}P^*\left(4-\frac{1}{x_2}-\frac{x_2}{x_3}-\frac{x_3}{x_4}x_6-\frac{x_4}{x_6}\right)\\
&+a_4\eta\beta_2\frac{\theta_P}{L^*}P^*Q^*\left(4-\frac{x_1}{x_3}-\frac{1}{x_1}-\frac{x_3}{x_4}x_5-\frac{x_4}{x_5}\right)\\
&+a_4\beta_2\eta_4\frac{S_L^*}{L^*}T^*\left(4-\frac{x_1}{x_4}x_9-\frac{1}{x_1}-\frac{x_4}{x_6}-\frac{x_6}{x_9}\right)\\
&+a_4\beta_2\eta_2 I_2^*\frac{S_L^*}{L^*}\left(4-\frac{x_1}{x_4}x_7-\frac{1}{x_1}-\frac{x_4}{x_6}-\frac{x_6}{x_7}\right)+a_4\beta_2\theta_H I_1^*\frac{S_H^*}{L^*}\left(3-\frac{x_2}{x_4}x_6-\frac{1}{x_2}-\frac{x_4}{x_6}\right)\\
&+a_4\eta\beta_2\frac{S_L^*}{L^*}Q^*\left(3-\frac{x_1}{x_4}x_5-\frac{1}{x_1}-\frac{x_4}{x_5}\right)+a_4\beta_2\eta_3 S_L^*\frac{H^*}{L^*}\left(4-\frac{x_1}{x_4}x_8-\frac{1}{x_1}-\frac{x_5}{x_8}-\frac{x_4}{x_5}\right)\\
&+a_4\eta\beta_2\theta_H\frac{S_H^*}{L^*}Q^*\left(3-\frac{x_2}{x_4}x_5-\frac{1}{x_2}-\frac{x_4}{x_5}\right)\\
&+a_4\beta_2\eta_3\theta_H S_H^*\frac{H^*}{L^*}\left(5-\frac{x_2}{x_4}x_8-\frac{x_7}{x_8}-\frac{1}{x_2}-\frac{x_6}{x_7}-\frac{x_4}{x_6}\right)\\
&+a_4\beta_2\eta_3\theta_P\frac{H^*}{L^*}P^*\left(6-\frac{x_3}{x_4}x_8-\frac{x_2}{x_3}-\frac{1}{x_2}-\frac{x_7}{x_8}-\frac{x_6}{x_7}-\frac{x_4}{x_6}\right)\\
&+a_4\beta_2\eta_4\frac{\theta_P}{L^*}P^*T^*\left(6-\frac{x_3}{x_4}x_9-\frac{x_2}{x_3}--\frac{1}{x_2}-\frac{x_7}{x_9}-\frac{x_6}{x_7}-\frac{x_4}{x_6}\right)\\
&+a_4\beta_2\eta_2\theta_P\frac{I_2^*}{L^*}P^*\left(5-\frac{x_3}{x_4}x_7-\frac{x_2}{x_3}-\frac{1}{x_2}-\frac{x_6}{x_7}-\frac{x_4}{x_6}\right)\\
&+a_4\beta_2\eta_4\theta_H\frac{S_H^*}{L^*}T^*\left(5-\frac{x_2}{x_4}x_9-\frac{1}{x_2}-\frac{x_7}{x_9}-\frac{x_6}{x_7}-\frac{x_4}{x_6}\right)\\
&+a_4\beta_2\eta_2\theta_H I_2^*\frac{S_H^*}{L^*}\left(4-\frac{x_2}{x_4}x_7-\frac{1}{x_2}-\frac{x_6}{x_7}-\frac{x_4}{x_6}\right).
\end{aligned}$$

Since the arithmetic mean is greater than or equal to the geometric mean, we have $\frac{1}{x_1}+\frac{x_1}{x_4}x_6+\frac{x_4}{x_6}\geq 3$, $\frac{1}{x_2}+\frac{x_2}{x_3}+\frac{x_3}{x_4}x_6+\frac{x_4}{x_6}\geq 4$, $\frac{x_1}{x_3}+\frac{1}{x_1}+\frac{x_3}{x_4}x_5+\frac{x_4}{x_5}\geq 4$, $\frac{x_1}{x_4}x_9+\frac{1}{x_1}+\frac{x_4}{x_6}+\frac{x_6}{x_9}\geq 4$, $\frac{x_1}{x_4}x_7+\frac{1}{x_1}+\frac{x_4}{x_6}+\frac{x_6}{x_7}\geq 4$, $\frac{x_2}{x_4}x_6+\frac{1}{x_2}+\frac{x_4}{x_6}\geq 3$, $\frac{x_1}{x_4}x_5+\frac{1}{x_1}+\frac{x_4}{x_5}\geq 3$, $\frac{x_1}{x_4}x_8+\frac{1}{x_1}+\frac{x_5}{x_8}+\frac{x_4}{x_5}\geq 4$, $\frac{x_2}{x_4}x_5+\frac{1}{x_2}+\frac{x_4}{x_5}\geq 3$, $\frac{x_2}{x_4}x_8+\frac{x_7}{x_8}+\frac{1}{x_2}+\frac{x_6}{x_7}+\frac{x_4}{x_6}\geq 5$, $\frac{x_3}{x_4}x_8+\frac{x_2}{x_3}+\frac{1}{x_2}+\frac{x_7}{x_8}+\frac{x_6}{x_7}+\frac{x_4}{x_6}\geq 6$, $\frac{x_3}{x_4}x_9+\frac{x_2}{x_3}+\frac{1}{x_2}+\frac{x_7}{x_9}+\frac{x_6}{x_7}+\frac{x_4}{x_6}\geq 6$, $\frac{x_3}{x_4}x_7+\frac{x_2}{x_3}+\frac{1}{x_2}+\frac{x_6}{x_7}+\frac{x_4}{x_6}\geq 5$, $\frac{x_2}{x_4}x_9+\frac{1}{x_2}+\frac{x_7}{x_9}+\frac{x_6}{x_7}+\frac{x_4}{x_6}\geq 5$, $\frac{x_2}{x_4}x_7+\frac{1}{x_2}+\frac{x_6}{x_7}+\frac{x_4}{x_6}\geq 4$.

In this manner, we have $\mathtt{L}'\leq 0$ in D. The equality $\mathtt{L}'=0$ exists if and only if $\{x_i=1, i=1,2,\ldots,9\}$. That is, $S_L=S_L^*, S_H=S_H^*, P=P^*, L=L^*, Q=Q^*, I_1=I_1^*, I_2=I_2^*, H=H^*, T=T^*$ in D. The maximal compact invariant set in $\{(S_L,S_H,P,L,Q,I_1,I_2,H,T,R)\in D:\mathtt{L}'=0\}$ is "E_1" whenever $R_C>1$. By LaSalle's invariance principle [28], "E_1" is globally asymptotically stable for $R_C>1$.

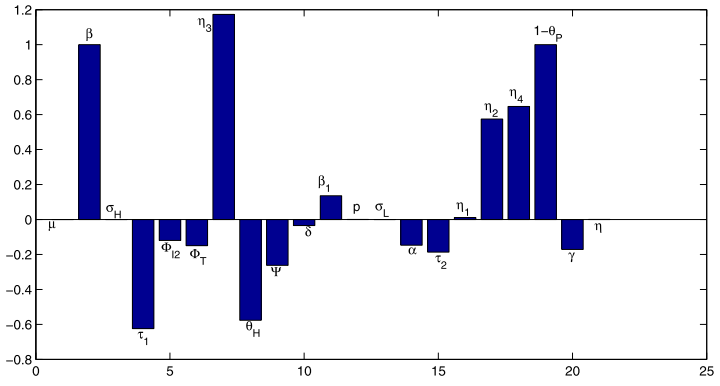

Figure 2 Sensitivity of R_C with respect to model parameters

4 Sensitivity analysis

The main thing for an infectious disease is to study its capability to enter a population. To check which variables are in charge of the expanse and existence of disease, we carry out the sensitivity analysis of R_C w.r.t. different parameters involved in R_C. It helps us in controlling the disease. We computed the sensitivity indices of the reproduction number R_C with respect to the model parameters given in Table 1. The sensitivity of the reproduction number R_C is shown in Fig. 2.

From Fig. 2, it is observed that sensitive parameters are $\eta_3, \beta, (1-\theta_P), \tau_1$, and θ_H. Parameters $\eta_3, \beta, (1-\theta_P)$ are directly proportional to R_C and τ_1 and θ_H are inversely proportional. It can be easily seen that by increasing (decreasing) the values of $\eta_3, \beta, (1-\theta_P)$ by 10%, the values of R_C increase (decrease) by almost 11%, 10%, and 9%, respectively. Similarly, by increment (reduction) in the values of τ_1 and θ_H by 10%, reduction (increment) of almost 6% and 5% occurs in the values of R_C, respectively. It means that we should focus on the isolation of hospitalized and infectious people. Sensitive parameter τ_1 indicates that the more the people will be treated at an early stage, the less the infection will spread.

5 Optimal control

Control strategies include prevention, vaccination or antiviral drugs, quarantine, and treatment. To estimate the effect of controlling strategies, we modify our model as follows:

$$\begin{aligned}
&\frac{dS_L}{dt} = \pi(1-p) - (1-u_1)\lambda S_L - \sigma_L S_L - r_2 u_2 S_L - \mu S_L,\\
&\frac{dS_H}{dt} = \pi p - (1-u_1)\theta_H \lambda S_H - \sigma_H S_H - r_2 u_2 S_H - \mu S_H,\\
&\frac{dP}{dt} = (\sigma_L S_L + \sigma_H S_H) + r_2 u_2 (S_L + S_H) - (1-u_1)\theta_P \lambda P - \mu P,\\
&\frac{dL}{dt} = (1-u_1)\lambda(S_L + \theta_H S_H + \theta_P P) - (\alpha+\mu)L - \beta_1 L - r_3 u_3 L,\\
&\frac{dQ}{dt} = \beta_1 L + r_3 u_3 L - (\mu+\eta)Q,\\
&\frac{dI_1}{dt} = \alpha L - \tau_1 I_1 - (\gamma+\mu)I_1,
\end{aligned} \tag{6}$$

$$\frac{dI_2}{dt} = \gamma I_1 - r_4 u_4 I_2 - (\tau_2 + \psi + \phi I_2 + \mu + \delta) I_2,$$
$$\frac{dH}{dt} = \psi I_2 + \eta Q - (\phi_H + \mu + \theta_1 \delta) H,$$
$$\frac{dT}{dt} = \tau_1 I_1 + \tau_2 I_2 + r_4 u_4 I_2 - (\phi_T + \mu) T,$$
$$\frac{dR}{dt} = \phi_{I_2} I_2 + \phi_H H + \phi_T T - \mu R.$$

The parameter $u_1(t)$ represents the awareness campaign of using the medical mask through the media transmission to reduce the force of infection, $u_2(t)$ portrays the vaccination or usage of antiviral drugs, $u_3(t)$ represents the quarantine of exposed individuals, and $u_4(t)$ denotes the treatment of infectious people. To inspect the optimal level of endeavors required to control the disease, we define the objective functional J. It helps to limit the number of infectious as well as minimize the cost of applied controls u_1, u_2, u_3, and u_4. One has

$$J(u_1, u_2, u_3, u_4) = \int_0^T \left(f_1 L + f_2 Q + f_3 I_1 + f_4 I_2 + \frac{1}{2} \sum_{i=1}^{4} B_i u_i^2 \right) dt,$$

where f_1, f_2, f_3, and f_4 represent the positive weights. The number of infected people and cost of controls $u_1(t), u_2(t), u_3(t)$, and $u_4(t)$ are reduced with the aid of the above mentioned objective functional. For this, we find an optimal control u_1^*, u_2^*, u_3^*, and u_4^* such that

$$J\left(u_1^*, u_2^*, u_3^*, u_4^*\right) = \min\left\{J(u_1, u_2, u_3, u_4), (u_1, u_2, u_3, u_4) \in U\right\},$$

where

$$U = \left\{(u_1, u_2, u_3, u_4) | u_i(t) \in [0, 1] \text{ and } u_i(t) \right.$$
$$\left. \text{is Lebesgue measurable on } [0, 1], i = 1, 2, 3, 4\right\}$$

is the control set. This OC problem is solved using Pontryagin's maximum principle [29] along with the derivation of necessary conditions.

6 Existence of an optimal control

Optimal control existence can be proved through a well-known classical result: according to [30], we have to check the following hypotheses:

- (H_1) The set consisting of controls and state variables is nonempty.
- (H_2) The admissible control set U is convex and closed.
- (H_3) R.H.S of system (6) is bounded by a linear function in the state and control.
- (H_4) The objective functional J has a convex integrand on U. This integrand is bounded below by $c_1(|u_1|^2 + |u_2|^2)^{\frac{\beta}{2}} - c_2$, where $c_1, c_2 > 0$ and $\beta > 1$.

The existence of solutions of ODEs (6) is established by using the result given by Lukes ([31], Th. 9.2.1, p. 182). In this way, we confirm the above hypotheses. (H_1) is satisfied because the coefficients are bounded. The boundedness of solutions shows that (H_2) has been satisfied by the control set. Since the solutions are bounded and we have bilinearity

of the system in u_1, u_2, u_3, u_4, hence R.H.S of (6) fulfills hypothesis (H_3). The last condition is also satisfied as the integrand of objective functional is convex.

$$f_1L + f_2Q + f_3I_1 + f_4I_2 + \frac{1}{2}\sum_{i=1}^{4} B_i {u_i}^2 \geq c_1\left(\sum_{i=1}^{4}|u_i|^2\right)^{\frac{\beta}{2}} - c_2,$$

where $f_1, f_2, f_3, f_4, B_1, B_2, B_3, B_4, c_1, c_2 > 0$ and $\beta > 1$. Hence we have the following theorem.

Theorem 6.1 *For* $U = \{(u_1, u_2, u_3, u_4) | 0 \leq u_i(t) \leq 1, i = 1, 2, 3, 4,$ *and* $t \in [0, T]\}$ *subject to Eqs.* (6) *having the initial conditions and*

$$J = \int_0^T \left(f_1L + f_2Q + f_3I_1 + f_4I_2 + \frac{1}{2}\sum_{i=1}^{4} B_i u_i^2\right) dt,$$

there is an optimal control $u = (u_1^*, u_2^*, u_3^*, u_4^*)$ *such that* $J(u_1^*, u_2^*, u_3^*, u_4^*) = \min\{J(u_1, u_2, u_3, u_4) : (u_1, u_2, u_3, u_4) \in U\}$.

For the solution of system (6), its Lagrangian and Hamiltonian have to be defined. Its Lagrangian is

$$\mathbb{L}(L, Q, I_1, I_2, u_1, u_2, u_3, u_4) = f_1L + f_2Q + f_3I_1 + f_4I_2 + \frac{1}{2}\left(B_1u_1^2 + B_2u_2^2 + B_3u_3^2 + B_4u_4^2\right).$$

We have to set up the minimal value of the Lagrangian. For this purpose, we construct the Hamiltonian H for the OC problem as follows:

Let us take $X = (S_L, S_H, P, L, Q, I_1, I_2, H, T, R)$, $U = (u_1, u_2, u_3, u_4)$, and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{10})$, then we have

$$\begin{aligned}
&H(X, U, \lambda)\\
&= f_1L + f_2Q + f_3I_1 + f_4I_2 + \frac{1}{2}\left(B_1u_1^2 + B_2u_2^2 + B_3u_3^2 + B_4u_4^2\right)\\
&\quad + \lambda_1\left(\pi(1-p) - (1-u_1)\lambda S_L - \sigma_L S_L - r_2u_2S_L - \mu S_L\right)\\
&\quad + \lambda_2\left(\pi p - (1-u_1)\theta_H\lambda S_H - \sigma_H S_H - r_2u_2S_H - \mu S_H\right)\\
&\quad + \lambda_3\left((\sigma_L S_L + \sigma_H S_H) + r_2u_2(S_L + S_H) - (1-u_1)\theta_P\lambda P - \mu P\right)\\
&\quad + \lambda_4\left((1-u_1)\lambda(S_L + \theta_H S_H + \theta_P P) - (\alpha + \mu)L - \beta_1 L - r_3u_3L\right)\\
&\quad + \lambda_5\left(\beta_1 L + r_3u_3L - (\mu + \eta)Q\right) + \lambda_6\left(\alpha L - \tau_1 I_1 - (\gamma + \mu)I_1\right)\\
&\quad + \lambda_7\left(\gamma I_1 - r_4u_4I_2 - (\tau_2 + \psi + \phi I_2 + \mu + \delta)I_2\right) + \lambda_8\left(\psi I_2 + \eta Q - (\phi_H + \mu + \theta_1\delta)H\right)\\
&\quad + \lambda_9\left(\tau_1 I_1 + \tau_2 I_2 + r_4u_4I_2 - (\phi_T + \mu)T\right) + \lambda_{10}(\phi_{I_2}I_2 + \phi_H H + \phi_T T - \mu R).
\end{aligned}$$

6.1 The optimality system

We apply Pontryagin's maximum principle [14] for finding the necessary conditions for the OC. It is discussed as follows:

There exists a nontrivial vector function $\lambda(t) = (\lambda_1(t), \lambda_2(t), \ldots, \lambda_{10}(t))$ provided $(u_1^*, u_2^*, u_3^*, u_4^*)$ is an optimal solution of the OC problem. This function satisfies the following conditions. The state equation is

$$\frac{dx}{dt} = \frac{\partial}{\partial \lambda}\big(H\big(t, u_1^*, u_2^*, \lambda(t)\big)\big),$$

the condition of optimality is given by

$$\frac{\partial}{\partial u}\big(H\big(t, u_1^*, u_2^*, u_3^*, u_4^*, \lambda(t)\big)\big) = 0,$$

and the equation containing the adjoint variables is given by

$$\frac{d\lambda}{dt} = -\frac{\partial}{\partial x}\big(H\big(t, u_1^*, u_2^*, u_3^*, u_4^*, \lambda(t)\big)\big).$$

Now, essential conditions are applied to the Hamiltonian H.

Theorem 6.2 *For the optimal controls $u_1^*, u_2^*, u_3^*, u_4^*$ and solutions $\hat{S_L}, \hat{S_H}, \hat{P}, \hat{L}, \hat{Q}, \hat{I_1}, \hat{I_2}, \hat{H}, \hat{T}, \hat{R}$ of the corresponding state system* (6)*, there are adjoint variables $\lambda_1, \lambda_2, \ldots, \lambda_{10}$ satisfying the following equations*:

$$\begin{aligned}
\frac{d\lambda_1}{dt} &= (\lambda_1 - \lambda_4)(1 - u_1)\lambda + (\lambda_1 - \lambda_3)r_2u_2 + \lambda_1\mu - \lambda_3\sigma_L,\\
\frac{d\lambda_2}{dt} &= (\lambda_2 - \lambda_4)(1 - u_1)\lambda\theta_H + (\lambda_2 - \lambda_3)\sigma_H + (\lambda_2 - \lambda_3)r_2u_2 + \lambda_2\mu,\\
\frac{d\lambda_3}{dt} &= (\lambda_3 - \lambda_4)(1 - u_1)\lambda\theta_P + \lambda_3\mu,\\
\frac{d\lambda_4}{dt} &= -f_1 + (\lambda_4 - \lambda_6)\alpha + (\lambda_4 - \lambda_5)\beta_1 + (\lambda_4 - \lambda_5)r_3u_3 + \lambda_4\mu,\\
\frac{d\lambda_5}{dt} &= -f_2 + \lambda_5\mu + (\lambda_5 - \lambda_8)\eta,\\
\frac{d\lambda_6}{dt} &= -f_3 + (\lambda_6 - \lambda_9)\tau_1 + (\lambda_6 - \lambda_7)\gamma + \lambda_6\mu,\\
\frac{d\lambda_7}{dt} &= -f_4 + (\lambda_7 - \lambda_9)r_4u_4 + (\lambda_7 - \lambda_9)\tau_2 + (\lambda_7 - \lambda_8)\psi + (\lambda_7 - \lambda_{10})\phi_{I_2} + \lambda_7(\mu + \delta),\\
\frac{d\lambda_8}{dt} &= (\lambda_8 - \lambda_{10})\phi_H + \lambda_8(\mu + \theta_1\delta),\\
\frac{d\lambda_9}{dt} &= (\lambda_9 - \lambda_{10})\phi_T + \lambda_9\mu,\\
\frac{d\lambda_{10}}{dt} &= \lambda_{10}\mu,
\end{aligned}$$

with transversality conditions $\lambda_1(T) = \lambda_2(T) = \cdots = \lambda_{10}(T) = 0$. Additionally, $u_1^*, u_2^*, u_3^*, u_4^*$ are given by

$$\begin{aligned}
u_1^* &= \frac{(\lambda_4 - \lambda_1)\lambda S_L + (\lambda_4 - \lambda_2)\theta_H\lambda S_H + (\lambda_4 - \lambda_3)\theta_P\lambda P}{B_1},\\
u_2^* &= \frac{(\lambda_1 - \lambda_3)r_2S_L + (\lambda_2 - \lambda_3)r_2S_H}{B_2},
\end{aligned}$$

$$u_3^* = \frac{\lambda_4 r_3 L - \lambda_5 r_3 L}{B_3},$$
$$u_4^* = \frac{(\lambda_7 - \lambda_9) r_4 I}{B_4}.$$

Proof Hamiltonian H is used for determining the adjoint equations and transversality conditions. Let us consider $S_L = \hat{S_L}, S_H = \hat{S_H}, P = \hat{P}, L = \hat{L}, Q = \hat{Q}, I_1 = \hat{I_1}, I_2 = \hat{I_2}, H = \hat{H}, T = \hat{T}, R = \hat{R}$, and differentiating H w.r.t. $(S_L, S_H, P, L, Q, I_1, I_2, H, T, R)$, we obtain

$$\frac{d\lambda_1}{dt} = (\lambda_1 - \lambda_4)(1 - u_1)\lambda + (\lambda_1 - \lambda_3) r_2 u_2 + \lambda_1 \mu - \lambda_3 \sigma_L,$$
$$\frac{d\lambda_2}{dt} = (\lambda_2 - \lambda_4)(1 - u_1)\lambda\theta_H + (\lambda_2 - \lambda_3)\sigma_H + (\lambda_2 - \lambda_3) r_2 u_2 + \lambda_2 \mu,$$
$$\frac{d\lambda_3}{dt} = (\lambda_3 - \lambda_4)(1 - u_1)\lambda\theta_P + \lambda_3 \mu,$$
$$\frac{d\lambda_4}{dt} = -f_1 + (\lambda_4 - \lambda_6)\alpha + (\lambda_4 - \lambda_5)\beta_1 + (\lambda_4 - \lambda_5) r_3 u_3 + \lambda_4 \mu,$$
$$\frac{d\lambda_5}{dt} = -f_2 + \lambda_5 \mu + (\lambda_5 - \lambda_8)\eta,$$
$$\frac{d\lambda_6}{dt} = -f_3 + (\lambda_6 - \lambda_9)\tau_1 + (\lambda_6 - \lambda_7)\gamma + \lambda_6 \mu,$$
$$\frac{d\lambda_7}{dt} = -f_4 + (\lambda_7 - \lambda_9) r_4 u_4 + (\lambda_7 - \lambda_9)\tau_2 + (\lambda_7 - \lambda_8)\psi + (\lambda_7 - \lambda_{10})\phi_{I_2} + \lambda_7(\mu + \delta),$$
$$\frac{d\lambda_8}{dt} = (\lambda_8 - \lambda_{10})\phi_H + \lambda_8(\mu + \theta_1 \delta),$$
$$\frac{d\lambda_9}{dt} = (\lambda_9 - \lambda_{10})\phi_T + \lambda_9 \mu,$$
$$\frac{d\lambda_{10}}{dt} = \lambda_{10} \mu,$$

with transversality conditions $\lambda_1(T) = \lambda_2(T) = \cdots = \lambda_{10}(T) = 0$. With the help of control space U and conditions of optimality, we can write

$$u_1^* = \frac{(\lambda_4 - \lambda_1)\lambda S_L + (\lambda_4 - \lambda_2)\theta_H \lambda S_H + (\lambda_4 - \lambda_3)\theta_P \lambda P}{B_1},$$
$$u_2^* = \frac{(\lambda_1 - \lambda_3) r_2 S_L + (\lambda_2 - \lambda_3) r_2 S_H}{B_2},$$
$$u_3^* = \frac{\lambda_4 r_3 L - \lambda_5 r_3 L}{B_3},$$
$$u_4^* = \frac{(\lambda_7 - \lambda_9) r_4 I}{B_4}.$$

□

7 Numerical simulations

In this section, the model is solved numerically. The values of weight constants in the objective functional are $f_1 = 1, f_2 = 5, f_3 = 10, f_4 = 8, B_1 = 3, B_2 = 7, B_3 = 8, B_4 = 9$. Other parameter values are given in Table 1. It is observed that the numerical outcomes are in great concurrence with the obtained hypothetical outcomes. Figure 3 demonstrates that the population approaches DFE when R_C is less than 1, while Fig. 4 demonstrates that

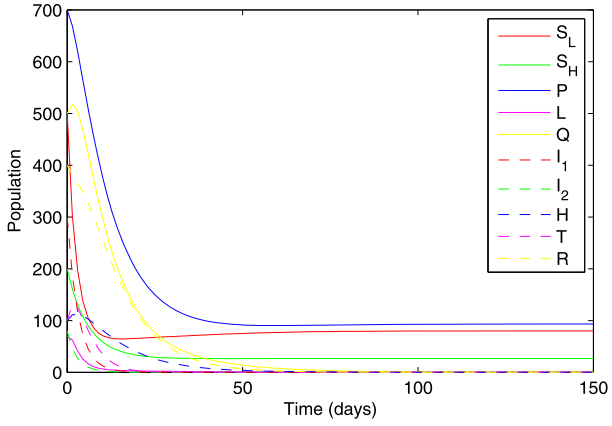

Figure 3 The population approaches DFE when $R_C = 0.238$

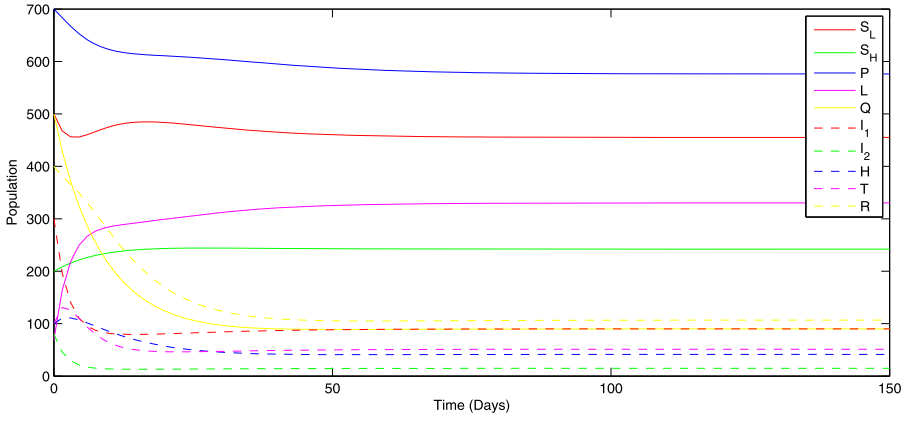

Figure 4 The population approaches EE when $R_C = 1.4$

the population approaches EE when the reproductive number exceeds unity even $\delta \neq 0$. It means that the condition $\delta = 0$ is the weaker condition for the global stability of endemic equilibrium. We also numerically investigated the influence of applied control strategies on the spread of swine flu in a population. Individuals having no control are represented by red lines in the graphs, while blue lines indicate the individuals with control. In Fig. 5, we observed that the endemic level of $L(t)$ (latent individuals), $Q(t)$ (quarantined individuals), $I_1(t)$ (symptomatic individuals at initial stage), $I_2(t)$ (symptomatic individuals at later stage), and $H(t)$ (hospitalized individuals) is reduced by applying these control strategies.

8 Conclusions

The deterministic model of swine influenza pandemic is rigorously analyzed in this article. The model consists of ten mutually exclusive compartments. It is shown that disease-free equilibrium is globally asymptotically stable whenever $R_C \leq R^* = \frac{\Omega}{\theta_h}$. The existence of unique endemic equilibrium is proved for $\frac{R_C}{\Omega} > \frac{1}{\theta_P}$, and its global stability is computed analytically when $\delta = 0$. Numerically, it is shown that the population approaches the endemic level even if $\delta \neq 0$. Sensitivity analysis suggests that the parameters β, η_3, τ_1 greatly influence the control reproduction number. Optimal control problem is analyzed. It consists

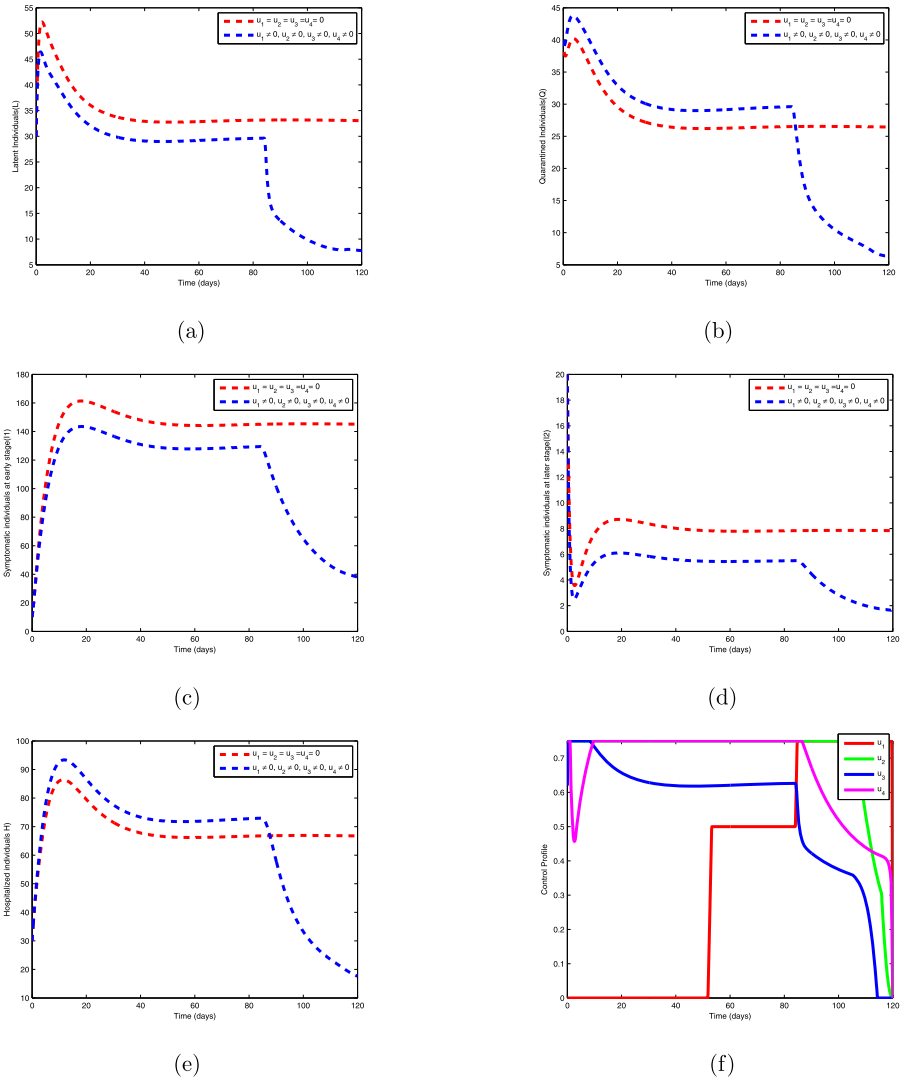

Figure 5 The plots represent the population with and without control

of four controls: the awareness campaign of using the medical mask through the media transmission to reduce the force of infection, vaccination or the use of antiviral drugs, the quarantine of exposed individuals, and the treatment of infectious people. Graphical results verify the usefulness of these control measures.

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Authors' contributions
TH and MO formulated the problem and determined the control reproduction number. MI and AUA calculated equilibria of the model and proved their global stability. AA performed the sensitivity analysis. TH and MO designed the optimal control problem. KOO did the numerical simulation. All authors read and approved the final manuscript.

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