Zhang and Zhang *Advances in Difference Equations* (2019) 2019:496
https://doi.org/10.1186/s13662-019-2422-8

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Some invariant solutions and conservation laws of a type of long-water wave system

Xiangzhi Zhang[1*] and Yufeng Zhang[1]

*Correspondence: zz301@163.com
[1]School of Mathematics, China University of Mining and Technology, Xuzhou, P.R. China

Abstract

We propose a generalized long-water wave system that reduces to the standard water wave system. We also obtain the Lax pair and symmetries of the generalized shallow-water wave system and single out some their similarity reductions, group-invariant solutions, and series solutions. We further investigate the corresponding self-adjointness and the conservation laws of the generalized system.

PACS Codes: 05.45.Yv; 02.30.Jr; 02.30.Ik

Keywords: Similarity solution; Conservation law; Symmetry

1 Introduction

The classical dispersiveless long wave equations

$$\begin{cases} u_t + uu_x + h_x = 0, \\ h_t + (uh)_x = 0, \end{cases} \tag{1}$$

have a number of dispersive generalizations [1]. Kupershmidt [2] considered the following extension of (1):

$$\begin{cases} u_t = (\frac{1}{2}u^2 + h - \beta u_x)_x, \\ h_t = (uh + \alpha u_{xx} - \beta h_x)_x, \end{cases} \tag{2}$$

where α, β are arbitrary constants. The invertible change of variables $u = \bar{u}, h = \bar{h} + \gamma \bar{u}x$, turns (2) into

$$\begin{cases} \bar{u}_t = (\frac{1}{2}\bar{u}^2 + \bar{h} + \mu \bar{u}_x)_x, \\ \bar{h}_t = (\bar{u}\bar{h} - \mu \bar{h}_x)_x, \mu = \gamma + \beta = \pm\sqrt{\alpha + \beta^2}. \end{cases}$$

Broer [1] derived system (2) for $\alpha = \frac{1}{3}, \beta = 0$, for which it is the proper Boussinesq equation. In terms of the potential $\varphi : u = \varphi_x$, system (2) was derived by Kaup [3]. Later, Matveev and Yavor [4] found algebrogeometrically a large class of almost periodic solutions. Li, Ma, and Zhang [5] used a scaling transformation to transfer a nonlinear long wave equation of Boussinesq class to the Broer–Kaup (BK) system, a type of long water wave equations:

$$\begin{cases} v_t = \frac{1}{2}(v^2 + 2w - v_x)_x, \\ w_t = (vw + \frac{1}{2}w_x)_x. \end{cases}$$

Furthermore, some exact solutions and Darboux transformations of (1) were obtained by applying the Lax-pair method. In terms of [5], we can study the similarity reduction, exact solutions, and conservation laws of the Boussinesq system through the scalar transformation

$$v = -u, w = \xi + 1 + \frac{v_x}{2},$$

that is, we can transform the BK system to the Boussinesq system

$$\begin{cases} \xi_t + [(1+\xi)u]_x = -\frac{1}{4}u_{xxx}, \\ u_t + uu_x + \xi_x = 0, \end{cases}$$

where ξ is the elevation of the water wave, u is the surface velocity of water along the x-direction. Hence, the results of the paper have certain physical sense.

In the paper, we construct a generalized BK system as follows:

$$\begin{cases} v_t = \frac{\alpha}{2}(v_x - v^2 - 2w)_x - \beta v_x, \\ w_t = -\frac{\alpha}{2}(w_x + 2wv)_x - \beta w_x, \end{cases}$$

where α, β are constants, so that some symmetries of (2) are produced by the symmetry group method [6]. It follows that some similarity solutions, group-invariant solutions, and series solutions are produced. In addition, Ibragimov and Avdonina [7] showed how to apply the symmetries of differential equations to study the self-adjointness and conservation laws. Thus we would like to follow the approach to investigate the quasiself-adjointness and conservation laws of the generalized BK system (2).

2 Integrability of (2)

Set

$$\varphi_x = U\varphi, \qquad \varphi_t = V\varphi, \tag{3}$$

where

$$U = \begin{pmatrix} -\lambda + \frac{v}{2} & 1 \\ -w & \lambda - \frac{v}{2} \end{pmatrix},$$

$$V = \begin{pmatrix} \alpha\lambda^2 + \beta\lambda + \frac{\alpha}{4}v_x - \frac{\alpha}{4}v^2 - \frac{\beta}{2}v & -\alpha\lambda - \frac{\alpha}{2}v - \beta \\ \alpha w\lambda + \frac{\alpha}{2}w_x + \frac{\alpha}{2}wv + \beta w & -\alpha\lambda^2 - \beta\lambda - \frac{\alpha}{4}v_x + \frac{\alpha}{4}v^2 + \frac{\beta}{2}v \end{pmatrix}.$$

Then the compatibility condition of (3)

$$V_t - U_x + UV - VU = 0$$

admits the generalized BK system, which can be directly verified. Hence the generalized BK system (2) is Lax integrable. Using (3), we can get some Darboux transformations for deducing solutions of the system. Here we omit them.

3 Similarity solutions and group-invariant solutions

Applying the Lie symmetry analysis, we can get the symmetry of system (2):

$$X_1 = \frac{\partial}{\partial t}, \qquad X_2 = \frac{\partial}{\partial x}, \qquad X_3 = \left(\frac{1}{2}x + \beta t\right)\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - \frac{1}{2}v\frac{\partial}{\partial v} - w\frac{\partial}{\partial w}. \tag{4}$$

The vector field X_3 has the following characteristic equation:

$$\frac{dt}{t} = \frac{dx}{\frac{1}{2}x + \beta t} = \frac{dv}{-\frac{1}{2}v} = \frac{dw}{-w}, \tag{5}$$

which gives rise to

$$\left(\beta t + \frac{1}{2}x\right)dt - t\,dx = 0. \tag{6}$$

One integration factor of (6) is given by

$$\mu = e^{-\int \frac{3}{2t}\,dt} = t^{-\frac{3}{2}},$$

which transfers (6) to the complete integration equation

$$\beta t^{-\frac{1}{2}}\,dt + d\left(-t^{-\frac{1}{2}}x\right) = 0,$$

from which we have the invariant variable $\xi = 2\beta t^{\frac{1}{2}} - t^{-\frac{1}{2}}x$. In terms of Eq. (5), we have the formal invariants

$$v = t^{-\frac{1}{2}}f(\xi), \qquad w = t^{-1}g(\xi), \tag{7}$$

where $f(\xi)$ and $g(\xi)$ are arbitrary smooth functions of ξ. Substituting (7) into system (2) yields the ordinary differential system

$$\begin{cases} -\frac{1}{2}f(\xi) + \frac{1}{2}xt^{-\frac{1}{2}}f'(\xi) = \frac{\alpha}{2}[f''(\xi) + 2f(\xi)f'(\xi) + 2g'(\xi)], \\ -g(\xi) + \frac{1}{2}xt^{-\frac{1}{2}}g'(\xi) = -\frac{\alpha}{2}[g''(\xi) - 2g'(\xi)f(\xi) - 2g(\xi)f'(\xi)]. \end{cases} \tag{8}$$

Let $\beta = 0$, Then system (8) reduces to

$$\begin{cases} f(\tau) + \tau f'(\tau) = -\alpha[f''(\tau) + 2f(\tau)f'(\tau) + 2g'(\tau)], \\ g(\tau) + \frac{1}{2}\tau g'(\tau) = \frac{\alpha}{2}[g''(\tau) - 2g'(\tau)f(\tau) - 2g(\tau)f'(\tau)], \end{cases} \tag{9}$$

where $\tau = -t^{-\frac{1}{2}}x$, which is a reduction of ξ. In facr, system (9) is an ordinary differential system corresponding to the BK system (1).

The group-invariant transformations of the generalized BK system are as follows:

$$\begin{cases} g_1 : (x,t,v,w) \to (x,t+\epsilon,v,w), \\ g_2 : (x,t,v,w) \to (x+\epsilon,t,v,w), \\ g_3 : (x,t,v,w) \to (2\beta te^{\epsilon} + (x-2\beta t)e^{\frac{1}{2}\epsilon}, te^{\epsilon}, e^{-\frac{1}{2}\epsilon}v, we^{-\epsilon}). \end{cases} \tag{10}$$

In what follows, we consider solutions to the BK system. Set $v = V(\rho)$, $w = W(\rho)$, $\rho = x + lt$. Then (2) becomes

$$\begin{cases} lV' = \frac{\alpha}{2}(V'' - 2VV' - 2W') - \beta V', \\ lW' = -\frac{\alpha}{2}(W'' + 2W'V + 2WV') - \beta W', \end{cases}$$

from which we have

$$\begin{cases} lV - \frac{\alpha}{2}(V' - V^2 - 2W) + \beta V = c_1, \\ lW + \frac{\alpha}{2}(W' + 2WV) - \beta W' = c_2. \end{cases} \tag{11}$$

A special solution to (11) is given by

$$V = \frac{1}{c+\xi}, \qquad W = \frac{1}{(c+\xi)^2} \tag{12}$$

in the case of $l = -\beta$, $c_1 = c_2 = 0$. Hence we get a set of solutions to the generalized BK system (2):

$$v = \frac{1}{c+x-\beta t}, \qquad w = \frac{1}{(c+x-\beta t)^2}. \tag{13}$$

Applying the group-invariant transformation (10), we can deduce some other new solutions to system (2):

$$\begin{cases} g_1: \quad v = \frac{1}{c+x-\beta(t+\epsilon)}, \qquad w = \frac{1}{[c+x-\beta(t+\epsilon)]^2}, \\ g_2: \quad v = \frac{1}{c+x-\beta t+\epsilon}, \qquad w = \frac{1}{(c+x-\beta t+\epsilon)^2}, \\ g_3: \quad v = \frac{e^{-\frac{1}{2}\epsilon}}{c+\beta te^{\epsilon}+(x-2\beta t)e^{\frac{1}{2}\epsilon}}, \qquad w = \frac{e^{-\epsilon}}{[c+\beta te^{\epsilon}+(x-2\beta t)e^{\frac{1}{2}\epsilon}]^2}. \end{cases}$$

Taking $\beta = 0$, we can obtain group-invariant solutions to the BK system (1). In particular, we can get the series solutions to the BK system. Indeed, let

$$f(\tau) = \sum_{n=0}^{\infty} c_n\tau^n, \qquad g(\tau) = \sum_{m=0}^{\infty} c_m\tau^m, \tag{14}$$

and substituting into (9), we have that

$$
\begin{aligned}
& c_0 + \sum_{n=1}^{\infty} c_n \tau^n + \tau c_1 + \sum_{n=1}^{\infty} (n+1) c_{n+1} \tau^{n+1} \\
&\quad = -\alpha \left[2c_2 + \sum_{n=1}^{\infty} (n+2)(n+1) c_{n+2} \tau^n \right] + 2 \left(c_0 + \sum_{n=1}^{\infty} c_n \tau^n \right) \left(c_1 + \sum_{n=1}^{\infty} (n+1) c_{n+1} \tau^n \right) \\
&\qquad + 2d_1 + 2 \sum_{m=1}^{\infty} (m+1) d_{m+1} \tau^m, \\
& d_0 + \sum_{m=1}^{\infty} d_m \tau^m + \frac{1}{2} \tau d_1 + \frac{1}{2} \sum_{m=1}^{\infty} (m+1) d_{m+1} \tau^{m+1} \\
&\quad = \frac{\alpha}{2} [2d_2 + \sum_{m=1}^{\infty} (m+2)(m+1) d_{m+2} \tau^m - 2 \left(c_0 + \sum_{n=1}^{\infty} c_n \tau^n \right) \left(d_1 + \sum_{m=1}^{\infty} (m+1) d_{m+1} \tau^m \right) \\
&\qquad - 2 \left(d_0 + \sum_{m=1}^{\infty} d_m \tau^m \right) \left(c_1 + \sum_{n=1}^{\infty} (n+1) c_{n+1} \tau^n \right),
\end{aligned}
$$

from which we infer that

$$
\begin{aligned}
& c_2 = -c_0 c_1 - \frac{1}{2\alpha} c_0 - d_1, \\
& d_2 = c_0 d_1 + d_0 c_1 + \frac{1}{\alpha} d_0, \\
& c_3 = -\frac{1}{3\alpha} c_1 - \frac{2}{3} c_0 c_2 - \frac{1}{6} c_1^2 - \frac{2}{3} d_1, \\
& d_3 = \frac{1}{2\alpha} d_1 + \frac{2}{3} c_0 d_2 - \frac{1}{3} d_1 c_1 + \frac{2}{3} d_0 c_2, \\
& \cdots\cdots, \\
& c_{n+2} = \frac{1}{\alpha(n+1)(n+2)} \Bigg[-c_n - 2\alpha c_0 (n+1) c_{n+1} - 2\alpha c_1 c_n \\
&\qquad - \alpha \sum_{i,j=2}^{n} c_i c_{j+1} (j+1) \tau^{i+j} - 2\alpha (n+1) d_{n+1} \Bigg], \\
& d_{n+2} = \frac{1}{(n+1)(n+2)} \Bigg[\frac{2}{\alpha} d_n + 2(n+1) c_0 d_{n+1} + 2d_1 c_n + 2 \sum_{i,j=2}^{n} c_i d_{j+1} \tau^{i+j} \\
&\qquad + 2(n+1) d_0 c_{n+1} + 2c_1 d_n + 2 \sum_{i,j=2}^{n} (j+1) d_i c_{j+1} \tau^{i+j} \Bigg],
\end{aligned}
$$

where c_0, d_0, c_1, d_1 are arbitrary parameters. Inserting these expressions into (14), we get the series solutions of the BK system. The second equation of system (9) can be reduced to

$$
g''(\tau) - \frac{1}{\alpha} \tau g'(\tau) - \frac{2}{\alpha} g(\tau) = 0 \tag{15}
$$

under the condition

$$(fg)' = 0 \quad \Rightarrow \quad fg = c. \tag{16}$$

As long as the solution of (15) is obtained, we can get the solution $f(\tau)$ from (16). If $g_1(\tau)$ is the known solution of (15), then we assume that $g(\tau) = u(\tau)g_1(\tau)$. If $u(\tau)$ is known, then the solution $g(\tau)$ to Eq. (15) can be presented. It is easy to see that

$$g''(\tau) = g_1(\tau)u''(\tau) + 2u'(\tau)g_1'(\tau) + u(\tau)g_1''(\tau). \tag{17}$$

Substituting (17) into Eq. (15) yields

$$g_1(\tau)u''(\tau) + \left(2g_1'(\tau) - \frac{1}{\alpha}\tau g_1(\tau)\right)u'(\tau) + \left(g_1''(\tau) - \frac{1}{\alpha}\tau g_1'(\tau) - \frac{2}{\alpha}g_1(\tau)\right)u(\tau) = 0.$$

Since

$$g_1''(\tau) - \frac{1}{\alpha}\tau g_1'(\tau) - \frac{2}{\alpha}g_1(\tau) = 0,$$

we have

$$g_1(\tau)u''(\tau) + \left(2g_1'(\tau) - \frac{1}{\alpha}\tau g_1(\tau)\right)u'(\tau) = 0. \tag{18}$$

Assume that $u'(\tau) = z(\tau)$. Then Eq. (18) becomes

$$g_1(\tau)z'(\tau) + \left(2g_1'(\tau) - \frac{1}{\alpha}\tau g_1(\tau)\right)z(\tau) = 0,$$

which has the solution

$$z = \frac{c}{g_1^2(\tau)}e^{\int \frac{1}{\alpha}\tau\, d\tau} = \frac{c}{g_1^2(\tau)}e^{\frac{1}{2\alpha}\tau^2},$$

where c is a constant. Thus we have

$$\begin{aligned} u(\tau) &= c\int^{\tau} \frac{1}{g_1^2(\tau)}e^{\frac{\tau^2}{2\alpha}}\, d\tau + \bar{c}, \\ g(\tau) &= g_1(\tau)\left[c\int^{\tau} \frac{1}{g_1^2(\tau)}e^{\frac{\tau^2}{2\alpha}}\, d\tau + \bar{c}\right]. \end{aligned} \tag{19}$$

Substituting (19) into Eq. (16), we can get $f(\tau)$. Thus a type of special solutions to system (9) can be obtained.

4 The self-adjointness of system (2)

Ibragimov [8] introduced a few related notations of the strict self-adjointness, the nonlinear self-adjointness, and the quasiself-adjointness. Let us recall them.

Let H be a Hilbert space with the scalar product (u, v) defined by

$$(Fu, v) = \big(u, F^*v\big), \quad u, v \in H, \tag{20}$$

where F^* is the adjoint operator to a linear operator F. A special Hilbert space is given by

$$H = \left\{ \int_{R^n} \left| f(x) \right|^2 dx \right\}$$

along with an inner product

$$(u, v) = \int_{R^n} u(x)v(x)\,dx.$$

Let F be a linear differential operator in H whose action on the function u is expressed by $F[u]$. Then Eq. (20) becomes

$$\left(F[u], v\right) = \left(u, F^*[v]\right),$$

which means that

$$vF[u] - uF^*[v] = D_i\left(\xi^i\right), \tag{21}$$

where $D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \partial_{u^\alpha} + u_{ij}^\alpha \partial_{u_j^\alpha} + \cdots$.

For the differential equations

$$F_\alpha(x, u, u_{x_i}, u_{x_i x_j}, \ldots) = 0, \quad \alpha = 1, \ldots, m, \tag{22}$$

where $u = (u^1, \ldots, u^m)$. The adjoint equations to (22) are as follows:

$$F_\alpha^*(x, u, v, u_{x_i}, v_{x_i}, \ldots) = 0, \quad \alpha = 1, \ldots, m, \tag{23}$$

with $F_\alpha^* = \frac{\delta \varphi}{\delta u^\alpha}$. The Lagrangian φ for (22) is defined by

$$\begin{aligned} &\varphi = v^\beta F_\beta =: \sum_{\beta=1}^m v^\beta F_\beta, \\ &\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{j=1}^\infty (-1)^j D_{i_1} \cdots D_{i_j} \frac{\partial}{\partial u_{i_1 \cdots i_j}^\alpha}. \end{aligned} \tag{24}$$

Definition 1 ([7, 8]) The differential Eqs. (22) are said to be strictly self-adjoint if their adjoint Eqs. (23) are equivalent to (23) upon the substitution $v = u$. That is, the equation

$$F^*(x, u, u, u_{x_i}, u_{x_i}, \ldots) = \lambda F(x, u, u_x, \ldots)$$

holds with a coefficient λ.

Definition 2 ([8]) Upon a substitution

$$v = \varphi(u), \tag{25}$$

if (23) becomes (22), then we call (22) is quasiself-adjoint.

Definition 3 ([7, 8]) Upon a substitution

$$v = \varphi(x,u) \neq 0, \tag{26}$$

if (26) solves the adjoint Eqs. (23) for all the solutions of (22), then we call system (22) nonlinearly self-adjoint, that is, we have the following equations:

$$F_\alpha^*(x,u,\varphi,\dots) = \lambda_\alpha^\beta F_\beta(x,u,\dots). \tag{27}$$

It is easy to find that the strictly self-adjoint and quasiself-adjoint equations both are particular cases of the nonlinear self-adjoint equations.

For the generalized BK system (2), denoted by

$$\begin{cases} F = v_t - \frac{\alpha}{2}(v_x - v^2 - 2w)_x + \beta v_x, \\ G = w_t + \frac{\alpha}{2}(w_x + 2wv)_x + \beta w_x, \end{cases}$$

the formal Lagrangian $\mathcal{L}$ can be written as $\mathcal{L} = pF + qG$, and the adjoint system of (2) is as follows:

$$\begin{cases} \frac{\delta\mathcal{L}}{\delta v} = 2\alpha p v_x - p_t - \frac{\alpha}{2}p_{xx} + \alpha(pv)_x - \beta p_x - \alpha w q_x = 0, \\ \frac{\delta\mathcal{L}}{\delta w} = -\alpha p_x - q_t - \alpha(qv)_x + \frac{\alpha}{2}q_{xx} - \beta q_x = 0. \end{cases} \tag{28}$$

Setting $p = \varphi(v,w)$ and $q = \psi(v,w)$ and substituting into (27), along with (28), we have

$$\frac{\delta\mathcal{L}}{\delta v}\bigg|_{p=\varphi,q=\psi} = \lambda_1 F + \mu_1 G, \qquad \frac{\delta\mathcal{L}}{\delta w}\bigg|_{p=\varphi,q=\psi} = \lambda_2 F + \mu_2 G, \tag{29}$$

where λ_1, λ_2, μ_1, μ_2 are undetermined functions. It is easy to get

$$\begin{cases} p_t = \varphi_v v_t + \varphi_w w_t, \qquad p_x = \varphi_v v_x + \varphi_w w_x, \\ p_{xx} = \varphi_{vv} v_x^2 + 2\varphi_{vw} v_x w_x + \varphi_{ww} w_x^2 + \varphi_v v_{xx} + \varphi_w w_{xx}, \\ q_t = \psi_v v_t + \psi_w w_t, \qquad q_x = \psi_v v_x + \psi_w w_x, \\ q_{xx} = \psi_{vv} v_x^2 + 2\psi_{vw} v_x w_x + \psi_{ww} w_x^2 + \psi_v v_{xx} + \psi_w w_{xx}. \end{cases}$$

Inserting all these results into (29) yields that

$$\lambda_1 = \mu_1 = \lambda_2 = \mu_2 = 0.$$

Therefore, for all solutions of system (2), (28) holds. Thus system (2) is nonlinearly self-adjoint.

5 Another expression of system (2) and some properties

Set

$$v(x,t) = V\left(x, \frac{\alpha}{2}t\right) - \frac{\beta}{\alpha}, \qquad w(x,t) = W\left(x, \frac{\alpha}{2}t\right).$$

Then system (2) becomes

$$\begin{cases} V_t = V_{xx} - 2VV_x - 2W_x, \\ W_t = -W_{xx} - 2W_xV - 2WV_x, \end{cases} \tag{30}$$

which has the infinitesimal symmetries

$$X = (2c_1t + c_2)\partial_t + (c_1x + c_3t + c_4)\partial_x + \left(c_1v - \frac{1}{2}c_3\right)\partial_V + 2c_1\partial_W,$$

where c_1, c_2, c_3, c_4 are constants. Obviously, when $c_1 = c_2 = c_3 = 0$ and $c_4 = 1$, we get $X_1 = \partial_x$. When $c_1 = c_3 = c_4 = 0$ and $c_2 = 1$, we have $X_2 = \partial_t$. When $c_2 = c_3 = c_4 = 0$ and $c_1 = 1$, we find $X_3 = 2t\partial_t + x\partial_x + \partial_V + 2\partial_W$; X_i $(i = 1, 2, 3)$ all are particular cases of X.

Next, we consider the characteristic equation of X so that we can obtain the similarity reductions of system (30). The characteristic equation of X reads as

$$\frac{dt}{2c_1t + c_2} = \frac{dx}{c_1x + c_3t + c_4} = \frac{dV}{-c_1v + \frac{1}{2}c_3} = \frac{dW}{-2c_1W}. \tag{31}$$

Case 1: $c_1 = 1$.

$$\xi = \frac{x - c_3t + c_4 - c_2c_3}{\sqrt{2t + c_2}}, \qquad V = \frac{1}{2}c_3 + \frac{f(\xi)}{\sqrt{2t + c_2}}, \qquad W = \frac{g(\xi)}{2(2t + c_2)}. \tag{32}$$

System (30) reduces to

$$\begin{cases} -f(\xi) - \xi f'(\xi) + 2f(\xi)f'(\xi) - f''(\xi) + g'(\xi) = 0, \\ -2g(\xi) - \xi g'(\xi) + g''(\xi) + 2g(\xi)f'(\xi) + 2g'(\xi)f(\xi) = 0. \end{cases} \tag{33}$$

Case 2: $c_1 = c_2 = 0$. Equation (31) becomes

$$\frac{dt}{0} = \frac{dx}{c_3t + c_4} = \frac{dV}{\frac{1}{2}c_3} = \frac{dW}{0}.$$

We take

$$\xi = t, \qquad W = W(t), \qquad V = \frac{c_3x}{2(c_3t + c_4)} - \frac{1}{2}c_3f(t).$$

Then system (30) reduces to

$$\begin{cases} c_3\xi f'(\xi) + c_3f(\xi) + c_4f'(\xi) = 0, \\ c_3\xi W'(\xi) + c_4W'(\xi) + c_3W(\xi) = 0. \end{cases} \tag{34}$$

The two equations are in fact the same.

Case 3: $c_1 = 0$, $c_2 \neq 0$. Equation (31) reduces to

$$\frac{dt}{c_2} = \frac{dx}{c_3t + c_4} = \frac{dV}{\frac{1}{2}c_3} = \frac{dW}{0}.$$

We choose

$$\xi = c_2x - \frac{1}{2}c_3t^2 - c_4t, \qquad V = \frac{c_3t}{2c_3} + \frac{f(\xi)}{c_2}, \qquad W = g(\xi).$$

Thus system (30) turns to

$$\begin{cases} -2c_4f'(\xi) + c_3 - 2c_2^2f''(\xi) + 4f(\xi)f'(\xi) + 4c_2^2g'(\xi) = 0, \\ -c_4g'(\xi) + c_2^2g''(\xi) + 2g'(\xi)f(\xi) + 2g(\xi)f'(\xi) = 0. \end{cases} \tag{35}$$

System (35) has the particular solutions

$$f(\xi) = \frac{1}{2}\xi^2 - \xi^{-1}, \qquad g(\xi) = c\xi,$$

where ξ satisfies the constraint

$$\xi^3 - \frac{3}{2}\xi^2 + c - 2 = 0.$$

Thus from (32) we get a set of new solutions of system (2):

$$\begin{cases} v(x,t) = \frac{1}{2}c_3 + \frac{1}{\sqrt{\alpha t+c_2}}(\frac{1}{2}\frac{(x-c_3t+c_4-c_2c_3)^2}{2t+c_2} - \frac{\sqrt{2t+c_3}}{x-c_3t+c_4-c_2c_3}) - \frac{\beta}{\alpha}, \\ w(x,t) = \frac{c}{2}\frac{x-c_3t+c_4-c_2c_3}{(\alpha t+c_2)^{\frac{3}{2}}}. \end{cases}$$

In what follows, we consider the series solutions of (33).
Setting

$$f(\xi) = \sum_{i=0}^{\infty} a_i\xi^i, \qquad g(\xi) = \sum_{i=0}^{\infty} b_i\xi^i$$

and substituting into system (33), we infer that

$$\begin{cases} a_0 + 2a_0a_1 - 2a_2 + b_1 = 0, \\ 4a_0a_2 + 2a_1^2 - 6a_3 + 2b_2 = 0, \\ -a_2 + 2(3a_0a_3 + 3a_1a_2) - 12a_4 + 3b_3 = 0, \end{cases}$$

$$a_n - na_n + 2\sum_{i,j=1}^{n} a_i(j+1)a_{j+1} - (n+2)!a_{n+2} + (n+1)b_{n+1} = 0, \tag{36}$$

$$\begin{cases} -2b_0 + 2b_2 + 2b_0a_1 + 2b_1a_0 = 0, \\ -3b_1 + 6b_3 + 2(2b_0a_2 + b_1a_1) + 2(b_1a_1 + 2b_2a_0) = 0, \\ -4b_2 + 12b_4 + 2(3b_0a_3 + 2b_1a_2 + a_1b_2) + 2(b_1a_2 + 2b_2a_1 + 3b_3a_0) = 0, \\ \dots \end{cases}$$

$$-2b_n - (n+1)b_{n+1} + (n+2)!b_{n+2} + 2\sum_{i,j=1}^{n} b_i(j+1)a_j + 2\sum_{i,j=1}^{n} a_i(j+1)b_{j+1} = 0, \tag{37}$$

from which we get

$$\begin{cases} a_2 = \frac{1}{2}a_0 + a_0a_1 + \frac{1}{2}b_1, \\ b_2 = b_0 - a_1b_0 - a_0b_1, \\ a_3 = \frac{1}{3}(2a_0a_2 + a_1^2 + b_2), \\ b_3 = \frac{1}{2}b_1 - \frac{1}{3}(a_1b_1 + 2a_2b_0) - \frac{1}{3}(b_1a_1 + 2b_2a_0), \\ \dots \end{cases}$$

where a_0, b_0, a_1, b_1 are arbitrary parameters. Thus we obtain the following formal series solutions of system (33):

$$f(\xi) = a_0 + a_1\xi + \left(\frac{1}{2}a_0 + a_0a_1 + \frac{1}{2}b_1\right)\xi^2 + \frac{1}{3}\left(2a_0a_2 + a_1^2 + b_2\right)\xi^3 + \sum_{i=4}^{\infty} a_i\xi^i, \tag{38}$$

$$\begin{aligned} g(\xi) &= b_0 + b_1\xi + (b_0 - a_1b_0 - a_0b_1)\xi^2 \\ &\quad + \left[\frac{1}{2}b_1 - \frac{1}{3}(2a_1b_1 + 2a_2b_0 + 2b_2a_0)\right]\xi^3 + \sum_{i=4}^{\infty} b_i\xi^i, \end{aligned} \tag{39}$$

where a_i, b_i $(i = 4, 5, \dots)$ satisfy (36) and (37). Substituting (38)and (39) into (32), we can get the series solutions of the generalized BK system.

Next, we consider the solutions to system (34). It is easy to see that

$$g(\xi) = f(\xi) = -\xi - \frac{c_4}{c_3} \quad \text{or} \quad g(\xi) = f(\xi) = \frac{\hat{c}}{\xi + \frac{c_4}{c_3}}, \tag{40}$$

where $\hat{c}$ is an integration constant.

System (35) is solvable similarly to system (33), and we omit the computations.

6 Conservation laws

In this section, we consider the conservation laws of the generalized BK system by using the method in [7, 8]. From the identity

$$X + D_i(\xi^i) = W^\alpha \frac{\delta}{\delta u^\alpha} + D_iN^i$$

we find that

$$X(\mathcal{L}) + D_i(\xi^i)\mathcal{L} = W^\alpha \frac{\delta \mathcal{L}}{\delta u^\alpha} + D_i[N^i(\mathcal{L})], \tag{41}$$

where

$$\begin{cases} X = \xi^i\partial_{x_i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \xi_i \frac{\partial}{\partial u_i^\alpha} + \cdots, \\ N^i = \xi^i + W^\alpha \frac{\delta}{\delta u_i^\alpha} + \sum_{s=1}^{\infty} D_{i_1}\cdots D_{i_s}(w_\alpha)\frac{\delta}{\delta u_{ii_1\cdots i_s}}, \quad i = 1, 2, \dots, n, \\ W^\alpha = \eta^\alpha - \xi^j u_j^\alpha, \quad \alpha = 1, \dots, m, \end{cases}$$

and $\mathcal{L}$ is the Euler–Lagrange function, which satisfies

$$\frac{\delta \mathcal{L}}{\delta u^\alpha} = 0, \quad \alpha = 1, \dots, m.$$

Since system (28) holds, we can investigate the conservation laws by using (41), where the components of the conservation laws are the following:

$$C^i = N^i(\mathcal{L}), \quad i = 1, \ldots, n, \tag{42}$$

which satisfy the conservation equations

$$D_i\big(C^i\big)_{(22)} = 0. \tag{43}$$

For $X_1 = \frac{\partial}{\partial x}$, we find that

$$W^{1,1} = -v_x, \qquad W^{1,2} = -w_x. \tag{44}$$

Substituting (44) into (42) yields

$$\begin{cases} C_v^1 = -\alpha v v_x(p+q) - (\alpha+\beta)v_x p - \alpha q w v_x - \beta q v_x + \frac{\alpha}{2}v_x(q_x - p_x) \\ \qquad + \frac{\alpha}{2}v_{xx} - \frac{\alpha}{2}q v_{xx}, \\ C_w^1 = -\alpha p v w_x - \beta p w_x - \alpha q w w_x - \alpha p w_x - \alpha q v w_x \\ \qquad - \beta q w_x - \frac{\alpha}{2}p_x w_x + \frac{\alpha}{2}w_x q_x + \frac{\alpha}{2}p w_{xx} - \frac{\alpha}{2}q w_{xx}. \end{cases}$$

For $X_2 = \frac{\partial}{\partial t}$, we get

$$\begin{cases} C_v^2 = -v_t(p+q) = -(p+q)[-\beta v_x + \frac{\alpha}{2}(v_x - v^2 - 2w)_x], \\ C_w^2 = (p+q)[\beta w_x + \frac{\alpha}{2}(w_x + 2wv)_x]. \end{cases}$$

For $X_3 = (\frac{1}{2}x + \beta t)\partial_x + t\partial_t - \frac{1}{2}v\partial_v - w\partial_w$, we infer

$$\begin{cases} W^{3,1} = -\frac{1}{2}v - tv_t - (\frac{1}{2}x + \beta t)v_x, \\ W^{3,2} = -w - tw_t - (\frac{1}{2}x + \beta t)w_x, \\ C_v^3 = [-\frac{1}{2}v - tv_t - (\frac{1}{2}x + \beta t)v_x][\alpha p v + \beta p + 2wq + \alpha p + \alpha q v + \beta \\ \qquad + \frac{\alpha}{2}(p+q)[-\frac{1}{2}v_x - v_{xt} - \frac{1}{2}v_x - (\frac{1}{2}x + \beta t)v_{xt}], \\ C_w^3 = [-w - tw_t - (\frac{1}{2}x + \beta t)w_x](\alpha p v + \beta p + 2wq + \alpha p + \alpha q v + \beta) \\ \qquad + \frac{\alpha}{2}(p+q)[-w_x - tw_{xt} - \frac{1}{2}w_x - (\frac{1}{2}x + \beta t)w_{xx}], \end{cases}$$

where v_t, w_t are given by system (2).

Remark Anco and Bluman [9] proposed a method for constructing conservation laws of differential equations, which uses a formula directly generating the conservation laws and independent of the system having a Lagrangian formulation, in contrast to Noether's theorem, which requires a Lagrangian. They adopted the linear equations and the adjoint equations of the original differential equations to study conservation laws. Essentially, the algorithm presented by Ibragimov et al. is the same as that of Anco and Bluman. Besides, Anco [10] also gave some comments on the work of Ibragimov.

7 Conclusions

In the paper, we have investigated various similarity reductions and exact solutions of the generalized BK system and various its conservation laws by the Lie group analysis. We have pointed out that the standard BK system is only a paticular case of the generalized BK system (2) when $\alpha = -1$ and $\beta = 0$. In addition, Lou [11, 12] applied the symmetry group method to study some coherent solutions of nonlocal KdV systems and primary branch solutions of a first-order autonomous system. We hope to extend the methods to the systems presented in the paper in the forthcoming days. In addition, Ma [13] obtained some new conservation laws of some discrete evolution equation by symmetries and adjoint symmetries. Zhang, et al. [14, 15] considered symmetry properties of some fractional equations. Therefore there is an open problem how we can look for the fractional systems that correspond to the systems presented in the paper and how we can to solve them. Besides, Liu, Zhang, and Zhou [16] constructed the fractional Volterra hierarchy, gave a definition of the hierarchy in terms of Lax pair and Hamiltonian formalisms, and constructed its tau functions and multisoliton solutions. Bridgman, Hereman, Quispel, and Kamp [17] and El-Nabulsi [18] studied the peakon and Toda lattice. The approaches adopted in [16–18] can lead us to investigate some related properties of the generalized BK system presented in the paper. These questions will be discussed in the future.

Acknowledgements
The authors wish to thank the anonymous referees for their valuable suggestions.

Funding
This work is supported by the Fundamental Research Funds for the Central University (No. 2017XKZD11).

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors declare that the study was realized in collaboration with the same responsibility. Both authors read and approved the final manuscript.

Publisher's Note

Received: 16 September 2019 Accepted: 18 November 2019 Published online: 04 December 2019

References

1. Broer, L.J.F.: Approximate equations for long water waves. Appl. Sci. Res. **31**, 377–395 (1975)
2. Kupershmidt, B.A.: Mathematics of dispersive water waves. Commun. Math. Phys. **99**, 51–73 (1985)
3. Kaup, D.J.: A higher-order water-wave equation and the method for solving it. Prog. Theor. Phys. **54**, 396–408 (1975)
4. Matveev, V.B., Yavor, M.I.: Solutions presque périodiques et a N-solitons de l'équation hydrodynamique non linéaire de Kaup. Ann. Inst. Henri Poincaré A **XXXI** (1), 25–41 (1979)
5. Li, Y.S., Ma, W.X., Zhang, J.E.: Darboux transformations of classical Boussinesq system and its new solutions. Phys. Lett. A **275**, 60–66 (2000)
6. Olver, P.J.: Applications of Lie Groups to Differential Equations. Springer, New York (1993)
7. Ibragimov, N.H., Avdonina, E.D.: Nonlinear self-adjointness, conservation laws, and the construction of solutions of partial differential equations using conservation laws. Russ. Math. Surv. **68**, 889–921 (2013)
8. Ibagimov, N.H.: Nonlinear self-adjointness in constructing conservation laws pp. 1–104 (2011) arXiv:1109.1728vl [math-ph]
9. Anco, S., Bluman, G.: Direct construction of conservation laws from field equations. Phys. Rev. Lett. **78**, 2869–2873 (1997)
10. Anco, S.: On the incompleteness of Ibragimov's conservation law theorem and its equivalence to a standard formula using symmetries and adjoint-symmetries. Symmetry **9**, 33 (2017)
11. Lou, S.Y., Huang, F.: Alice–Bob physics: coherent solutions of nonlocal KdV systems. Sci. Rep. **7**, 1–13 (2016)
12. Lou, S.Y., Yao, R.X.: Invariant functions, symmetries and primary branch solutions of first order autonomous systems. Commun. Theor. Phys. **68**, 21–28 (2017)

13. Ma, W.X.: Conservation laws of discrete evolution equations by symmetries and adjoint symmetries. Symmetry **7**, 714–725 (2015)
14. Zhang, Y.F., et al.: Symmetry properties and explicit solutions of some nonlinear differential and fractional equations. Appl. Math. Comput. **337**, 408–418 (2018)
15. Zhang, X.Z., Zhang, Y.F.: Some similarity solutions and numerical solutions to the time-fractional Burgers system. Symmetry **11**, 112 (2019). https://doi.org/10.3390/sym11010112
16. Liu, S.Q., Zhang, Y., Zhou, C.: Fractional Volterra hierarchy. Lett. Math. Phys. **108**, 261–283 (2018)
17. Bridgman, T.J., Hereman, W., Quispel, G.R.W., van der Kamp, P.: Symbolic computation of Lax pairs of partial difference equations using consistency around the cube. Found. Comput. Math. **13**, 517–544 (2012)
18. El-Nabulsi, R.A.: Non-standard higher-order G-strand partial differential equations on matrix Lie algebra. J. Niger. Math. Soc. **36**, 101–112 (2017)