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Positive solutions of beam equations under nonlocal boundary value conditions

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Abstract

In this article, we study the fourth-order problem with the first and second derivatives in nonlinearity under nonlocal boundary value conditions

$$\begin{cases} u^{(4)}(t) = h(t)f(t, u(t), u'(t), u''(t)), \quad t \in (0,1), \\ u(0) = u(1) = \beta_1[u], \qquad u''(0) + \beta_2[u] = 0, \qquad u''(1) + \beta_3[u] = 0, \end{cases}$$

where $f : [0,1] \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_- \to \mathbb{R}_+$ is continuous, $h \in L^1(0,1)$ and $\beta_i[u]$ is Stieltjes integral ($i = 1,2,3$). This equation describes the deflection of an elastic beam. Some inequality conditions on nonlinearity f are presented that guarantee the existence of positive solutions to the problem by the theory of fixed point index on a special cone in $C^2[0,1]$. Two examples are provided to support the main results under mixed boundary conditions involving multi-point with sign-changing coefficients and integral with sign-changing kernel.

MSC: Primary 34B18; 34B10; secondary 34B15

Keywords: Positive solution; Fixed point index; Cone

1 Introduction

In this article, we study the existence of positive solutions for fourth-order boundary value problem (BVP) with dependence on the first and second derivatives in nonlinearity subject to boundary conditions of Stieltjes integral type

$$\begin{cases} u^{(4)}(t) = h(t)f(t, u(t), u'(t), u''(t)), \quad t \in (0,1), \\ u(0) = u(1) = \beta_1[u], \qquad u''(0) + \beta_2[u] = 0, \qquad u''(1) + \beta_3[u] = 0, \end{cases} \tag{1.1}$$

where $\beta_i[u] = \int_0^1 u(t)\,d\mathcal{B}_i(t)$ is Stieltjes integral with $\mathcal{B}_i$ of bounded variation ($i = 1,2,3$). This equation describes the deflection of an elastic beam.

Alves et al. [1] established the existence of positive solutions for the beam equation

$$u^{(4)}(t) = f\bigl(t, u(t), u'(t)\bigr)$$

under boundary conditions

$$u(0) = u'(0) = 0, \qquad u'''(1) = g\bigl(u(1)\bigr), \qquad u'(1) = 0 \quad \text{or} \quad u''(1) = 0,$$

where g is a continuous function. Using of the monotonically iterative technique, Yao [2] investigated the positive solution for fourth-order two-point boundary value problem

$$\begin{cases} u^{(4)}(t) = f(t, u(t), u'(t)), \quad t \in (0,1), \\ u(0) = u'(0) = u''(1) = u'''(1) = 0. \end{cases}$$

Li [3] and Ma [4] dealt with the existence of positive solutions for the fourth-order boundary value problem

$$\begin{cases} u^{(4)}(t) = f(t, u(t), u''(t)), \quad t \in (0,1), \\ u(0) = u''(0) = u(1) = u''(1) = 0. \end{cases}$$

Their methods are respectively based on fixed point index theory on cones and global bifurcation techniques. Bai [5] and Guo et al. [6] explored the existence of positive solutions respectively for the nonlocal fourth-order problems

$$u^{(4)}(t) + \beta u''(t) = \lambda f\big(t, u(t), u''(t)\big)$$

and

$$u^{(4)}(t) + \beta u''(t) = \lambda f\big(t, u(t), u'(t), u''(t), u'''(t)\big)$$

subject to the same boundary conditions

$$u(0) = u(1) = \int_0^1 p(s)u(s)\,ds, \qquad u''(0) = u''(1) = \int_0^1 q(s)u''(s)\,ds,$$

where $p, q \in L[0,1]$ are nonnegative. Li [7] discussed the existence of positive solutions for a local fully nonlinear problem

$$\begin{cases} u^{(4)}(t) = f(t, u(t), u'(t), u''(t), u'''(t)), \quad t \in [0,1], \\ u(0) = u'(0) = u''(1) = u'''(1) = 0, \end{cases}$$

where $f : [0,1] \times \mathbb{R}_+^3 \times \mathbb{R}_- \to \mathbb{R}_+$ is continuous. Under the conditions that the nonlinearity $f(t, x_1, x_2, x_3, x_4)$ may have superlinear or sublinear growth in x_1, x_2, x_3, x_4, the existence of positive solutions is obtained. We also refer to some previous studies, for instance, [8–12]. Recently the existence of positive solutions was proved in [13] to the following problems:

$$\begin{cases} u^{(4)}(t) = f(t, u(t), u'(t), u''(t), u'''(t)), \quad t \in [0,1], \\ u'(0) + \beta_1[u] = 0, \qquad u''(0) + \beta_2[u] = 0, \qquad u(1) = \beta_3[u], \qquad u'''(1) = 0, \end{cases} \tag{1.2}$$

and

$$\begin{cases} -u^{(4)}(t) = g(t, u(t), u'(t), u''(t), u'''(t)), \quad t \in [0,1], \\ u(0) = \alpha_1[u], \qquad u'(0) = \alpha_2[u], \qquad u''(0) = \alpha_3[u], \qquad u'''(1) = 0, \end{cases} \tag{1.3}$$

where $\beta_i[u]$ and $\alpha_i[u]$ $(i = 1, 2, 3)$ are Stieltjes integrals of signed measures. All the signs of the derivatives from the first to the third with respect to t of the Green's functions corresponding to (1.2) and (1.3) do not change, which plays an essential role in [13] when estimating the norms. The readers are referred to [14, 15] for more information and techniques about the issue considered.

Note that the boundary conditions in (1.1) are different from those in (1.2) and (1.3), and both the first and third derivatives with respect to t of the Green's function corresponding to (1.1) may be sign-changing. We reformulate BVP (1.1) as an integral equation by the method due to Webb and Infante [16], see also [17, 18]. If $u(0) = u(1)$, the existence of positive solutions to the resulting integral equation is tackled by the theory of fixed point index on a special cone in $C^2[0,1]$ under the inequality conditions posed on the nonlinearity. In particular, the fixed point indexes are computed via the cone expansion and compression conditions of functional type. Two examples are provided to support the main results under mixed boundary conditions involving multi-point with sign-changing coefficients and integral with sign-changing kernel.

2 Preliminaries

In order to prove the main theorems, we need the notion of a fixed point index; see, for example, [19, 20]. Let X be a Banach space, a nonempty subset K is called a cone in X if it is a closed convex set and satisfies the properties that $\lambda x \in K$ for any $\lambda > 0$, $x \in K$, and that $\pm x \in K$ implies $x = 0$ (the zero element in X). We say that $\alpha : K \to [0,+\infty)$ is a sublinear functional if $\alpha(tx) \le t\alpha(x)$ for all $x \in K$, $t \in [0,1]$. The following lemmas come from [21].

Lemma 2.1 *Let K be a cone in Banach space X and Ω be a bounded open subset relative to K with $0 \in \Omega$, $S : \overline{\Omega} \to K$ is a completely continuous operator. Suppose that $\alpha : K \to [0,+\infty)$ is a continuous and sublinear functional with $\alpha(0) = 0$, $\alpha(x) \neq 0$ for $x \neq 0$. If $Sx \neq x$ and $\alpha(Sx) \le \alpha(x)$ for all $x \in \partial\Omega$, then the fixed point index $i(S,\Omega,K) = 1$.*

Lemma 2.2 *Let K be a cone in Banach space X and Ω be a bounded open subset relative to K with $0 \in \Omega$, $S : \overline{\Omega} \to K$ is a completely continuous operator. Suppose that $\alpha : K \to [0,+\infty)$ is a continuous and sublinear functional with $\alpha(0) = 0$, $\alpha(x) \neq 0$ for $x \neq 0$, and $\inf_{x\in\partial\Omega} \alpha(x) > 0$. If $Sx \neq x$, $\alpha(Sx) \ge \alpha(x)$ for all $x \in \partial\Omega$, then the fixed point index $i(S,\Omega,K) = 0$.*

Let $X = C^2[0,1]$ be the Banach space consisting of all twice continuously differentiable functions on $[0,1]$ with the norm

$$\|u\|_{C^2} = \max\left\{\|u\|_C, \left\|u'\right\|_C, \left\|u''\right\|_C\right\},$$

where $\|u\|_C = \max\{|u(t)| : t \in [0,1]\}$ for $u \in C[0,1]$. Define an operator in $C^2[0,1]$ as

$$(Tu)(t) = \sum_{i=1}^{3} \beta_i[u]\gamma_i(t) + \int_0^1 k_0(t,s)h(s)f\left(s,u(s),u'(s),u''(s)\right)ds, \tag{2.1}$$

where $\gamma_1(t) = 1$, $\gamma_2(t) = \frac{1}{6}t(1-t)(5-t)$, $\gamma_3(t) = \frac{1}{6}t(1-t)(1+t)$,

$$k_0(t,s) = \begin{cases} \frac{1}{6}t(1-s)(2s-t^2-s^2), & 0 \le t \le s \le 1, \\ \frac{1}{6}s(1-t)(2t-s^2-t^2), & 0 \le s \le t \le 1, \end{cases} \tag{2.2}$$

in which $\beta_i[u] = \int_0^1 u(t)\,d\mathcal{B}_i(t)$ $(i = 1,2,3)$. We set

$$(Bu)(t) =: \sum_{i=1}^{3} \beta_i[u]\gamma_i(t), \qquad (Fu)(t) =: \int_0^1 k_0(t,s)h(s)f\bigl(s,u(s),u'(s),u''(s)\bigr)\,ds,$$

so $(Tu)(t) = (Bu)(t) + (Fu)(t)$.

We assume throughout this paper that

(C_1) $f : [0,1] \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_- \to \mathbb{R}_+$ is continuous; here $\mathbb{R}_+ = [0,\infty)$ and $\mathbb{R}_- = (-\infty,0]$, $h \in L^1(0,1)$ with $h(t) \ge 0$ and $\int_0^1 h(t)\,dt > 0$.

(C_2) For each $i \in \{1,2,3\}$, $\mathcal{B}_i$ is of bounded variation and

$$\mathcal{K}_i(s) := \int_0^1 k_0(t,s)\,d\mathcal{B}_i(t) \ge 0, \quad \forall s \in [0,1].$$

(C_3) $\beta_i[\gamma_j] \ge 0$ $(i,j = 1,2,3)$ and for the 3×3 matrix

$$[B] = \begin{pmatrix} \beta_1[\gamma_1] & \beta_1[\gamma_2] & \beta_1[\gamma_3] \\ \beta_2[\gamma_1] & \beta_2[\gamma_2] & \beta_2[\gamma_3] \\ \beta_3[\gamma_1] & \beta_3[\gamma_2] & \beta_3[\gamma_3] \end{pmatrix},$$

its spectral radius $r([B]) < 1$.

Writing $\langle \beta,\gamma\rangle = \sum_{i=1}^{3} \beta_i\gamma_i$ for the inner product in $\mathbb{R}^3$, we define the operator S in $C^2[0,1]$ as

$$(Su)(t) = \bigl\langle \bigl(I - [B]\bigr)^{-1}\beta[Fu], \gamma(t)\bigr\rangle + (Fu)(t), \tag{2.3}$$

where $\beta[Fu] = (\beta_1[Fu], \beta_2[Fu], \beta_3[Fu])^T$ is the transposed vector. Similar to [16] we have the following lemmas.

Lemma 2.3 *Suppose that* (C_1) *holds. Then BVP* (1.1) *has a solution if and only if there exists a fixed point of* T *in* $C^2[0,1]$.

Lemma 2.4 *Suppose that* (C_1)–(C_3) *hold. Then* S *can be written as*

$$\begin{aligned} (Su)(t) &= \bigl((I - B)^{-1}Fu\bigr)(t) \\ &= \int_0^1 \Bigl(\bigl\langle \bigl(I - [B]\bigr)^{-1}\mathcal{K}(s), \gamma(t)\bigr\rangle + k_0(t,s)\Bigr)h(s)f\bigl(s,u(s),u'(s),u''(s)\bigr)\,ds \\ &=: \int_0^1 k_S(t,s)h(s)f\bigl(s,u(s),u'(s),u''(s)\bigr)\,ds, \end{aligned} \tag{2.4}$$

where $\mathcal{K}(s) = (\mathcal{K}_1(s), \mathcal{K}_2(s), \mathcal{K}_2(s))^T$, *i.e.*,

$$k_S(t,s) = \bigl\langle \bigl(I - [B]\bigr)^{-1}\mathcal{K}(s), \gamma(t)\bigr\rangle + k_0(t,s) = \sum_{i=1}^{3} \kappa_i(s)\gamma_i(t) + k_0(t,s) \tag{2.5}$$

and $\kappa_i(s)$ *is the ith component of* $(I - [B])^{-1}\mathcal{K}(s)$.

Lemma 2.5 *If* (C_2) *and* (C_3) *hold, then* $\kappa_i(s) \geq 0$ $(i = 1, 2)$,

$$k_S(0,s) = k_S(1,s) = \kappa_1(s),$$

and for $t, s \in [0,1]$,

$$c_0(t)\Phi_0(s) \leq k_S(t,s) \leq \Phi_0(s), \tag{2.6}$$

where

$$\Phi_0(s) = \sum_{i=1}^{3} \kappa_i(s) + \widetilde{\Phi}_0(s), \qquad c_0(t) = \widetilde{c}_0(t) + \gamma_3(t),$$

$$\widetilde{c}_0(t) = \begin{cases} \frac{3\sqrt{3}}{2} t(1-t^2), & 0 \leq t \leq \frac{1}{2}, \\ \frac{3\sqrt{3}}{2} t(1-t)(2-t), & \frac{1}{2} < t \leq 1, \end{cases}$$

$$\widetilde{\Phi}_0(s) = \begin{cases} \frac{\sqrt{3}}{27} s(1-s^2)^{3/2}, & 0 \leq s \leq \frac{1}{2}, \\ \frac{\sqrt{3}}{27} (1-s)s^{3/2}(2-s)^{3/2}, & \frac{1}{2} < s \leq 1; \end{cases}$$

and

$$c_1(t)\Phi_1(s) \leq -\frac{\partial^2 k_S(t,s)}{\partial t^2} \leq \Phi_1(s), \tag{2.7}$$

where $\Phi_1(s) = 2\kappa_2(s) + \kappa_3(s) + s(1-s)$, $c_1(t) = \min\{t, (1-t)/2\}$.

Proof Inequality $\kappa_i(s) \geq 0$ is due to [16] and we can find in [18] the inequalities

$$\widetilde{c}_0(t)\widetilde{\Phi}_0(s) \leq k_0(t,s) \leq \widetilde{\Phi}_0(s).$$

As for (2.7), it can be checked easily. □

Define a cone K in $C^2[0,1]$ as follows:

$$P = \{u \in C^2[0,1] : u(t) \geq 0, u''(t) \leq 0, \forall t \in [0,1]\},$$

$$K = \{u \in P : u(0) = u(1), u(t) \geq c_0(t)\|u\|_C,$$
$$-u''(t) \geq c_1(t)\|u''\|_C, \forall t \in [0,1]; \beta_i[u] \geq 0\ (i = 1,2,3)\}. \tag{2.8}$$

Lemma 2.6 *If* (C_1)–(C_3) *hold, then* $S : P \to K$ *is a completely continuous operator.*

Proof For $u \in P$ and $t \in [0,1]$, it is easy to see that $Su \in C^2[0,1]$, $(Su)(t) \geq 0$ and $(Su)''(t) \leq 0$. By Lemma 2.5,

$$(Su)(0) = \int_0^1 k_S(0,s)h(s)f(s,u(s),u'(s),u''(s))\,ds$$
$$= \int_0^1 k_S(1,s)h(s)f(s,u(s),u'(s),u''(s))\,ds = (Su)(1).$$

Also by Lemma 2.5,

$$\begin{aligned}(Su)(t) &= \int_0^1 k_S(t,s)h(s)f\big(s,u(s),u'(s),u''(s)\big)\,ds\\ &\geq c_0(t)\int_0^1 \Phi_0(s)h(s)f\big(s,u(s),u'(s),u''(s)\big)\,ds\end{aligned}$$

and

$$\begin{aligned}-(Su)''(t) &= -\int_0^1 \frac{\partial^2 k_S(t,s)}{\partial t^2}h(s)f\big(s,u(s),u'(s),u''(s)\big)\,ds\\ &\geq c_1(t)\int_0^1 \Phi_1(s)h(s)f\big(s,u(s),u'(s),u''(s)\big)\,ds,\end{aligned}$$

hence we have

$$\begin{aligned}(Su)(t) &= \int_0^1 k_S(t,s)h(s)f\big(s,u(s),u'(s),u''(s)\big)\,ds\\ &\leq \int_0^1 \Phi_0(s)h(s)f\big(s,u(s),u'(s),u''(s)\big)\,ds\end{aligned}$$

and

$$\begin{aligned}-(Su)''(t) &= -\int_0^1 \frac{\partial^2 k_S(t,s)}{\partial t^2}h(s)f\big(s,u(s),u'(s),u''(s)\big)\,ds\\ &\leq \int_0^1 \Phi_1(s)h(s)f\big(s,u(s),u'(s),u''(s)\big)\,ds,\end{aligned}$$

therefore $(Su)(t) \geq c_0(t)\|Su\|_C$ and $-(Su)''(t) \geq c_1(t)\|(Su)''\|_C$ for $t \in [0,1]$. Moreover, it follows from (C_2) that

$$\begin{aligned}\beta_i[Su] &= \int_0^1 \left(\int_0^1 k_S(t,s)h(s)f\big(s,u(s),u'(s),u''(s)\big)\,ds\right)d\mathcal{B}_i(t)\\ &= \int_0^1 \left(\int_0^1 k_S(t,s)\,d\mathcal{B}_i(t)\right)h(s)f\big(s,u(s),u'(s),u''(s)\big)\,ds\\ &= \int_0^1 \mathcal{K}_i(s)h(s)f\big(s,u(s),u'(s),u''(s)\big) \geq 0 \quad (i=1,2,3),\end{aligned}$$

that is, $Su \in K$. The complete continuity of S is obvious. □

Lemma 2.7 *If (C_1)–(C_3) hold, then S and T have the same fixed points in K. As a result, BVP (1.1) has a positive solution if and only if S has a fixed point in K.*

3 Positive solutions of BVP

Take $\tau \in (0,1/3)$ such that $\int_\tau^{1-\tau} h(t)\,dt > 0$ and denote

$$h_0 = \max\left\{\int_0^1 \Phi_0(t)h(t)\,dt, \int_0^1 \Phi_1(t)h(t)\,dt\right\},$$

$$h_\tau = \min\left\{\int_\tau^{1-\tau} \Phi_0(t)h(t)\,dt, \int_\tau^{1-\tau} \Phi_1(t)h(t)\,dt\right\}.$$

Define a functional $\alpha : K \to [0,+\infty)$ as

$$\alpha(u) = \max\left\{\max_{\tau\le t\le 1-\tau}\left|u(t)\right|, \max_{\tau\le t\le 1-\tau}\left|u''(t)\right|\right\}.$$

Clearly, α is a continuous and sublinear functional with $\alpha(0) = 0$. Moreover, since

$$\alpha(u) \ge \max_{\tau\le t\le 1-\tau}\left|u(t)\right| \ge \Bigl(\max_{\tau\le t\le 1-\tau} c_0(t)\Bigr)\|u\|_C \ge \frac{1}{16}(9\sqrt{3}+1)\|u\|_C, \quad u \in K,$$

it is easy to see that $\alpha(u) \ne 0$ for $u \ne 0$.

Theorem 3.1 *Suppose that* (C_1)*–*(C_3) *are satisfied. If there exist constants* a *and* b *with* $0 < b < a$ *satisfying* $3b < \tau a$,

$$f(t,x_1,x_2,x_3) \le h_0^{-1}b \tag{3.1}$$

for $(t,x_1,x_2,x_3) \in D_1 = [0,1] \times [0,3b] \times [-3b,3b] \times [-3b,0]$ *and*

$$f(t,x_1,x_2,x_3) \ge 3h_\tau^{-1}a \tag{3.2}$$

for $(t,x_1,x_2,x_3) \in D_2 \cup D_3$, *where*

$$D_2 = [0,1] \times [\tau a, a] \times [-3a,3a] \times [-3a,0],$$
$$D_3 = [0,1] \times [0,3a] \times [-3a,3a] \times [-a,-\tau a],$$

then BVP (1.1) *has at least one positive solution.*

Proof Obviously, $D_1 \cap (D_2 \cup D_3) = \emptyset$ since $3b < \tau a$. Let

$$\Omega_1 = \bigl\{u \in K : \alpha(u) < b\bigr\}, \qquad \Omega_2 = \bigl\{u \in K : \alpha(u) < a\bigr\},$$

then it is clear that Ω_1 and Ω_2 are open sets in K with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$.

If $u \in \Omega_2$, by Lemma 2.5, we have

$$a > \max_{\tau\le t\le 1-\tau}\left|u(t)\right| \ge \Bigl(\max_{\tau\le t\le 1-\tau} c_0(t)\Bigr)\|u\|_C \ge \frac{1}{16}(9\sqrt{3}+1)\|u\|_C \ge \frac{1}{3}\|u\|_C$$

and

$$a > \max_{\tau\le t\le 1-\tau}\left|u''(t)\right| \ge \Bigl(\max_{\tau\le t\le 1-\tau} c_1(t)\Bigr)\left\|u''\right\|_C = \frac{1}{3}\left\|u''\right\|_C.$$

Since $u(0) = u(1)$, there exists $\xi \in (0,1)$ such that $u'(\xi) = 0$ and thus

$$\left\|u'\right\|_C = \max_{0\le t\le 1}\left|u'(t)\right| \le \max_{0\le t\le 1}\left|\int_\xi^t \left|u''(s)\right|ds\right| \le \left\|u''\right\|_C < 3a.$$

Therefore, Ω_2 is bounded and $\|u\|_{C^2} < 3a$, $\forall u \in \Omega_2$. Similarly, Ω_1 is bounded and $\|u\|_{C^2} < 3b$, $\forall u \in \Omega_1$.

If $u \in \partial\Omega_1$, then $\alpha(u) = b$ and $\|u\|_{C^2} \le 3b$. From Lemma 2.5 and (3.1) it follows that

$$\begin{aligned}\max_{\tau\le t\le 1-\tau}\bigl|(Su)(t)\bigr| &= \max_{\tau\le t\le 1-\tau}\left|\int_0^1 k_S(t,s)h(s)f\bigl(s,u(s),u'(s),u''(s)\bigr)\,ds\right|\\ &\le h_0^{-1}b\int_0^1 \Phi_0(s)h(s)\,ds \le b,\\ \max_{\tau\le t\le 1-\tau}\bigl|(Su)''(t)\bigr| &= \max_{\tau\le t\le 1-\tau}\left|\int_0^1 \frac{\partial^2 k_S(t,s)}{\partial t^2}h(s)f\bigl(s,u(s),u'(s),u''(s)\bigr)\,ds\right|\\ &\le h_0^{-1}b\int_0^1 \Phi_1(s)h(s)\,ds \le b,\end{aligned}$$

and hence $\alpha(Su) \le \alpha(u)$. So by Lemma 2.1 the fixed point index

$$i(S,\Omega_1,K) = 1, \tag{3.3}$$

provided $Su \ne u$ for $u \in \partial\Omega_1$.

If $u \in \partial\Omega_2$, then $\alpha(u) = a$ and, by Lemma 2.5 for $t \in [\tau, 1-\tau]$,

$$\begin{aligned} a &\ge u(t) \ge c_0(t)\|u\|_C \ge \Bigl(\min_{\tau\le t\le 1-\tau} c_0(t)\Bigr)\|u\|_C\\ &\ge \left(\frac{3\sqrt{3}}{2}+\frac{1}{6}\right)\tau\bigl(1-\tau^2\bigr)\|u\|_C \ge \tau\|u\|_C \ge \tau\max_{\tau\le t\le 1-\tau}\bigl|u(t)\bigr| \end{aligned}\tag{3.4}$$

and

$$\begin{aligned} a &\ge -u''(t) \ge c_1(t)\bigl\|u''\bigr\|_C \ge \Bigl(\min_{\tau\le t\le 1-\tau} c_1(t)\Bigr)\|u\|_C\\ &= \tau\bigl\|u''\bigr\|_C \ge \tau\max_{\tau\le t\le 1-\tau}\bigl|u''(t)\bigr|. \end{aligned}\tag{3.5}$$

When $\alpha(u) = a = \max_{\tau\le t\le 1-\tau}|u(t)|$, it follows from Lemma 2.5, together with (3.2) and (3.4), that

$$\begin{aligned}\max_{\tau\le t\le 1-\tau}\bigl|(Su)(t)\bigr| &= \max_{\tau\le t\le 1-\tau}\left|\int_0^1 k_S(t,s)h(s)f\bigl(s,u(s),u'(s),u''(s)\bigr)\,ds\right|\\ &\ge \Bigl(\max_{\tau\le t\le 1-\tau} c_0(t)\Bigr)\int_\tau^{1-\tau}\Phi_0(s)h(s)f\bigl(s,u(s),u'(s),u''(s)\bigr)\,ds\\ &\ge \frac{1}{3}3h_\tau^{-1}a\int_\tau^{1-\tau}\Phi_0(s)h(s)\,ds \ge a,\\ \max_{\tau\le t\le 1-\tau}\bigl|(Su)''(t)\bigr| &= \max_{\tau\le t\le 1-\tau}\left|\int_0^1 \frac{\partial^2 k_S(t,s)}{\partial t^2}h(s)f\bigl(s,u(s),u'(s),u''(s)\bigr)\,ds\right|\\ &\ge \Bigl(\max_{\tau\le t\le 1-\tau} c_1(t)\Bigr)\int_\tau^{1-\tau}\Phi_1(s)h(s)f\bigl(s,u(s),u'(s),u''(s)\bigr)\,ds\\ &\ge \frac{1}{3}3h_\tau^{-1}a\int_\tau^{1-\tau}\Phi_1(s)h(s)\,ds \ge a,\end{aligned}$$

and hence $\alpha(Su) \geq \alpha(u)$; when $\alpha(u) = a = \max_{\tau\leq t\leq 1-\tau} |u''(t)|$, it similarly follows from Lemma 2.5, together with (3.2) and (3.5), that $\alpha(Su) \geq \alpha(u)$. So by Lemma 2.2 and since $\inf_{x\in\partial\Omega_2} \alpha(u) = a > 0$, the fixed point index

$$i(S,\Omega_2,K) = 0, \tag{3.6}$$

provided $Su \neq u$ for $u \in \partial\Omega_2$.

From (3.3) and (3.6) it follows that the fixed point index

$$i(S,\Omega_2 \setminus \overline{\Omega}_1,K) = i(S,\Omega_2,K) - i(S,\Omega_1,K) = -1,$$

hence S has at least one fixed solution and BVP (1.1) has at least one positive solution. □

Theorem 3.2 *Suppose that* (C_1)–(C_3) *are satisfied. If there exist constants a and b with* $0 < b < a$ *satisfying* $a > 3h_0h_\tau^{-1}b$,

$$f(t,x_1,x_2,x_3) \geq 3h_\tau^{-1}b \tag{3.7}$$

for $(t,x_1,x_2,x_3) \in D_1 \cup D_2$, *where*

$$D_1 = [0,1] \times [\tau b, b] \times [-3b, 3b] \times [-3b, 0],$$
$$D_2 = [0,1] \times [0,3b] \times [-3b, 3b] \times [-b, -\tau b],$$

and

$$f(t,x_1,x_2,x_3) \leq h_0^{-1}a \tag{3.8}$$

for $(t,x_1,x_2,x_3) \in D_3 = [0,1] \times [0,3a] \times [-3a,3a] \times [-3a,0]$, *then BVP* (1.1) *has at least one positive solution.*

Proof Obviously, $D_1 \cup D_2 \subset D_3$; however, (3.7) and (3.8) are well-posed since $a > 3h_0h_\tau^{-1}b$. Letting

$$\Omega_1 = \{u \in K : \alpha(u) < b\}, \qquad \Omega_2 = \{u \in K : \alpha(u) < a\},$$

we know form the proof of Theorem 3.1 that Ω_1 and Ω_2 are bounded open sets in K with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$; moreover, $\|u\|_{C^2} < 3b$ for $u \in \Omega_1$ and $\|u\|_{C^2} < 3a$ for $u \in \Omega_2$.

If $u \in \partial\Omega_1$, then $\alpha(u) = b$ and, by Lemma 2.5 for $t \in [\tau, 1-\tau]$,

$$\begin{aligned} b &\geq u(t) \geq c_0(t)\|u\|_C \geq \Big(\min_{\tau\leq t\leq 1-\tau} c_0(t)\Big)\|u\|_C \\ &\geq \left(\frac{3\sqrt{3}}{2} + \frac{1}{6}\right)\tau\left(1-\tau^2\right)\|u\|_C \geq \tau\|u\|_C \geq \tau \max_{\tau\leq t\leq 1-\tau}\left|u(t)\right| \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} b &\geq -u''(t) \geq c_1(t)\left\|u''\right\|_C \geq \Big(\min_{\tau\leq t\leq 1-\tau} c_1(t)\Big)\|u\|_C \\ &= \tau\left\|u''\right\|_C \geq \tau \max_{\tau\leq t\leq 1-\tau}\left|u''(t)\right|. \end{aligned} \tag{3.10}$$

When $\alpha(u) = b = \max_{\tau\le t\le 1-\tau} |u(t)|$, it follows from Lemma 2.5, as well as (3.7) and (3.9), that

$$\begin{aligned}
\max_{\tau\le t\le 1-\tau} \bigl|(Su)(t)\bigr| &= \max_{\tau\le t\le 1-\tau} \left| \int_0^1 k_S(t,s)h(s)f\bigl(s,u(s),u'(s),u''(s)\bigr)\,ds \right| \\
&\ge \Bigl(\max_{\tau\le t\le 1-\tau} c_0(t)\Bigr) \int_\tau^{1-\tau} \Phi_0(s)h(s)f\bigl(s,u(s),u'(s),u''(s)\bigr)\,ds \\
&\ge \frac{1}{3} 3h_\tau^{-1} b \int_\tau^{1-\tau} \Phi_0(s)h(s)\,ds \ge b, \\
\max_{\tau\le t\le 1-\tau} \bigl|(Su)''(t)\bigr| &= \max_{\tau\le t\le 1-\tau} \left| \int_0^1 \frac{\partial^2 k_S(t,s)}{\partial t^2} h(s)f\bigl(s,u(s),u'(s),u''(s)\bigr)\,ds \right| \\
&\ge \Bigl(\max_{\tau\le t\le 1-\tau} c_1(t)\Bigr) \int_\tau^{1-\tau} \Phi_1(s)h(s)f\bigl(s,u(s),u'(s),u''(s)\bigr)\,ds \\
&\ge \frac{1}{3} 3h_\tau^{-1} b \int_\tau^{1-\tau} \Phi_1(s)h(s)\,ds \ge b,
\end{aligned}$$

and hence $\alpha(Su) \ge \alpha(u)$; when $\alpha(u) = b = \max_{\tau\le t\le 1-\tau} |u''(t)|$, it similarly follows from Lemma 2.5, together with (3.7) and (3.10), that $\alpha(Su) \ge \alpha(u)$. So by Lemma 2.2 and since $\inf_{x\in\partial\Omega_1} \alpha(u) = b > 0$, the fixed point index

$$i(S,\Omega_1,K) = 0, \tag{3.11}$$

provided $Su \ne u$ for $u \in \partial\Omega_1$.

If $u \in \partial\Omega_2$, then $\alpha(u) = a$ and $\|u\|_{C^2} \le 3a$. From Lemma 2.5 and (3.8) it follows that

$$\begin{aligned}
\max_{\tau\le t\le 1-\tau} \bigl|(Su)(t)\bigr| &= \max_{\tau\le t\le 1-\tau} \left| \int_0^1 k_S(t,s)h(s)f\bigl(s,u(s),u'(s),u''(s)\bigr)\,ds \right| \\
&\le h_0^{-1} a \int_0^1 \Phi_0(s)h(s)\,ds \le a, \\
\max_{\tau\le t\le 1-\tau} \bigl|(Su)''(t)\bigr| &= \max_{\tau\le t\le 1-\tau} \left| \int_0^1 \frac{\partial^2 k_S(t,s)}{\partial t^2} h(s)f\bigl(s,u(s),u'(s),u''(s)\bigr)\,ds \right| \\
&\le h_0^{-1} a \int_0^1 \Phi_1(s)h(s)\,ds \le a,
\end{aligned}$$

and hence $\alpha(Su) \le \alpha(u)$. So by Lemma 2.1 the fixed point index

$$i(S,\Omega_2,K) = 1, \tag{3.12}$$

provided $Su \ne u$ for $u \in \partial\Omega_2$.

From (3.11) and (3.12) it follows that the fixed point index

$$i(S,\Omega_2 \setminus \overline{\Omega}_1, K) = i(S,\Omega_2,K) - i(S,\Omega_1,K) = 1,$$

hence S has at least one fixed solution and BVP (1.1) has at least one positive solution. □

4 Examples

We consider fourth-order problems under mixed boundary conditions involving multi-point with sign-changing coefficients and integral with sign-changing kernel

$$\begin{cases} u^{(4)}(t) = \frac{1}{\sqrt{t(1-t)}} f(t, u(t), u'(t), u''(t)), \quad t \in (0,1), \\ u(0) = u(1) = \frac{1}{4}u(\frac{1}{4}) - \frac{1}{12}u(\frac{3}{4}), \\ u''(0) + \int_0^1 u(t)(2t - \frac{1}{2})\,dt = 0, \qquad u''(1) + \frac{1}{2}u(\frac{1}{2}) - \frac{1}{4}u(\frac{3}{4}) = 0, \end{cases} \tag{4.1}$$

that is, $\beta_1[u] = \frac{1}{4}u(\frac{1}{4}) - \frac{1}{12}u(\frac{3}{4})$, $\beta_2[u] = \int_0^1 u(t)(2t - \frac{1}{2})\,dt$, $\beta_3[u] = \frac{1}{2}u(\frac{1}{2}) - \frac{1}{4}u(\frac{3}{4})$. Hence for $s \in [0,1]$,

$$\begin{aligned} 0 \le \mathcal{K}_1(s) &= \frac{1}{4}k_0\left(\frac{1}{4}, s\right) - \frac{1}{12}k_0\left(\frac{3}{4}, s\right) \\ &= \begin{cases} -\frac{1}{36}s^3 + \frac{1}{96}s, & 0 \le s \le \frac{1}{4}, \\ \frac{1}{72}s^3 - \frac{1}{32}s^2 + \frac{7}{384}s - \frac{1}{1536}, & \frac{1}{4} < s \le \frac{3}{4}, \\ -\frac{1}{192}s + \frac{1}{192}, & \frac{3}{4} < s \le 1, \end{cases} \\ \mathcal{K}_2(s) &= \int_0^1 k_0(t,s)\left(2t - \frac{1}{2}\right)dt = \frac{1}{60}s^5 - \frac{1}{48}s^4 - \frac{1}{72}s^3 + \frac{13}{720}s \ge 0, \\ 0 \le \mathcal{K}_3(s) &= \frac{1}{2}k_0\left(\frac{1}{2}, s\right) - \frac{1}{4}k_0\left(\frac{3}{4}, s\right) \\ &= \begin{cases} -\frac{1}{32}s^3 + \frac{11}{512}s, & 0 \le s \le \frac{1}{2}, \\ \frac{5}{96}s^3 - \frac{1}{8}s^2 + \frac{43}{512}s - \frac{1}{96}, & \frac{1}{2} < s \le \frac{3}{4}, \\ \frac{1}{96}s^3 - \frac{1}{32}s^2 + \frac{7}{512}s + \frac{11}{1536}, & \frac{3}{4} < s \le 1, \end{cases} \end{aligned}$$

and the 3×3 matrix

$$[B] = \begin{pmatrix} \beta_1[\gamma_1] & \beta_1[\gamma_2] & \beta_1[\gamma_3] \\ \beta_2[\gamma_1] & \beta_2[\gamma_2] & \beta_2[\gamma_3] \\ \beta_3[\gamma_1] & \beta_3[\gamma_2] & \beta_3[\gamma_3] \end{pmatrix} = \begin{pmatrix} \frac{1}{6} & \frac{5}{192} & \frac{1}{192} \\ 0 & \frac{43}{720} & \frac{17}{720} \\ \frac{1}{4} & \frac{31}{512} & \frac{9}{512} \end{pmatrix}.$$

Its spectral radius is $r([B]) \approx 0.1832 < 1$. This means that (C_2) and (C_3) are satisfied. Moreover,

$$\kappa_1(s) = 1.2022\mathcal{K}_1(s) + 0.0338\mathcal{K}_2(s) + 0.0072\mathcal{K}_3(s),$$

$$\kappa_2(s) = 0.0077\mathcal{K}_1(s) + 1.0654\mathcal{K}_2(s) + 0.0256\mathcal{K}_3(s),$$

$$\kappa_3(s) = 0.3064\mathcal{K}_1(s) + 0.0742\mathcal{K}_2(s) + 1.0213\mathcal{K}_3(s).$$

Take $\tau = 1/4$ and then

$$h_0 = \max\left\{\int_0^1 \Phi_0(t)h(t)\,dt, \int_0^1 \Phi_1(t)h(t)\,dt\right\} = \max\{0.0578, 0.4257\} = 0.4257,$$

$$h_\tau = \min\left\{\int_{1/4}^{3/4} \Phi_0(t)h(t)\,dt, \int_{1/4}^{3/4} \Phi_1(t)h(t)\,dt\right\} = \min\{0.0366, 0.2600\} = 0.0366.$$

Example 4.1 If $f(t, x_1, x_2, x_3) = x_1^2 + \frac{1+t}{2}x_2^2 + x_3^2$, then BVP (4.1) has a positive solution.

Proof For $a = 1600$, $b = 0.01$, it is clear that $3b < a/4$. Moreover,

$$f(t,x_1,x_2,x_3) \le 3 \times \left(\frac{3}{100}\right)^2 = 0.0027 < h_0^{-1} b = 0.0235$$

for $(t,x_1,x_2,x_3) \in D_1 = [0,1] \times [0,0.03] \times [-0.03,0.03] \times [-0.03,0]$, and

$$f(t,x_1,x_2,x_3) \ge 400^2 > 3h_\tau^{-1} a$$

for $(t,x_1,x_2,x_3) \in ([0,1] \times [400,1600] \times [-4800,4800] \times [-4800,0]) \cup ([0,1] \times [0,4800] \times [-4800,4800] \times [-1600,-400])$. Then BVP (4.1) has a positive solution by Theorem 3.1. □

Example 4.2 If $f(t,x_1,x_2,x_3) = 2000(1 - \frac{1}{1+x_1^2+(1+t)x_2^4+x_3^2})$, then BVP (4.1) has a positive solution.

Proof For $a = 1000$, $b = 1$, it is clear that $a > 3h_0 h_\tau^{-1} b$. Moreover,

$$f(t,x_1,x_2,x_3) \le 2000 \le h_0^{-1} a = 2350$$

for $(t,x_1,x_2,x_3) \in [0,1] \times [0,3000] \times [-3000,3000] \times [-3000,0]$, and

$$f(t,x_1,x_2,x_3) \ge 2000\left(1 - \frac{1}{1 + (\frac{1}{4})^2}\right) \ge 3h_\tau^{-1} b$$

for $(t,x_1,x_2,x_3) \in ([0,1] \times [1/4,1] \times [-3,3] \times [-3,0]) \cup ([0,1] \times [0,3] \times [-3,3] \times [-1,-1/4])$. Then BVP (4.1) has a positive solution by Theorem 3.2. □

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