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The weak solutions of a doubly nonlinear parabolic equation related to the $p(x)$-Laplacian

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Abstract

A nonlinear degenerate parabolic equation related to the $p(x)$-Laplacian

$$u_t = \operatorname{div}\left(b(x)\left|\nabla a(u)\right|^{p(x)-2}\nabla a(u)\right) + \sum_{i=1}^{N}\frac{\partial b_i(u)}{\partial x_i} + c(x,t) - b_0 a(u)$$

is considered in this paper, where $b(x)|_{x\in\Omega} > 0$, $b(x)|_{x\in\partial\Omega} = 0$, $a(s) \geq 0$ is a strictly increasing function with $a(0) = 0$, $c(x,t) \geq 0$ and $b_0 > 0$. If $\int_\Omega b(x)^{-\frac{1}{p^- -1}}\,dx \leq c$ and $|\sum_{i=1}^{N} b_i'(s)| \leq ca'(s)$, then the solutions of the initial-boundary value problem is well-posedness. When $\int_\Omega b(x)^{-(p(x)-1)}\,dx < \infty$, without the boundary value condition, the stability of weak solutions can be proved.

MSC: 35K55; 35K92; 35K85; 35R35

Keywords: $p(x)$-Laplacian; The initial-boundary value problem; Stability

1 Introduction

The evolutionary $p(x)$-Laplacian equation

$$u_t = \operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right), \quad (x,t) \in Q_T = \Omega \times (0,T), \tag{1.1}$$

with the initial value

$$u|_{t=0} = u_0(x), \quad x \in \Omega, \tag{1.2}$$

and the homogeneous boundary value

$$u|_{\Gamma_T} = 0, \quad (x,t) \in \Gamma_T = \partial\Omega \times (0,T), \tag{1.3}$$

has been subject of a profound study from the beginning of this century [1–9], where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $p(x)$ is a measurable function. In 2013, Guo–Gao [10] and Gao–Gao [11] had considered the more general equation

$$u_t = \operatorname{div}\left(\left(|u|^{\sigma(x,t)} + d_0\right)|\nabla u|^{p(x,t)-2}\nabla u\right) + c(x,t) - b_0 u(x,t), \tag{1.4}$$

where $\sigma(x,t) > 1$, $d_0 > 0$, $c(x,t) \geq 0$ and $b_0 > 0$. This model may describe some properties of image restoration in space and time, the functions $u(x,t)$, $p(x,t)$ represent a recovering image and its observed noisy image, respectively. In [12], the authors obtained the existence and uniqueness of weak solutions with the assumption that the exponent $\sigma(x,t) \equiv 0$, $1 < p^- < p^+ < 2$. In [10], when $\sigma(x,t) \equiv 0$ and $b_0 = 0$, the authors applied the method of parabolic regularization and Galerkin's method to prove the existence of weak solutions. In [11], the authors generalized the results obtained in [10], moreover, they proved the existence and uniqueness of weak solution not only in the case when $\sigma(x,t) \in (2, \frac{2p^+}{p^+-1})$, but also in the case when $\sigma(x,t) \in (1,2)$, $1 < p^- < p^+ \leq 1 + \sqrt{2}$. They applied energy estimates and Gronwall's inequality to obtain the extinction of solutions when the exponents p^- and p^+ belong to different intervals.

If $\sigma(x,t) = \sigma$ and $p(x,t) = p$ are constants, Eq. (1.3) can be transformed to

$$\begin{aligned} u_t = {} & \operatorname{div}\left(\left|\nabla \frac{u^m}{m}\right|^{p-2} \nabla \frac{u^m}{m}\right) + d_0 \operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right) \\ & + c(x,t) - m b_0 \frac{u^m}{m}, \quad (x,t) \in Q_T, \end{aligned} \tag{1.5}$$

where $\sigma = (m-1)(p-1)$ or $m = 1 + \frac{\sigma}{p-1}$. For this equation, whether $d_0 = 0$ or $d_0 > 0$, it is well-known that the well-posedness problem of weak solutions had been solved perfectly. However, since Eq. (1.4) is with nonstandard growth, it cannot been transformed to another equation which has a similar type as Eq. (1.5). In fact, both in the uniformly estimates related to the existence and in the proof of the uniqueness of weak solution, the condition $d_0 > 0$ acts as a very important role in [10–12]. In other words, if $d_0 = 0$, how to obtain the well-posedness of weak solutions is an important subject deserving to be pursued in further research. In this paper, we will study a more general equation than Eq. (1.5),

$$u_t = \operatorname{div}\left(b(x)\left|\nabla a(u)\right|^{p(x)-2} \nabla a(u)\right) + \sum_{i=1}^{N} \frac{\partial b_i(u)}{\partial x_i} + c(x,t) - b_0 a(u), \quad (x,t) \in Q_T, \tag{1.6}$$

where $1 < p(x) \in C^1(\overline{\Omega})$, $a(s) \geq 0$, $a(0) = 0$ and $a(s)$ is a strictly increasing function, $b_0 > 0$ is a constant. Meanwhile, $b(x) \in C^1(\overline{\Omega})$ satisfies

$$b(x) > 0, \quad x \in \Omega, \qquad b(x) = 0, \quad x \in \partial\Omega, \tag{1.7}$$

and $b_i(s) \in C^1(\mathbb{R})$. We set

$$p^+ = \max_{\bar{\Omega}} p(x), \qquad 1 < p^- = \min_{\bar{\Omega}} p(x),$$

as usual.

A special case of Eq. (1.6) is $a(u) = u^m$, the equation reflects a polytropic filtration process if $p(x) = p$ is a constant. In this case, a lot of important results about the existence, the uniqueness, the Harnack inequality, the regularity, the extinction and the large time behavior of weak solutions have been obtained by many scholars; one can refer to [13–15] and the references therein. Also, it is worth noting that the constant $b_0 > 0$ is essential, if $b_0 < 0$, the weak solutions may blow up in a finite time [16–18]. While $p(x)$ is a $C^1(\overline{\Omega})$

function, only a few references could be found (for example, [19]). Moreover, since we only require that $a(s)$ is strictly increasing, it can be chosen as $a(s) = s^{m(x)}$ with $m(x) > 0$, and it even can be chosen as

$$a(s) = \begin{cases} s^{m_1}, & \text{if } 0 \leq s < 1, \\ s^{m_2}, & \text{if } s \geq 1, \end{cases} \tag{1.8}$$

with $m_1 \neq m_2$. Such a form is more appropriate to represent the model of image processing.

In this paper, we will use the parabolically regularized method to prove the existence of the weak solution, and we use some ideas of [7, 20–22] to prove the stability of weak solutions.

2 The definitions of weak solution and the main results

For completeness of the paper, we review the basic functional spaces firstly. For every fixed $t \in [0, T]$, we define

$$V_t(\Omega) = \left\{u(x) : u(x) \in L^2(\Omega) \cap W_0^{1,1}(\Omega), \left|\nabla u(x)\right|^{p(x)} \in L^1(\Omega)\right\},$$
$$\|u\|_{V_t(\Omega)} = \|u\|_{2,\Omega} + \|\nabla u\|_{p(x),\Omega},$$

and define $V_t'(\Omega)$ to be its dual space. At the same time, we denote the Banach space

$$\begin{cases} \mathbf{W}(Q_T) = \{u : [0,T] \to V_t(\Omega) | u \in L^2(Q_T), |\nabla u|^{p(x)} \in L^1(Q_T), u = 0 \text{ on } \Gamma_T\}, \\ \|u\|_{\mathbf{W}(Q_T)} = \|\nabla u\|_{p(x),Q_T} + \|u\|_{2,Q_T}, \end{cases}$$

and define $\mathbf{W}'(Q_T)$ to be its dual space.

$$w \in \mathbf{W}'(Q_T) \quad \Longleftrightarrow \quad \begin{cases} w = w_0 + \sum_{i=1}^n D_i w_i, \quad w_0 \in L^2(Q_T), w_i \in L^{p'(x,t)}(Q_T), \\ \forall \phi \in \mathbf{W}(Q_T), \quad \ll w, \phi \gg = \iint_{Q_T} (w_0 \phi + \sum_i w_i D_i \phi)\, dx\, dt. \end{cases}$$

One can refer to [19, 20] for more information.

Definition 2.1 If $0 \leq u(x,t) \in L^\infty(Q_T)$ satisfies

$$u_t \in \mathbf{W}'(Q_T), \qquad b(x)\left|\nabla a(u)\right|^{p(x)} \in L^1(Q_T), \tag{2.1}$$

and, for any function $\varphi \in L^\infty(0, T; W_0^{1,p(x)}(\Omega)) \cap \mathbf{W}(Q_T)$,

$$\begin{aligned} &\iint_{Q_T} \left[u_t \varphi + b(x)\left|\nabla a(u)\right|^{p(x)-2} \nabla a(u) \cdot \nabla \varphi + \sum_{i=1}^N b_i(u) \varphi_{x_i}\right] dx\, dt \\ &\quad = \iint_{Q_T} \left[c(x,t) - b_0 a(u)\right] \varphi(x,t)\, dx\, dt, \end{aligned} \tag{2.2}$$

then $u(x,t)$ is said to be a weak solution of Eq. (1.6) with the initial value (1.2), provided that

$$\lim_{t \to 0} \int_\Omega u(x,t) \phi(x)\, dx = \int_\Omega u_0(x) \phi(x)\, dx, \quad \forall \phi(x) \in C_0^\infty(\Omega). \tag{2.3}$$

Here, $W^{1,p(x)}(\Omega)$ is the variable exponent Sobolev space, $W_0^{1,p(x)}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$, one can refer to [23–25] for the details. The following basic lemma reflects some important characters of variable exponent Sobolev spaces [23–25].

Lemma 2.2

(i) *The space* $(L^{p(x)}(\Omega), \|\cdot\|_{L^{p(x)}(\Omega)})$, $(W^{1,p(x)}(\Omega), \|\cdot\|_{W^{1,p(x)}(\Omega)})$ *and* $W_0^{1,p(x)}(\Omega)$ *are reflexive Banach spaces.*

(ii) *$p(x)$-Hölder's inequality. Let $q_1(x)$ and $q_2(x)$ be real functions with $\frac{1}{q_1(x)} + \frac{1}{q_2(x)} = 1$. Then, the conjugate space of $L^{q_1(x)}(\Omega)$ is $L^{q_2(x)}(\Omega)$. And for any $u \in L^{q_1(x)}(\Omega)$ and $v \in L^{q_2(x)}(\Omega)$, we have*

$$\left| \int_\Omega uv\,dx \right| \le 2\|u\|_{L^{q_1(x)}(\Omega)} \|v\|_{L^{q_2(x)}(\Omega)}. \tag{2.4}$$

(iii) $\|u\|_{L^{p(x)}(\Omega)}$ *and* $\int_\Omega |u|^{p(x)}\,dx$ *satisfy*

$$\begin{aligned}
&\textit{If } \|u\|_{L^{p(x)}(\Omega)} = 1, \quad \textit{then } \int_\Omega |u|^{p(x)}\,dx = 1. \\
&\textit{If } \|u\|_{L^{p(x)}(\Omega)} > 1, \quad \textit{then } |u|^{p^-}_{L^{p(x)}(\Omega)} \le \int_\Omega |u|^{p(x)}\,dx \le |u|^{p^+}_{L^{p(x)}(\Omega)}. \\
&\textit{If } \|u\|_{L^{p(x)}(\Omega)} < 1, \quad \textit{then } |u|^{p^+}_{L^{p(x)}(\Omega)} \le \int_\Omega |u|^{p(x)}\,dx \le |u|^{p^-}_{L^{p(x)}(\Omega)}.
\end{aligned}$$

(iv) *If the exponent $p(x)$ is required to satisfy a logarithmic Hölder continuity condition, then*

$$W_0^{1,p(x)}(\Omega) = \mathring{W}^{1,p(x)}(\Omega). \tag{2.5}$$

The main results are the following theorems.

Theorem 2.3 *If $0 \le u_0(x) \in L^\infty(\Omega)$ satisfies*

$$b(x)|\nabla u_0|^{p(x)} \in L^1(\Omega), \tag{2.6}$$

then Eq. (1.6) with initial value (1.2) has a weak solution $u(x,t)$. If

$$\int_\Omega b(x)^{-\frac{1}{p^- - 1}}\,dx < \infty, \tag{2.7}$$

then Eq. (1.6) with the initial-boundary values (1.2)–(1.3) has a solution u. Moreover, let $u(x,t)$ and $v(x,t)$ be two weak solutions of Eq. (1.6) with

$$u(x,t) = v(x,t) = 0, \quad (x,t) \in \Gamma_T,$$

and with the initial values $u(x,0)$ and $v(x,0)$, respectively, $b_i(s)$ and $a(s)$ satisfy

$$\left| \sum_{i=1}^N \frac{b_i(s_1) - b_i(s_2)}{a(s_1) - a(s_2)} \right| \le c, \quad i = 1, 2, \ldots, N. \tag{2.8}$$

Then

$$\int_\Omega \left|u(x,t)-v(x,t)\right| dx \le c \int_\Omega \left|u_0(x)-v_0(x)\right| dx. \tag{2.9}$$

In fact, only if $b(x)$ satisfies (1.7) and the condition (2.8) is true, even without boundary value condition (1.3), by a similar method of [26], we can show that

$$\int_\Omega b(x)^\alpha \left|u(x,t)-v(x,t)\right| dx \le \int_\Omega b(x)^\alpha \left|u_0(x)-v_0(x)\right| dx, \tag{2.10}$$

where $\alpha \ge 2$ is a constant. This inequality implies that uniqueness of weak solution to Eq. (1.6) with the initial value (1.2) is always true only if (2.8) is true, no matter whether there is the condition (2.7) or not.

Based on this fact, we are able to improve the stability theorem to the case without boundary value condition (1.3).

Theorem 2.4 *Let $u(x,t)$ and $v(x,t)$ be two weak solutions of Eq.* (1.6) *with the initial values $u(x,0)$ and $v(x,0)$, respectively, the variable exponent $p(x)$ satisfies the logarithmic Hölder continuity condition. If $b(x)$ satisfies* (1.7), (2.8) *and*

$$\int_\Omega b(x)^{1-p(x)}\, dx < \infty, \tag{2.11}$$

then the stability (2.9) *is true.*

If $a(s)=s$ and $1<p^- \le p^+ < 2$ and

$$\int_\Omega b(x)^{-1}\, dx < \infty, \tag{2.12}$$

a similar result as Theorem 2.4 had been obtained in [22]. Clearly, (2.11) has a broader sense than (2.12). Comparing Theorem 2.3 with Theorem 2.4, the essential improvements lies in that, if $b(x)$ only satisfies (2.11), the weak solutions u may lack the regularity to be defined the trace on the boundary generally. Thus, we cannot impose the usual boundary value condition (1.3), except for the case $p(x)\equiv 2$ (in which (2.11) is equivalent to (2.7)). Theorem 2.4 tells us that the stability of the weak solutions is controlled by the initial value completely, only if (2.11) is true.

At the end of this section, comparing with our previous work [21, 22] and [26] etc., we give a comprehensive overview of this paper.

It is well known that there are essential differences between the non-Newtonian fluid equation

$$u_t = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right), \quad (x,t)\in\Omega\times(0,T), \tag{2.13}$$

and the polytropic diffusion equation

$$u_t = \operatorname{div}\left(\left|\nabla u^m\right|^{p-2}\nabla u^m\right), \quad (x,t)\in\Omega\times(0,T). \tag{2.14}$$

Inspired by this fact, roughly speaking, our original jumping-off point is to show the essential differences between the electrorheological fluid equation

$$u_t = \operatorname{div}\big(a(x)|\nabla u|^{p(x)-2}\nabla u\big) + f(x,t,u,\nabla u), \quad (x,t) \in \Omega \times (0,T), \tag{2.15}$$

and the polytropic electrorheological fluid equation

$$u_t = \operatorname{div}\big(a(x)\big|\nabla u^m\big|^{p(x)-2}\nabla u^m\big) + f(x,t,u,\nabla u), \quad (x,t) \in \Omega \times (0,T). \tag{2.16}$$

The well-posedness of solutions to Eq. (2.15) was considered in [21, 22] etc.: that the degeneracy of $a(x)$ on the boundary $\partial\Omega$ can take place of the boundary value condition (1.3) had been shown in some special cases. But very few papers on the well-posedness of solutions to Eq. (2.16) can be found. In this paper, we directly study a much more general equation,

$$u_t = \operatorname{div}\big(b(x)\big|\nabla a(u)\big|^{p(x)-2}\nabla a(u)\big) + \sum_{i=1}^{N}\frac{\partial b_i(u)}{\partial x_i} + c(x,t) - b_0 a(u), \quad (x,t) \in Q_T, \tag{2.17}$$

$a(s) \geq 0$, $a(0) = 0$ and $a(s)$ is a strictly increasing function. As we have said before, Eq. (2.17) admits $a(s)$ satisfying (1.8) and has a wider applications.

In addition, condition (2.8) implies that equation (2.17) cannot be of the hyperbolic characteristic, usually, such a restriction has demonstrated a strong preference for being unnatural before. However, a model of strong degenerate parabolic equation arises in mathematical finance, which indicates that condition (2.8) is important and indispensable in the decision theory under the risk [27]. We have given more details in our previous work [28], so it is not appropriate to repeat the details here.

3 The proof of Theorem 2.3

Lemma 3.1 *Let $q \geq 1$. If $u_\varepsilon \in L^\infty(0,T;L^2(\Omega)) \cap \mathbf{W}(Q_T)$, $\|u_{\varepsilon t}\|_{\mathbf{W}'(Q_T)} \leq c$, $\|\nabla(|u_\varepsilon|^{q-1}u_\varepsilon)\|_{p^-,Q_T} \leq c$, then there is a subsequence of $\{u_\varepsilon\}$ which shows relatively compactness in $L^s(Q_T)$ with $s \in (1,\infty)$.*

This lemma can be found in [19].

Since $a(s)$ is a strictly increasing function, by a limit process, we can assume that $a(s)$ is a C^1 function in the proof. Consider the following regularized system:

$$\begin{aligned} u_{\varepsilon t} &= \operatorname{div}\Big(\big(b(x)+\varepsilon\big)\big(\big|\nabla a(u_\varepsilon)\big|^2 + \varepsilon\big)^{\frac{p(x)-2}{2}}\nabla a(u_\varepsilon)\Big) \\ &\quad + \sum_{i=1}^{N}\frac{\partial b_i(u_\varepsilon)}{\partial x_i} + c(x,t) - b_0 a(u), \quad (x,t) \in Q_T, \end{aligned} \tag{3.1}$$

$$u_\varepsilon(x,t) = \varepsilon, \quad (x,t) \in \partial\Omega \times (0,T), \tag{3.2}$$

$$u_\varepsilon(x,0) = u_{0\varepsilon}(x) + \varepsilon, \quad x \in \Omega, \tag{3.3}$$

where $u_{\varepsilon,0} \in C_0^\infty(\Omega)$ and $(b(x)+\varepsilon)|\nabla a(u_{\varepsilon,0})|^{p(x)} \in L^1(\Omega)$ are uniformly bounded, and $u_{\varepsilon,0}$ converges to u_0 in $W_0^{1,p(x)}(\Omega)$. Since we assume that $a(s)$ is a strictly increasing function, by the monotone convergence method, according to the classical parabolic equation theory [29, 30], there is a unique classical solution u_ε of the initial-boundary value problem (3.1)–(3.3), and

$$\|u_\varepsilon\|_{L^\infty(Q_T)} \le c. \tag{3.4}$$

Throughout this paper, the constants c may be different from one place to another.

Theorem 3.2 *There is a weak solution u of Eq.* (1.6) *with the initial value* (1.2) *in the sense of Definition* 2.1.

Proof For any $t \in [0, T)$, we multiply (3.1) by $a(u_\varepsilon) - a(\varepsilon)$ and integrate it over $Q_t = \Omega \times [0, t)$. By (3.3), (3.4) and

$$\begin{aligned}\iint_{Q_t} \bigl[a(u_\varepsilon) - a(\varepsilon)\bigr]\frac{\partial b_i(u_\varepsilon)}{\partial x_i}\,dx\,dt &= -\iint_{Q_t} \frac{\partial u_\varepsilon}{\partial x_i} a'(u_\varepsilon) b_i(u_\varepsilon)\,dx\,dt \\ &= -\iint_{Q_t} \frac{\partial}{\partial x_i}\int_\varepsilon^{u_\varepsilon} b_i(s)a'(s)\,ds\,dx\,dt \\ &= 0, \quad i = 1, 2, \dots, N,\end{aligned} \tag{3.5}$$

we have

$$\begin{aligned}&\int_\Omega A(u_\varepsilon)\,dx + \iint_{Q_t} \bigl(b(x) + \varepsilon\bigr)\bigl(\bigl|\nabla a(u_\varepsilon)\bigr|^2 + \varepsilon\bigr)^{\frac{p(x)-2}{2}} \bigl|\nabla a(u_\varepsilon)\bigr|^2\,dx\,dt \\ &\quad\le \int_\Omega A\bigl(u_0(x)\bigr)\,dx + a(\varepsilon)\int_\Omega \bigl|u(x, t) - u_0(x)\bigr|\,dx + c \\ &\quad\le c,\end{aligned} \tag{3.6}$$

where $A'(s) = a(s)$.

Since $b(x) > 0$ in Ω, for any $\Omega_1 \subset\subset \Omega$, (3.6) yields

$$\int_0^T \int_{\Omega_1} \bigl(b(x) + \varepsilon\bigr)\bigl(\bigl|\nabla a(u_\varepsilon)\bigr|^2 + \varepsilon\bigr)^{\frac{p(x)-2}{2}} \bigl|\nabla a(u_\varepsilon)\bigr|^2\,dx\,dt \le c \tag{3.7}$$

and

$$\int_0^T \int_{\Omega_1} \bigl|\nabla a(u_\varepsilon)\bigr|\,dx\,dt \le c\biggl(\int_0^T \int_{\Omega_1} \bigl|\nabla a(u_\varepsilon)\bigr|^{p^-}\,dx\,dt\biggr)^{\frac{1}{p^-}} \le c(\Omega_1). \tag{3.8}$$

Now, for any $v \in \mathbf{W}(Q_T)$, $\|v\|_{W(Q_T)} = 1$,

$$\begin{aligned}&\langle u_{\varepsilon t}, v\rangle \\ &\quad= -\iint_{Q_T} b(x)\bigl(\bigl|\nabla a(u_\varepsilon)\bigr|^2 + \varepsilon\bigr)^{\frac{p(x)-2}{2}} \nabla a(u_\varepsilon)\nabla v\,dx\,dt - \sum_{i=1}^N \iint_{Q_T} \frac{\partial v}{\partial x_i} b_i(u_\varepsilon)\,dx\,dt \\ &\qquad+ \iint_{Q_T} \bigl[c(x, t) - b_0 a(u_\varepsilon)(x, t)\bigr]v\,dx\,dt.\end{aligned}$$

By the Young inequality

$$\begin{aligned}
&\iint_{Q_T} b(x)\big(\left|\nabla a(u_\varepsilon)\right|^2+\varepsilon\big)^{\frac{p(x)-2}{2}}\nabla a(u_\varepsilon)\nabla v\,dx\,dt \\
&\quad\le c\iint_{Q_T} b(x)\Big[\big(\left|\nabla a(u_\varepsilon)\right|^2+\varepsilon\big)^{\frac{p(x)-2}{2}\frac{p(x)}{p(x)-1}}\left|\nabla a(u_\varepsilon)\right|^{\frac{p(x)}{p(x)-1}}+|\nabla v|^{p(x)}\Big]\,dx\,dt \\
&\quad\le c\iint_{Q_T} b(x)\big[\left|\nabla a(u_\varepsilon)\right|^{p(x)}+|\nabla v|^{p(x)}+1\big]\,dx\,dt \\
&\quad\le c,
\end{aligned}$$

we easily obtain

$$\left|\langle u_{\varepsilon t}, v\rangle\right|\le c,$$

which implies

$$\left\|(u_\varepsilon)_t\right\|_{\mathbf{W}'(Q_T)}\le c \tag{3.9}$$

and

$$\left\|a(u_\varepsilon)_t\right\|_{\mathbf{W}'(Q_T)}=\left\|a'(u_\varepsilon)u_{\varepsilon t}\right\|_{\mathbf{W}'(Q_T)}\le c. \tag{3.10}$$

Now, let $D_\lambda=\{x\in\Omega: d(x)>\lambda\}$ and $d(x)=\operatorname{dist}(x,\partial\Omega)$ be the distance function from $\partial\Omega$. For any given $\varphi\in C_0^1(\Omega)$, $0\le\varphi\le 1$, which satisfies

$$\varphi\,|_{D_{2\lambda}}=1,\qquad \varphi\,|_{\Omega\setminus D_\lambda}=0,$$

then

$$\left|\left\langle\left[\varphi a(u_\varepsilon)\right]_t, v\right\rangle\right|=\left|\left\langle\varphi a(u_\varepsilon)_t, v\right\rangle\right|$$

and

$$\left\|\left[\varphi a(u_\varepsilon)\right]_t\right\|_{\mathbf{W}'(Q_T)}\le\left\|a(u_\varepsilon)_t\right\|_{\mathbf{W}'(Q_T)}\le c.$$

If we denote $u_{1\varepsilon}=a(u_\varepsilon)$, then

$$\left\|(u_{1\varepsilon})_t\right\|_{\mathbf{W}'(Q_T)}\le c. \tag{3.11}$$

At the same time, from (3.8),

$$\iint_{Q_T}\left|\nabla\left[\varphi a(u_\varepsilon)\right]\right|^{p^-}dx\,dt\le c(\lambda)\left(1+\int_0^T\int_{\Omega_\lambda}\left|\nabla a(u_\varepsilon)\right|^{p^-}dx\,dt\right)\le c(\lambda),$$

i.e.

$$\left\|\nabla(|\varphi u_{1\varepsilon})\right\|_{p^-,Q_T}=\left\|\nabla\left[|\varphi a(u_\varepsilon)\right]\right\|_{p^-,Q_T}\le c(\lambda). \tag{3.12}$$

Thus $\varphi u_{1\varepsilon}$ shows relative compactness in $L^s(Q_T)$ with $s \in (1,\infty)$ by Lemma 3.1. Accordingly, $\varphi u_{1\varepsilon} \to \varphi u_1$ a.e. in Q_T and so $u_{1\varepsilon} \to u_1$ a.e. in Q_T.

Since $a'(s) \geq 0$ and $a(s)$ is a strictly monotone increasing function, $u_\varepsilon = a^{-1}(u_{1\varepsilon})$, setting $u = a^{-1}(u)$, we know that $u_\varepsilon \to u$ a.e. in Q_T.

From (3.4), there exists a function u such that

$$u_\varepsilon \rightharpoonup *u, \quad \text{in } L^\infty(Q_T),$$

and

$$u \in L^\infty(Q_T), \qquad u_t \in \mathbf{W}'(Q_T).$$

From (3.6), (3.8), there is a n-dimensional vector function $\overrightarrow{\zeta} = (\zeta_1, \ldots, \zeta_n)$ satisfying

$$|\overrightarrow{\zeta}| \in L^1\big(0,T;L^{\frac{p(x)}{p(x)-1}}(\Omega)\big),$$

such that

$$\big(b(x)+\varepsilon\big)\big|\nabla a(u_\varepsilon)\big|^{p(x)-2}\nabla a(u_\varepsilon) \rightharpoonup \overrightarrow{\zeta} \quad \text{in } L^1\big(0,T;L^{\frac{p(x)}{p(x)-1}}(\Omega)\big).$$

In what follows, we want to prove that u satisfies Eq. (1.6). At first,

$$\begin{aligned}&\iint_{Q_T}\Bigg[u_{\varepsilon t}\varphi + \big(b(x)+\varepsilon\big)\big(\big|\nabla a(u_\varepsilon)\big|^2+\varepsilon\big)^{\frac{p(x)-2}{2}}\nabla a(u_\varepsilon)\cdot\nabla\varphi + \sum_{i=1}^N b_i(u_\varepsilon)\cdot\varphi_{x_i}\Bigg]\,dx\,dt\\ &\quad = \iint_{Q_T}\big[c(x,t) - b_0 a(u_\varepsilon)\big]\varphi\,dx\,dt,\end{aligned} \tag{3.13}$$

for any function $\varphi \in L^\infty(0,T;W_0^{1,p(x)}(\Omega)) \cap \mathbf{W}(Q_T)$. Since $u_\varepsilon \to u$ almost everywhere, $b_i(u_\varepsilon) \to b_i(u)$ and $a(u_\varepsilon) \to a(u)$. Letting $\varepsilon \to 0$ in (3.13) yields

$$\begin{aligned}&\iint_{Q_T}\Bigg[\frac{\partial u}{\partial t}\varphi + \vec{\zeta}\cdot\nabla\varphi + \sum_{i=1}^N b_i(u)\cdot\varphi_{x_i}\Bigg]\,dx\,dt\\ &\quad = \iint_{Q_T}\big[c(x,t) - b_0 a(u)\big]\varphi\,dx\,dt.\end{aligned} \tag{3.14}$$

Secondly, we will prove that

$$\iint_{Q_T} b(x)\big|\nabla a(u)\big|^{p(x)-2}\nabla a(u)\cdot\nabla\varphi\,dx\,dt = \iint_{Q_T}\overrightarrow{\zeta}\cdot\nabla\varphi\,dx\,dt, \tag{3.15}$$

for any function $\varphi \in C_0^\infty(Q_T)$.

Let $0 \leq \psi \in C_0^\infty(Q_T)$ and $\psi = 1$ in $\operatorname{supp}\varphi$, and let $v \in L^\infty(Q_T)$, $b(x)|\nabla v|^{p(x)} \in L^1(Q_T)$. Then

$$\begin{aligned}&\iint_{Q_T}\psi\big(b(x)+\varepsilon\big)\big[\big|\nabla a(u_\varepsilon)\big|^{p(x)-2}\nabla a(u_\varepsilon) - |\nabla v|^{p(x)-2}\nabla v\big]\cdot\big(\nabla a(u_\varepsilon)-\nabla v\big)\,dx\,dt\\ &\quad \geq 0.\end{aligned} \tag{3.16}$$

We choose $\varphi = \psi a(u_\varepsilon)$ in (3.13), then

$$\begin{aligned}
&\iint_{Q_T} \psi\big(b(x)+\varepsilon\big)\big(\big|\nabla a(u_\varepsilon)\big|^2+\varepsilon\big)^{\frac{p(x)-2}{2}}\big|\nabla a(u_\varepsilon)\big|^2\,dx\,dt \\
&\quad= \iint_{Q_T} \psi_t A(u_\varepsilon)\,dx\,dt - \iint_{Q_T}\big(b(x)+\varepsilon\big)a(u_\varepsilon)\big(\big|\nabla a(u_\varepsilon)\big|^2+\varepsilon\big)^{\frac{p(x)-2}{2}}\nabla a(u_\varepsilon)\cdot\nabla\psi\,dx\,dt \\
&\qquad- \sum_{i=1}^{N}\iint_{Q_T} b_i(u_\varepsilon)\big(a'(u_\varepsilon)u_{\varepsilon x_i}\psi + a(u_\varepsilon)\psi_{x_i}\big)\,dx\,dt \\
&\qquad+ \iint_{Q_T}\big[c(x,t)-b_0 a(u_\varepsilon)\big]\psi a(u_\varepsilon)\,dx\,dt.
\end{aligned} \tag{3.17}$$

By (3.16), we can extrapolate to

$$\begin{aligned}
&\iint_{Q_T} \psi_t A(u_\varepsilon)\,dx\,dt - \iint_{Q_T}\big(b(x)+\varepsilon\big)a(u_\varepsilon)\big(\big|\nabla a(u_\varepsilon)\big|^2+\varepsilon\big)^{\frac{p(x)-2}{2}}\nabla a(u_\varepsilon)\cdot\nabla\psi\,dx\,dt \\
&\qquad- \sum_{i=1}^{N}\iint_{Q_T} b_i(u_\varepsilon)\big(a'(u_\varepsilon)u_{\varepsilon x_i}\psi + a(u_\varepsilon)\psi_{x_i}\big)\,dx\,dt \\
&\qquad+ \varepsilon^{\frac{p^-}{2}}c(\Omega) - \iint_{Q_T}\big(b(x)+\varepsilon\big)\psi\big|\nabla a(u_\varepsilon)\big|^{p(x)-2}\nabla a(u_\varepsilon)\nabla v\,dx\,dt \\
&\qquad- \iint_{Q_T}\big(b(x)+\varepsilon\big)\psi|\nabla v|^{p(x)-2}\nabla v\cdot\nabla\big(a(u_\varepsilon)-v\big)\,dx\,dt \\
&\qquad+ \iint_{Q_T}\big[c(x,t)-b_0 a(u_\varepsilon)\big]\psi a(u_\varepsilon)\,dx\,dt \\
&\quad\geq 0.
\end{aligned} \tag{3.18}$$

Accordingly,

$$\begin{aligned}
&\iint_{Q_T} \psi_t A(u_\varepsilon)\,dx\,dt - \iint_{Q_T}\big(b(x)+\varepsilon\big)a(u_\varepsilon)\big(\big|\nabla a(u_\varepsilon)\big|^2+\varepsilon\big)^{\frac{p(x)-2}{2}}\nabla a(u_\varepsilon)\cdot\nabla\psi\,dx\,dt \\
&\qquad- \sum_{i=1}^{N}\iint_{Q_T} b_i(u_\varepsilon)\big(a'(u_\varepsilon)u_{\varepsilon x_i}\psi + a(u_\varepsilon)\psi_{x_i}\big)\,dx\,dt \\
&\qquad+ \varepsilon^{\frac{p^-}{2}}c(\Omega) - \iint_{Q_T}\big(b(x)+\varepsilon\big)\psi\big|\nabla a(u_\varepsilon)\big|^{p(x)-2}\nabla a(u_\varepsilon)\nabla v\,dx\,dt \\
&\qquad- \iint_{Q_T}\psi b(x)|\nabla v|^{p(x)-2}\nabla v\cdot\big(\nabla a(u_\varepsilon)-\nabla v\big)\,dx\,dt \\
&\qquad- \varepsilon\iint_{Q_T}\psi|\nabla v|^{p(x)-2}\nabla v\cdot\big(\nabla a(u_\varepsilon)-\nabla v\big)\,dx\,dt \\
&\qquad+ \iint_{Q_T}\big[c(x,t)-b_0 a(u_\varepsilon)\big]\psi a(u_\varepsilon)\,dx\,dt \\
&\quad\geq 0.
\end{aligned} \tag{3.19}$$

Now, since

$$
\begin{aligned}
&\left(\left|\nabla a(u_\varepsilon)\right|^2+\varepsilon\right)^{\frac{p(x)-2}{2}}\nabla a(u_\varepsilon)\\
&\quad=\left|\nabla a(u_\varepsilon)\right|^{p(x)-2}\nabla a(u_\varepsilon)+\frac{p(x)-2}{2}\varepsilon\int_0^1\left(\left|\nabla a(u_\varepsilon)\right|^2+\varepsilon s\right)^{\frac{p(x)-4}{2}}\,ds\nabla a(u_\varepsilon),
\end{aligned}
$$

we have

$$
\lim_{\varepsilon\to0}\iint_{Q_T}\frac{p(x)-2}{2}\varepsilon\int_0^1\left(\left|\nabla a(u_\varepsilon)\right|^2+\varepsilon s\right)^{\frac{p(x)-4}{2}}\,ds\nabla a(u_\varepsilon)\nabla\psi a(u_\varepsilon)\,dx\,dt=0. \tag{3.20}
$$

At the same time, using the Hölder inequality

$$
\int_\Omega b(x)|\nabla v|^{p(x)-1}\left|\nabla a(u_\varepsilon)\right|dx\le\left\|b^{\frac{1}{s(x)}}|\nabla v|^{p(x)-1}\right\|_{L^{s(x)}(\Omega)}\left\|b^{\frac{1}{p(x)}}\left|\nabla a(u_\varepsilon)\right|\right\|_{L^{p(x)}(\Omega)},
$$

we have

$$
\iint_{Q_T}b(x)|\nabla v|^{p(x)}\,dx\,dt+\iint_{Q_T}b(x)|\nabla v|^{p(x)-1}\left|\nabla a(u_\varepsilon)\right|dx\,dt\le c. \tag{3.21}
$$

Here $s(x)=\frac{p(x)}{p(x)-1}$.

By (3.20)–(3.21), we have

$$
\begin{aligned}
&\lim_{\varepsilon\to0}\varepsilon\left|\iint_{Q_T}\psi|\nabla v|^{p(x)-2}\nabla v\cdot\left(\nabla a(u_\varepsilon)-\nabla v\right)dx\,dt\right|\\
&\quad\le\lim_{\varepsilon\to0}\varepsilon\sup_{(x,t)\in Q_T}\frac{|\psi|}{b(x)}\iint_{Q_T}b(x)|\nabla v|^{p(x)-1}\left|\nabla a(u_\varepsilon)-\nabla v\right|dx\,dt\\
&\quad\le\lim_{\varepsilon\to0}\varepsilon\sup_{(x,t)\in Q_T}\frac{|\psi|}{b(x)}\left(\iint_{Q_T}b(x)|\nabla v|^{p(x)}\,dx\,dt\right.\\
&\qquad\left.+\iint_{Q_T}b(x)|\nabla v|^{p(x)-1}\left|\nabla a(u_\varepsilon)\right|dx\,dt\right)\\
&\quad=0.
\end{aligned} \tag{3.22}
$$

Let $\varepsilon\to0$. By (3.19) and (3.22), we have

$$
\begin{aligned}
&\iint_{Q_T}\psi_tA(u)\,dx\,dt-\iint_{Q_T}a(u)\overrightarrow{\zeta}\cdot\nabla\psi\,dx\,dt\\
&\qquad-\sum_{i=1}^N\iint_{Q_T}b_i(u)\left(a'(u)u_{x_i}\psi+a(u)\psi_{x_i}\right)dx\,dt\\
&\qquad-\iint_{Q_T}\psi\overrightarrow{\zeta}\cdot\nabla v\,dx\,dt-\iint_{Q_T}\psi b(x)|\nabla v|^{p(x)-2}\nabla v\cdot\left(\nabla a(u)-\nabla v\right)dx\,dt\\
&\qquad+\iint_{Q_T}\left[c(x,t)-b_0a(u)\right]\psi a(u)\,dx\,dt\\
&\quad\ge0.
\end{aligned}
$$

Let $\varphi = \psi u$ in (3.14). We get

$$\begin{aligned}&\iint_{Q_T} \psi \overrightarrow{\zeta} \cdot \nabla a(u)\,dx\,dt - \iint_{Q_T} a(u)\psi_t\,dx\,dt\\&\quad + \iint_{Q_T} a(u)\overrightarrow{\zeta} \cdot \nabla \psi\,dx\,dt\\&\quad + \sum_{i=1}^{N} \iint_{Q_T} b_i(u)(a'(u)u_{x_i}\psi + a(u)\psi_{x_i}\,dx\,dt\\&\quad + \iint_{Q_T} \bigl[c(x,t) - b_0 a(u)\bigr]\psi a(u)\,dx\,dt\\&= 0.\end{aligned}$$

From the above formulas, we can extrapolate to

$$\iint_{Q_T} \psi\bigl(\overrightarrow{\zeta} - b(x)|\nabla v|^{p(x)-2}\nabla v\bigr) \cdot \bigl(\nabla a(u) - \nabla v\bigr)\,dx\,dt \ge 0. \tag{3.23}$$

If we choose $v = a(u) - \lambda\varphi$ and choose $\lambda > 0$ or $\lambda < 0$, respectively, letting $\lambda \to 0$, we can deduce

$$\iint_{Q_T} \psi\bigl(\overrightarrow{\zeta} - b(x)\bigl|\nabla a(u)\bigr|^{p(x)-2}\nabla a(u)\bigr) \cdot \nabla\varphi\,dx\,dt = 0.$$

Since $\psi = 1$ on $\operatorname{supp}\varphi$, we know that (3.15) is true.

At last, (2.3) can be showed as in [19], the proof of Theorem 3.2 finishes. □

Lemma 3.3 *Let $u(x,t)$ be a solution of Eq. (1.6) with the initial value (1.2). If $\int_\Omega b(x)^{-\frac{1}{p^- -1}}\,dx < \infty$, then*

$$\int_\Omega \bigl|\nabla a(u)\bigr|\,dx < \infty.$$

Proof

$$\begin{aligned}&\int_\Omega \bigl|\nabla a(u)\bigr|\,dx\\&\quad = \int_{\{x\in\Omega: b(x)^{\frac{1}{p^- -1}}|\nabla a(u)|\le 1\}} \bigl|\nabla a(u)\bigr|\,dx + \int_{\{x\in\Omega: b(x)^{\frac{1}{p^- -1}}|\nabla a(u)|>1\}} \bigl|\nabla a(u)\bigr|\,dx\\&\quad \le \int_\Omega b(x)^{-\frac{1}{p^- -1}}\,dx + \int_\Omega b(x)\bigl|\nabla a(u)\bigr|^{p^-}\,dx\\&\quad \le c.\end{aligned}$$

□

For small $\eta > 0$, we define

$$S_\eta(s) = \int_0^s h_\eta(\tau)\,d\tau,$$

where $h_\eta(s) = \frac{2}{\eta}(1 - \frac{|s|}{\eta})_+$, and clearly

$$\lim_{\eta\to 0^+} sS'_\eta(s) = \lim_{\eta\to 0} sh_\eta(s) = 0,$$

$$\lim_{\eta\to 0^+} S_\eta(s) = \operatorname{sgn}(s),$$

where $\operatorname{sgn}(s)$ is the sign function.

Theorem 3.4 *Suppose $\int_\Omega b(x)^{-\frac{1}{p^- -1}}\,dx < \infty$, $a(s)$ and $b_i(s)$ satisfying (1.7) and (2.8). If $u(x,t)$ and $v(x,t)$ are two weak solutions with the same homogeneous boundary value (1.3) and with different initial values $u(x,0)$, $v(x,0)$, respectively, we have*

$$\int_\Omega \bigl|u(x,t) - v(x,t)\bigr|\,dx \le c\int_\Omega \bigl|u_0(x) - v_0(x)\bigr|\,dx, \quad \forall t \in [0,T). \tag{3.24}$$

Proof By Definition 2.1, $b(x)|\nabla a(u)|^{p(x)}, b(x)|\nabla a(v)|^{p(x)} \in L^1(Q_T)$, and for any

$$\varphi \in L^\infty\bigl(0,T; W_0^{1,p(x)}(\Omega)\bigr) \cap \mathbf{W}(Q_T)$$

we have

$$\begin{aligned}
&\iint_{Q_t} \varphi \frac{\partial(u-v)}{\partial t}\,dx\,dt \\
&\quad = -\iint_{Q_t} b(x)\bigl(\bigl|\nabla a(u)\bigr|^{p(x)-2}\nabla a(u) - \bigl|\nabla a(v)\bigr|^{p(x)-2}\nabla a(v)\bigr)\cdot \nabla\varphi\,dx\,dt \\
&\qquad - \sum_{i=1}^N \iint_{Q_t} \bigl[b_i(u) - b_i(v)\bigr]\cdot \varphi_{x_i}\,dx\,dt - \iint_{Q_t} b_0\bigl[a(u) - a(v)\bigr]\varphi\,dx\,dt,
\end{aligned} \tag{3.25}$$

where $Q_t = \Omega \times (0,t)$.

Thus, if we choose $S_\eta(a(u) - a(v))$ as the test function, we have

$$\begin{aligned}
&\iint_{Q_t} S_\eta\bigl(a(u) - a(v)\bigr)\frac{\partial(u-v)}{\partial t}\,dx\,dt \\
&\qquad + \iint_{Q_t} b(x)\bigl[\bigl|\nabla a(u)\bigr|^{p(x)-2}\nabla a(u) - \bigl|\nabla a(v)\bigr|^{p(x)-2}\nabla a(v)\bigr] \\
&\qquad \cdot \nabla\bigl(a(u) - a(v)\bigr)h_\eta\bigl(a(u) - a(v)\bigr)\,dx\,dt \\
&\qquad + \sum_{i=1}^N \iint_{Q_t} \bigl[b_i(u) - b_i(v)\bigr]\cdot \bigl(a(u) - a(v)\bigr)_{x_i} h_\eta(u-v)\,dx\,dt \\
&\quad = -\iint_{Q_t} b_0\bigl[a(u) - a(v)\bigr]S_\eta\bigl(a(u) - a(v)\bigr)\,dx\,dt.
\end{aligned} \tag{3.26}$$

Since $a(s)$ is a monotone increasing function, we can easily show that

$$\lim_{\eta\to 0^+} \int_\Omega S_\eta\bigl(a(u) - a(v)\bigr)\frac{\partial(u-v)}{\partial t}\,dx = \frac{d}{dt}\|u - v\|_{L^1(\Omega)}, \tag{3.27}$$

and clearly

$$\iint_{Q_t} b(x)\big[\big|\nabla a(u)\big|^{p(x)-2}\nabla a(u) - \big|\nabla a(v)\big|^{p(x)-2}\nabla a(v)\big] \\ \cdot \nabla\big(a(u)-a(v)\big)h_\eta\big(a(u)-a(v)\big)\,dx\,dt \geq 0. \tag{3.28}$$

Now, by that $|sh_\eta(s)| \leq 1$, we have

$$\begin{aligned} &\left|\iint_{Q_t\cap\{|a(u)-a(v)|<\eta\}} \sum_{i=1}^N \big[b_i(u)-b_i(v)\big]\big[S_\eta\big(a(u)-a(v)\big)\big]_{x_i}\,dx\,dt\right| \\ &\quad= \left|\iint_{Q_t\cap\{|a(u)-a(v)|<\eta\}} \sum_{i=1}^N \big[b_i(u)-b_i(v)\big]h_\eta\big(a(u)-a(v)\big)\big(a(u)-a(v)\big)_{x_i}\,dx\,dt\right| \\ &\quad\leq c\iint_{Q_t\cap\{|a(u)-a(v)|<\eta\}} \sum_{i=1}^N \left|\frac{b_i(u)-b_i(v)}{a(u)-a(v)}\right|\left|\big(a(u)-a(v)\big)_{x_i}\right|\,dx\,dt \\ &\quad= c\iint_{Q_t\cap\{|a(u)-a(v)|<\eta\}} \left|b(x)^{-\frac{1}{p^-}}\sum_{i=1}^N \frac{b_i(u)-b_i(v)}{a(u)-a(v)}\right| b(x)^{\frac{1}{p^-}}\left|\big(a(u)-a(v)\big)_{x_i}\right|\,dx\,dt \\ &\quad\leq c\left[\iint_{Q_t\cap\{|a(u)-a(v)|<\eta\}} \left(\left|b(x)^{-\frac{1}{p^-}}\sum_{i=1}^N \frac{b_i(u)-b_i(v)}{a(u)-a(v)}\right|\right)^{\frac{p^-}{p^--1}}\,dx\,dt\right]^{\frac{p^--1}{p^-}} \\ &\qquad\cdot\left(\iint_{Q_t\cap\{|a(u)-a(v)|<\eta\}} \big|b(x)\nabla\big(a(u)-a(v)\big)\big|^{p^-}\,dx\,dt\right)^{\frac{1}{p^-}}. \end{aligned} \tag{3.29}$$

Since $\int_\Omega b(x)^{-\frac{1}{p^--1}}\,dx < \infty$, by the assumption (2.8), we have

$$\begin{aligned} &\iint_{Q_t\cap\{|a(u)-a(v)|<\eta\}} \left(\left|b(x)^{-\frac{1}{p^-}}\sum_{i=1}^N \frac{b_i(u)-b_i(v)}{a(u)-a(v)}\right|\right)^{\frac{p}{p^--1}}\,dx\,dt \\ &\quad\leq c\iint_{Q_t} b(x)^{-\frac{1}{p^--1}}\,dx\,dt \leq c. \end{aligned} \tag{3.30}$$

Let $\eta \to 0^+$ in (3.29). If $\{x\in\Omega : |a(u)-a(v)| = 0\}$ is a set with 0 measure, then

$$\lim_{\eta\to 0^+}\iint_{Q_t\cap\{|a(u)-a(v)|<\eta\}} \left|b(x)^{\frac{-1}{p^--1}}\right|\,dx\,dt = \iint_{Q_t\cap\{|a(u)-a(v)|=0\}} \left|b(x)^{\frac{-1}{p^--1}}\right|\,dx\,dt = 0. \tag{3.31}$$

If the set $\{x\in\Omega : |a(u)-a(v)| = 0\}$ has a positive measure, then

$$\begin{aligned} &\lim_{\eta\to 0^+}\iint_{Q_t\cap\{|a(u)-a(v)|<\eta\}} b(x)\big|\nabla\big(a(u)-a(v)\big)\big|^{p^-}\,dx\,dt \\ &\quad= \iint_{Q_t\cap\{|a(u)-a(v)|=0\}} b(x)\big|\nabla\big(a(u)-a(v)\big)\big|^{p^-}\,dx\,dt \\ &\quad= 0. \end{aligned} \tag{3.32}$$

Therefore, in both cases, (3.29) tends to 0 as $\eta \to 0^+$.

Thus,

$$\lim_{\eta\to 0^+}\iint_{Q_t}\bigl[b_i(u)-b_i(v)\bigr]h_\eta\bigl(a(u)-a(v)\bigr)\bigl(a(u)-a(v)\bigr)_{x_i}\,dx\,dt=0, \tag{3.33}$$

$$\begin{aligned}
&-\lim_{\eta\to 0^+}\iint_{Q_t} b_0\bigl[a(u)-a(v)\bigr]S_\eta\bigl(a(u)-a(v)\bigr)\,dx\,dt\\
&\quad=-\iint_{Q_t} b_0\bigl|a(u)-a(v)\bigr|\,dx\,dt\le 0.
\end{aligned} \tag{3.34}$$

Let $\eta\to 0^+$ in (3.26). Then, by (3.27)–(3.34), we have

$$\int_\Omega\bigl|u(x,t)-v(x,t)\bigr|\,dx-\int_\Omega\bigl|u_0(x)-v_0(x)\bigr|\,dx=\int_0^t\frac{d}{dt}\|u-v\|_{L^1(\Omega)}\,dt\le 0.$$

Then

$$\int_\Omega\bigl|u(x,t)-v(x,t)\bigr|\,dx\le c\int_\Omega\bigl|u_0(x)-v_0(x)\bigr|\,dx,\quad \forall t\in[0,T).$$

Theorem 3.4 is proved. □

Theorem 2.3 is the directly corollary of Theorem 3.2, Lemma 3.3 and Theorem 3.4.

4 The proof of Theorem 2.4

Proof of Theorem 2.4 For any small $\lambda>0$, denote

$$\Omega_\lambda=\bigl\{x\in\Omega:b(x)>\lambda\bigr\}, \tag{4.1}$$

let $\beta>0$ and

$$\phi(x)=\bigl(b(x)-\lambda\bigr)_+^\beta. \tag{4.2}$$

Let u_ε and v_ε be the mollified function of the solutions u and v, respectively, $\chi_{[s,t]}$ be the characteristic function of $[s,t]\subset(0,T)$ and let us choose $\chi_{[s,t]}S_\eta(\phi(a(u_\varepsilon)-a(v_\varepsilon)))$ as a test function. Then

$$\begin{aligned}
&\int_s^t\int_{\Omega_\lambda}S_\eta\bigl(\phi\bigl(a(u_\varepsilon)-a(v_\varepsilon)\bigr)\bigr)\frac{\partial(u-v)}{\partial t}\,dx\,dt\\
&\quad+\int_s^t\int_{\Omega_\lambda}b(x)\bigl[\bigl|\nabla a(u)\bigr|^{p(x)-2}\nabla a(u)-\bigl|\nabla a(v)\bigr|^{p(x)-2}\nabla a(v)\bigr]\\
&\quad\cdot\phi\nabla\bigl(a(u_\varepsilon)-a(v_\varepsilon)\bigr)h_\eta\bigl(\phi\bigl(a(u_\varepsilon)-a(v_\varepsilon)\bigr)\bigr)\,dx\,dt\\
&\quad+\int_s^t\int_{\Omega_\lambda}b(x)\bigl[\bigl|\nabla a(u)\bigr|^{p(x)-2}\nabla a(u)-\bigl|\nabla a(v)\bigr|^{p(x)-2}\nabla a(v)\bigr]\\
&\quad\cdot\nabla\phi\bigl(a(u_\varepsilon)-a(v_\varepsilon)\bigr)h_\eta\bigl(\phi\bigl(a(u_\varepsilon)-a(v_\varepsilon)\bigr)\bigr)\,dx\,dt\\
&\quad+\sum_{i=1}^N\int_s^t\int_{\Omega_\lambda}\bigl[b_i(u)-b_i(v)\bigr]\phi\bigl(a(u_\varepsilon)-a(v_\varepsilon)\bigr)_{x_i}h_\eta\bigl(\phi\bigl(a(u_\varepsilon)-a(v_\varepsilon)\bigr)\bigr)\,dx\,dt
\end{aligned}$$

$$+\sum_{i=1}^{N}\int_s^t\int_{\Omega_\lambda}\bigl[b_i(u)-b_i(v)\bigr]\phi_{x_i}\bigl(a(u_\varepsilon)-a(v_\varepsilon)\bigr)h_\eta\bigl(\phi\bigl(a(u_\varepsilon)-a(v_\varepsilon)\bigr)\bigr)\,dx\,dt$$
$$=-\int_s^t\int_{\Omega_\lambda}b_0\bigl(a(u_\varepsilon)-a(v_\varepsilon)\bigr)S_\eta\bigl(\phi\bigl(a(u_\varepsilon)-a(v_\varepsilon)\bigr)\bigr)\,dx\,dt. \tag{4.3}$$

For any given $\lambda>0$, by (2.1) in Definition 2.1, $|\nabla a(u)|\in L^{p(x)}(\Omega_\lambda)$, $|\nabla a(v)|^{p(x)}\in L^{p(x)}(\Omega_\lambda)$. Thus according to the definition of the mollified function, since the exponent $p(x)$ is required to satisfy the logarithmic Hölder continuity condition, we have

$$\begin{aligned}&a(u_\varepsilon)\in L^\infty(Q_T),\qquad a(v_\varepsilon)\in L^\infty(Q_T),\qquad a(u_\varepsilon)\to a(u),\\&a(v_\varepsilon)\to a(v),\quad \text{a.e. in } Q_T,\end{aligned} \tag{4.4}$$

$$\begin{aligned}&\bigl\||\nabla a(u_\varepsilon)|^{p(x)}\bigr\|_{1,\Omega_\lambda}\le\bigl\||\nabla a(u)|^{p(x)}\bigr\|_{1,\Omega_\lambda},\\&\bigl\||\nabla a(v_\varepsilon)|^{p(x)}\bigr\|_{1,\Omega_\lambda}\le\bigl\||\nabla a(v)|^{p(x)}\bigr\|_{1,\Omega_\lambda},\end{aligned} \tag{4.5}$$

$$\nabla a(u_\varepsilon)\to\nabla a(u),\qquad \nabla a(v_\varepsilon)\to\nabla a(v),\quad \text{in } L^{p(x)}(\Omega_\lambda). \tag{4.6}$$

We give some explanations. Denoting $w=a(u)\in W^{1,p(x)}(\Omega_\lambda)$, there is a series $w_\varepsilon\in W^{1,p(x)}(\Omega_\lambda)$ such that

$$w_\varepsilon\to w=a(u),\quad \text{in } W^{1,p(x)}(\Omega_\lambda). \tag{4.7}$$

Since $a(s)$ is a strictly monotone increasing function, by (4.7), it is easy to show that

$$a^{-1}(w_\varepsilon)\to a^{-1}(w)=u,\quad \text{in } W^{1,p(x)}(\Omega_\lambda) \tag{4.8}$$

by the uniqueness of the limit, then $w_\varepsilon=a(u_\varepsilon)$, accordingly, we have (4.4)–(4.6).

By

$$0\le h_\eta\bigl(\phi\bigl(a(u_\varepsilon)-a(v_\varepsilon)\bigr)\bigr)\le\frac{2}{\eta}, \tag{4.9}$$

we have

$$\begin{aligned}&\bigl|\nabla\bigl(a(u_\varepsilon)-a(v_\varepsilon)\bigr)h_\eta\bigl(\phi\bigl(a(u_\varepsilon)-a(v_\varepsilon)\bigr)\bigr)\bigr|_{L^{p(x)}(\Omega_\lambda)}\\&\quad\le c(\eta)\bigl|\nabla\bigl(a(u_\varepsilon)-a(v_\varepsilon)\bigr)\bigr|_{L^{p(x)}(\Omega_\lambda)}\le c(\eta).\end{aligned} \tag{4.10}$$

If we denote

$$\begin{aligned}&\int_{\Omega_\lambda}\nabla\bigl(a(u_\varepsilon)-a(v_\varepsilon)\bigr)h_\eta\bigl(\phi\bigl(a(u_\varepsilon)-a(v_\varepsilon)\bigr)\bigr)\varphi\,dx\\&\qquad-\int_{\Omega_\lambda}\nabla\bigl(a(u)-a(v)\bigr)h_\eta\bigl(\phi\bigl(a(u)-a(v)\bigr)\bigr)\varphi\,dx\\&\quad=\int_{\Omega_\lambda}\nabla\bigl(a(u_\varepsilon)-a(v_\varepsilon)\bigr)\bigl[h_\eta\bigl(\phi\bigl(a(u_\varepsilon)-a(v_\varepsilon)\bigr)\bigr)-h_\eta\bigl(\phi\bigl(a(u)-a(v)\bigr)\bigr)\bigr]\varphi\,dx\end{aligned}$$

$$\begin{aligned}&+\int_{\Omega_\lambda}\left[\nabla\left(a(u_\varepsilon)-a(v_\varepsilon)\right)-\nabla\left(a(u)-a(v)\right)\right]h_\eta\left(\phi\left(a(u)-a(v)\right)\right)\varphi\,dx\\&=I_1+I_2,\end{aligned}\tag{4.11}$$

for any $\varphi\in L^{\frac{p(x)}{p(x)-1}}(\Omega_\lambda)$, by $\nabla a(u_\varepsilon)\to\nabla a(u)$, $\nabla a(v_\varepsilon)\to\nabla a(v)$, in $L^{p(x)}(\Omega_\lambda)$, we obtain

$$\lim_{\varepsilon\to0}I_2=0,\tag{4.12}$$

Moreover,

$$\begin{aligned}\lim_{\varepsilon\to0}I_1&\le\lim_{\varepsilon\to0}\left\|\nabla\left(a(u_\varepsilon)-a(v_\varepsilon)\right)\right\|_{L^{p(x)}(\Omega_\lambda)}\\&\quad\cdot\left\|\left[h_\eta\left(\phi\left(a(u_\varepsilon)-a(v_\varepsilon)\right)\right)-h_\eta\left(\phi\left(a(u)-a(v)\right)\right)\right]\varphi\right\|_{L^{\frac{p(x)}{p(x)-1}}(\Omega_\lambda)}\\&\le\lim_{\varepsilon\to0}\left\|\nabla\left(a(u)-a(v)\right)\right\|_{L^{p(x)}(\Omega_\lambda)}\\&\quad\cdot\left\|\left[h_\eta\left(\phi\left(a(u_\varepsilon)-a(v_\varepsilon)\right)\right)-h_\eta\left(\phi\left(a(u)-a(v)\right)\right)\right]\varphi\right\|_{L^{\frac{p(x)}{p(x)-1}}(\Omega_\lambda)}\\&=0,\end{aligned}\tag{4.13}$$

by the Lebesgue dominated convergence theorem. By (4.11)–(4.13), we obtain

$$\begin{aligned}&\nabla\left(a(u_\varepsilon)-a(v_\varepsilon)\right)h_\eta\left(\phi\left(a(u_\varepsilon)-a(v_\varepsilon)\right)\right)\\&\quad\rightharpoonup\nabla\left(a(u)-a(v)\right)h_\eta\left(\phi\left(a(u)-a(v)\right)\right),\quad\text{in }L^{p(x)}(\Omega_\lambda).\end{aligned}\tag{4.14}$$

By (4.14)

$$\begin{aligned}&\lim_{\varepsilon\to0}\int_{\Omega_\lambda}b(x)\left[\left|\nabla a(u)\right|^{p(x)-2}\nabla a(u)-\left|\nabla a(v)\right|^{p(x)-2}\nabla a(v)\right]\\&\qquad\cdot\phi\nabla\left(a(u_\varepsilon)-a(v_\varepsilon)\right)h_\eta\left(\phi\left(a(u_\varepsilon)-a(v_\varepsilon)\right)\right)dx\\&\quad=\int_{\Omega_\lambda}b(x)\left[\left|\nabla a(u)\right|^{p(x)-2}\nabla a(u)-\left|\nabla a(v)\right|^{p(x)-2}\nabla a(v)\right]\\&\qquad\cdot\phi\nabla\left(a(u)-a(v)\right)h_\eta\left(\phi\left(a(u)-a(v)\right)\right)dx,\end{aligned}\tag{4.15}$$

due to

$$\left|b(x)\phi\left[\left|\nabla a(u)\right|^{p(x)-2}\nabla a(u)-\left|\nabla a(v)\right|^{p(x)-2}\nabla a(v)\right]\right|\in L^{\frac{p(x)}{p(x)-1}}(\Omega_\lambda).$$

At the same time, clearly

$$\begin{aligned}&\lim_{\varepsilon\to0}\int_{\Omega_\lambda}b(x)\left[\left|\nabla a(u)\right|^{p(x)-2}\nabla a(u)-\left|\nabla a(v)\right|^{p(x)-2}\nabla(v)\right]\\&\qquad\cdot\nabla\phi\left(a(u_\varepsilon)-a(v_\varepsilon)\right)h_\eta\left(\phi\left(a(u_\varepsilon)-a(v_\varepsilon)\right)\right)dx\\&\quad=\int_{\Omega_\lambda}b(x)\left[\left|\nabla a(u)\right|^{p(x)-2}\nabla a(u)-\left|\nabla a(v)\right|^{p(x)-2}\nabla a(v)\right]\\&\qquad\cdot\nabla\phi\left(a(u)-a(v)\right)h_\eta\left(\phi\left(a(u)-a(v)\right)\right)dx,\end{aligned}\tag{4.16}$$

by the Lebesgue dominated convergence theorem.

Once more, we can obtain

$$\begin{aligned}&\big(a(u_\varepsilon)-a(v_\varepsilon)\big)_{x_i}h_\eta\big(\phi\big(a(u_\varepsilon)-a(v_\varepsilon)\big)\big)\\&\quad\rightharpoonup\big(a(u)-a(v)\big)_{x_i}h_\eta\big(\phi\big(a(u)-a(v)\big)\big),\quad\text{in } L^{p(x)}(\Omega_\lambda),\end{aligned}\tag{4.17}$$

by a similar method to (4.14). Thus

$$\begin{aligned}&\lim_{\varepsilon\to0}\int_\Omega\big[b_i(u)-b_i(v)\big]\phi\big(a(u_\varepsilon)-a(v_\varepsilon)\big)_{x_i}h_\eta\big(\phi\big(a(u_\varepsilon)-a(v_\varepsilon)\big)\big)\,dx\\&\quad=\int_\Omega\big[b_i(u)-b_i(v)\big]\phi\big(a(u)-a(v)\big)_{x_i}h_\eta\big(\phi\big(a(u)-a(v)\big)\big)\,dx.\end{aligned}\tag{4.18}$$

Meanwhile, we have

$$\begin{aligned}&\lim_{\varepsilon\to0}\int_\Omega\big[b_i(u)-b_i(v)\big]\phi_{x_i}\big(a(u_\varepsilon)-a(v_\varepsilon)\big)h_\eta\big(\phi\big(a(u_\varepsilon)-a(v_\varepsilon)\big)\big)\,dx\\&\quad=\int_\Omega\big[b_i(u)-b_i(v)\big]\phi_{x_i}\big(a(u)-a(v)\big)h_\eta\big(\phi\big(a(u)-a(v)\big)\big)\,dx.\end{aligned}\tag{4.19}$$

In addition, since

$$u_t,v_t\in\mathbf{W}'(Q_T),\tag{4.20}$$

according to [3], we have

$$\begin{aligned}&\lim_{\varepsilon\to0}\int_{\Omega_\lambda}S_\eta\big(\phi\big(a(u_\varepsilon)-a(v_\varepsilon)\big)\big)\frac{\partial(a(u)-a(v))}{\partial t}\,dx\\&\quad=\int_{\Omega_\lambda}S_\eta\big(\phi\big(a(u)-a(v)\big)\big)\frac{\partial(a(u)-a(v))}{\partial t}\,dx.\end{aligned}\tag{4.21}$$

Now, only if we let $\varepsilon\to0$, and let $\lambda\to0$ in (4.3), we have

$$\begin{aligned}&\int_s^t\int_\Omega S_\eta\big(b^\beta\big(a(u)-a(v)\big)\big)\frac{\partial(u-v)}{\partial t}\,dx\,dt\\&\qquad+\int_s^t\int_\Omega b(x)\big[\big|\nabla a(u)\big|^{p(x)-2}\nabla a(u)-\big|\nabla a(v)\big|^{p(x)-2}\nabla a(v)\big]\\&\qquad\cdot b^\beta\nabla\big(a(u)-a(v)\big)h_\eta\big(b^\beta\big(a(u)-a(v)\big)\big)\,dx\,dt\\&\qquad+\int_s^t\int_\Omega b(x)\big[\big|\nabla a(u)\big|^{p(x)-2}\nabla a(u)-\big|\nabla a(v)\big|^{p(x)-2}\nabla a(v)\big]\\&\qquad\cdot\nabla b^\beta\big(a(u)-a(v)\big)h_\eta\big(b^\beta\big(a(u)-a(v)\big)\big)\,dx\,dt\\&\qquad+\sum_{i=1}^N\int_s^t\int_\Omega\big[b_i(u)-b_i(v)\big]b^\beta\big(a(u)-a(v)\big)_{x_i}h_\eta\big(b^\beta\big(a(u)-a(v)\big)\big)\,dx\,dt\\&\qquad+\sum_{i=1}^N\int_s^t\int_\Omega\big[b_i(u)-b_i(v)\big]b^\beta_{x_i}\big(a(u)-a(v)\big)h_\eta\big(b^\beta\big(a(u)-a(v)\big)\big)\,dx\,dt\\&\quad=0.\end{aligned}\tag{4.22}$$

Let us analyze every term on the left hand side of (4.22).

For the first term, by $a(s)$ being strictly increasing,

$$\begin{aligned}
&\lim_{\eta\to 0^+}\int_\Omega S_\eta\bigl(b^\beta\bigl(a(u)-a(v)\bigr)\bigr)\frac{\partial(u-v)}{\partial t}\,dx\\
&=\int_\Omega \operatorname{sgn}\bigl(b^\beta\bigl(a(u)-a(v)\bigr)\bigr)\frac{\partial(u-v)}{\partial t}\,dx\\
&=\int_\Omega \operatorname{sgn}(u-v)\frac{\partial(u-v)}{\partial t}\,dx\\
&=\frac{d}{dt}\|u-v\|_{L^1(\Omega)}. \qquad (4.23)
\end{aligned}$$

For the second term,

$$\begin{aligned}
&\int_\Omega b(x)\bigl[\bigl|\nabla a(u)\bigr|^{p(x)-2}\nabla a(u)-\bigl|\nabla a(v)\bigr|^{p(x)-2}\nabla a(v)\bigr]\\
&\quad\cdot\nabla\bigl(a(u)-a(v)\bigr)h_\eta\bigl(b^\beta\bigl(a(u)-a(v)\bigr)\bigr)\phi(x)\,dx\geq 0. \qquad (4.24)
\end{aligned}$$

For the third term, from (iii) of Lemma 2.2, since $\int_\Omega b(x)^{-(p(x)-1)}\,dx<\infty$, and using the Lebesgue dominated convergence theorem, we have

$$\begin{aligned}
&\bigl\|b(x)^{-\frac{p(x)-1}{p(x)}}\bigl|b^\beta\bigl(a(u)-a(v)\bigr)h_\eta\bigl(b^\beta\bigl(a(u)-a(v)\bigr)\bigr)\bigr|\bigr\|_{L^{p(x)}(\{x:b^\beta|a(u)-a(v)|<\eta\})}\\
&\leq\biggl(\int_\Omega b(x)^{-(p(x)-1)}b^\beta\bigl(a(u)-a(v)\bigr)h_\eta\bigl(b^\beta\bigl(a(u)-a(v)\bigr)\bigr)\,dx\biggr)^{\frac{1}{p^+}}, \qquad (4.25)
\end{aligned}$$

which goes to zero as $\eta\to 0^+$.

By (4.2), we have

$$\begin{aligned}
&\lim_{\eta\to 0^+}\biggl|\int_\Omega b(x)\bigl[\bigl|\nabla a(u)\bigr|^{p(x)-2}\nabla a(u)-\bigl|\nabla a(v)\bigr|^{p(x)-2}\nabla a(v)\bigr]\\
&\qquad\cdot\nabla b^\beta\bigl(a(u)-a(v)\bigr)h_\eta\bigl(b^\beta\bigl(a(u)-a(v)\bigr)\bigr)\,dx\biggr|\\
&\leq c\lim_{\eta\to 0}\int_{\{x:b^\beta|a(u)-a(v)|<\eta\}}\bigl|\bigl|\nabla a(u)\bigr|^{p(x)-2}\nabla a(u)-\bigl|\nabla a(v)\bigr|^{p(x)-2}\nabla a(v)\bigr|\\
&\qquad\cdot\bigl|b^\beta\bigl(a(u)-a(v)\bigr)h_\eta\bigl(b^\beta\bigl(a(u)-a(v)\bigr)\bigr)\bigr|\,dx\\
&\leq c\lim_{\eta\to 0}\bigl\|b(x)^{-\frac{p(x)-1}{p(x)}}b^\beta\bigl(a(u)-a(v)\bigr)h_\eta\bigl(b^\beta\bigl(a(u)-a(v)\bigr)\bigr)\bigr\|_{L^{p(x)}(\{x:b^\beta|a(u)-a(v)|<\eta\})}\\
&\qquad\cdot\bigl\|b(x)^{\frac{p(x)-1}{p(x)}}\bigl[\bigl|\nabla a(u)\bigr|^{p(x)-2}\nabla a(u)-\bigl|\nabla a(v)\bigr|^{p(x)-2}\nabla a(v)\bigr]\bigr\|_{L^{\frac{p(x)}{p(x)-1}}(\{x:b^\beta|a(u)-a(v)|<\eta\})}\\
&=0. \qquad (4.26)
\end{aligned}$$

For the fourth term, we have

$$\begin{aligned}
&\biggl|\int_\Omega\bigl[b_i(u)-b_i(v)\bigr]b^\beta\bigl(a(u)-a(v)\bigr)_{x_i}h_\eta\bigl(b^\beta\bigl(a(u)-a(v)\bigr)\bigr)\,dx\biggr|\\
&\leq\int_\Omega\bigl|b(x)^{\frac{1}{p(x)}}\bigl(a(u)-a(v)\bigr)_{x_i}\bigr|
\end{aligned}$$

$$
\begin{aligned}
&\cdot \left| b(x)^{-\frac{1}{p(x)}} \frac{b_i(u) - b_i(v)}{a(u) - a(v)} b^\beta \big(a(u) - a(v)\big) h_\eta \big(b^\beta \big(a(u) - a(v)\big)\big) \right| dx \\
&\le c \big\| b(x)^{\frac{1}{p(x)}} \big(|\nabla a(u)| + |\nabla a(v)| \big) \big\|_{L^{p(x)}(\Omega)} \\
&\quad \cdot \big\| b(x)^{-\frac{1}{p(x)}} b^\beta \big(a(u) - a(v)\big) h_\eta \big(b^\beta \big(a(u) - a(v)\big)\big) \big\|_{L^{\frac{p(x)}{p(x)-1}}(\Omega)}, \qquad (4.27)
\end{aligned}
$$

which goes to 0 as $\eta \to 0^+$. Moreover, for the last term, since $u, v \in L^\infty(Q_T)$, $|b_i(u) - b_i(v)| \le c$, by the dominated convergence theorem, we have

$$
\begin{aligned}
&\left| \int_\Omega \big[b_i(u) - b_i(v)\big] b^\beta_{x_i} \big(a(u) - a(v)\big) S'_\eta \big(b^\beta \big(a(u) - a(v)\big)\big) \, dx \right| \\
&\le c \int_{\Omega_\lambda} b^{-1}(x) \big| b^\beta \big(a(u) - a(v)\big) S'_\eta \big(b^\beta \big(a(u) - a(v)\big)\big) \big| \, dx \\
&\le \frac{c}{\lambda} \int_{\Omega_\lambda} \big| b^\beta \big(a(u) - a(v)\big) S'_\eta \big(b^\beta \big(a(u) - a(v)\big)\big) \big| \, dx \\
&\to 0, \qquad (4.28)
\end{aligned}
$$

as $\eta \to 0^+$. Here, $\Omega_\lambda = \{x \in \Omega : b(x) > \lambda\}$.

Then, by (4.23)–(4.28),

$$
\int_0^t \frac{d}{dt} \|u - v\|_{L^1(\Omega)} \, dt \le c \int_0^t \|u - v\|_1 \, dt.
$$

It implies that

$$
\int_\Omega \big| u(x,t) - v(x,t) \big| \, dx \le c(T) \int_\Omega \big| u_0(x) - v_0(x) \big| \, dx.
$$

Theorem 2.4 is proved. □

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References

1. Acerbi, E., Mingione, G.: Regularity results for stationary electro-rheological fluids. Arch. Ration. Mech. Anal. **164**, 213–259 (2002)
2. Antontsev, S., Rodrigues, J.F.: On stationary thermo-rheological viscous flows. Ann. Univ. Ferrara, Sez. 7: Sci. Mat. **52**, 19–36 (2006)
3. Rajagopal, K., Ruzicka, M.: Mathematical modelling of electro-rheological fluids. Contin. Mech. Thermodyn. **13**, 59–78 (2001)
4. Aboulaich, R., Meskine, D., Souissi, A.: New diffusion models in image processing. Comput. Math. Appl. **56**, 874–882 (2008)
5. Levine, S., Chen, Y.M., Stanich, J.: Image Restoration Via NonstandArd Diffusion, Department of Mathematics and Computer Science, Duquesne University (2004)
6. Guo, B., Li, Y.J., Gao, W.J.: Singular phenomena of solutions for nonlinear diffusion equations involving $p(x)$-Laplace operator and nonlinear source. Z. Angew. Math. Phys. **66**, 989–1005 (2015)
7. Antontsev, S., Chipot, M., Shmarev, S.: Uniqueness and comparison theorems for solutions of doubly nonlinear parabolic equations with nonstandard growth conditions. Commun. Pure Appl. Anal. **12**, 1527–1546 (2013)
8. Gao, Y.C., Ch, Y., Gao, W.J.: Existence, uniqueness, and nonexistence of solution to nonlinear diffusion equations with $p(x,t)$-Laplacian operator. Bound. Value Probl. **2016**, 149 (2016)
9. Liu, B., Dong, M.: A nonlinear diffusion problem with convection and anisotropic nonstandard growth conditions. Nonlinear Anal., Real World Appl. **48**, 383–409 (2019)
10. Guo, B., Gao, W.: Study of weak solutions for parabolic equations with nonstandard growth conditions. J. Math. Anal. Appl. **374**(2), 374–384 (2011)
11. Gao, Y.C., Gao, W.J.: Extinction and asymptotic behavior of solutions for nonlinear parabolic equations with variable exponent of nonlinearity. Bound. Value Probl. **2013**, 164 (2013)
12. Chen, Y., Levine, S., Rao, M.: Variable exponent, linear growth functionals in image restoration. SIAM J. Appl. Math. **66**, 1383–1406 (2006)
13. Manfredi, J., Vespri, V.: Large time behavior of solutions to a class of doubly nonlinear parabolic equations. Electron. J. Differ. Equ. **1994**, 2 (1994)
14. Lee, K., Petrosyan, A., Vázquez, J.L.: Large time geometric properties of solutions of the evolution p-Laplacian equation. J. Differ. Equ. **229**, 389–411 (2006)
15. Zhan, H.: Infiltration equation with degeneracy on the boundary. Acta Appl. Math. **153**(1), 147–161 (2018)
16. Mcleod, J.B.: Nonlinear Diffusion Equations 152(4), 608–615 (2006)
17. Liu, W., Wang, M.: Blow-up of the solution for a p-Laplace equation with positive initial energy. Acta Appl. Math. **103**(2), 141–146 (2008)
18. Antontsev, S., Shmarev, S.: Doubly degenerate parabolic equations with variable nonlinearity II: blow-up and extinction in a finite time. Nonlinear Anal. **95**(1), 483–498 (2014)
19. Antontsev, S., Shmarev, S.: Parabolic equations with double variable nonlinearity. Math. Comput. Simul. **81**, 2018–2032 (2011)
20. Antontsev, S., Shmarev, S.: Parabolic equations with anisotropic nonstandard growth conditions. Int. Ser. Numer. Math. **154**, 33–44 (2007)
21. Zhan, H., Wen, J.: Evolutionary $p(x)$-Laplacian equation free from the limitation of the boundary value. Electron. J. Differ. Equ. **2016**, 143 (2016)
22. Zhan, H.: The weak solutions of an evolutionary $p(x)$-Laplacian equation are controlled by the initial value. Comput. Math. Appl. **76**, 2272–2285 (2018)
23. Zhikov, V.V.: On the density of smooth functions in Sobolev–Orlicz spaces. Otdel. Mat. Inst. Steklov. (POMI) **310**, 67–81 (2004). Translation in J. Math. Sci. (N.Y.) 132, 285–294 (2006)
24. Fan, X.L., Zhao, D.: On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}$. J. Math. Anal. Appl. **263**, 424–446 (2001)
25. Kovácik, O., Rákosník, J.: On spaces $L^{p(x)}$ and $W^{k,p(x)}$. Czechoslov. Math. J. **41**, 592–618 (1991)
26. Zhan, H.: The uniqueness of a nonlinear diffusion equation related to the p-Laplacian. J. Inequal. Appl. **2018**, 7 (2018)
27. Antonelli, F., Barucci, E., Mancino, M.E.: A comparison result for FBSDE with applications to decisions theory. Math. Methods Oper. Res. **54**, 407–423 (2001)
28. Zhan, H., Feng, Z.: The stability of the solutions to a degenerate parabolic equation. J. Differ. Equ. **267**(5), 2874–2890 (2019)
29. Gu, L.: Second Order Parabolic Partial Differential Equations. The Publishing Company of Xiamen University, Xiamen (2002) (in Chinese)
30. Taylor, M.E.: Partial Differential Equations III. Springer, Berlin (1999)