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Existence and multiplicity of solutions for fractional Schödinger equation involving a critical nonlinearity

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Abstract

In this paper, we investigate the fractional Schödinger equation involving a critical growth. By using the principle of concentration compactness and the variational method, we obtain some new existence results for the above equation, which improve the related results on this topic.

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Keywords: Fractional Laplacian; Principle of concentration compactness; Fractional Schödinger equation; Weak solution; Variational method

1 Introduction

This paper is concerned with the following fractional Schödinger equation involving a critical nonlinearity:

$$(-\Delta)^\alpha u(x) + V(x)u(x) = f\big(x,u(x)\big) + \lambda\big|u(x)\big|^{2_\alpha^*-2}u(x), \quad x \in \mathbb{R}^N, \tag{1.1}$$

where $\alpha \in (0,1)$, $2\alpha < N$, $2_\alpha^* = \frac{2N}{N-2\alpha}$, $\lambda > 0$ is a parameter, V is potential and continuous function, while f is a continuous function, both of them satisfying some conditions; $(-\Delta)^\alpha$ is the fractional Laplacian operator of order α, which can be defined as $(-\Delta)^\alpha u = \mathscr{F}^{-1}(|\xi|^{2\alpha}\mathscr{F}u)$, where $\mathscr{F}$ is the usual Fourier transform in $\mathbb{R}^N$.

Recently, fractional calculus attained more importance due to its wide applications in various field, such as diffusion, electrical circuits, control theory, blood flow phenomena, electro-analytical chemistry, etc.; for details, see [1–12] and the references therein. Problem (1.1) is related to the standing wave solutions of the following fractional nonlinear Schödinger equation of the form:

$$i\frac{\partial\psi}{\partial t}(x,t) = (-\Delta)^\alpha\psi(x,t) + V(x)\psi(x,t) + g(x,t), \quad (x,t) \in \mathbb{R}^N \times \mathbb{R}. \tag{1.2}$$

It was discovered by Laskin [13] when expanding the Feynman path integral, from Brownian-like to Lévy-like quantum mechanical paths. Recently, the fractional Schödinger equation has became a fundamental equation in fractional quantum mechanics, where, when $\varepsilon \to 0$ is taken in (1.2), the existence of solutions is very important; see [14] and the references therein. In the past few years, many works were devoted to establishing the existence and multiplicity of solutions of fractional Schödinger equation, see [15–22] and the references therein.

Recently, the existence of solutions of fractional Schödinger equation with perturbation was investigated, see [18–21]. Shang and Zhang [18] used the Ljusternik–Schnirelemann category theory to study the following fractional Schödinger equation with critical growth:

$$\varepsilon^{2\alpha}(-\Delta)^{\alpha}u(x) + V(x)u(x) = \left|u(x)\right|^{2_{\alpha}^{*}-2}u(x) + \lambda f\left(u(x)\right), \quad x \in \mathbb{R}^{N}. \tag{1.3}$$

They proved that equation (1.3) has a nonnegative ground state solution in Nehari manifold. There are also some contributions considering the different forms of fractional Schödinger equation with critical growth, see [19–21]. It is worth mentioning that [19] studied the following fractional Schödinger equation:

$$(-\Delta)^{\alpha}u(x) + V(x)u(x) = k(x)f\left(u(x)\right) + \lambda\left|u(x)\right|^{2_{\alpha}^{*}-2}u(x), \quad x \in \mathbb{R}^{N}. \tag{1.4}$$

By the principle of concentration compactness and the variational method, a nontrivial radially symmetric weak solution was obtained for (1.4).

It is well known that in the case of critical growth, the compactness of related embedding is lost, and then certain difficulties arise in the proving of the existence of solutions. In [18], Shang and Zhang obtained a nonnegative ground state solution for problem (1.3) in Nehari manifold by using the Ljusternik–Schnirelemann category theory. It seems that it is difficult to obtain the multiplicity result for problem (1.3). In [19], Zhang et al. used the fractional version of the principle of concentration compactness [23] and obtained new existence results for nontrivial radially symmetric weak solution of problem (1.4).

Motivated by these results, in this paper, we are interested in the fractional Schödinger equation involving critical exponent (1.1), which is more general than (1.4). By the Ekeland's variational principle [24] and the Mountain Pass Theorem [25], we obtain a new multiplicity result, which is different from the work of [18, 19]. To the best of our knowledge, our work is the first attempt to use the principle of concentration compactness to study the existence and multiplicity of solutions for fractional Schödinger equation involving a critical nonlinearity as in (1.1).

To obtain our main result, we consider the potential function $V \in C(\mathbb{R}^{N}, \mathbb{R})$ satisfying

$$0 < \inf_{x\in\mathbb{R}^{N}} V(x) := V_0 < \lim_{|x|d\to\infty} V(x) := V(\infty) = +\infty. \tag{V_0}$$

The hypothesis (V_0) was first introduced by Rabinowitz in [26]. Furthermore, we impose the following assumptions for the nonlinearity f:

(f1) $f \in C(\mathbb{R}^{N} \times \mathbb{R}, \mathbb{R})$, $f(x,t) = o(|t|)$ as $t \to 0$ uniformly in $x \in \mathbb{R}^{N}$;

(f2) There exists a constant $C_1 > 0$ such that $|f(x,t)| \le C_1(1 + |t|^{2_{\alpha}^{*}-1})$ for all $(x,t) \in \mathbb{R}^{N} \times \mathbb{R}$.

(f3) $F(x,t) \ge 0$ and $\frac{F(x,t)}{t^2} \to +\infty$ as $t \to \infty$ uniformly in $x \in \mathbb{R}^{N}$, where $F(x,t) = \int_0^t f(x,s)\,ds$.

(f4) There exist $\theta > 0$ such that $0 < \theta F(x,t) \le tf(x,t)$ for all $(x,t) \in \mathbb{R}^{N} \times \mathbb{R}$.

Now we state the main result.

Theorem 1.1 *Assume that V and f satisfy* (V_0) *and* (f1)–(f4). *There exists* $\lambda_1 > 0$ *such that for any* $0 < \lambda < \lambda_1$, *the problem* (1.1) *has three solutions.*

This paper organized as follows. In Sect. 2, we give some preliminaries of fractional Sobolev spaces and prove some lemmas, which will be used later. Section 3 presents the proof of Theorem 1.1.

2 Preliminaries

Throughout this paper, $\|\cdot\|_p$ is the usual norm of $L^p(\mathbb{R}^N)$, $B(x,r)$ is the open ball centered at x with radius r, C_i $(i = 1,2,\dots)$ and C denote positive constants, $\langle\cdot\rangle$ is the inner product at Hilbert space. We first provide a short review of fractional Sobolev spaces. A complete introduction can be found in [14, 27].

For $\alpha \in (0,1)$, the fractional Sobolev space $H^\alpha(\mathbb{R}^N)$ is defined by

$$H^\alpha(\mathbb{R}^N) := \left\{u \in L^2(\mathbb{R}^N) : \frac{|u(x)-u(y)|^2}{|x-y|^{N+2\alpha}} \in L^2(\mathbb{R}^N \times \mathbb{R}^N)\right\},$$

endowed with the natural norm

$$\|u\|_{H^\alpha(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N} u^2(x)\,dx + \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2\alpha}}\,dx\,dy\right)^{\frac{1}{2}}.$$

Moreover, by Proposition 3.6 in [14], we have

$$\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2\alpha}}\,dx\,dy = \int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{\alpha}{2}} u(x)\right|^2 dx. \tag{2.1}$$

Therefore, for any $v \in H^\alpha(\mathbb{R}^N)$, we have

$$\iint_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2\alpha}}\,dx\,dy = \int_{\mathbb{R}^N} (-\Delta)^{\frac{\alpha}{2}} u(x) \cdot (-\Delta)^{\frac{\alpha}{2}} v(x)\,dx. \tag{2.2}$$

The working space E is defined by

$$E := \left\{u \in H^\alpha(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)u^2(x)\,dx < +\infty\right\}.$$

Then E is a Hilbert space with the norm

$$\|u\|^2 = \int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{\alpha}{2}} u(x)\right|^2 dx + \int_{\mathbb{R}^N} V(x)u^2(x)\,dx.$$

We say that $u \in E$ is a weak solution of problem (1.1) if for any $v \in E$,

$$\begin{aligned}&\int_{\mathbb{R}^N} (-\Delta)^{\frac{\alpha}{2}} u(x) \cdot (-\Delta)^{\frac{\alpha}{2}} v(x)\,dx + \int_{\mathbb{R}^N} V(x)u(x)v(x)\,dx \\ &\quad = \int_{\mathbb{R}^N} \left(f(x,u(x)) + \lambda\left|u(x)\right|^{2_\alpha^*-2} u(x)\right)v(x)\,dx.\end{aligned}$$

The energy functional $I_\lambda : E \to \mathbb{R}$ associated with problem (1.1) is defined by

$$I_\lambda(u) := \frac{1}{2}\int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{\alpha}{2}} u(x)\right|^2 dx + \int_{\mathbb{R}^N} V(x)u^2(x)\,dx - \int_{\mathbb{R}^N} F\big(x,u(x)\big)\,dx - \frac{\lambda}{2_\alpha^*}\int_{\mathbb{R}^N} \left|u(x)\right|^{2_\alpha^*} dx.$$

Under our assumptions, I_λ is well defined in E and $I_\lambda \in C^1(E,\mathbb{R})$. It is easy to obtain that I_λ is Gâteaux-differentiable in E and the critical points of I_λ are solutions to problem (1.1).

Lemma 2.1 ([14]) *Let $\alpha \in (0,1)$ and $N \geq 1$ be such that $2\alpha < N$. Then there exists $C = C(N,\alpha) > 0$ such that*

$$\|u\|_{L^{2_\alpha^*}(\mathbb{R}^N)} \leq C\|u\|_{H^\alpha(\mathbb{R}^N)}$$

for every $u \in H^\alpha(\mathbb{R}^N)$, where $2_\alpha^* = \frac{2N}{N-2\alpha}$ is the fractional critical exponent. Moreover, the embedding $H^\alpha(\mathbb{R}^N) \hookrightarrow L^\gamma(\mathbb{R}^N)$ is continuous for each $\gamma \in [2,2_\alpha^*]$ and is locally compact for $\gamma \in [2,2_\alpha^*)$.

Lemma 2.2 ([15, 28]) *Assume that (V_0) holds, then the embedding $E \hookrightarrow L^p(\mathbb{R}^N)$ is compact for all $2 \leq p < 2_\alpha^*$.*

Remark 2.3 From Lemmas 2.1 and 2.2, we know that the embedding $E \hookrightarrow L^\gamma(\mathbb{R}^N)$ is continuous for each $\gamma \in [2,2_\alpha^*]$, thus

$$\|u\|_{L^\gamma(\mathbb{R}^N)}^\gamma \leq C_\gamma \|u\|^\gamma \quad \text{for each } \gamma \in \left[2,2_\alpha^*\right], \tag{2.3}$$

where C_γ is the best constant for the embedding of E into $L^\gamma(\mathbb{R}^N)$.

In the following lemma, we show that I_λ satisfies the geometric structure of the Mountain Pass Theorem [25].

Lemma 2.4 *The functional I_λ satisfies the following condition*:

(i) *There exists $\lambda_0 > 0$ such that for any $\lambda \in (0,\lambda_0)$ and $\|u\| = \rho$, $I_\lambda(u) \geq \beta$ for some constants $\beta, \rho > 0$.*
(ii) *There exists $e \in E$ satisfying $\|e\| > \rho$ such that $I_\lambda(e) < 0$.*

Proof (i) From conditions (f1) and (f2), for any $\varepsilon > 0$, there exists a constant C_ε depending on ε such that

$$\left|f(x,t)\right| \leq \varepsilon|t| + C_\varepsilon|t|^{2_\alpha^*-1}, \tag{2.4}$$

and

$$\left|F(x,t)\right| \leq \frac{\varepsilon}{2}|t|^2 + \frac{C_\varepsilon}{2_\alpha^*}|t|^{2_\alpha^*}. \tag{2.5}$$

Therefore, for any $u \in E$, by (2.3) and (2.5), we have

$$\begin{aligned} I_\lambda(u) &= \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} F\bigl(x,u(x)\bigr)\,dx - \frac{\lambda}{2_\alpha^*}\int_{\mathbb{R}^N}\bigl|u(x)\bigr|^{2_\alpha^*}\,dx \\ &\geq \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N}\biggl[\frac{\varepsilon}{2}\bigl|u(x)\bigr|^2 + \frac{C_\varepsilon}{2_\alpha^*}\bigl|u(x)\bigr|^{2_\alpha^*}\biggr]\,dx - \frac{\lambda}{2_\alpha^*}\int_{\mathbb{R}^N}\bigl|u(x)\bigr|^{2_\alpha^*}\,dx \\ &= \frac{1}{2}\|u\|^2 - \frac{\varepsilon}{2}\int_{\mathbb{R}^N}\bigl|u(x)\bigr|^2\,dx - \frac{C_\varepsilon}{2_\alpha^*}\int_{\mathbb{R}^N}\bigl|u(x)\bigr|^{2_\alpha^*}\,dx - \frac{\lambda}{2_\alpha^*}\int_{\mathbb{R}^N}\bigl|u(x)\bigr|^{2_\alpha^*}\,dx \\ &\geq \frac{1}{2}\|u\|^2 - \frac{\varepsilon C_2}{2}\|u\|^2 - \frac{C_\varepsilon C_{2_\alpha^*}}{2_\alpha^*}\|u\|^{2_\alpha^*} - \frac{\lambda C_{2_\alpha^*}}{2_\alpha^*}\|u\|^{2_\alpha^*} \\ &= \frac{1-\varepsilon C_2}{2}\|u\|^2 - \frac{(C_\varepsilon+\lambda)C_{2_\alpha^*}}{2_\alpha^*}\|u\|^{2_\alpha^*}. \end{aligned}$$

Then there exists $\lambda_0 > 0$ such that for any $\lambda \in (0,\lambda_0)$, we can choose two positive constants $\beta, \rho > 0$ such that $I_\lambda(u) \geq \beta$ for $\|u\| = \rho$.

(ii) For any $u \in E$, by condition (f3), we have

$$\frac{I_\lambda(tu)}{t^2} = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N}\frac{F(x,tu(x))}{t^2u^2(x)}u^2(x)\,dx - \frac{\lambda t^{2_\alpha^*}}{2_\alpha^*}\int_{\mathbb{R}^N}\bigl|u(x)\bigr|^{2_\alpha^*}\,dx,$$

then

$$\lim_{t\to+\infty}\frac{I_\lambda(tu)}{t^2} \leq \frac{1}{2}\|u\|^2 - \lim_{t\to+\infty}\frac{\lambda t^{2_\alpha^*}}{2_\alpha^*}\int_{\mathbb{R}^N}\bigl|u(x)\bigr|^{2_\alpha^*}\,dx.$$

So, $I_\lambda(tu) \to -\infty$ as $t \to +\infty$, which means that there exists a large enough $t_0 > 0$ such that $I_\lambda(t_0u) \leq 0$. Therefore, we can choose $e := t_0u \in E$ with $\|e\| > \rho$ such that $I_\lambda(e) < 0$. □

Lemma 2.5 *Let* $\lambda_0 > 0$ *be given in Lemma* 2.4. *Then* $\{u_n\}$ *is a* (PS) *sequence of* I_λ *which is bounded in E for all* $\lambda \in (0,\lambda_0)$.

Proof By condition (f4) and (2.3), for any $\{u_n\} \subset E$, we have

$$\begin{aligned} 1 + C + \|u_n\| &\geq I_\lambda(u_n) - \frac{1}{\theta}\bigl\langle I'_\lambda(u_n), u_n\bigr\rangle \\ &= \biggl(\frac{1}{2}-\frac{1}{\theta}\biggr)\|u_n\|^2 + \int_{\mathbb{R}^N}\bigl(u_n(x)f\bigl(x,u_n(x)\bigr) - \theta F\bigl(x,u_n(x)\bigr)\bigr)\,dx \\ &\quad + \lambda\biggl(\frac{1}{\theta}-\frac{1}{2_\alpha^*}\biggr)\int_{\mathbb{R}^N}\bigl|u_n(x)\bigr|^{2_\alpha^*}\,dx \\ &\geq \biggl(\frac{1}{2}-\frac{1}{\theta}\biggr)\|u_n\|^2 + \lambda\biggl(\frac{1}{\theta}-\frac{1}{2_\alpha^*}\biggr)C_{2_\alpha^*}\|u_n\|^{2_\alpha^*}, \end{aligned}$$

which means that $\{u_n\}$ is bounded in E. □

Let $\Omega_c(\mathbb{R}^N) := \{u \in C(\mathbb{R}^N) : \operatorname{supp}(u) \text{ is a compact subset of } \mathbb{R}^N\}$ and denote by $\Omega_0(\mathbb{R}^N)$ the closure of $\Omega_c(\mathbb{R}^N)$ endowed with the norm $\|\eta\|_\infty = \sup_{x\in\mathbb{R}^N}|\eta(x)|$. For a measure μ, we define

$$\|\mu\|_{\Omega_0} := \sup_{\Omega_0(\mathbb{R}^N),\|\eta\|_\infty=1} |(\mu,\eta)|,$$

where $(\mu,\eta) = \int_{\mathbb{R}^N} \eta\, d\mu$. We assume that J is a countable set.

Definition 2.6 ([16]) Let $\mathcal{M}(\mathbb{R}^N)$ denote the space of finite nonnegative Borel measures on $\mathbb{R}^N$. For any $\mu \in \mathcal{M}(\mathbb{R}^N)$, the equation $\mu(\mathbb{R}^N) = \|\mu\|_{\Omega_0}$ holds. We define that $\mu_n \rightharpoonup \mu$ weakly-$*$ in $\mathcal{M}(\mathbb{R}^N)$ if $(\mu_n,\eta) \to (\mu,\eta)$ holds for all $\eta \in \bar{C}(\mathbb{R}^N)$ as $n \to \infty$.

Lemma 2.7 ([23]) *Let $\{u_n\} \subset \dot{H}^\alpha(\mathbb{R}^N)$ be a sequence and $u_n \rightharpoonup u$ weakly as $n \to \infty$ and such that $|(-\Delta)^{\frac{\alpha}{2}} u_n|^2 \rightharpoonup \mu$ and $|u_n|^{2_\alpha^*} \rightharpoonup \nu$ weakly-$*$ in $\mathcal{M}(\mathbb{R}^N)$. Then, either $u_n \to u$ in $L^{2_\alpha^*}_{\mathrm{loc}}(\mathbb{R}^N)$ or there exists a set of distinct points $\{x_j\}_{j\in J} \subset \bar{\Omega}$ (at most countable) and positive numbers $\{\nu_j\}_{j\in J}$ such that $\nu = |u|^{2_\alpha^*} + \sum_{j\in J} \nu_j \delta_{x_j}$, where $\Omega \subseteq \mathbb{R}^N$ is an open subset. If Ω is bounded, then there exist a positive measure $\tilde{\mu} \in \mathcal{M}(\mathbb{R}^N)$ with $\operatorname{supp}\tilde{\mu} \subset \bar{\Omega}$ and positive numbers $\{\mu_j\}_{j\in J}$ such that $\mu = |(-\Delta)^{\frac{\alpha}{2}} u_n|^2 + \tilde{\mu} + \sum_{j\in J} \mu_j \delta_{x_j}$.*

Lemma 2.8 ([19]) *Define*

$$\mu_\infty = \lim_{R\to\infty} \limsup_{n\to\infty} \int_{\{x\in\mathbb{R}^N:|x|>R\}} \left|(-\Delta)^{\frac{\alpha}{2}} u_n(x)\right|^2 dx,$$
$$\nu_\infty = \lim_{R\to\infty} \limsup_{n\to\infty} \int_{\{x\in\mathbb{R}^N:|x|>R\}} \left|u_n(x)\right|^{2_\alpha^*} dx.$$

Then the quantities μ_∞ and ν_∞ are well defined and satisfy

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{\alpha}{2}} u_n(x)\right|^2 dx = \int_{\mathbb{R}^N} d\mu + \mu_\infty,$$
$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} \left|u_n(x)\right|^{2_\alpha^*} dx = \int_{\mathbb{R}^N} d\nu + \nu_\infty.$$

Lemma 2.9 ([19]) *Let $\{u_n\} \subset \dot{H}^\alpha(\mathbb{R}^N)$ be such that $u_n \rightharpoonup u$ weakly-$*$ in $\dot{H}^\alpha(\mathbb{R}^N)$, $|(-\Delta)^{\frac{\alpha}{2}} u_n|^2 \rightharpoonup \mu$ and $|u_n|^{2_\alpha^*} \rightharpoonup \nu$ weakly-$*$ in $\mathcal{M}(\mathbb{R}^N)$ as $n \to \infty$. Then $\nu_i \le (S_\alpha^{-1}\mu\{x_i\})^{\frac{2_\alpha^*}{2}}$ for $i \in J$ and $\nu_\infty \le (S_\alpha^{-1}\mu_\infty)^{\frac{2_\alpha^*}{2}}$, where x_i, ν_i are from Lemma 2.7 and μ_∞, ν_∞ are given in Lemma 2.8, S_α is the best Sobolev constant of $\dot{H}^\alpha(\mathbb{R}^N) \hookrightarrow L^{2_\alpha^*}(\mathbb{R}^N)$, i.e.,*

$$S_\alpha = \inf_{u\in\dot{H}^\alpha(\mathbb{R}^N)} \frac{\int_{\mathbb{R}^N} |(-\Delta)^{\frac{\alpha}{2}} u(x)|^2\, dx}{\|u\|^2_{L^{2_\alpha^*}(\mathbb{R}^N)}}. \tag{2.6}$$

Now, we are ready to establish the following compactness result, which is used to prove our main result.

Lemma 2.10 *There exists $\lambda_* > 0$ such that for any $\lambda \in (0,\lambda_*)$, each bounded (PS) sequence for functional I_λ contains a convergent subsequence.*

Proof Let $\{u_n\} \subset E$ be a bounded (PS) sequence, i.e., there exists $C_3 > 0$ such that

$$I_\lambda(u_n) \le C_3 \quad \text{and} \quad I_\lambda'(u_n) \to 0 \quad \text{in } E, \text{ as } n \to \infty.$$

Passing to a subsequence, we still denote by $\{u_n\}$. Assume that $u_n \rightharpoonup u_0$ weakly in E. According to Lemma 2.2, we have $u_n \to u_0$ in $L^p(\mathbb{R}^N)$ and $u_n \to u_0$ a.e. in $\mathbb{R}^N$ as $n \to \infty$. Therefore, by Prokhorov's Theorem [29], there exist $\mu, \nu \in \mathcal{M}(\mathbb{R}^N)$ such that

$$\left|(-\Delta)^{\frac{\alpha}{2}} u_n\right|^2 \to \mu \quad \text{and} \quad |u_n|^{2_\alpha^*} \to \nu \quad \text{weakly-}* \text{ in } \mathcal{M}(\mathbb{R}^N) \text{ as } n \to \infty.$$

By Lemma 2.7, we have $u_n \to u_0$ in $L^{2_\alpha^*}_{\mathrm{loc}}(\mathbb{R}^N)$ or $\nu = |u_0|^{2_\alpha^*} + \sum_{j\in J} \nu_j \delta_{x_j}$ as $n \to \infty$.

For any $\phi \in E$, we have

$$\begin{aligned}
&\langle I_\lambda'(u_n), \phi\rangle - \langle I_\lambda'(u_0), \phi\rangle \\
&\quad = \int_{\mathbb{R}^N} (-\Delta)^{\frac{\alpha}{2}} \big(u_n(x) - u_0(x)\big)(-\Delta)^{\frac{\alpha}{2}} \phi(x)\, dx + \int_{\mathbb{R}^N} V(x)\big(u_n(x) - u_0(x)\big)\phi(x)\, dx \\
&\qquad - \int_{\mathbb{R}^N} \big(f\big(x, u_n(x)\big) - f\big(x, u_0(x)\big)\big)\phi(x)\, dx \\
&\qquad - \lambda \int_{\mathbb{R}^N} \big(\left|u_n(x)\right|^{2_\alpha^*-2} u_n(x) - \left|u_0(x)\right|^{2_\alpha^*-2} u_0(x)\big)\phi(x)\, dx.
\end{aligned}$$

Since $u_n \rightharpoonup u_0$ weakly in E,

$$\int_{\mathbb{R}^N} (-\Delta)^{\frac{\alpha}{2}} \big(u_n(x) - u_0(x)\big)(-\Delta)^{\frac{\alpha}{2}} \phi(x)\, dx + \int_{\mathbb{R}^N} V(x)\big(u_n(x) - u_0(x)\big)\phi(x)\, dx \to 0 \quad \text{as } n \to \infty.$$

Since

$$\left\{|u_n|^{2_\alpha^*-2} u_n - |u_0|^{2_\alpha^*-2} u_0\right\}_n \quad \text{is bounded in } L^{\frac{2_\alpha^*}{2_\alpha^*-1}}(\mathbb{R}^N)$$

and

$$|u_n|^{2_\alpha^*-2} u_n - |u_0|^{2_\alpha^*-2} u_0 \to 0 \quad \text{a.e. in } \mathbb{R}^N,$$

then

$$|u_n|^{2_\alpha^*-2} u_n - |u_0|^{2_\alpha^*-2} u_0 \to 0 \quad \text{weakly in } L^{\frac{2_\alpha^*}{2_\alpha^*-1}}(\mathbb{R}^N),$$

we obtain that

$$\int_{\mathbb{R}^N} \big(\left|u_n(x)\right|^{2_\alpha^*-2} u_n(x) - \left|u_0(x)\right|^{2_\alpha^*-2} u_0(x)\big)\phi(x)\, dx \to 0.$$

In the following, we prove that

$$\int_{\mathbb{R}^N} \big(f\big(x, u_n(x)\big) - f\big(x, u_0(x)\big)\big)\phi(x)\, dx \to 0.$$

As $\langle I_\lambda'(u_n), \phi\rangle \to 0$, we get $\langle I_\lambda'(u_0), \phi\rangle = 0$, i.e., $I_\lambda'(u_0) = 0$. Thus,

$$\int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{\alpha}{2}} u_0(x)\right|^2 dx + \int_{\mathbb{R}^N} V(x) u_0^2(x)\, dx$$

$$= \int_{\mathbb{R}^N} f\big(x, u_0(x)\big) u_0(x)\, dx + \frac{\lambda}{2_\alpha^*} \int_{\mathbb{R}^N} \big|u_0(x)\big|^{2_\alpha^*}\, dx.$$

By (2.4) and Young inequality, we have

$$\begin{aligned}
&\big|\big[f(x,u_n) - f(x,u_0)\big]\phi\big| \\
&\quad\le |u_n||\phi| + C_1|u_n|^{2_\alpha^*-1}|\phi| + |u_0||\phi| + C_2|u_0|^{2_\alpha^*-1}|\phi| \\
&\quad\le |u_n - u_0||\phi| + 2|u_0||\phi| + C_1|u_n - u_0|^{2_\alpha^*-1}|\phi| + C_2|u_n - u_0|^{2_\alpha^*-1}|\phi| \\
&\quad\le \varepsilon|u_n - u_0|^2 + C_\varepsilon|\phi|^2 + 2|u_0||\phi| + \varepsilon|u_n - u_0|^{2_\alpha^*} + C_1C_\varepsilon|\phi|^{2_\alpha^*} + C_2|u_0|^{2_\alpha^*-1}|\phi|.
\end{aligned}$$

Let

$$\begin{aligned}
G_{\varepsilon,n}(x) := \max\Big\{&\big|\big[f\big(x,u_n(x)\big) - f\big(x,u_0(x)\big)\big]\phi(x)\big]\big| - \varepsilon\big|u_n(x) - u_0(x)\big|^2 \\
&- \varepsilon\big|u_n(x) - u_0(x)\big|^{2_\alpha^*}, 0\Big\}.
\end{aligned}$$

Then

$$0 \le G_{\varepsilon,n}(x) \le C_\varepsilon|\phi|^2 + 2|u_0||\phi| + C_1C_\varepsilon|\phi|^{2_\alpha^*} + C_2|u_0|^{2_\alpha^*-1}|\phi| \in L^1\big(\mathbb{R}^N\big),$$

and $G_{\varepsilon,n}(x) \to 0$ a.e. on $\mathbb{R}^N$. By the Lebesgue dominated convergence theorem, we have

$$\int_{\mathbb{R}^N} G_{\varepsilon,n}(x)\, dx \to 0 \quad \text{as } n \to \infty.$$

Therefore,

$$\begin{aligned}
&\limsup_{n\to\infty}\left|\int_{\mathbb{R}^N} \big(f\big(x,u_n(x)\big) - f\big(x,u_0(x)\big)\big)\phi\, dx\right| \\
&\quad\le \limsup_{n\to\infty}\int_{\mathbb{R}^N} G_{\varepsilon,n}(x)\, dx + \varepsilon\limsup_{n\to\infty}\int_{\mathbb{R}^N}\big|u_n(x) - u_0(x)\big|^2\, dx \\
&\qquad + \varepsilon\limsup_{n\to\infty}\int_{\mathbb{R}^N}\big|u_n(x) - u_0(x)\big|^{2_\alpha^*}\, dx \\
&\quad\le C_3\varepsilon.
\end{aligned}$$

By the arbitrariness of ε, we have

$$\int_{\mathbb{R}^N} \big(f\big(x,u_n(x)\big) - f\big(x,u_0(x)\big)\big)\phi(x)\, dx \to 0.$$

Next we will verify that $u_n \to u_0$ in $L^{2_\alpha^*}(\mathbb{R}^N)$. We claim that there exists a constant $\lambda_* > 0$ such that $\nu_i = 0$ and $\nu_\infty = 0$ holds for any $0 < \lambda < \lambda_*$ and $i \in J$. We argue by contradiction. Suppose that there exists $i_0 \in J$ such that $\nu_{i_0} > 0$ or $\nu_\infty > 0$, then, by Lemma 2.9, we have

$$\nu_{i_0} \le \big(S_\alpha^{-1}\mu(x_{i_0})\big)^{\frac{2_\alpha^*}{2}}. \tag{2.7}$$

Let $\varphi \in C_0^\infty(\mathbb{R}^N)$ be such that $\varphi \in (0,1)$, $\varphi = 1$ in $B(0,1)$ and $\varphi = 0$ in $\mathbb{R}^N \setminus B(0,2)$. For any $\varepsilon > 0$, we define $\varphi_\varepsilon(x) := \varphi(\frac{x-x_{i_0}}{\varepsilon})$, where $i_0 \in J$. Using (2.1) and (2.6), one has

$$\int_{\mathbb{R}^N} \left|u_n(x)\varphi_\varepsilon(x)\right|^{2_\alpha^*} dx \le \left(S_\alpha^{-1} \iint_{\mathbb{R}^{2N}} \frac{|u_n(x)\varphi_\varepsilon(x) - u_n(y)\varphi_\varepsilon(y)|^2}{|x-y|^{N+2\alpha}} \,dx\,dy\right)^{\frac{2_\alpha^*}{2}},$$

which means that

$$\int_{\mathbb{R}^N} \left|u_n(x)\varphi_\varepsilon(x)\right|^{2_\alpha^*} dx \to \int_{\mathbb{R}^N} \varphi_\varepsilon^{2_\alpha^*} \,d\nu \quad \text{as } n \to \infty, \tag{2.8}$$

and

$$\int_{\mathbb{R}^N} \varphi_\varepsilon^{2_\alpha^*} \,d\nu \to \nu\left(\{x_{i_0}\}\right) = \nu_{i_0}, \quad \text{as } \varepsilon \to 0. \tag{2.9}$$

By (f4), we obtain

$$\begin{aligned}
&\theta I_\lambda(u_n) - \left\langle I_\lambda'(u_n), u_n\right\rangle \\
&\quad = \frac{\theta}{2}\|u_n\|^2 - \theta \int_{\mathbb{R}^N} F\left(x, u_n(x)\right) dx - \frac{\lambda\theta}{2_\alpha^*} \int_{\mathbb{R}^N} \left|u_n(x)\right|^{2_\alpha^*} dx \\
&\qquad - \|u_n\|^2 + \int_{\mathbb{R}^N} f\left(x, u_n(x)\right)u_n(x) \,dx + \lambda \int_{\mathbb{R}^N} \left|u_n(x)\right|^{2_\alpha^*} dx \\
&\quad = \left(\frac{\theta-2}{2}\right)\|u_n\|^2 + \int_{\mathbb{R}^N} \left[f\left(x, u_n(x)\right)u_n(x) - \theta F\left(x, u_n(x)\right)\right] dx \\
&\qquad + \lambda\left(1 - \frac{\theta}{2_\alpha^*}\right) \int_{\mathbb{R}^N} \left|u_n(x)\right|^{2_\alpha^*} dx \\
&\quad \ge \lambda\left(1 - \frac{\theta}{2_\alpha^*}\right) \int_{\mathbb{R}^N} \left|u_n(x)\right|^{2_\alpha^*} dx \\
&\quad \ge \lambda\left(\frac{2_\alpha^* - \theta}{2_\alpha^*}\right) \int_{\mathbb{R}^N} \left|u_n(x)\varphi_\varepsilon(x)\right|^{2_\alpha^*} dx.
\end{aligned}$$

Let $n \to \infty$, then we have $\theta C_3 \ge \lambda(\frac{2_\alpha^*-\theta}{2_\alpha^*}) \int_{\mathbb{R}^N} \varphi_\varepsilon^{2_\alpha^*}(x)\,dx$. By (2.8) and (2.9), we know that

$$2C_3 \ge \lambda\left(\frac{2_\alpha^* - \theta}{2_\alpha^*}\right)\nu_{i_0}. \tag{2.10}$$

Since $\{u_n\}$ is bounded in E, by the definition of φ_ε, we know that $\{u_n\varphi_\varepsilon\}$ is also bounded in E. Thus

$$\left\langle I_\lambda'(u_n), u_n\varphi_\varepsilon\right\rangle \to 0 \quad \text{as } n \to \infty,$$

which means that

$$\begin{aligned}
&\int_{\mathbb{R}^N} (-\Delta)^{\frac{\alpha}{2}} u_n(x) \cdot (-\Delta)^{\frac{\alpha}{2}} (u_n\varphi_\varepsilon)(x) \,dx \\
&\quad = \int_{\mathbb{R}^N} \left[f\left(x, u_n(x)\right)u_n(x) + \lambda\left|u_n(x)\right|^{2_\alpha^*} - V(x)\left|u_n(x)\right|^2\right]\varphi_\varepsilon(x) \,dx + o(1).
\end{aligned} \tag{2.11}$$

By (2.4), we have

$$tf(x,t) \le \varepsilon|t|^2 + C_\varepsilon |t|^{2_\alpha^*}.$$

Then

$$\begin{aligned}
&\int_{\mathbb{R}^N} \bigl[f\bigl(x,u_n(x)\bigr)u_n(x) + \lambda\bigl|u_n(x)\bigr|^{2_\alpha^*} - V(x)\bigl|u_n(x)\bigr|^2\bigr]\varphi_\varepsilon(x)\,dx \\
&\quad \le \int_{\mathbb{R}^N} \bigl[\varepsilon\bigl|u_n(x)\bigr|^2 + C_\varepsilon\bigl|u_n(x)\bigr|^{2_\alpha^*} + \lambda\bigl|u_n(x)\bigr|^{2_\alpha^*}\bigr]\varphi_\varepsilon(x)\,dx.
\end{aligned}$$

Since

$$\int_{\mathbb{R}^N} \bigl|u_n(x)\bigr|^2\varphi_\varepsilon(x)\,dx = \int_{B(x_i,2\varepsilon)} \bigl|u_n(x)\bigr|^2\varphi_\varepsilon(x)\,dx \to \int_{B(x_i,2\varepsilon)} \bigl|u_0(x)\bigr|^2\varphi_\varepsilon(x)\,dx \quad \text{as } n\to\infty,$$

and

$$\int_{B(x_i,2\varepsilon)} \bigl|u_0(x)\bigr|^2\varphi_\varepsilon(x)\,dx \to 0 \quad \text{as } \varepsilon\to 0,$$

we get

$$\begin{aligned}
&\limsup_{\varepsilon\to 0}\limsup_{n\to\infty} \int_{\mathbb{R}^N} \bigl[f\bigl(x,u_n(x)\bigr)u_n(x) + \lambda\bigl|u_n(x)\bigr|^{2_\alpha^*} - V(x)\bigl|u_n(x)\bigr|^2\bigr]\varphi_\varepsilon(x)\,dx \\
&\quad \le (C_\varepsilon + \lambda)\limsup_{\varepsilon\to 0}\limsup_{n\to\infty} \int_{\mathbb{R}^N} \bigl|u_n(x)\bigr|^{2_\alpha^*}\varphi_\varepsilon(x)\,dx \\
&\quad \le (C_\varepsilon + \lambda)\limsup_{\varepsilon\to 0} \int_{\mathbb{R}^N} \varphi_\varepsilon^{2_\alpha^*}(x)\,dx \\
&\quad = (C_\varepsilon + \lambda)\nu_{i_0}.
\end{aligned}$$

In order to construct a contradiction with our claim, we choose λ large enough satisfies $\lambda > C_\varepsilon$, then

$$\begin{aligned}
&\limsup_{\varepsilon\to 0}\limsup_{n\to\infty} \int_{\mathbb{R}^N} \bigl[f\bigl(x,u_n(x)\bigr)u_n(x) + \lambda\bigl|u_n(x)\bigr|^{2_\alpha^*} - V(x)\bigl|u_n(x)\bigr|^2\bigr]\varphi_\varepsilon(x)\,dx \\
&\quad \le 2\lambda\nu_{i_0}.
\end{aligned} \tag{2.12}$$

By (2.2), we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} (-\Delta)^{\frac{\alpha}{2}}u_n(x)\cdot(-\Delta)^{\frac{\alpha}{2}}(u_n\varphi_\varepsilon)(x)\,dx \\
&\quad = \iint_{\mathbb{R}^{2N}} \frac{(u_n(x)-u_n(y))(u_n(x)\varphi_\varepsilon(x)-u_n(y)\varphi_\varepsilon(y))}{|x-y|^{N+2\alpha}}\,dx\,dy \\
&\quad = \iint_{\mathbb{R}^{2N}} \frac{(u_n(x)-u_n(y))^2\varphi_\varepsilon(y)}{|x-y|^{N+2\alpha}}\,dx\,dy \\
&\qquad + \iint_{\mathbb{R}^{2N}} \frac{(u_n(x)-u_n(y))(\varphi_\varepsilon(x)-\varphi_\varepsilon(y))u_n(x)}{|x-y|^{N+2\alpha}}\,dx\,dy.
\end{aligned}$$

It is easy to get that

$$\iint_{\mathbb{R}^{2N}} \frac{(u_n(x)-u_n(y))^2\varphi_\varepsilon(y)}{|x-y|^{N+2\alpha}}\,dx\,dy \to \int_{\mathbb{R}^N} \varphi_\varepsilon\,d\mu \quad \text{as } n\to\infty$$

and

$$\int_{\mathbb{R}^N} \varphi_\varepsilon\,d\mu \to \mu(\{x_{i_0}\}) \quad \text{as } \varepsilon \to 0.$$

Using Hölder inequality, one has

$$\begin{aligned} &\iint_{\mathbb{R}^{2N}} \frac{(u_n(x)-u_n(y))(\varphi_\varepsilon(x)-\varphi_\varepsilon(y))u_n(x)}{|x-y|^{N+2\alpha}}\,dx\,dy \\ &\quad\le \iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)||\varphi_\varepsilon(x)-\varphi_\varepsilon(y)||u_n(x)|}{|x-y|^{N+2\alpha}}\,dx\,dy \\ &\quad\le C\left(\iint_{\mathbb{R}^{2N}} \frac{u_n^2(x)|\varphi_\varepsilon(x)-\varphi_\varepsilon(y)|^2}{|x-y|^{N+2\alpha}}\,dx\,dy\right)^{\frac{1}{2}}. \end{aligned}$$

By the following equality (see (3.7) in [19])

$$\limsup_{\varepsilon\to 0}\limsup_{n\to\infty} \iint_{\mathbb{R}^{2N}} \frac{u_n^2(x)|\varphi_\varepsilon(x)-\varphi_\varepsilon(y)|^2}{|x-y|^{N+2\alpha}}\,dx\,dy = 0,$$

as well as (2.11) and (2.12), we obtain that

$$\mu(\{x_{i_0}\}) \le 2\lambda\nu_{i_0} \quad \text{for any } i_0 \in J.$$

From (2.7) and (2.10), one has

$$2C_3 \ge \lambda\left(\frac{2_\alpha^*-\theta}{2_\alpha^*}\right)\left(\frac{\lambda^{-1}S_\alpha}{2}\right)^{\frac{2_\alpha^*}{2_\alpha^*-2}} = \left(\frac{2_\alpha^*-\theta}{2_\alpha^*}\right)\left(\frac{S_\alpha}{2}\right)^{\frac{N}{2\alpha}}\lambda^{\frac{2\alpha-N}{2\alpha}},$$

which implies

$$\lambda \ge \left(\frac{2_\alpha^*-\theta}{2C_3 2_\alpha^*}\right)\left(\frac{S_\alpha}{2}\right)^{\frac{2_\alpha^*}{2}} := \lambda_*,$$

in contradiction with our assumption $0<\lambda<\lambda_*$. Then, for any $i \in J$, $\nu_i = 0$ and $\nu_\infty = 0$ holds. Using Lemma 2.8, we have

$$\limsup_{n\to\infty}\int_{\mathbb{R}^N} |u_n(x)|^{2_\alpha^*}\,dx = \int_{\mathbb{R}^N} |u_0(x)|^{2_\alpha^*}\,dx.$$

Since $|u_n - u_0|^{2_\alpha^*} \le 2^{2_\alpha^*}(|u_n|^{2_\alpha^*} + |u_0|^{2_\alpha^*})$, by Fatou's Lemma, we obtain

$$\begin{aligned} &\int_{\mathbb{R}^N} 2^{2_\alpha^*+1}|u_0(x)|^{2_\alpha^*}\,dx \\ &\quad= \int_{\mathbb{R}^N} \liminf_{n\to\infty}\left(2^{2_\alpha^*}|u_n(x)|^{2_\alpha^*} + 2^{2_\alpha^*}|u_0(x)|^{2_\alpha^*} - |u_n(x)-u_0(x)|^{2_\alpha^*}\right)dx \end{aligned}$$

$$\leq \liminf_{n\to\infty} \int_{\mathbb{R}^N} \left(2^{2_\alpha^*} |u_n(x)|^{2_\alpha^*} + 2^{2_\alpha^*} |u_0(x)|^{2_\alpha^*} - |u_n(x) - u_0(x)|^{2_\alpha^*}\right) dx$$
$$+ 2^{2_\alpha^*+1} \int_{\mathbb{R}^N} |u_0(x)|^{2_\alpha^*} dx - \limsup_{n\to\infty} \int_{\mathbb{R}^N} |u_n(x) - u_0(x)|^{2_\alpha^*} dx,$$

which implies that

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} |u_n(x) - u_0(x)|^{2_\alpha^*} dx = 0,$$

then $u_n \to u_0$ in $L^{2_\alpha^*}(\mathbb{R}^N)$ as $n \to \infty$.

Note that $I'_\lambda(u_n) \to 0$, and therefore

$$\begin{aligned}
&\limsup_{n\to\infty} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{\alpha}{2}} u_n(x)|^2 dx \\
&= \limsup_{n\to\infty} \left(\int_{\mathbb{R}^N} f(x, u_n(x)) u_n(x)\, dx + \lambda \int_{\mathbb{R}^N} |u_n(x)|^{2_\alpha^*} dx - \int_{\mathbb{R}^N} V(x) u_n^2(x)\, dx \right) \\
&\leq \int_{\mathbb{R}^N} f(x, u_0(x)) u_0(x)\, dx + \lambda \int_{\mathbb{R}^N} |u_0(x)|^{2_\alpha^*} dx - \int_{\mathbb{R}^N} V(x) u_0^2(x)\, dx \\
&\leq \liminf_{n\to\infty} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{\alpha}{2}} u_n(x)|^2 dx,
\end{aligned}$$

which means that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{\alpha}{2}} u_n(x)|^2 dx = \int_{\mathbb{R}^N} |(-\Delta)^{\frac{\alpha}{2}} u_0(x)|^2 dx. \tag{2.13}$$

Thus,

$$\begin{aligned}
&\limsup_{n\to\infty} \int_{\mathbb{R}^N} V(x) u_n^2(x)\, dx \\
&= \limsup_{n\to\infty} \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} f(x, u_n(x)) u_n(x)\, dx + \lambda \int_{\mathbb{R}^N} |u_n(x)|^{2_\alpha^*} dx \right. \\
&\quad \left. - \int_{\mathbb{R}^N} |(-\Delta)^{\frac{\alpha}{2}} u_n(x)|^2 dx \right) \\
&= \int_{\mathbb{R}^N} f(x, u_0(x)) u_0(x)\, dx + \lambda \int_{\mathbb{R}^N} |u_0(x)|^{2_\alpha^*} dx - \int_{\mathbb{R}^N} |(-\Delta)^{\frac{\alpha}{2}} u_0(x)|^2 dx \\
&= \int_{\mathbb{R}^N} V(x) u_0^2(x)\, dx.
\end{aligned}$$

Since $u_n \rightharpoonup u_0$ weakly in E, then by (2.13), we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{\alpha}{2}} (u_n - u_0)(x)|^2 dx + \int_{\mathbb{R}^N} V(x) (u_n(x) - u_0(x))^2 dx = 0,$$

which implies that $u_n \to u_0$ in E. □

Denote $B_\rho = \{u \in E : \|u\| \leq \rho\}$, where ρ is given in Lemma 2.4. The following lemma expresses that the functional I_λ has a local minimum in $\bar{B_\rho}$.

Lemma 2.11 *Assume that* (V_0) *and* (f4) *hold. Let* $\lambda_0 > 0$ *be as in Lemma* 2.4, *then for every* $\lambda \in (0, \lambda_0)$, *there exists* $u_0 \in E$ *such that*

$$I_\lambda(u_0) = \inf\{I_\lambda(u) : u \in \bar{B_\rho}\} < 0.$$

Proof Choose $\xi \in E$ such that $\int_{\mathbb{R}^N} |\xi(x)|^{2^*_\alpha}\,dx > 0$. By (f4), for $t > 0$ sufficiently large, we have

$$\begin{aligned} I_\lambda(t\xi) &= \frac{t^2}{2}\|\xi\|^2 - \int_{\mathbb{R}^N} F\bigl(x, t\xi(x)\bigr)\,dx - \frac{\lambda t^{2^*_\alpha}}{2^*_\alpha}\int_{\mathbb{R}^N}\bigl|\xi(x)\bigr|^{2^*_\alpha}\,dx \\ &\le \frac{t^2}{2}\|\xi\|^2 - \frac{\lambda t^{2^*_\alpha}}{2^*_\alpha}\int_{\mathbb{R}^N}\bigl|\xi(x)\bigr|^{2^*_\alpha}\,dx \\ &< 0. \end{aligned}$$

Therefore, letting $\rho > 0$ be given as in Lemma 2.4, one has

$$I_\lambda(u_0) = \inf\{I_\lambda(u) : u \in \bar{B_\rho}\} < 0. \qquad \square$$

3 Proof of the main results

In this section, we use the Ekeland's variational principle [24] and the Mountain Pass Theorem [25] to prove our main result.

Proof of Theorem 1.1 Due to (f1)–(f4), we easily know that 0 is a trivial solution of problem (1.1). Using the Ekeland's variational principle and Lemma 2.11, there exists a minimizing sequence $u_n \in \bar{B_\rho}$ such that $I_\lambda(u_n) \to \inf_{u\in\bar{B_\rho}} I_\lambda(u_0)$ and $I'_\lambda(u_n) \to 0$ when $n \to \infty$. Therefore, by Lemma 2.10, we have that problem (1.1) has a nontrivial solution u_* which satisfies $I_\lambda(u_*) < 0$ and $\|u_*\| \le \rho$.

Choose $\lambda_1 := \min\{\lambda_0, \lambda_*\}$, where λ_0, λ_* are given in Lemmas 2.4 and 2.10. By the Mountain Pass Theorem and Lemma 2.4, there exists a (PS) sequence $\{u_n\} \subset E$ for I_λ in E for any $0 < \lambda < \lambda_1$. According to Lemma 2.10, there exist a subsequence of $\{u_n\} \subset E$, still denoted by $\{u_n\}$, and $u_{**} \in E$ such that $u_n \to u_{**}$ in E as $n \to \infty$. Moreover, $I'_\lambda(u_{**}) = 0$ and $I_\lambda(u_{**}) \ge \beta > 0$. Hence, u_{**} is another nontrivial solution of problem (1.1).

From the above argument, we can conclude that problem (1.1) possesses three solutions such that $I_\lambda(u_*) < 0 = I_\lambda(0) < I_\lambda(u_{**})$ for all $0 < \lambda < \lambda_1$. The proof is completed. $\square$

4 Conclusion

The fractional Schödinger equation is particularly important equation in fractional quantum mechanics. Recently, lots of papers have been published on the existence of solutions for fractional Schödinger equation. However, there are few of papers consider the existence of solutions for the fractional Schödinger equation with perturbation. In this paper, we study a new form of fractional Schödinger equation involving a critical nonlinearity (1.1). By using the principle of concentration compactness and the variational method, we obtain the existence and multiplicity of solutions for it; our work differs from earlier studies. Therefore, our results improve the corresponding results for this problem. In the future, we think that an accurate numerical solution for the fractional Schödinger equation can be obtained by using numerical methods, such as that developed in [30, 31].

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