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Discontinuous finite volume element method of two-dimensional unsaturated soil water movement problem

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Abstract

In this paper, a numerical approximation method for the two-dimensional unsaturated soil water movement problem is established by using the discontinuous finite volume method. We prove the optimal error estimate for the fully discrete format. Finally, the reliability of the method is verified by numerical experiments. This method is not only simple to calculate, but also stable and reliable.

Keywords: Unsaturated soil water movement; Discontinuous finite volume element method; Error estimate; Numerical experiments

1 Introduction

The movement of water in soil is a very complicated problem. This paper mainly studies the water movement in furrow irrigation, that is, the water movement in the trapezoidal region to the soil diffusion on both sides and the infiltration of the underground pipeline into the surrounding soil. Unsaturated soil water movement refers to the movement of water in the soil when the water is not full of pores. It is an important form of fluid movement in porous media. We assume that the soil is homogeneous and isotropic. Let the x-axis be horizontal to the right and the z-axis vertically downward. According to Darcy's law and the continuity principle, the problem of unsaturated soil water movement can be reduced to the following model (see [1]):

$$\frac{\partial Q}{\partial t}-\frac{\partial}{\partial x}\left(D(Q)\frac{\partial Q}{\partial x}\right)-\frac{\partial}{\partial z}\left(D(Q)\frac{\partial Q}{\partial z}\right)+\frac{\partial K(Q)}{\partial z}=S_r. \tag{1}$$

where $Q(x,z,t)$ is the soil moisture volume water content, $D(Q)$ indicates the diffusion rate of soil water, $K(Q)$ indicates the hydraulic conductivity, $-S_r$ is the absorption rate of the root zone, the relationship between $K(Q)$, $D(Q)$ and Q is as follows:

$$\begin{cases} K(Q)=K_s(\frac{Q}{Q_s})^{2b+3}, \\ D(Q)=-\frac{bK_s\psi_s}{Q_s}(\frac{Q}{Q_s})^{b+2}, \quad Q_r\le Q(z,t)\le Q_s, \end{cases} \tag{2}$$

where Q_s is the soil water saturated water content, Q_r is the residual moisture content of the soil moisture, where $0<Q_s<1$, the saturated water conductivity K_s, the soil parameter

b and the saturated soil water potential ψ_s are all related to the soil structure and are known constants. Therefore, it can be determined that $K(Q)$, $D(Q)$, $\frac{\partial K(Q)}{\partial Q}$, $\frac{\partial K(Q)}{\partial z}$, $\frac{\partial D(Q)}{\partial Q}$ are bounded, that is, there are two constants K_1, K_2, such that: $K_1 \leq K(Q)$, $\frac{\partial K(Q)}{\partial Q}$, $\frac{\partial K(Q)}{\partial z}$, $\frac{\partial D(Q)}{\partial Q}$, $D(Q) \leq K_2$.

The following conditions are given for (1):

(1) Initial condition: $Q(x, z, 0) = Q_0$;

(2) Boundary condition:

$$\begin{cases} Q(0,0,t) = Q_s, & t \in [0,T], \\ Q(L,z,t) = Q_0, & L \to \pm\infty, t \in [0,T], \\ Q(x,M,t) = Q_0, & M \to \pm\infty, t \in [0,T], \end{cases}$$

where Q_0 represents the initial water content and Q_s represents the saturated water content.

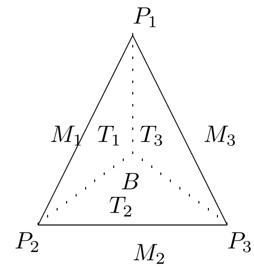
According to the literature [2], the solution to the problem is existing and unique. Based on the reliability of this problem and its practical significance in meteorology, agricultural environmental engineering, hydrodynamics, etc., in recent years, many scholars have proposed numerical methods to solve it. The numerical solutions of one-dimensional and two-dimensional soil water movement problems are given by the finite difference method in Ref. [2, 3]. However, because the finite difference method is very sensitive to boundary conditions and soil parameters, the error is large. The authors of Ref. [4, 5] used the finite volume element method to simulate the two-dimensional soil water flow problem and overcome the weakness of the finite difference method. To the best of our knowledge, there is no report on the discontinuous finite volume element method to deal with two-dimensional unsaturated water movement. In this paper, we focus on the mathematical model characteristics of the two-dimensional unsaturated water motion problem, and we mainly discuss the discontinuous finite volume element method of the problem. This method not only inherits the advantages of the format of the finite volume element method, that is, a simple structure, high precision, simple calculation and local conservation between physical quantities, but it also has the characteristics of discontinuous finite element, the finite element space does not need to meet any continuity requirements, the space structure is simple, and there is good locality and parallelism.

This paper is organized as follows, in Sect. 2, we derive a discontinuous finite volume element format for unsaturated soil water movement problems. In Sect. 3, we give some lemmas related to error analysis. In Sect. 4, we obtain the optimal estimate of L^2-norm and $|||\cdot|||_{1,h}$-norm. In Sect. 5, numerical experiments are given to verify the validity of the theoretical analysis.

2 Discontinuous finite volume element format for unsaturated soil water movement problems

For convenience, it is assumed that the region $\Omega \in R^2$ is a suitably smooth and sufficiently large bounded region. On the boundary $\partial\Omega$ of the region Ω, the initial moisture content Q_0 remains unchanged. We define the dual partition T_h^* of T_h for the test function space as follows. Let T_h be a triangulation of Ω with diam(Ω), where h is the set of all the triangular elements K, $h(K)$ is the side length of the unit $K \in T_h$. Let Γ denote the union of the boundaries of the triangle K of T_h, $\Gamma_0 = \Gamma \backslash \partial\Omega$, $h = \max_{K \in T_h} h(K)$. We divide each K into

Figure 1 Original subdivision and dual subdivision



three triangles by connecting the barycenter B and three corners of the triangle as shown in Fig. 1. Let T_h^* consist of all these triangles T_j.

On the original split, we define the following broken Sobolev space:

$$H^m(T_h) = \{v \in L^2(\Omega) : v|_k \in H^m(K), \forall K \in T_h\}$$

and its norm

$$\|v\|_{m,h} = \left(\sum_{i=0}^{m} |v|_{i,h}^2\right)^{1/2}, \qquad |v|_{i,h} = \left(\sum_{K\in T_h} |v|_{H^i(K)}^2\right)^{1/2},$$

where $H^i(K)$ is the standard Sobolev space defined on unit K, m is a positive integer.

We define a finite dimensional trial function space U_h with the original partition T_h:

$$U_h = \{u_h \in L^2(\Omega) : u_h|_K \in P_1(K), \forall K \in T_h\}.$$

Define the finite dimensional test function space on the dual partition T_h^*:

$$V_h = \{v_h \in L^2(\Omega) : v_h|_T \in P_0(T), \forall T \in T_h^*\},$$

where P_l denotes a polynomial with degree less than or equal to l ($l = 0, 1$) defined on $K(T)$.

Let $U(h) = U_h + H^2(\Omega) \cap H_0^1(\Omega)$. Define a mapping $\gamma_h : U(h) \to V_h$ as

$$\gamma_h v_h|_T = \frac{1}{h_e}\int_e v_h|_T \, ds, \quad \forall T \in T_h^*,$$

where h_e is the length of the boundary e of the unit K.

In order to facilitate theoretical analysis, take $Q_0 = 0$, and let $F(Q) = \mathbf{p} \cdot K(Q)$, we have

$$\frac{\partial K(Q)}{\partial z} = \nabla \cdot F(Q), \quad \mathbf{p} = (0, 1).$$

So (1) can be written as

$$\frac{\partial Q}{\partial t} - \frac{\partial}{\partial x}\left(D(Q)\frac{\partial Q}{\partial x}\right) - \frac{\partial}{\partial z}\left(D(Q)\frac{\partial Q}{\partial z}\right) + \nabla \cdot F(Q) = S_r, \tag{3}$$

Multiplying (3) by $v_h \in V_h$, integrating on the dual unit, summing over T, and using the Green formula, we obtain

$$\left(\frac{\partial Q}{\partial t}, v_h\right) - \sum_{T\in T_h^*} \int_{\partial T} D(Q)\nabla Q \cdot \mathbf{n} v_h \, ds = \left(S_r - \nabla \cdot F(Q), v_h\right), \tag{4}$$

where $\mathbf{n}$ is the unit outward normal vector on ∂T.

So

$$\begin{aligned}\sum_{T\in T_h^*} \int_{\partial T} D(Q)\nabla Q \cdot \mathbf{n} v_h \, ds ={}& \sum_{K\in T_h} \sum_{i=1}^{3} \int_{P_{i+1}BP_i} D(Q)\nabla Q \cdot \mathbf{n} v_h \, ds \\ &+ \sum_{K\in T_h} \int_{\partial T} D(Q)\nabla Q \cdot \mathbf{n} v_h \, ds,\end{aligned}$$

where $P_4 = P_1$, $P_5 = P_2$, $P_6 = P_3$.

Assume $e = \partial K_1 \cap \partial K_2$, where K_1, K_2 are the adjacent two units in T_h, and let $\mathbf{n}_1$ and $\mathbf{n}_2$ be unit normal vectors on e pointing exterior to K_1 and K_2. For scalar functions p and vector functions $\mathbf{q}$, we define the average $\{\cdot\}$ and the jump $[\cdot]$ on e, as follows (see [6]).

If $e \in \Gamma_0$ and $e \subset \partial K_1 \cap \partial K_2$, then

$$\begin{aligned}&[v]|_e = v|_{\partial K_1}\mathbf{n}_1 + v|_{\partial K_2}\mathbf{n}_2, \qquad \{v\}|_e = \frac{1}{2}(v|_{\partial K_1} + v|_{\partial K_2}),\\ &[\mathbf{w}]|_e = \mathbf{w}|_{\partial K_1}\cdot\mathbf{n}_1 + \mathbf{w}|_{\partial K_2}\cdot\mathbf{n}_2, \qquad \{\mathbf{w}\}|_e = \frac{1}{2}(\mathbf{w}|_{\partial K_1} + \mathbf{w}|_{\partial K_2}).\end{aligned}$$

If $e \in \Gamma\backslash\Gamma_0$, and $e \subset \partial K$, then

$$\begin{aligned}&[v]|_e = v|_{\partial K}\mathbf{n}_k, \qquad \{v\}|_e = v|_{\partial K},\\ &[\mathbf{w}]|_e = \mathbf{w}|_{\partial K}\cdot\mathbf{n}_k, \qquad \{\mathbf{w}\}|_e = \mathbf{w}|_{\partial K}.\end{aligned}$$

According to the above average and the definition of the jump, a straightforward computation gives

$$\sum_{K\in T_h} \int_{\partial K} p\mathbf{w}\cdot\mathbf{n}\, ds = \sum_{e\in\Gamma} \int_e [p]\cdot\{\mathbf{w}\}\, ds + \sum_{e\in\Gamma_0} \int_e \{p\}[\mathbf{w}]\, ds. \tag{5}$$

Using (5) and the fact that $[D(Q)\nabla Q\cdot\mathbf{n}v_h]|_e = 0$, $\forall e \in \Gamma_0$, we have

$$\sum_{K\in T_h} \int_{\partial K} D(Q)\nabla Q\cdot\mathbf{n}v_h\, ds = \sum_{e\in\Gamma} \int_e \left\{D(Q)\nabla Q\right\}\cdot[v_h]\, ds.$$

Define the following bilinear form:

$$\begin{aligned}A(Q_h;Q_h,\gamma_h v_h)={}&-\sum_{K\in T_h}\sum_{i=1}^{3}\int_{P_{i+1}BP_i}D(Q_h)\nabla Q_h\cdot\mathbf{n}\gamma_h v_h\,ds\\&-\sum_{e\in\Gamma}\int_e\{D(Q_h)\nabla Q_h\}[\gamma_h v_h]\,ds\\&-\sum_{e\in\Gamma}\int_e\{D(Q_h)\nabla v_h\}[\gamma_h Q_h]\,ds\\&+\alpha\sum_{e\in\Gamma}[\gamma_h Q_h][\gamma_h v_h],\end{aligned}$$

where α is a real constant (see [7]).

The semi-discrete discontinuous finite volume element format of problem (1) is to find $Q_h\in U_h$, such that

$$\begin{cases}\text{(a)}\quad (\frac{\partial Q_h}{\partial t},\gamma_h v_h)+A(Q_h;Q_h,\gamma_h v_h)=(S_r-\nabla\cdot F(Q),\gamma_h v_h),\quad \forall v_h\in U_h,\\ \text{(b)}\quad Q_h(0)=0.\end{cases}\tag{6}$$

Since Q is the solution to the problem (1), and $[\gamma_h Q]|_e=0$, the true solution Q satisfies

$$\begin{cases}\text{(a)}\quad (\frac{\partial Q}{\partial t},\gamma_h v_h)+A(Q;Q,\gamma_h v_h)=(S_r-\nabla\cdot F(Q),\gamma_h v_h),\quad \forall v_h\in U_h,\\ \text{(b)}\quad Q(0)=0.\end{cases}\tag{7}$$

Next, we present the fully discrete discontinuous finite volume element method. Take Δt as the time step, recorded as $\Delta t=\frac{T}{N}$, $N>0$, and N is a positive integer, $t^j=j\Delta t$,when $t=t^j$. If we use the backward difference quotient $\partial_t Q_h^j=\frac{Q_h^j-Q_h^{j-1}}{\Delta t}$ to approximate the differential quotient in the semi-discrete scheme, then we get the backward Euler fully discrete discontinuous finite volume element format: find $Q_h^j\in U_h$, $(j=1,\dots,N)$, $\forall v_h\in U_h$, satisfying

$$\begin{cases}\text{(a)}\quad (\frac{Q_h^j-Q_h^{j-1}}{\Delta t},\gamma_h v_h)+A(Q_h^j;Q_h^j,\gamma_h v_h)=(S_r^j-\nabla\cdot F(Q_h^j),\gamma_h v_h),\\ \text{(b)}\quad Q_h(0)=0.\end{cases}\tag{8}$$

3 Some lemmas

We define a norm $|||\cdot|||$ for U_h as follows:

$$|||u|||_{1,h}^2=|u|_{1,h}^2+\sum_{e\in\Gamma}[\gamma_h u]_e^2+\sum_{K\in T_h}h_k^2|u|_{2,K}^2,\quad \forall u\in U(h),$$

where $|u|_{1,h}^2=\sum_{K\in T_h}|u|_{1,K}^2$.

The following trace inequality can be found in [8]. If e is the edge of unit K with length h_e, then

$$\|u\|_e^2\le C\bigl(h_e^{-1}\|u\|_K^2+h_e|u|_{1,K}^2\bigr),\quad \forall u\in H^2(K).$$

For $\forall Q \in H^1(\Omega)$, introduce the original equation solution Q to geta Ritz projection $R_h(t): H^1(\Omega) \to U_h$ which satisfies

$$A(Q;Q,\gamma_h v_h) = A(Q;R_h Q,\gamma_h v_h), \quad \forall v_h \in V_h. \tag{9}$$

According to the relevant theory of the Ritz projection R_h (see [3]), the following interpolation properties are obtained:

$$\begin{cases} \text{(a)} & \|Q - R_h Q\| \le Ch^2 \|Q\|_3, \\ \text{(b)} & \|(Q - R_h Q)_t\| \le Ch^2 \|Q\|_{1,3,2}, \\ \text{(c)} & |||Q - R_h Q|||_{1,h} \le Ch\|Q\|_2, \\ \text{(d)} & |||(Q - R_h Q)_t|||_{1,h} \le Ch\|Q\|_{1,2,2}. \end{cases} \tag{10}$$

Lemma 1 (see [9]) *The operator γ_h is self-adjoint with respect to the L^2-inner product, that is,*

$$(u_h, \gamma_h v_h) = (v_h, \gamma_h u_h), \quad \forall u_h, v_h \in U_h,$$

and if we define

$$|||u_h|||_0 = (u_h, \gamma_h u_h)^{1/2},$$

then $|||\cdot|||_0$ and $\|\cdot\|$ are equivalent, and

$$\|\gamma_h u_h\| = \|u_h\|, \quad \forall u_h \in U_h.$$

Lemma 2 (see [9]) *The operator γ_h satisfies the following properties*:

$$\int_K (w_h - \gamma_h w_h)\,dx = 0, \quad \forall w_h \in U(h), \forall K \in T_h,$$

$$\int_e (w_h - \gamma_h w_h)\,ds = 0, \quad \forall w_h \in U(h), \forall e \in \partial K, \forall K \in T_h,$$

$$[w_h] = 0 \quad \Longrightarrow \quad [\gamma_h w_h] = 0, \quad \forall w_h \in U(h),$$

$$\|\gamma_h w_h - w_h\|_{0,K} \le Ch_K |w_h|_{1,K}, \quad \forall w_h \in U(h), \forall K \in T_h.$$

Lemma 3 (see [10]) *There is a normal number β that is independent of h, such that*

$$\beta |||u_h|||_{1,h}^2 \le A(q;u_h,\gamma_h u_h), \quad \forall u_h \in U_h. \tag{11}$$

Lemma 4 (see [6]) *For $\forall u_h, v_h \in U_h$, there exists a positive constant C independent of h, and we have*

$$A(q;u_h,\gamma_h v_h) \le C|||u_h|||_{1,h} |||v_h|||_{1,h}. \tag{12}$$

Lemma 5 (see [2, 5, 6]) *There is a constant C independent of h such that*

$$\left|A(q;u_h,\gamma_h v_h)-A(q;v_h,\gamma_h u_h)\right| \le Ch|||u_h|||_{1,h}|||v_h|||_{1,h}, \quad \forall u_h \in U_h. \tag{13}$$

$$\left|A(p;u_h,\gamma_h v_h)-A(q;u_h,\gamma_h v_h)\right| \le C|u_h|_{1,\infty}\big(\|p-q\|+h|||p-q|||_{1,h}\big)\big)|||v_h|||_{1,h}. \tag{14}$$

Lemma 6 (see [9]) *There is a constant C independent of h such that*

$$h|||u_h|||_{1,h} \le C\|u_h\|, \quad \forall u_h \in U_h. \tag{15}$$

$$\|u_h\| \le C|||u_h|||_{1,h}, \quad \forall u_h \in U_h. \tag{16}$$

Lemma 7 *If $Q, Q_t \in W^{2,\infty}(\Omega)\cap H^3(\Omega)$, there is a constant C_0, C_1 independent of h such that, for small enough, we have*

$$|R_hQ|_{1,\infty} \le C_0, \tag{17}$$

$$|R_hQ_t|_{1,\infty} \le C_1. \tag{18}$$

Proof Let $Q_I \in U_h$ be the interpolation of Q, it is well known that

$$|Q-Q_I|_{s,p,k} \le Ch^{2-s}|Q|_{2,p,k} \quad \forall K \in T_h, s=0,1,1 \le p \le \infty. \tag{19}$$

By the definition of $|||\cdot|||_{1,h}$, we have

$$|||Q-Q_I|||_{1,h} \le Ch|Q|_2. \tag{20}$$

Using Lemma 3 and Lemma 4, we have

$$\begin{aligned}\beta|||R_hQ-Q_I|||_{1,h}^2 &\le A\big(Q;R_hQ-Q_I,\gamma_h(R_hQ-Q_I)\big)\\ &= A\big(Q;Q-Q_I,\gamma_h(R_hQ-Q_I)\big)\\ &\le C|||Q-Q_I|||_{1,h}\cdot|||R_hQ-Q_I|||_{1,h}.\end{aligned}$$

Using triangular inequalities, inverse inequalities and (19), we obtain

$$|R_hQ|_{1,\infty} \le |R_hQ-Q_I|_{1,\infty}+|Q_I-Q|_{1,\infty}+|Q|_{1,\infty}.$$

So

$$|||R_hQ-Q_I|||_{1,h} \le Ch|Q|_2.$$

Therefore

$$|R_hQ|_{1,\infty} \le C\big(|Q|_2+|Q|_1\big).$$

For (18), the proof is similar to that of (17). □

4 Convergence analysis

Theorem 1 *Let Q, Q_h^n be the solutions of* (1) *and* (8), *if $Q \in L^2(0,T;H^3(\Omega))$, $Q_t \in L^2(0,T;L^2(\Omega))$, $Q_{tt} \in L^2(0,T;L^2(\Omega))$, $Q_h(0) = R_hQ(0) = 0$, then there is a constant C independent of h and Δt, such that*

$$\|Q(t^n) - Q_h^n\| \le Ch^2\left(\int_0^{t^n} \|Q_t\|_3^2\,dt\right)^{\frac{1}{2}} + C\Delta t\left(\int_0^{t^n} \|Q_{tt}\|^2\,dt\right)^{\frac{1}{2}}, \tag{21}$$

$$\begin{aligned}\|\!|Q(t^n) - Q_h^n\|\!|_{1,h} &\le Ch\left(\|Q^n\|_3 + \|Q^n\|_2 + \int_0^{t^n} \|Q_t\|_3^2\,dt\right)^{\frac{1}{2}} \\ &\quad + C\Delta t\left(\int_0^{t^n} \|Q_{tt}\|^2\,dt\right)^{\frac{1}{2}}.\end{aligned} \tag{22}$$

Proof Let $Q_h^n = Q(t^n)$, $R_hQ^n = R_hQ(t^n)$, $Q_t^n = \frac{\partial Q^n}{\partial t}$, $\rho^n = Q^n - R_hQ^n$, $\theta^n = R_hQ^n - Q_h^n$. Now we estimate $\|\theta^n\|$.

Subtracting (8) from (7) gives the error equation

$$\begin{aligned}&\left(Q_t^j - \frac{Q^j - Q^{j-1}}{\Delta t}, \gamma_h v_h\right) + A\left(Q^j; Q^j, \gamma_h v_h\right) - A\left(Q_h^j; Q_h^j, \gamma_h v_h\right) \\ &\quad = \left(\nabla \cdot \left(F\left(Q_h^j\right) - F\left(Q^j\right)\right), \gamma_h v_h\right),\end{aligned} \tag{23}$$

where

$$\begin{aligned}&A\left(Q^j; Q^j, \gamma_h v_h\right) - A\left({Q_h}^j; {Q_h}^j, \gamma_h v_h\right) \\ &\quad = A\left(Q^j; \rho^j + \theta^j, \gamma_h v_h\right) + A\left(Q^j; Q_h^j, \gamma_h v_h\right) - A\left(Q_h^j; Q_h^j, \gamma_h v_h\right), \\ &Q_t^j - \frac{Q^j - Q^{j-1}}{\Delta t} = Q_t^j + \partial t\theta^j - \partial t R_h Q^j.\end{aligned} \tag{24}$$

Using (10), the error equation is equivalent to

$$\begin{aligned}&\left(\partial_t\theta^j, \gamma_h v_h\right) + A\left(Q^j; \theta^j, \gamma_h v_h\right) \\ &\quad = \left(\partial_t R_h Q^j - Q_t^j, \gamma_h v_h\right) \\ &\qquad + A\left(Q_h^j; Q_h^j, \gamma_h v_h\right) - A\left(Q^j; Q_h^j, \gamma_h v_h\right) + \left(\nabla \cdot \left(F\left(Q_h^j\right) - F\left(Q^j\right)\right), \gamma_h v_h\right).\end{aligned} \tag{25}$$

Choosing $v_h = \theta^j$ in (25), using Lemma 3, we have

$$\begin{aligned}\left(\partial_t\theta^j, \gamma_h\theta^j\right) + A\left(Q^j; \theta^j, \gamma_h\theta^j\right) &= \left(\frac{\theta^j - \theta^{j-1}}{\Delta t}, \gamma_h\theta^j\right) + A\left(Q^j; \theta^j, \gamma_h\theta^j\right) \\ &\ge \frac{\|\theta^j\|^2 - \|\theta^{j-1}\|^2}{2\Delta t} + \beta\|\!|\theta^{j-1}\|\!|_{1,h}^2.\end{aligned} \tag{26}$$

The right side of Eq. (25) is recorded as I_1, I_2, I_3, and is estimated item by item.

Using the Hölder inequalities and ε inequalities, we get

$$|I_1| \le C\|\!|\partial_t R_h Q^j - Q_t^j\|\!|_0^2 + C\|\theta^j\|^2.$$

Using Lemma 5, the triangular inequalities and the ε inequalities, we have

$$
\begin{aligned}
|I_2| &\le C|Q_h^j|_{1,\infty}\big(\|Q^j - Q_h^j\| + h\|\!|Q^j - Q_h^j|\!\|_{1,h}\big)\|\!|\theta^j|\!\|_{1,h} \\
&\le C\big(\|\rho^j\| + \|\theta^j\| + h\|\!|\rho^j|\!\|_{1,h} + h\|\!|\theta^j|\!\|_{1,h}\big)\|\!|\theta^j|\!\|_{1,h} \\
&\le C\big(\|\rho^j\|^2 + \|\theta^j\|^2 + h^2\|\!|\rho^j|\!\|_{1,h}^2\big) + \frac{\beta}{2}\|\!|\theta^j|\!\|_{1,h}^2.
\end{aligned}
$$

Assume $|Q_h^j|_{1,\infty} \le C_0$, $j = 0, 1, \dots, N$. Proof of it will be given later.

From the Hölder inequalities and the ε inequalities, we have

$$
|I_3| \le C\|Q^j - Q_h^j\|^2 + C\|\theta^j\|^2 \le C\|\rho^j\|^2 + \|\theta^j\|^2.
$$

Let $\xi^j = \partial_t R_h Q^j - Q_t^j$ and use the above estimate; (25) is transformed into

$$
\begin{aligned}
&\frac{\|\theta^j\|^2 - \|\theta^{j-1}\|^2}{2\triangle t} + \beta\|\!|\theta^j|\!\|_{1,h}^2 \\
&\quad\le C\|\!|\xi^j|\!\|_0^2 + C\|\theta^j\|^2 + C\|\rho^j\|^2 + C\|\theta^j\|^2 + Ch^2\|\!|\rho^j|\!\|_{1,h}^2.
\end{aligned}
\tag{27}
$$

Multiplying $2\Delta t$ on both sides of type (27), summing over j from 1 to n at both sides of (27) and noting that $\theta^0 = 0$,

$$
\begin{aligned}
&\|\theta^n\|^2 + \Delta t\beta\sum_{j=1}^{n}\|\!|\theta^j|\!\|_{1,h}^2 \\
&\quad\le C\Delta t\sum_{j=1}^{n}\|\!|\xi^j|\!\|_0^2 + C\Delta t\sum_{j=1}^{n}\big(\|\rho^j\|^2 + h^2\|\!|\rho^j|\!\|_{1,h}^2\big) + C\Delta t\sum_{j=1}^{n}\|\theta^j\|^2,
\end{aligned}
\tag{28}
$$

where

$$
\begin{aligned}
\sum_{j=1}^{n}\|\!|\xi^j|\!\|_0^2 &\le \frac{1}{\Delta t}\sum_{j=1}^{n}\int_{t^{j-1}}^{t^j}\|Q_t - R_hQ_t\|^2\,dt + \Delta t\sum_{j=1}^{n}\int_{t^{j-1}}^{t^j}\|Q_{tt}\|^2\,dt \\
&\le C\frac{h^4}{\Delta t}\int_0^{t^n}\|Q_t\|_3^2\,dt + C\Delta t\int_0^{t^n}\|Q_{tt}\|^2\,dt.
\end{aligned}
\tag{29}
$$

Taking (10) and the discrete Gronwall lemma,

$$
\begin{aligned}
&\|\theta^n\|^2 + \Delta t\beta\sum_{j=1}^{n}\|\!|\theta^j|\!\|_{1,h}^2 \\
&\quad\le Ch^4\int_0^{t^n}\|Q_t\|_3^2\,dt + C\Delta t^2\int_0^{t^n}\|Q_{tt}\|^2\,dt + Ch^4\big(\|Q^n\|_3^2 + \|Q^n\|_2^2\big) \\
&\quad\le Ch^4\int_0^{t^n}\|Q_t\|_3^2\,dt + C\Delta t^2\int_0^{t^n}\|Q_{tt}\|^2\,dt,
\end{aligned}
$$

which is

$$
\|\theta^n\| \le Ch^2\left(\int_0^{t^n}\|Q_t\|_3^2\,dt\right)^{\frac{1}{2}} + \Delta t\left(\int_0^{t^n}\|Q_{tt}\|^2\,dt\right)^{\frac{1}{2}}.
\tag{30}
$$

Combining (10) with the triangle inequality, we obtain the desired result (21).

The following proves $|Q_h^j|_{1,\infty} \le C_0$, $j = 0, 1, \dots, N$.

Assume $C_0 = 1 + \sup_{[0,T]} |R_h Q|_{1,\infty}$, actually, when $t = 0$, $Q_h^0 = R_h Q^0 = 0$, obviously

$$\left|Q_h^0\right|_{1,\infty} \le \sup_{[0,T]} |R_h Q_t|_{1,\infty} < C_0.$$

Let us assume that $j = 0, 1, \dots, k-1$, $|Q_h^j|_{1,\infty} \le C_0$ is established. So when $\Delta t = O(h)$ and h is full,

$$\begin{aligned}
\left|Q_h^j\right|_{1,\infty} &\le \left|Q_h^j - R_h Q_t^j\right|_{1,\infty} + \left|R_h Q^j\right|_{1,\infty} \\
&\le C|\ln h|^{\frac{1}{2}} \left|Q_h^j - R_h Q^j\right|_{1,\infty} + \left|R_h Q^j\right|_{1,\infty} \\
&\le C_0.
\end{aligned}$$

Next we estimate $|||\theta^n|||_{1,h}$.

By letting $v_h = \partial_t \theta^j$ in (24), we have

$$\begin{aligned}
&\left(\partial_t\theta^j, \gamma_h \partial_t \theta^j\right) + A\left(Q^j; \theta^j, \gamma_h \partial_t \theta^j\right) \\
&\quad = \left(\partial_t R_h Q^j - Q_t^j, \gamma_h \partial_t \theta^j\right) + A\left(Q_h^j; Q_h^j, \gamma_h \partial_t \theta^j\right) - A\left(Q^j; Q_h^j, \gamma_h \partial_t \theta^j\right) \\
&\qquad + \left(\nabla \cdot \left(F\left(Q_h^j\right) - F\left(Q^j\right)\right), \gamma_h \partial_t \theta^j\right).
\end{aligned} \tag{31}$$

The left end item is obtained by Lemma 1

$$\begin{aligned}
\left(\partial_t\theta^j, \gamma_h \partial_t \theta^j\right) &= |||\partial_t \theta^j|||_0^2, \\
A\left(Q^j; \theta^j, \gamma_h \partial_t \theta^j\right) &= \frac{1}{2\Delta t}\Big[A\left(Q^j; \theta^j + \theta^{j-1}, \gamma_h\left(\theta^j - \theta^{j-1}\right)\right) \\
&\qquad + A\left(Q^j; \theta^j - \theta^{j-1}, \gamma_h\left(\theta^j - \theta^{j-1}\right)\right)\Big] \\
&\ge \frac{1}{2\Delta t} A\left(Q^j; \theta^j + \theta^{j-1}, \gamma_h\left(\theta^j - \theta^{j-1}\right)\right) \\
&= \frac{1}{2\Delta t}\Big[A\left(Q^j; \theta^j, \gamma_h \theta^j\right) - A\left(Q^j; \theta^{j-1}, \gamma_h \theta^{j-1}\right)\Big] \\
&\qquad - \frac{1}{2}\Big[A\left(Q^j; \partial_t \theta^j, \gamma_h \theta^j\right) - A\left(Q^j; \theta^j, \gamma_h \partial_t \theta^j\right)\Big].
\end{aligned}$$

Therefore, the error equation is transformed:

$$\begin{aligned}
&|||\partial_t \theta^j|||_0^2 + \frac{1}{2\Delta t}\Big[A\left(Q^j; \theta^j, \gamma_h \theta^j\right) - A\left(Q^j; \theta^{j-1}, \gamma_h \theta^{j-1}\right)\Big] \\
&\quad \le \left(\partial_t R_h Q^j - Q_t^j, \gamma_h \partial_t \theta^j\right) + A\left(Q_h^j; Q_h^j, \gamma_h \partial_t \theta^j\right) - A\left(Q^j; Q_h^j, \gamma_h \partial_t \theta^j\right) \\
&\qquad + \left(\nabla \cdot \left(F\left(Q_h^j\right) - F\left(Q^j\right)\right), \gamma_h \partial_t \theta^j\right) \\
&\qquad + \frac{1}{2}\Big[A\left(Q^j; \partial_t \theta^j, \gamma_h \theta^j\right) - A\left(Q^j; \theta^j, \gamma_h \partial_t \theta^j\right)\Big] \\
&\quad = J_1 + J_2 + J_3 + J_4.
\end{aligned} \tag{32}$$

Let $\xi^j = \partial_t R_h Q(t^j) - Q_t(t^j)$, similar to the previous estimates

$$
\begin{aligned}
|J_1| &\le C|||\xi^j|||_0^2 + \frac{1}{4}|||\partial_t\theta^j|||_0^2, \\
|J_2| &\le C|Q_h^j|_{1,\infty}\left(\|Q(t^j) - Q_h^j\| + h|||\rho^j|||_{1,h} + h|||\theta^j|||_{1,h}\right)|||\partial_t\theta^j|||_{1,h} \\
&\le C|Q_h^j|_{1,\infty}\left(\|\rho^j\| + \|\theta^j\| + h|||\rho^j|||_{1,h} + h|||\theta^j|||_{1,h}\right)Ch^{-1}\|\partial_t\theta^j\| \\
&\le Ch^{-2}\|\rho^j\|^2 + Ch^{-2}\|\theta^j\|^2 + C|||\rho^j|||_{1,h}^2 + C|||\theta^j|||_{1,h}^2 + \frac{1}{4}|||\partial_t\theta^j|||_0^2, \\
|J_3| &\le C\|Q(t^j) - Q_h^j\|^2 + \frac{1}{4}|||\partial_t\theta^j|||_0^2.
\end{aligned}
$$

For J_4, using Lemma 5, the ε inequality and the boundedness of $D(Q_h)$, we get

$$
|J_4| \le Ch|||\partial_t\theta^j|||_{1,h}|||\theta^j|||_{1,h} \le C\|\partial_t\theta^j\|\,|||\theta^j|||_{1,h} \le C|||\theta^j|||_{1,h}^2 + \frac{1}{4}|||\partial_t\theta^j|||_0^2.
$$

In summary,

$$
\begin{aligned}
&A\left(Q^j;\theta^j,\gamma_h\theta^j\right) - A\left(Q^j;\theta^{j-1},\gamma_h\theta^{j-1}\right) \\
&\quad\le C\Delta t|||\xi^j|||_0^2 + C\Delta t|||\theta^j|||_{1,h}^2 + C\Delta t\left(|||\rho^j|||_{1,h}^2 + h^{-2}\|\rho^j\|^2 + \|\rho^j\|^2\right).
\end{aligned} \tag{33}
$$

Summing over j from 1 to n at both sides of (32) and, noting that $\theta^0 = 0$, we have

$$
\begin{aligned}
\beta|||\theta^n|||_{1,h}^2 &\le C\Delta t\sum_{j=1}^{n}|||\xi^j|||_0^2 + C\Delta t\sum_{j=1}^{n}|||\theta^j|||_{1,h}^2 \\
&\quad + C\Delta t\sum_{j=1}^{n}\left(|||\rho^j|||_{1,h}^2 + h^{-2}\|\rho^j\|^2 + \|\rho^j\|^2\right).
\end{aligned}
$$

It follows from (10) and the Gronwall lemma that

$$
\begin{aligned}
&|||\theta^n|||_{1,h}^2 + \Delta t\beta\sum_{j=1}^{n}|||\theta^j|||_{1,h}^2 \\
&\quad\le Ch^4\int_0^{t^n}\|Q_t\|_3^2\,dt + C\Delta t^2\int_0^{t^n}\|Q_{tt}\|^2\,dt + Ch^2\left(\|Q^n\|_3^2 + \|Q^n\|_2^2\right).
\end{aligned}
$$

So

$$
|||\theta^n|||_{1,h} \le Ch\left(\|Q^n\|_3 + \|Q^n\|_2 + \int_0^{t^n}\|Q_t\|_3^2\,dt\right)^{\frac{1}{2}} + C\Delta t\left(\int_0^{t^n}\|Q_{tt}\|^2\,dt\right)^{\frac{1}{2}}. \tag{34}
$$

Finally, the conclusions are proved by (10), (34) and the triangular inequalities. □

Table 1 $t = 1$

Δt	h	L^2 error	L^2 error order	H^1 error	H^1 error order
1/2	1/2	0.2563		1.5192	
1/4	1/8	0.0683	1.9079	0.7697	0.9809
1/8	1/32	0.0180	1.9239	0.3795	1.0202

Table 2 $c = \frac{1}{40}$ $t = 1$

Δt	h	L^2 error	L^2 error order	H^1 error	H^1 error order
1/2	1/2	0.2139		1.5188	
1/4	1/8	0.0516	2.0515	0.7759	0.9690
1/8	1/32	0.0105	2.2970	0.3628	1.0967
1/16	1/128	0.0023	2.1907	0.1724	1.0734

5 Numerical experiments

5.1 Experiment 1

First consider the following questions:

$$\begin{cases} u_t - \nabla \cdot \nabla u = t \sin x \sin y, & (x,y,t) \in [0,\pi] \times [0,\pi] \times [0,1), \\ u(x,y,0) = 0, & (x,y) \in \Omega, \\ u(x,y,t) = 0, & (x,y) \in \partial\Omega. \end{cases} \tag{35}$$

where Ω is our solution area, $T = 1$, let h be the space step, t be the time step, and the numerical solution be u_h. The calculation results are shown in Table 1. We use Matlab to calculate the numerical solution.

5.2 Experiment 2

In order to verify the correctness of the theoretical analysis results, consider the following questions:

$$\begin{cases} u_t - \nabla \cdot (A(u)\nabla u)) + \nabla \cdot F(u) = f(x), & (x,y,t) \in [0,1] \times [0,1] \times [0,1), \\ u(x,y,0) = 0, & (x,y) \in \Omega, \\ u(x,y,t) = 0, & (x,y) \in \partial\Omega. \end{cases} \tag{36}$$

Take a true solution $u = t\sin(pix)\sin(piy)$, $F(u) = \mathbf{p} \cdot K(u)$, $\mathbf{p} = (0,1)$, $K(u) = cu^{10}$. Obviously, the true solution satisfies $0 \le u \le 1$, the solution interval is $\Omega = (0,1) \times (0,1)$, and the time interval is $[0,1]$. h is the space step, Δt is the time step and f is the source and sink item. We take $A(u)$, $K(u)$ with different values, calculate the numerical solution u_h, and give the corresponding L^2 mode error and H^1 mode error and the corresponding error order.

Take $A(u) = u + 1$, $K(u) = cu^{10}$, the results are shown in Tables 2–4.

When we take $A(u) = u + 1$, $K(u) = cu^{10}$, we find that: when the nonlinear property of the nonlinear term A(u)is not strong and the nonlinear property of the nonlinear term K(u) is strong, if the convection term is not dominant, as shown in Table 2, the error orders of L^2-mode and H^1-mode obtained are approximately equal to 2 and 1, respectively, consistent with the theoretical analysis; if the convection term is dominant, as shown in Tables 3 and 4, the discontinuous finite volume element method is far from the expected result.

Table 3 $c = 1$ $t = 1$

Δt	h	L^2 error	L^2 error order	H^1 error	H^1 error order
1/2	1/2	0.2144		1.5207	
1/4	1/8	0.0534	2.0054	0.7765	0.9697
1/8	1/32	0.0109	2.2925	0.3697	1.0706
1/16	1/128	0.0053	1.0403	0.1976	0.9038

Table 4 $c = 2$ $t = 1$

Δt	h	L^2 error	L^2 error order	H^1 error	H^1 error order
1/2	1/2	0.2156		1.5263	
1/4	1/8	0.0561	1.9423	0.7822	0.9644
1/8	1/32	0.0138	2.0233	0.3983	0.9737
1/16	1/128	0.0781		0.7738	

Table 5 $c = \frac{1}{40}$ $t = 1$

Δt	h	L^2 error	L^2 error order	H^1 error	H^1 error order
1/2	1/2	0.2508		1.5153	
1/4	1/8	0.0629	1.9954	0.7779	0.9619
1/8	1/32	0.0125	2.3311	0.3644	1.0941
1/16	1/128	0.0025	2.3219	0.1726	1.0781

Table 6 $c = 1$ $t = 1$

Δt	h	L^2 error	L^2 error order	H^1 error	H^1 error order
1/2	1/2	0.2512		1.5178	
1/4	1/8	0.0654	1.9415	0.7817	0.9573
1/8	1/32	0.0130	2.3308	0.3769	1.0524
1/16	1/128	0.0079	0.7186	0.2103	0.8417

Table 7 $c = 2$ $t = 1$

Δt	h	L^2 error	L^2 error order	H^1 error	H^1 error order
1/2	1/2	0.2526		1.5252	
1/4	1/8	0.0698	1.8556	0.7977	0.9351
1/8	1/32	0.0213	1.7124	0.4391	0.8613
1/16	1/128	0.9234		1.0149	

Take $A(u) = u^8 + 1$, $K(u) = c \cdot u^{10}$, the results are shown in Tables 5–7.

Take $A(u) = u^9 + 1$, $K(u) = c \cdot u^{10}$, the results are shown in Tables 8–10.

When we take $A(u) = u^8 + 1$ and $A(u) = u^9 + 1$, $K(u) = cu^{10}$, we find that: when the nonlinear property of the nonlinear term A(u) and K(u) is strong, if the convection term is not dominant, as shown in Tables 5 and 8, the error orders of the L^2-mode and H^1-mode obtained are approximately equal to 2 and 1, respectively, which are consistent with the theoretical analysis; if the convection term is dominant, from Tables 6 and 7 and 9 and 10, the discontinuous finite volume element method is far from the theoretical analysis.

6 Conclusion

In this paper, we mainly apply discontinuous finite volume element method to study two-dimensional unsaturated soil water movement. Firstly, we give semi-discrete discontinuous finite volume element scheme. Secondly, the convergence analysis is carried out on the basis of the fully discrete scheme. It is proved that the L^2-modulus estimation and

Table 8 $c = \frac{1}{40}$ $t = 1$

Δt	h	L^2 error	L^2 error order	H^1 error	H^1 error order
1/2	1/2	0.2509		1.5154	
1/4	1/8	0.0651	1.9464	0.7737	0.9699
1/8	1/32	0.0138	2.2380	1.3764	1.0395
1/16	1/128	0.0099	0.4792	0.2169	0.7952

Table 9 $c = 1$ $t = 1$

Δt	h	L^2 error	L^2 error order	H^1 error	H^1 error order
1/2	1/2	0.2514		1.5179	
1/4	1/8	0.0662	1.9251	0.7793	0.9618
1/8	1/32	0.0131	2.3372	0.3744	1.0576
1/16	1/128	0.0058	1.1754	0.2049	0.8697

Table 10 $c = 2$ $t = 1$

Δt	h	L^2 error	L^2 error order	H^1 error	H^1 error order
1/2	1/2	0.2527		1.5252	
1/4	1/8	0.0699	1.8541	0.7998	0.9313
1/8	1/32	0.0193	1.8567	0.3919	1.0292
1/16	1/128	0.0135		0.2266	0.7903

the H^1-modulus estimation of the scheme reach 2 and 1, respectively. Finally, the validity of the theoretical analysis is verified by numerical experiments. Through numerical experiments, it is found that: when the convection term of the nonlinear problem is not dominant, the discontinuous finite volume element method can be considered to deal with such problems.

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