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Boundary linear stabilization of the modified generalized Korteweg–de Vries–Burgers equation

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Abstract

The linear stabilization problem of the modified generalized Korteweg–de Vries–Burgers equation (MGKdVB) is considered when the spatial variable lies in [0, 1]. First, the existence and uniqueness of global solutions are proved. Next, the exponential stability of the equation is established in $L^2(0,1)$. Then, a linear adaptive boundary control is put forward. Finally, numerical simulations for both non-adaptive and adaptive problems are provided to illustrate the analytical outcomes.

Keywords: Modified generalized Korteweg–de Vries–Burgers equation; Well-posedness; Exponential stability; Boundary control

1 Introduction

This paper deals with the well-posedness and exponential stability of the modified generalized Korteweg–de Vries–Burgers (MGKdVB) equation

$$u_t + \gamma_1 u^\alpha u_x - \sigma u_{xx} + \mu u_{xxx} + \gamma_2 u_{xxxx} = 0, \quad x \in (0,1), t > 0, \tag{1.1}$$

with the following boundary and initial conditions:

$$\begin{cases} u(0,t) = u(1,t) = u_{xx}(0,t) = 0, & t > 0, \\ u_{xx}(1,t) = \mathcal{F}(t), & t > 0, \\ u(x,0) = u_0(x), & x \in (0,1), \end{cases} \tag{1.2}$$

where γ_1, γ_2, σ, and μ are positive physical parameters, whereas α is a positive integer. Furthermore, $\mathcal{F}(t)$ is the linear boundary control to be proposed so that the solutions of the system exponentially decay.

The MGKdVB equation has been extensively studied in literature but in some very special cases of the physical parameters. Indeed, when $\alpha = \gamma_1 = 1$, $\mu = 0$, and $\sigma < 0$ in (1.1), the MGKdVB equation becomes the Kuramoto–Sivashinsky (KS) equation which was derived by Sivashinsky [33] and Kuramoto [23] with the purpose of describing the thermo-diffusive instability in flame fronts. Due to its crucial physical aspects, numerous research investigations were devoted to studying this equation. For instance, many scientists have considered the existence and uniqueness of global solutions to the Cauchy problem related to this equation [6, 18, 19, 24, 40]. Moreover, He et al. [20] considered the KS equation when the boundary conditions are periodic and proved the stability of this equation. In [14], the authors proved that the mixed problem for the KS equation is globally well-posed in a bounded domain with moving boundaries. They also proved the exponential decay of the solutions with L^2-norm. In [24], the author considered the generalized Kuramoto-Sivashinsky equation (i.e., when $\alpha = \gamma_1 = 1$, $\mu > 0$, and $\gamma_2 = -\sigma > 0$ in (1.1)). He proved the well-posedness of the problem and the exponential decay of the solution provided that σ and the norm of the initial conditions are sufficiently small.

In case $\sigma = 0$, $\gamma_2 = 0$, $\gamma_1 = 1$, and $\alpha = 1$, (1.1) reduces to the well-known Korteweg and de Vries (KdV) equation which was derived by Korteweg and de Vries to describe the dynamics of water waves [29]. The KdV equation was also used to describe many phenomena such as modeling waves in a rotating atmosphere or ion-acoustic waves in plasma [15]. Many scientists paid a lot of attention to the well-posedness of this equation [9, 11, 21]. In fact, Bona et al. [9] studied the global well-posedness of the KdV equation in appropriate functional spaces, and Hublov [21] studied the solvability of the KdV equation with dissipation in a bounded domain. Also, the well-posedness of the mixed problem for the KdV equation when $x > 0$ and $t \in (0, t)$ was discussed by Bui [11]. The main idea of the proof is based on the approximation of the KdV equation by the KS type equations.

On the other hand, setting $\alpha = \gamma_1 = 1$ and $\mu = \gamma_2 = 0$ in (1.1), the MGKdVB equation becomes Burgers equation. This equation was first derived by Burgers [12] in 1948 as a prototype model for turbulent liquid flow. Due to its importance in describing many real life phenomena, many scientists have studied this equation [3–5, 16, 22, 32, 34–36].

Furthermore, when $\alpha = \gamma_1 = 1$ and $\gamma_2 = 0$, the MGKdVB equation reduces to the Korteweg–de Vries–Burgers (KdVB) equation. Many researchers used the KdVB equation to describe several phenomena [2, 7, 8].

In turn, if $\gamma_1 = 1$ and $\gamma_2 = 0$ in (1.1), the MGKdVB equation becomes the generalized Korteweg–de Vries–Burgers (GKdVB) equation. This equation is useful in modeling many physical phenomena such as the unidirectional propagation of planar waves [1]. It also models longitudinal deformations in nonlinear elastic rod [26]. The non-adaptive and adaptive control problems of this equation were studied in [37] and [38, 39].

In this article, we suggest the following linear boundary control:

$$\mathcal{F}(t) = -\frac{\mu}{\gamma_2} u_x(1, t), \quad t > 0. \tag{1.3}$$

Up to our knowledge, the well-posedness and stability of the of modified generalized Korteweg–de Vries–Burgers (MGKdVB) equation (1.1) have not been discussed in the literature. Thus, the principal objective of this paper is two-fold: First, to investigate the problem of existence and uniqueness of solutions to (1.1)–(1.3) as well as their long-time behavior. The main ingredient of the proof is the utilization of Faedo–Galerkin method. It is worth mentioning that this method has been used for many nonlinear equations (see [14, 17, 24, 27, 30, 31]). Secondly, the linear adaptive boundary control law of the dynamics of equation (1.1) is presented.

The remainder of the paper is organized as follows: In Sect. 2, notations and preliminaries are presented. Some a priori estimates will be proved in order to guarantee the existence and uniqueness of solutions in Sect. 3. Then, Sect. 4 is devoted to showing the exponential stability of the MGKdVB equation subject to the non-adaptive linear boundary control (1.3). Section 5 presents a linear adaptive boundary control law for the MGKdVB equation when both of γ_2 and μ are unknown. Section 6 shows the numerical simulations that illustrate the theoretical results. Lastly, concluding remarks are given in Sect. 7.

2 Notations and preliminaries

In this section, we provide notations to be used throughout the paper.

Let $L^2(0,1)$ be the Hilbert space endowed with its usual inner product. In our case,

$$\langle u,v\rangle(t)=\int_0^1 u(x,t)v(x,t)\,dx, \quad \text{and} \quad \|u(t)\|^2=\langle u,u\rangle(t).$$

In turn, $\|u\|^2=\|u\|^2_{L^2(Q)}$, where $Q=(0,1)\times(0,t)$. For convenience, we shall often denote by ∂_x^n the differential operator $\frac{\partial^n}{\partial x^n}$. Moreover, consider the Sobolev space (see [25] for more details)

$$H^n(0,1)=\{u:(0,1)\to\mathbb{R};\partial_x^n u\in L^2(0,1),\text{for } n\in\mathbb{N}\}$$

equipped with the usual norm

$$\|u\|_{H^n(0,1)}=\sum_{i=0}^{i=n}\|\partial_x^i u\|_{L^2(0,1)}.$$

Thereafter, as in Larkin [24], we have the following result.

Lemma 1 *Let* $\mu,\gamma_2>0$ *and consider the following eigenvalue problem*:

$$\begin{aligned}&\gamma_2\partial_x^4\varphi_j=\mu_j\varphi_j,\\&\varphi_j(0)=\varphi_j(1)=\varphi_{jxx}(0)=\varphi_{jxx}(1)+\frac{\mu}{\gamma_2}\varphi_{jx}(1)=0,\end{aligned}$$

where $\mu_j\in\mathbb{C}$ *is the eigenvalue and* $\varphi_j\in H^4(0,1)$ *is the corresponding eigenfunction for each* $j\in\mathbb{N}$. *Then* $\mu_j>0$ *for each* $j\in\mathbb{N}$ *and the eigenfunctions* $\{\varphi_j\}_j\in\mathbb{N}$ *form a basis in* $H^4(0,1)$. *Moreover,* $\{\varphi_j\}$ *is an orthonormal set in* $L^2(0,1)$.

Proof Consider the operator ∂_x^4 in $H^4(0,1)$ such that the boundary conditions of Lemma 1 hold. Then, let $u,\varphi\in H^4(0,1)$ that satisfy the boundary conditions of Lemma 1. Integrating by parts, we have

$$\langle\partial_x^4 u,\varphi\rangle=\int_0^1 u_{xxxx}\varphi\,dx=[u_{xxx}\varphi-u_{xx}\varphi_x+u_x\varphi_{xx}-u\varphi_{xxx}]_0^1+\int_0^1 u\varphi_{xxxx}\,dx=\langle u,\partial_x^4\varphi\rangle.$$

Hence, the above operator is self-adjoint. We also have

$$\gamma_2\langle\partial_x^4 u,u\rangle=\mu u_x^2(1)+\gamma_2\|\partial_x^2 u\|^2.$$

Whereupon, the operator is also positive, and the claims of the lemma are a direct consequence of the results in p. 78 of [28]. □

3 Existence and uniqueness of solutions to system (1.1)–(1.3)

The ultimate outcome of this section is to show the existence of solutions to (1.1)–(1.3). To do so, we will use the so-called Faedo–Galerkin method and some a priori estimates (see [24] for a similar situation).

Recall that α is a positive integer, whereas μ, σ, and γ_2 are positive real parameters. Then we have the following result.

Theorem 1 *Let $u_0 \in H^4(0,1) \cap H_0^1(0,1)$ such that*

$$u_{0xx}(0) = u_{0xx}(1) + \frac{\mu}{\gamma_2} u_{0x}(1) = 0.$$

Then there exists a unique solution to (1.1)–(1.3) satisfying, for any $T > 0$,

$$\begin{aligned}
&u \in L^\infty\bigl(0, T; H_0^1(0,1) \cap H^4(0,1)\bigr) \cap C\bigl(0, T; H_0^1(0,1) \cap H^2(0,1)\bigr), \\
&u_t \in L^2\bigl(0, T; H_0^1(0,1) \cap H^4(0,1)\bigr) \cap L^\infty\bigl(0, T; L^2(0,1)\bigr).
\end{aligned}$$

Proof First of all, let

$$\mathcal{P}u := u_t + \gamma_1 u^\alpha u_x - \sigma u_{xx} + \mu u_{xxx} + \gamma_2 u_{xxxx} = 0, \tag{3.1}$$

and define as in [14, 24] approximate solutions u^k to (1.1)–(1.3)

$$u^k(x,t) = \sum_{j=1}^{k} f_j^k(t) \varphi_j(x),$$

where $\varphi_j(x)$ are given in Lemma 1 and $f_j^k(t)$ are solutions to the initial-value problem, which consists of the system of k ordinary differential equations:

$$\begin{aligned}
\bigl\langle \mathcal{P}u^k, \varphi_j \bigr\rangle(t) = {}& \bigl\langle u_t^k, \varphi_j \bigr\rangle(t) + \gamma_1 \bigl\langle \bigl(u^k\bigr)^\alpha u_x^k, \varphi_j \bigr\rangle(t) + \mu \bigl\langle \partial_x^3 u^k, \varphi_j \bigr\rangle(t) - \sigma \bigl\langle u_{xx}^k, \varphi_j \bigr\rangle(t) \\
& + \gamma_2 \bigl\langle \partial_x^4 u^k, \varphi_j \bigr\rangle(t) = 0, \qquad (3.2) \\
f_j^k(0) = {}& \langle u_0, \varphi_j \rangle, \quad j = 1, \ldots, k. \qquad (3.3)
\end{aligned}$$

One can view the system (3.2)–(3.3) as a system of k nonlinear ordinary differential equations satisfying the existence result stated in Sect. 1.5 of [13], and hence there exist k differentiable solutions $f_j^k(t)$, for $j = 1, 2, \ldots, k$, of the system (3.2)–(3.3) on an interval $(0, T^k)$ for some $T^k > 0$ [13]. It remains to extend those solutions to any interval $(0, T)$ and pass to the limit as $k \to +\infty$. To do so, a number of a priori estimates must be established.

A priori Estimate 1:

Let us first replace φ_j by $2u^k$ in (3.2), which is possible since $u^k \in H^4(0,1)$ is a solution of (1.1)–(1.3). Then, integrating by parts and using boundary conditions (1.2)–(1.3), it follows that

$$\frac{d}{dt}\left\|u^k(t)\right\|^2 + \mu\left(u_x^k\right)^2(0,t) + \mu\left(u_x^k\right)^2(1,t) + 2\gamma_2\left\|u_{xx}^k(t)\right\|^2 + 2\sigma\left\|u_x^k(t)\right\|^2 = 0. \tag{3.4}$$

Clearly, (3.4) yields

$$\frac{d}{dt}\left\|u^k(t)\right\|^2 + \mu\left(u_x^k\right)^2(0,t) + \gamma_2\left\|u_{xx}^k(t)\right\|^2 \le 0,$$

and hence

$$\left\|u^k(t)\right\|^2 + \mu\int_0^t \left(u_x^k\right)^2(0,s)\,ds + \gamma_2\int_0^t \left\|u_{xx}^k(s)\right\|^2\,ds \le C\|u_0\|^2. \tag{3.5}$$

Here and in the sequel, we shall use for convenience the same letter C to represent a positive constant which is independent of t and k.

A priori Estimate 2:

Substituting φ_j by $\partial_x^4 u^k$ in (3.2) as $u^k \in H^4(0,1)$ is a solution of (1.1)–(1.3), we obtain

$$\sum_{i=1}^{i=5} I_i, \tag{3.6}$$

where

$$I_1 = \langle u_t^k, \partial_x^4 u^k\rangle(t), \qquad I_2 = \mu\langle \partial_x^3 u^k, \partial_x^4 u^k\rangle(t), \qquad I_3 = \gamma_1\langle \left(u^k\right)^\alpha u_x^k, \partial_x^4 u^k\rangle(t),$$
$$I_4 = -\sigma\langle \partial_x^2 u^k, \partial_x^4 u^k\rangle(t), \quad \text{and} \quad I_5 = \gamma_2\langle \partial_x^4 u^k, \partial_x^4 u^k\rangle(t) = \gamma_2\left\|\partial_x^4 u^k(t)\right\|^2.$$

The immediate task is to estimate I_i for $i = 1, \dots, 4$.

Integrating by parts and owing the boundary conditions in (1.1), we obtain

$$I_1 = \langle u_t^k, \partial_x^4 u^k\rangle(t) = -u_{tx}^k(1,t)u_{xx}^k(1,t) + \frac{1}{2}\int_0^1 \frac{\partial}{\partial t}\left(u_{xx}^k\right)^2\,dx.$$

Since $-\frac{\gamma_2}{\mu}u_{xx}^k(1,t) = u_x^k(1,t)$ (see (1.3)), we get

$$I_1 = \frac{\gamma_2}{\mu}u_{txx}^k(1,t)u_{xx}^k(1,t) + \frac{1}{2}\frac{d}{dt}\left\|u_{xx}^k\right\|^2.$$

Therefore, I_1 simplifies to

$$I_1 = \frac{1}{2}\frac{d}{dt}\left(\frac{\gamma_2}{\mu}\left[\partial_x^2 u^k(1,t)\right]^2 + \left\|\partial_x^2 u^k\right\|^2\right).$$

Concerning I_2, we have

$$I_2 = \mu\langle \partial_x^3 u^k, \partial_x^4 u^k\rangle(t) = \int_0^1 \frac{2\mu}{\sqrt{\gamma_2}}u_{xxx}^k \frac{\sqrt{\gamma_2}}{2}u_{xxxx}^k\,dx.$$

By using the following version of Young's inequality:

$$ab \geq \frac{-a^2}{4} - b^2, \tag{3.7}$$

with $a = \frac{\sqrt{\gamma_2}}{2} u^k_{xxxx}$ and $b = \frac{2\mu}{\sqrt{\gamma_2}} u^k_{xxx}$, we can estimate I_2 as follows:

$$I_2 \geq -\frac{\gamma_2}{16} \big\| \partial_x^4 u^k(t) \big\|^2 - \frac{4\mu^2}{\gamma_2} \big\| \partial_x^3 u^k(t) \big\|^2.$$

Exploiting the interpolation inequality (see, for instance, p. 233 in [10]) on the last term of the previous inequality, we get

$$I_2 \geq -\frac{\gamma_2}{16} \big\| \partial_x^4 u^k(t) \big\|^2 - \frac{4\mu^2}{\gamma_2} \big(\epsilon \big\| \partial_x^4 u^k(t) \big\|^2 + C(\epsilon) \big\| u^k(t) \big\|^2 \big),$$

in which ϵ is an arbitrary positive number. Thereby, choosing $\epsilon = \frac{\gamma_2^2}{64\mu^2}$ yields

$$I_2 \geq -\frac{\gamma_2}{8} \big\| \partial_x^4 u^k(t) \big\|^2 - C \big\| u^k(t) \big\|^2.$$

Next, using Cauchy–Schwarz's and Poincaré's inequalities, it follows from $I_3 = \gamma_1 \langle (u^k)^\alpha u^k_x, \partial_x^4 u^k \rangle(t)$ that

$$I_3 \geq -\gamma_1 \big\| u^k_x \big\|^2 \big\| u^k \big\|^{\alpha-1} \big\| \partial_x^4 u^k(t) \big\|.$$

Invoking the interpolation inequality $\|u^k_x\|^2 \leq K \|u^k\| \|u^k_{xx}\|$, where K is a positive constant [10], we get

$$I_3 \geq \left(K \frac{-\gamma_1 \sqrt{2}}{\sqrt{\gamma_2}} \big\| u^k \big\|^\alpha \big\| u^k_{xx} \big\| \right) \frac{\sqrt{\gamma_2}}{\sqrt{2}} \big\| \partial_x^4 u^k(t) \big\|.$$

Then, using once again Young's inequality (3.7) and (3.5), we deduce that

$$I_3 \geq -\frac{\gamma_2}{8} \big\| \partial_x^4 u^k(t) \big\|^2 - C \big\| \partial_x^2 u^k(t) \big\|^2$$

for $C > 0$.

Analogously, arguing as before, the term $I_4 = -\sigma \langle \partial_x^2 u^k, \partial_x^4 u^k \rangle(t)$ gives

$$I_4 \geq -\frac{\gamma_2}{8} \big\| \partial_x^4 u^k(t) \big\|^2 - \frac{2\sigma^2}{\gamma_2} \big\| \partial_x^2 u^k(t) \big\|^2.$$

Now substituting I_1–I_5 into (3.6), we obtain

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \left(\frac{\gamma_2}{\mu} \big[\partial_x^2 u^k(1,t) \big]^2 + \big\| \partial_x^2 u^k \big\|^2 \right) + \frac{5\gamma_2}{8} \big\| \partial_x^4 u^k(t) \big\|^2 \\ &\quad \leq \left(C + \frac{5\gamma_2}{8} \right) \big\| \partial_x^2 u^k(t) \big\|^2 + C \big\| u^k(t) \big\|^2. \end{aligned}$$

Taking into account Estimate 1 (see (3.5)) and exploiting Gronwall–Bellman's inequality, the last inequality gives

$$\begin{aligned} &\left\|\partial_x^2 u^k(t)\right\|^2+\frac{\gamma_2}{\mu}\left|\partial_x^2 u^k(1,t)\right|^2+\frac{5\gamma_2}{4}\int_0^t\left\|\partial_x^4 u^k(s)\right\|^2 ds \\ &\quad\le C\left[\frac{\gamma_2}{\mu}\left|u_{0xx}(1)\right|^2+\|u_0\|_{H^2(0,1)}^2\right], \end{aligned} \tag{3.8}$$

where $C>0$.

A priori Estimate 3:

As in [24], putting $t=0$ and $\varphi_j=u_t^k(0)$ in (3.2) since $u^k\in H^4(0,1)$ is a solution of (1.1)–(1.3), we obtain

$$\left\|u_t^k(0)\right\|^2+\left\langle\mu u_{xxx}^k+\gamma_2 u_{xxxx}^k-\sigma u_{xx}^k+\gamma_1\left(u^k\right)^\alpha u_x^k,u_t^k\right\rangle(0)=0,$$

which together with the triangle inequality and Estimate 1 (see (3.5)) imply that

$$\left\|u_t^k(0)\right\|\le C\|u_0\|_{H^4(0,1)\cap H_0^1(0,1)}. \tag{3.9}$$

On the other hand, using (3.5) and (3.8), we get

$$\max_{t>0}\left\|u^k(t)\right\|_{H^2(0,1)}\le C. \tag{3.10}$$

Now, let us differentiate (3.2) with respect to t. We obtain

$$\begin{aligned} &\left\langle u_{tt}^k,\varphi_j\right\rangle(t)+\left\langle u_t^k,\varphi_{jt}\right\rangle(t)+\left\langle\left(\gamma_1\left(u^k\right)^\alpha u_x^k\right)_t,\varphi_j\right\rangle(t)+\left\langle\gamma_1\left(u^k\right)^\alpha u_x^k,\varphi_{jt}\right\rangle(t) \\ &\quad+\left\langle\mu\partial_x^3 u_t^k,\varphi_j\right\rangle(t)+\left\langle\mu\partial_x^3 u^k,\varphi_{jt}\right\rangle(t)-\sigma\left\langle u_{xxt}^k,\varphi_j\right\rangle(t) \\ &\quad-\sigma\left\langle u_{xx}^k,\varphi_{jt}\right\rangle(t)+\gamma_2\left\langle\partial_x^4 u_t^k,\varphi_j\right\rangle(t)+\gamma_2\left\langle\partial_x^4 u^k,\varphi_{jt}\right\rangle(t)=0. \end{aligned}$$

Substituting for φ_j (which depends only on x) by $2u_t^k$, we get

$$\begin{aligned} &\left\langle u_{tt}^k,2u_t^k\right\rangle(t)+\gamma_1\left\langle\left(\left(u^k\right)^\alpha u_x^k\right)_t,2u_t^k\right\rangle(t)+\mu\left\langle\partial_x^3 u_t^k,2u_t^k\right\rangle(t)-\sigma\left\langle u_{xxt}^k,2u_t^k\right\rangle(t) \\ &\quad+\gamma_2\left\langle\partial_x^4 u_t^k,2u_t^k\right\rangle(t)=0. \end{aligned} \tag{3.11}$$

The first term in (3.11) can be written as

$$\left\langle u_{tt}^k,2u_t^k\right\rangle(t)=\frac{d}{dt}\left\|u_t^k(t)\right\|^2.$$

With regards to the second term in (3.11), we simply integrate by parts and use boundary conditions (1.2) to conclude

$$\begin{aligned} \gamma_1\left\langle\left(\left(u^k\right)^\alpha u_x^k\right)_t,2u_t^k\right\rangle(t)&=2\gamma_1\int_0^1\left(\left(u^k\right)^\alpha\right)_x\left(u_t^k\right)^2 dx+2\gamma_1\int_0^1\left(u^k\right)^\alpha u_{xt}^k u_t^k\,dx \\ &=-2\gamma_1\left\langle\left(u^k\right)^\alpha u_t^k,u_{xt}^k\right\rangle(t). \end{aligned}$$

Similarly, the last three terms in (3.11) can be expanded as follows:

$$\mu\langle\partial_x^3 u_t^k, 2u_t^k\rangle(t) = \mu\left(u_{xt}^k\right)^2(0,t) - \mu\left(u_{xt}^k\right)^2(1,t),$$
$$-\sigma\langle u_{xxt}^k, 2u_t^k\rangle(t) = 2\sigma\left\|u_{xt}^k(t)\right\|^2,$$
$$\gamma_2\langle\partial_x^4 u_t^k, 2u_t^k\rangle(t) = -\mu\left(u_{xt}^k\right)^2(1,t) + 2\gamma_2\left\|u_{xxt}^k(t)\right\|^2,$$

where we took into account the fact that boundary conditions (1.2)–(1.3) are invariant with respect to time-differentiation. Inserting the previous identities in (3.11) yields

$$\frac{d}{dt}\left\|u_t^k(t)\right\|^2 - 2\gamma_1\langle\left(u^k\right)^\alpha u_t^k, u_{xt}^k\rangle(t) + \mu\left(u_{xt}^k\right)^2(0,t) + \mu\left(u_{xt}^k\right)^2(1,t) + 2\sigma\left\|u_{xt}^k(t)\right\|^2 + 2\gamma_2\left\|u_{xxt}^k(t)\right\|^2 = 0. \tag{3.12}$$

Consequently,

$$\frac{d}{dt}\left\|u_t^k(t)\right\|^2 + \gamma_2\left\|u_{xxt}^k(t)\right\|^2 \le \left\|u_t^k(t)\right\|^2 + 2\gamma_1\langle\left(u^k\right)^\alpha u_t^k, u_{xt}^k\rangle(t). \tag{3.13}$$

A straightforward computation permits to write the last term in (3.13) as follows:

$$2\gamma_1\langle\left(u^k\right)^\alpha u_t^k, u_{xt}^k\rangle(t) = -\gamma_1\langle\left(\left(u^k\right)^\alpha\right)_x, \left(u_t^k\right)^2\rangle(t).$$

This together with Cauchy–Schwarz inequality and estimate (3.10) implies that

$$2\gamma_1\langle\left(u^k\right)^\alpha u_t^k, u_{xt}^k\rangle(t) \le C\left\|u_t^k(t)\right\|^2,$$

where $C > 0$ is independent of t and N. Whereupon, (3.13) gives

$$\frac{d}{dt}\left\|u_t^k(t)\right\|^2 + \gamma_2\left\|u_{xxt}^k(t)\right\|^2 \le C\left\|u_t^k(t)\right\|^2, \tag{3.14}$$

where $C > 0$.

Integrating (3.14) from 0 to t and using Gronwall–Bellman's inequality, we get

$$\left\|u_t^k(t)\right\|^2 + \gamma_2\int_0^t\left\|u_{xxs}^k(s)\right\|^2\,ds \le C\left\|u_t^k(0)\right\|^2,$$

where $C > 0$.

Referring to (3.9), the previous inequality can be written as

$$\left\|u_t^k(t)\right\|^2 + \gamma_2\int_0^t\left\|u_{xxs}^k(s)\right\|^2\,ds \le C\left\|u_t^k(0)\right\| \le C\|u_0\|_{H^4(0,1)\cap H_0^1(0,1)}, \tag{3.15}$$

where $C > 0$ does not depend on t and k.

Estimates (3.5), (3.8), (3.15) allow us to extend the approximate solution $u^k(x,t)$ for all $T \in (0,\infty)$ and also $u(x,t)$ converges as $N \to \infty$. Hence, taking the limit in (3.2), we conclude the existence of a global solution u that belongs to the classes stated in our theorem.

The immediate task is to show the uniqueness of solutions. To this end, assume that u_1 and u_2 are two solutions of system (1.1)–(1.3) and let $w = u_1 - u_2$. Then w is a solution of the following problem:

$$w_t + \gamma_1\big(u_1^\alpha u_{1x} - u_2^\alpha u_{2x}\big) - \sigma\partial_x^2 w + \mu\partial_x^3 w + \gamma_2\partial_x^4 w = 0, \tag{3.16}$$

$$w(x,0) = 0, \quad x \in (0,1), \tag{3.17}$$

$$w(1,t) = w(0,t) = \partial_x^2 w(0,t) = 0, \tag{3.18}$$

$$\partial_x^2 w(1,t) = -\frac{\mu}{\gamma_2} w_x(1,t), \quad t > 0. \tag{3.19}$$

Taking the inner product of (3.16) with $2w$ in $L^2(0,1)$, we get

$$\begin{aligned}&\langle w_t, 2w\rangle(t) + \gamma_1\big\langle u_1^\alpha u_{1x} - u_2^\alpha u_{2x}, 2w\big\rangle(t) - \sigma\big\langle\partial_x^2 w, 2w\big\rangle(t) + \mu\big\langle\partial_x^3 w, 2w\big\rangle(t)\\&\quad + \gamma_2\big\langle\partial_x^4 w, 2w\big\rangle(t) = 0.\end{aligned} \tag{3.20}$$

Integrating by parts and utilizing the boundary conditions (3.18)–(3.19), it follows that

$$\begin{aligned}&\frac{d}{dt}\big\|w(t)\big\|^2 + \gamma_1\big\langle u_1^\alpha u_{1x} - u_2^\alpha u_{2x}, 2w\big\rangle(t) + \mu(w_x)^2(0,t) + \mu(w_x)^2(1,t)\\&\quad + 2\gamma_2\big\|w_{xx}(t)\big\|^2 + 2\sigma\big\|w_x(t)\big\|^2 = 0,\end{aligned} \tag{3.21}$$

which gives

$$\frac{d}{dt}\big\|w(t)\big\|^2 + 2\sigma\big\|w_x(t)\big\|^2 \le -\gamma_1\big\langle u_1^\alpha u_{1x} - u_2^\alpha u_{2x}, 2w\big\rangle(t). \tag{3.22}$$

To estimate the right-hand side of inequality (3.22), we proceed as follows:

$$\begin{aligned}\gamma_1\big\langle u_1^\alpha u_{1x} - u_2^\alpha u_{2x}, 2w\big\rangle(t) &= \frac{2\gamma_1}{\alpha+1}\big\langle\big[u_1^{\alpha+1} - u_2^{\alpha+1}\big]_x, w\big\rangle(t)\\&= \frac{2\gamma_1}{\alpha+1}\big\langle\big[u_1^{\alpha+1} - u_2^{\alpha+1}\big], w_x\big\rangle(t)\\&= \frac{2\gamma_1}{\alpha+1}\big\langle w\big(u_1^\alpha + u_1^{\alpha-1}u_2 + \cdots + u_1u_2^{\alpha-1} + u_2^\alpha\big), w_x\big\rangle(t).\end{aligned} \tag{3.23}$$

Utilizing Cauchy–Schwarz's inequality, we have

$$\begin{aligned}\big|\gamma_1\big\langle u_1^\alpha u_{1x} - u_2^\alpha u_{2x}, 2w\big\rangle(t)\big| &\le \frac{2\gamma_1}{\alpha+1}\big\|w(t)\big\|K\big\|w_x(t)\big\|\\&\le \frac{\gamma_1 K}{\alpha+1}2\bigg(\frac{\|w(t)\|}{\sqrt{\delta}}\sqrt{\delta}\big\|w_x(t)\big\|\bigg),\end{aligned} \tag{3.24}$$

where $K = \|u_1\|_\infty^\alpha + \|u_1\|_\infty^{\alpha-1}\|u_2\|_\infty + \cdots + \|u_1\|_\infty\|u_2\|_\infty^{\alpha-1} + \|u_2\|_\infty^\alpha$ and $\|f\|_\infty = \|f\|_{C([0,T];L^2(0,1))}$. Utilizing Young's inequality, (3.24) yields

$$\big|\gamma_1\big\langle u_1^\alpha u_{1x} - u_2^\alpha u_{2x}, 2w\big\rangle(t)\big| \le \frac{\gamma_1 K}{\alpha+1}\bigg(\frac{\|w(t)\|^2}{\delta} + \delta\big\|w_x(t)\big\|^2\bigg). \tag{3.25}$$

Inserting (3.25) in (3.22), we obtain

$$\frac{d}{dt}\|w(x,t)\|^2 \le \frac{\gamma_1 K}{(\alpha+1)\delta}\|w(t)\|^2 + \left(\frac{\gamma_1 K}{\alpha+1}\delta - 2\sigma\right)\|w_x(t)\|^2. \tag{3.26}$$

Choosing $\delta = \frac{2\sigma(\alpha+1)}{\gamma_1 K}$, we get

$$\frac{d}{dt}\|w(x,t)\|^2 \le \frac{(\gamma_1 K)^2}{2\sigma(\alpha+1)^2}\|w(t)\|^2. \tag{3.27}$$

Lastly, using Gronwall–Bellman's inequality and noting that $w(0) = 0$, we conclude that $w = u_1 - u_2 = 0$, and thus $u_1 = u_2$. This proves the desired uniqueness result. □

4 Stability of system (1.1)–(1.3)

This section is concerned with the long-time behavior of solutions to system (1.1)–(1.3). Indeed, we have the following result.

Theorem 2 *Assume that α is a positive integer, while μ, σ, and γ_2 are positive real parameters. Given an initial condition $u_0 \in H^4(0,1) \cap H_0^1(0,1)$ and subject to boundary conditions (1.2)–(1.3), the corresponding solution of MGKdVB equation (1.1) is globally exponentially stable in $L^2(0,1)$.*

Proof Taking the inner product in $L^2(0,1)$ of (1.1) with $2u$, we get

$$\begin{aligned} &2\int_0^1 u(x,t)u_t(x,t)\,dx - 2\sigma\int_0^1 u(x,t)u_{xx}(x,t)\,dx + 2\mu\int_0^1 u(x,t)u_{xxx}(x,t)\,dx \\ &\quad + 2\gamma_1\int_0^1 u(x,t)u^\alpha(x,t)u_x(x,t)\,dx + 2\gamma_2\int_0^1 u(x,t)u_{xxxx}(x,t)\,dx = 0. \end{aligned} \tag{4.1}$$

Integrating by parts each term in (4.1) and using boundary conditions (1.2)–(1.3), we get

$$\frac{d}{dt}\|u(t)\|^2 + \mu u_x^2(0,t) + \mu u_x^2(1,t) + 2\gamma_2\|u_{xx}(t)\|^2 + 2\sigma\|u_x(t)\|^2 = 0. \tag{4.2}$$

Thereby

$$\frac{d}{dt}\|u(t)\|^2 + 2\sigma\|u_x(t)\|^2 \le 0,$$

which by means of Poincaré's inequality leads to

$$\frac{d}{dt}\|u(x,t)\|^2 \le -2\sigma\|u(x,t)\|^2.$$

Integrating the latter, we obtain

$$\|u(x,t)\| \le e^{-\sigma t}\|u_0(x)\|, \tag{4.3}$$

and hence the solution of (1.1)–(1.3) is exponentially stable. □

5 Linear adaptive boundary control law for MGKdVB equation (1.1)

In this section, we present a linear adaptive boundary control law for MGKdVB equation (1.1), subject to the same boundary conditions as in (1.2), that is,

$$u(1,t) = u(0,t) = u_{xx}(0,t) = 0, \tag{5.1}$$

$$u_{xx}(1,t) = \mathcal{M}(t), \tag{5.2}$$

where $\mathcal{M}(t)$ is a linear adaptive boundary control to be proposed when the parameters μ and γ_2 are unknowns. The following theorem states the findings of our linear adaptive boundary control law.

Theorem 3 *Consider α to be a positive integer. Given an initial condition $u_0 \in H^4(0,1) \cap H_0^1(0,1)$, the modified generalized Korteweg–de Vries–Burgers (MGKdVB) equation (1.1) subject to boundary conditions (5.1)–(5.2) is globally exponentially stable in $L^2(0,1)$ as long as the following linear adaptive control law is applied*:

$$\mathcal{M}(t) = -\eta_1 u_x(1,t), \tag{5.3}$$

where

$$\dot{\eta}_1 = r_1 u_x^2(1,t), \tag{5.4}$$

in which the feedback gain r_1 is a positive real number.

Proof Let

$$V(t) = \frac{1}{2}\int_0^1 u^2(x,t)\,dx \tag{5.5}$$

be the Lyapunov function candidate. Clearly, $V(t) \geq 0$ for all $t \geq 0$. Moreover, differentiating $V(t)$ with respect to time, and referring to (1.1), we have

$$\dot{V}(t) = \sigma\int_0^1 uu_{xx}\,dx - \mu\int_0^1 uu_{xxx}\,dx - \gamma_1\int_0^1 u^{\alpha+1}u_x\,dx - \gamma_2\int_0^1 uu_{xxxx}\,dx. \tag{5.6}$$

Integrating by parts and using the boundary conditions (5.1)–(5.2), we obtain from (5.6)

$$\dot{V}(t) \leq -\sigma\int_0^1 u_x^2(x,t)\,dx + \frac{\mu}{2}u_x^2(1,t) + \gamma_2 u_x(1,t)\mathcal{M}(t). \tag{5.7}$$

Inserting the control law (5.3) in (5.7), we get

$$\dot{V}(t) \leq -\sigma\int_0^1 u_x^2(x,t)\,dx + \frac{\mu}{2}u_x^2(1,t) - \gamma_2\eta_1 u_x^2(1,t). \tag{5.8}$$

Using Poincaré's inequality, (5.8) becomes

$$\dot{V}(t) \leq -\sigma\int_0^1 u^2(x,t)\,dx - \left(\gamma_2\eta_1 - \frac{\mu}{2}\right)u_x^2(1,t) \quad \text{for any } t \geq 0. \tag{5.9}$$

Now, we define an energy $E(t)$ as follows:

$$E(t) = \frac{1}{2\gamma_2 r_1}\left(\gamma_2\eta_1 - \frac{\mu}{2} - a\right)^2 + V(t), \quad \text{where } a \geq 0. \tag{5.10}$$

Obviously, $E(t) \geq 0$ for all $t \geq 0$. Differentiating $E(t)$ with respect to time yields

$$\dot{E}(t) = \dot{V}(t) + \frac{\dot{\eta}_1}{r_1}\left(\gamma_2\eta_1 - \frac{\mu}{2} - a\right). \tag{5.11}$$

Using (5.4) and (5.9), identity (5.11) leads to

$$\dot{E}(t) \leq -\sigma\int_0^1 u^2(x,t)\,dx - \left(\gamma_2\eta_1 - \frac{\mu}{2}\right)u_x^2(1,t) + \left(\gamma_2\eta_1 - \frac{\mu}{2} - a\right)u_x^2(1,t).$$

Therefore,

$$\begin{aligned}\dot{E}(t) \leq{}& -\sigma\int_0^1 u^2(x,t)\,dx - \gamma_2\eta_1 u_x^2(1,t) + \frac{\mu}{2}u_x^2(1,t) + \gamma_2\eta_1 u_x^2(1,t) - \frac{\mu}{2}u_x^2(1,t)\\ &- au_x^2(1,t).\end{aligned} \tag{5.12}$$

This reduces to

$$\dot{E}(t) \leq -\sigma\int_0^1 u^2(x,t)\,dx - au_x^2(1,t). \tag{5.13}$$

Since $a \geq 0$, the term $-au_x^2(1,t) \leq 0$ for all t ≥ 0, and hence (5.13) yields

$$\dot{E}(t) \leq -\sigma\int_0^1 u^2(x,t)\,dx \quad \text{for any } t \geq 0.$$

Thereby, $E(t) \leq E(0)$ for any $t \geq 0$, which means that η_1 is bounded for all $t > 0$. Thus, $u_x(1,t) \in L^2(0,\infty)$. Using (5.5), we can write (5.9) as follows:

$$\dot{V}(t) \leq -2\sigma V(t) - \left(\gamma_2\eta_1 - \frac{\mu}{2}\right)u_x^2(1,t) \quad \text{for any } t \geq 0. \tag{5.14}$$

Using Gronwall–Bellman's inequality, we get

$$V(t) \leq e^{-2\sigma t}V(0) + C\int_0^t e^{-2\sigma(t-\tau)}u_x^2(1,\tau)\,d\tau, \tag{5.15}$$

where $C = \sup|\gamma_2\eta_1 - \frac{\mu}{2}|$. Since

$$\int_0^t e^{-2\sigma(t-\tau)}u_x^2(1,\tau)\,d\tau \to 0, \quad \text{as } t \to \infty,$$

we conclude that $V(t)$ converges to zero as $t \longrightarrow \infty$. Thus, $\|u(x,t)\|$ exponentially approaches zero as t tends to infinity. □

Remark 1 It is worth mentioning that the choice of the feedback gain r_1 will definitely affect the value of η_1, and therefore the stability of the solution. Fixing the initial condition $\eta_1(0)$ and increasing the value of r_1 will increase the value of η_1, which in turn speeds up the convergence of the solution to zero.

6 Numerical simulations of the MGKdVB equation

In order to illustrate our theoretical results, we numerically present in this section the dynamical behavior of the MGKdVB equation when the non-adaptive boundary control (1.3) and adaptive boundary control law (5.3) are applied. To this end, numerical solutions for the (MGKdVB) equation will be simulated using COMSOL Multiphysics software.

6.1 Linear non-adaptive boundary control law for the MGKdVB equation (1.3)

This subsection is devoted to numerical simulations of solutions for the (MGKdVB) equation with the non-adaptive control law (1.3). The solutions are computed and simulated for $\alpha = 1, 2, 3$, and 4.

Let us pick up the initial condition $u_0(x) = \sin(\pi x)$. Figures 1(a)–1(d) depict the behavior of the solution $u(x,t)$ as it evolves in time. In the simulations, we set the parameters σ, μ, γ_1, and γ_2 to be 0.01, 0.001, 1, and 0.0005, respectively. The norm $\|u(x,t)\|$ versus time is presented in Figure 2. This figure indicates that the solution converges to zero as t goes to infinity. This confirms the theoretical results obtained in Sect. 4. It should be noted that Figure 2 also emphasizes that condition (1.3), that is, $u_{xx}(1,t) = -\frac{\mu}{\gamma_2} u_x(1,t)$, is acting as a linear boundary control where $-\frac{\mu}{\gamma_2} u_x(1,t)$ plays the role of the input control. A careful look at Figures 1 and 2 shows that the decay rate to the steady state solution decreases as α increases from 1 to 4.

6.2 Linear adaptive boundary control law for the MGKdVB equation (5.3)

In this subsection, we present numerically the dynamical behavior of the MGKdVB equation by applying the linear adaptive boundary control law presented in (5.3). Using COM-

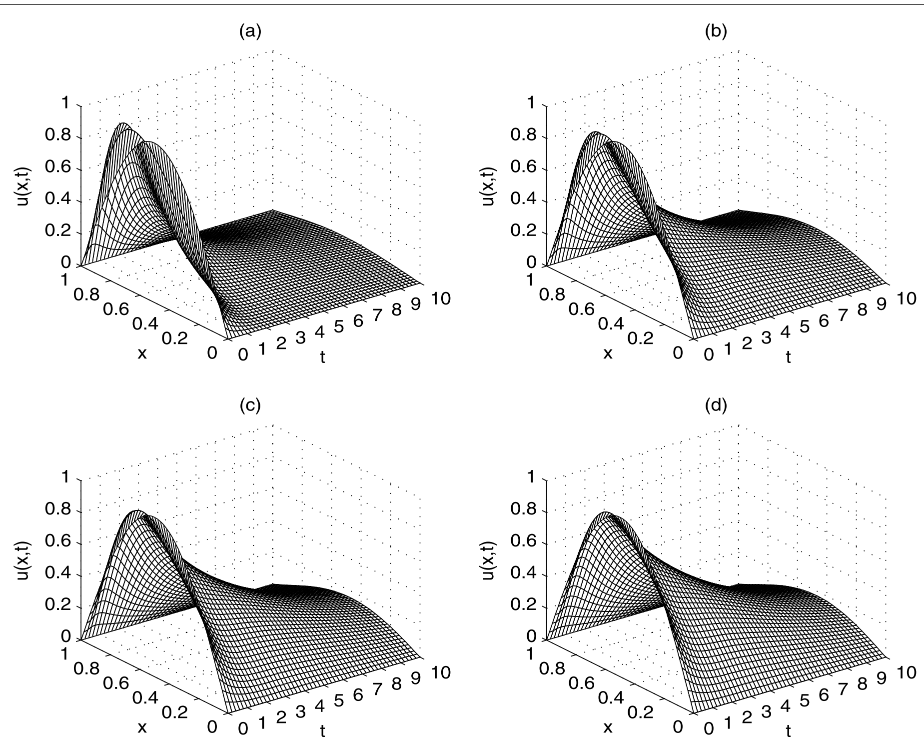

Figure 1 A 3-d landscape of the dynamics of the MGKdVB equation subject to boundary conditions (1.2)–(1.3); $\sigma = 0.01$, $\mu = 0.001$, $\gamma_1 = 1$, $\gamma_2 = 0.0005$, and $u_0(x) = \sin(\pi x)$; (**a**) $\alpha = 1$; (**b**) $\alpha = 2$; (**c**) $\alpha = 3$; (**d**) $\alpha = 4$

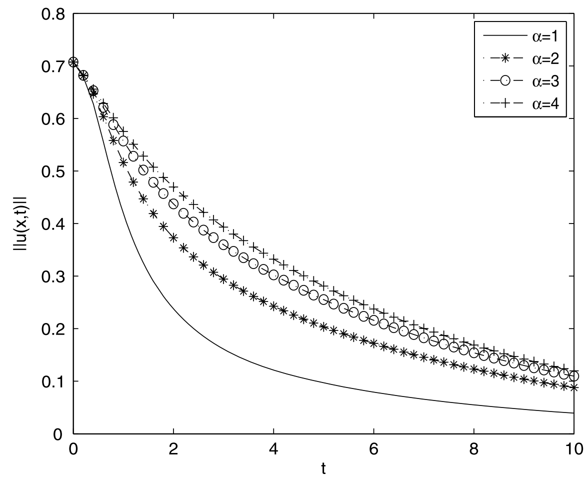

Figure 2 The L^2-norm of $u(x,t)$, $\|u(x,t)\|$ versus time for different values of α; $\sigma = 0.01$, $\mu = 0.001$, $\gamma_1 = 1$, $\gamma_2 = 0.0005$, and $u_0(x) = \sin(\pi x)$

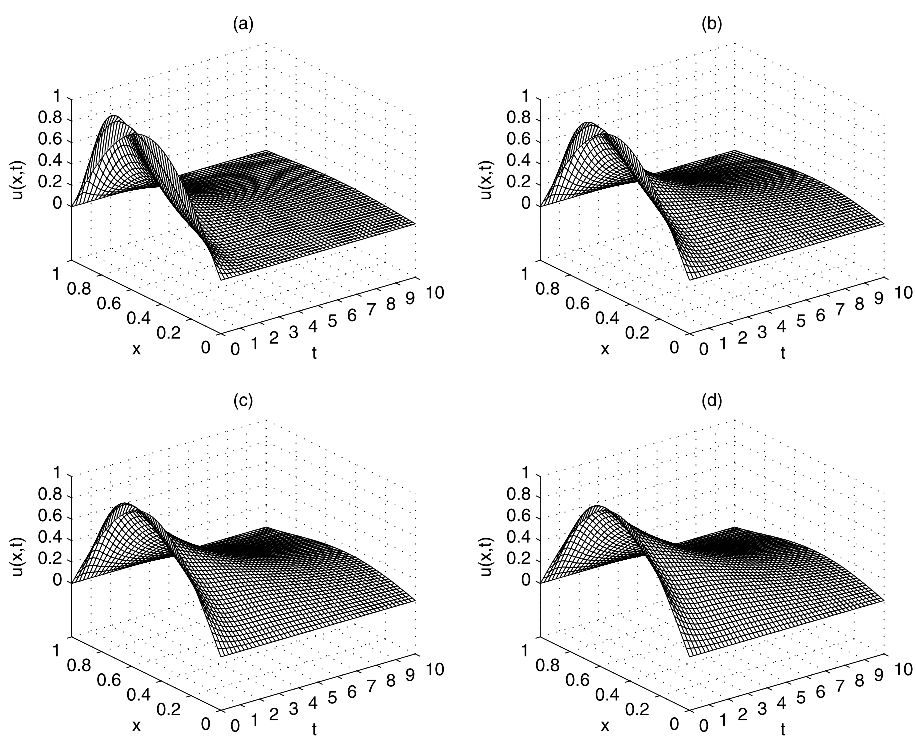

Figure 3 A 3-d landscape of the dynamics of the MGKdVB equation using the linear adaptive control law; $\sigma = 0.01$, $\gamma_1 = 1$, $r_1 = 0.2$, and $u_0(x) = \sin(\pi x)$; (**a**) $\alpha = 1$; (**b**) $\alpha = 2$; (**c**) $\alpha = 3$; (**d**) $\alpha = 4$

SOL Multiphysics software, the solutions are provided for several values of α. These values are 1, 2, 3, and 4.

As in the previous subsection, let $u_0(x) = \sin(\pi x)$. It is clear from (5.3) that the linear adaptive control law proposed in Theorem 3 does not require the pre-knowledge of μ and γ_2 which are assumed to be unknowns. Nevertheless, for simulation purposes, these values are set to be 0.001 and 0.0005, respectively. The parameters σ and γ_1 are set to be

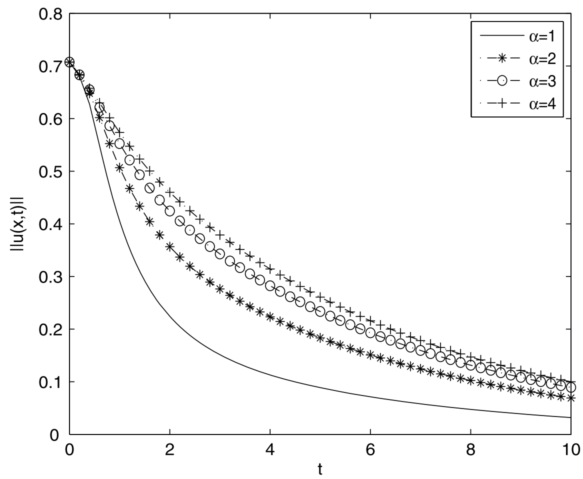

Figure 4 The L^2-norm of $u(x,t)$, $\|u(x,t)\|$ versus time for different values of α when applying the linear adaptive control law; $\sigma = 0.01$, $\gamma_1 = 1$, $r_1 = 0.2$, and $u_0(x) = \sin(\pi x)$

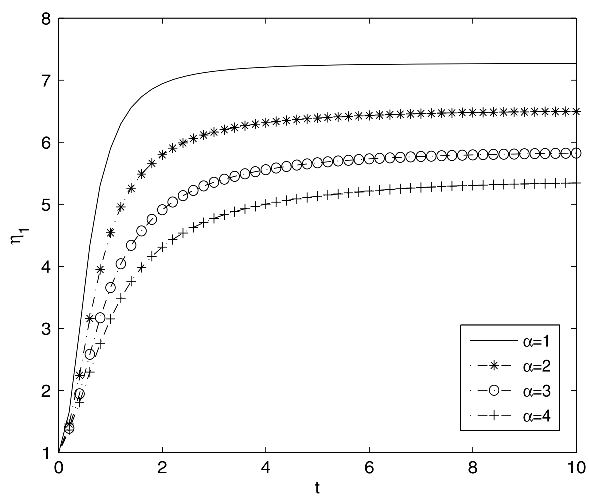

Figure 5 η_1 versus time for different values of α when applying the linear adaptive control law

0.01 and 1, respectively. Moreover, r_1 is chosen to equal 0.2. The initial value of η_1 is set to be such that $\eta_1(0) = 1$. Figures 3(a)–3(d) show a 3-d landscape of the solution of the MGKdVB equation when the control law is given in Theorem 3 for different values of α. Figure 4 shows the L^2-norm of these solutions. It can be noticed from the figures that as α increases, the decay rate of the solution to the steady state solution decreases. This is due to the effect of the nonlinear term which causes the instability in the behavior of the solutions of the MGKdVB equation.

Figure 5 depicts the behavior of the function η_1, which appears in the control. Figure 5 shows that η_1 decreases as α increases from 1 to 4.

7 Concluding remarks

In this paper, the MGKdVB equation is considered and a feedback boundary control is proposed. Then, the well-posedness of the system and the exponential stability of the solutions are investigated. Furthermore, a linear adaptive control law is put forward when the parameters γ_2 and μ are unknowns. In this case, the solutions are also shown to be exponentially stable. Finally, numerical simulations are presented to illustrate our results.

The control problem of the MGKdVB equation in the presence of a time delay will be the subject of future research studies.

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Authors' contributions
NS derived and analyzed the stability results, performed the numerical simulations, wrote and edited the manuscript. BC proved the well-posedness of the model, wrote and edited the manuscript. AA worked on the well-posedness, derived the stability results, performed the numerical simulations of the model, and wrote the manuscript. All authors read and approved the final manuscript.

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