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# Fixed points of differences of meromorphic functions

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**Abstract**

Let $f$ be a transcendental meromorphic function of finite order and $c$ be a nonzero complex number. Define $\Delta_c f = f(z+c) - f(z)$. The authors investigate the existence on the fixed points of $\Delta_c f$. The results obtained in this paper may be viewed as discrete analogues on the existing theorem on the fixed points of $f'$. The existing theorem on the fixed points of $\Delta_c f$ generalizes the relevant results obtained by (Chen in Ann. Pol. Math. 109(2):153–163, 2013; Zhang and Chen in Acta Math. Sin. New Ser. 32(10):1189–1202, 2016; Cui and Yang in Acta Math. Sci. 33B(3):773–780, 2013) et al.

**Keywords:** Difference operator; Fixed point; Borel exceptional values; Deficiency

## 1 Introduction

Let $f(z)$ be a function meromorphic in the complex plane $C$. We use the general notation of the Nevanlinna theory (see [12, 20, 23]) such as $m(r,f)$, $N(r,f)$, $T(r,f)$, $m(r,\frac{1}{f-a})$, $N(r,\frac{1}{f-a})$, …, and assume that the reader is familiar with these notations. We also use $S(r,f)$ to denote any quantity of $S(r,f) = o(T(r,f))$ $(r \to \infty)$, possibly outside a set with finite logarithmic measure. The order and the lower order of $f(z)$ are denoted by $\sigma(f)$ and $\mu(f)$ respectively.

For any $a \in C$, the exponent of convergence of zeros of $f(z) - a$ (or poles of $f(z)$) is denoted by $\lambda(f,a)$ (or $\lambda(\frac{1}{f})$). Especially, we denote $\lambda(f,0)$ by $\lambda(f)$. If $\lambda(f,a) < \sigma(f)$ (or $\lambda(\frac{1}{f}) < \sigma(f)$), then $a$ (or $\infty$) is said to be a Borel exceptional value of $f(z)$. Nevanlinna's deficiency of $f$ with respect to complex number $a \in C \cup \{\infty\}$ is defined by

$$\delta(a,f) = \liminf_{r\to\infty} \frac{m(r,\frac{1}{f-a})}{T(r,f)} = 1 - \limsup_{r\to\infty} \frac{N(r,\frac{1}{f-a})}{T(r,f)}.$$

If $a = \infty$, then one should replace $N(r,\frac{1}{f-a})$ in the above formula by $N(r,f)$.

A point $z_0 \in C \cup \{\infty\}$ is said to be a fixed point of $f(z)$ if $f(z_0) = z_0$. There is a considerable number of results on the fixed points of meromorphic functions, we refer the reader to Chuang and Yang [7]. It follows Chen and Shon [2, 4], we use the notation $\tau(f)$ to denote the exponent of convergence of fixed points of $f$, i.e.,

$$\tau(f) = \limsup_{r\to\infty} \frac{\log N(r,\frac{1}{f-z})}{\log r}.$$

In 1993, Lahiri [13] proved the following theorem.

**Theorem A** *Let $f$ be a transcendental meromorphic function in the plane. Suppose that there exists $a \in C$ with $\delta(a,f) > 0$ and $\delta(\infty,f) = 1$. Then $f$ has infinitely many fixed points.*

In this paper, we shall study the fixed points of the differences of meromorphic functions. For each $c \in C\backslash\{0\}$, the forward difference $\Delta_c^{k+1} f(z)$ is defined (see [1]) by

$$\Delta_c f(z) = f(z+c) - f(z), \Delta_c^2 f(z) = \Delta_c f(z+c) - \Delta_c f(z).$$

Especially, we denote $\Delta_1 f(z)$ by $\Delta f(z)$.

Recently, some well-known facts of the Nevanlinna theory have been extended for the differences of meromorphic functions (see [5, 6, 9–11, 14–18]). For the existence on the fixed points of differences, Cui and Yang [8] have proved the following theorems.

**Theorem B** ([8]) *Let $f$ be a function transcendental and meromorphic in the plane with the order $\sigma(f) = 1$. If $f$ has finitely many poles and infinitely many zeros with exponent of convergence of zeros $\lambda(f) \neq 1$, then $\Delta f$ has infinitely many zeros and fixed points.*

**Theorem C** ([8]) *Let $f$ be a non-periodic function transcendental and meromorphic in the plane with the order $\sigma(f) = 1$, $\max\{\lambda(f), \lambda(\frac{1}{f})\} \neq 1$. If $f$ has infinitely many zeros, then $\Delta f$ has infinitely many zeros and fixed points.*

The conditions of Theorems B and C imply that $0, \infty$ are Borel exceptional values. If $\infty$ and $d \in C$ are Borel exceptional values of $f$, Chen [3] obtains the following theorem.

**Theorem D** ([3]) *Let $f$ be a finite order meromorphic function such that $\lambda(\frac{1}{f}) < \sigma(f)$, and let $c \in C\backslash\{0\}$ be a constant such that $f(z+c) \not\equiv f(z)$. If $f(z)$ has a Borel exceptional value $d \in C$, then $\tau(\Delta_c f) = \sigma(f)$.*

In [22], Zhang and Chen showered that the condition $\lambda(\frac{1}{f}) < \sigma(f)$ in Theorem D cannot be omitted. Moreover, they obtained the following theorem.

**Theorem E** ([22]) *Let $f$ be a finite order meromorphic function, and let $c \in C\backslash\{0\}$ be a constant such that $f(z+c) \not\equiv f(z)$. If $f(z)$ has two Borel exceptional values, then $\tau(\Delta_c f) = \sigma(f)$.*

In [19], Yi and Yang have proved the following theorem.

**Theorem F** ([19]) *Let $f$ be a transcendental meromorphic function in $C$ with a positive order. If $f$ has two distinct Borel exceptional values, say $a_1$ and $a_2$, then the order of $f$ is a positive integer or $\infty$ and $\sigma(f) = \mu(f)$, $\delta(a_1,f) = \delta(a_2,f) = 1$.*

By Theorem F, we can derive that the order of $f$ in Theorems D and E is a positive integer. Is it necessary to ask if the order of $f$ is an integer?, i.e., Can we get similar results as those in Theorems B, C, D, and E if the order of $f$ is not a positive integer? The main purpose of this paper is to study this question. In fact, we shall prove the following theorems.

**Theorem 1.1** *Let $f$ be a transcendental meromorphic function of finite order in the plane. Suppose that $c \in C \setminus \{0\}$ such that $\Delta_c f \not\equiv 0$. If there is $a \in C$ with $\delta(a,f) > 0$ and $\delta(\infty,f) = 1$, then $\Delta_c f$ have infinitely many fixed points and $\tau(\Delta_c f) = \sigma(f)$.*

**Theorem 1.2** *Let $f$ be a transcendental meromorphic function of finite order in the plane. Suppose that $c \in C \setminus \{0\}$ such that $\Delta_c f \not\equiv 0$. If $\delta(\infty, f) = 1$, $\delta(0, f) = 1$, then*

$$T(r, \Delta_c f) \sim T(r, f) \sim N\left(r, \frac{1}{(\Delta_c f) - z}\right),$$

*as $r \to \infty$, $r \notin E$, where $E$ is a possible exception set of $r$ with finite logarithmic measure.*

Let $f(z) = \frac{e^z}{z}$, then $N(r, f) = \log r = S(r, f)$, $N(r, \frac{1}{f}) = 0$ and $\Delta_c f = \frac{(e^c-1)z-1}{z(z+c)} e^z \not\equiv 0$. By the second fundmental theorem, we have

$$T(r, \Delta_c f) \sim T(r, f) \sim N\left(r, \frac{1}{(\Delta_c f) - z}\right) \quad (r \to \infty),$$

and $\tau(\Delta_c f) = \sigma(f)$.

## 2 Proof of Theorems 1.1 and 1.2

**Lemma 2.1** ([6]) *Let $f(z)$ be a finite order meromorphic function, then, for each $k \in N$, $\sigma(\Delta_c^k f) \le \sigma(f)$.*

**Lemma 2.2** ([9]) *Let $f$ be a transcendental meromorphic function of finite order. Then, for any positive integer $n$, we have*

$$m\left(r, \frac{\Delta_c^n f(z)}{f(z)}\right) = S(r, f).$$

**Lemma 2.3** *Let $f$ be a transcendental meromorphic function of finite order. Suppose that $c \in C \setminus \{0\}$ such that $\Delta_c f \not\equiv 0$ and $\delta(0, f) > 0$. Then $\Delta_c f$ is a transcendental and meromorphic function of finite order.*

*Proof* From Lemma 2.1, we know that $\sigma(\Delta_c f) \le \sigma(f) < +\infty$. If $\Delta_c f$ is not a transcendental meromorphic function, then there is a rational function $R(z)$ such that $R(z)\Delta_c f \equiv 1$, i.e.,

$$\frac{1}{f} \equiv R(z) \frac{\Delta_c f}{f}.$$

Applying Lemma 2.2 and noticing that $f(z)$ is transcendental, we have

$$m\left(r, \frac{1}{f}\right) \le m\big(r, R(z)\big) + m\left(r, \frac{\Delta_c f}{f}\right) = S(r, f).$$

This contradicts $\delta(0, f) > 0$. Thus $\Delta_c f$ is a transcendental and meromorphic function of finite order. □

**Lemma 2.4** ([11]) *Let $f(z)$ be a transcendental meromorphic function of finite order, then*

$$m\left(r, \frac{f(z+c)}{f}\right) = S(r, f).$$

**Lemma 2.5** ([14, 21]) *Let f be a transcendental meromorphic function of finite order. Then*

$$N\bigl(r,f(z+c)\bigr) = N(r,f) + S(r,f),$$
$$T\bigl(r,f(z+c)\bigr) = T(r,f) + S(r,f).$$

**Lemma 2.6** *Let $f$ be a finite order transcendental meromorphic function. Suppose that $c \in C \setminus \{0\}$ such that $\Delta_c f \not\equiv 0$. If $\delta(0,f) > 0$, then*

$$T(r,f) \leq 4N(r,f) + N\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{(\Delta_c f) - z}\right) + S(r,f).$$

*Proof* By Lemma 2.3, we know that $\Delta_c f$ is a transcendental meromorphic function. Put $F = \Delta_c f$, then there is $\eta \in C \setminus \{0\}$ such that $z\Delta_\eta F - \eta F(z) \not\equiv 0$. If not, then

$$\frac{F(z)}{z} \equiv \frac{F(z+\eta)}{z+\eta}$$

holds for any $\eta \in C \setminus \{0\}$. Hence $\frac{F(z)}{z}$ is a constant, which contradicts $F = \Delta_c f$ is a transcendental meromorphic function. Hence there is $\eta \in \setminus\{0\}$ such that $z\Delta_\eta F - \eta F(z) \not\equiv 0$, i.e.,

$$\begin{aligned} &z\Delta_\eta F - \eta F(z) \\ &\quad = z\Delta_\eta(\Delta_c f) - \eta\Delta_c f \\ &\quad = z\Delta_\eta\bigl((\Delta_c f) - z\bigr) - \eta\bigl((\Delta_c f) - z\bigr) \\ &\quad = zf(z+c+\eta) - zf(z+\eta) - (z+\eta)f(z+c) + (z+\eta)f(z) \not\equiv 0. \end{aligned} \tag{1}$$

Noticing

$$\frac{1}{f} = \frac{\Delta_c f}{zf} - \frac{z\Delta_\eta(\Delta_c f) - \eta\Delta_c f}{zf}\frac{(\Delta_c f) - z}{z\Delta_\eta(\Delta_c f) - \eta\Delta_c f}. \tag{2}$$

Combining (1), (2) and Lemmas 2.2, 2.4, we can get

$$\begin{aligned} &m\left(r,\frac{1}{f}\right) \\ &\quad \leq m\left(r,\frac{\Delta_c f}{zf}\right) + m\left(r,\frac{z\Delta_\eta(\Delta_c f) - \eta\Delta_c f}{zf}\right) \\ &\qquad + m\left(r,\frac{(\Delta_c f) - z}{z\Delta_\eta(\Delta_c f) - \eta\Delta_c f}\right) + \log 2 \\ &\quad \leq m\left(r,\frac{\Delta_c f}{f}\right) + m\left(r,\frac{f(z+c+\eta)}{f}\right) + m\left(r,\frac{f(z+c)}{f}\right) \\ &\qquad + m\left(r,\frac{f(z+\eta)}{f}\right) + m\left(r,\frac{(\Delta_c f) - z}{z\Delta_\eta(\Delta_c f) - \eta\Delta_c f}\right) + O(\log r) \\ &\quad = m\left(r,\frac{(\Delta_c f) - z}{z\Delta_\eta(\Delta_c f) - \eta\Delta_c f}\right) + S(r,f). \end{aligned}$$

Applying the first fundamental theorem of Nevanlinna theory, we have

$$T(r,f) \le N\left(r,\frac{1}{f}\right) + m\left(r,\frac{(\Delta_c f)-z}{z\Delta_\eta(\Delta_c f)-\eta\Delta_c f}\right) + S(r,f), \tag{3}$$

and we get

$$\begin{aligned} & m\left(r,\frac{(\Delta_c f)-z}{z\Delta_\eta(\Delta_c f)-\eta\Delta_c f}\right) \\ &\quad \le m\left(r,\frac{z\Delta_\eta(\Delta_c f)-\eta\Delta_c f}{(\Delta_c f)-z}\right) + N\left(r,\frac{z\Delta_\eta(\Delta_c f)-\eta\Delta_c f}{(\Delta_c f)-z}\right) + O(1). \end{aligned} \tag{4}$$

It follows from (1) that

$$m\left(r,\frac{z\Delta_\eta(\Delta_c f)-\eta\Delta_c f}{(\Delta_c f)-z}\right) \le m\left(r,\frac{\Delta_\eta((\Delta_c f)-z)}{(\Delta_c f)-z}\right) + S(r,f). \tag{5}$$

Applying Lemma 2.3 and Lemma 2.5, we know that $(\Delta_c f)-z$ is a transcendental meromorphic function of finite order and

$$T\big(r,(\Delta_c f)-z\big) \le 2T(r,f) + S(r,f).$$

Therefore,

$$S\big(r,(\Delta_c f)-z\big) = S(r,f). \tag{6}$$

It follows from Lemma 2.2 and (6) that

$$m\left(r,\frac{z\Delta_\eta(\Delta_c f)-\eta\Delta_c f}{(\Delta_c f)-z}\right) = S(r,f). \tag{7}$$

By Lemma 2.5 and (1), we derive

$$\begin{aligned} & N\left(r,\frac{z\Delta_\eta(\Delta_c f)-\eta\Delta_c f}{(\Delta_c f)-z}\right) \\ &\quad \le N\big(r,z\Delta_\eta(\Delta_c f)-\eta\Delta_c f\big) + N\left(r,\frac{1}{(\Delta_c f)-z}\right) \\ &\quad \le N\left(r,\frac{1}{(\Delta_c f)-z}\right) + 4N(r,f) + S(r,f). \end{aligned} \tag{8}$$

Combining (3)–(5) and (7)–(8), we have

$$T(r,f) \le 4N(r,f) + N\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{(\Delta_c f)-z}\right) + S(r,f). \qquad \square$$

### 2.1 Proof of Theorem 1.1

Denoting $g = f - a$, by Lemma 2.6, we have

$$\begin{aligned} T(r,f) &= T(r,g) + O(1) \\ &\le 4N(r,g) + N\left(r, \frac{1}{g}\right) + N\left(r, \frac{1}{(\Delta_c g) - z}\right) + S(r,g) \\ &= 4N(r,f) + N\left(r, \frac{1}{f-a}\right) + N\left(r, \frac{1}{(\Delta_c f) - z}\right) + S(r,f). \end{aligned} \tag{9}$$

Since $\delta(a,f) > 0$ and $\delta(\infty,f) = 1$, then there is a positive number $\theta < 1$ such that

$$N\left(r, \frac{1}{f-a}\right) < \theta T(r,f), \tag{10}$$

$$N(r,f) \le o(1)T(r,f). \tag{11}$$

Combining (9)–(11), we can get

$$\bigl(1 - o(1) - \theta\bigr) T(r,f) \le N\left(r, \frac{1}{(\Delta_c f) - z}\right). \tag{12}$$

Note that $f$ is transcendental, we can get that $\Delta_c f$ has infinitely many fixed points and $\tau(\Delta_c f) = \sigma(f)$ from (12).

### 2.2 Proof of Theorem 1.2

Since

$$\begin{aligned} m\left(r, \frac{1}{f}\right) &= m\left(r, \frac{\Delta_c f}{f} \frac{1}{\Delta_c f}\right) \le m\left(r, \frac{\Delta_c f}{f}\right) + m\left(r, \frac{1}{\Delta_c f}\right) \\ &\le m\left(r, \frac{1}{\Delta_c f}\right) + S(r,f). \end{aligned} \tag{13}$$

By the first fundamental theorem of Nevanlinna theory and (13), we can get

$$T(r,f) \le T(r, \Delta_c f) + N\left(r, \frac{1}{f}\right) + S(r,f). \tag{14}$$

Hence

$$\begin{aligned} 1 &\le \liminf_{r\to\infty} \frac{T(r,\Delta_c f)}{T(r,f)} + \limsup_{r\to\infty} \frac{N(r,\frac{1}{f})}{T(r,f)} \\ &= \liminf_{r\to\infty} \frac{T(r,\Delta_c f)}{T(r,f)} + \bigl(1 - \delta(0,f)\bigr) \\ &= \liminf_{r\to\infty} \frac{T(r,\Delta_c f)}{T(r,f)}. \end{aligned} \tag{15}$$

On the other hand, we have

$$\begin{aligned} T(r,\Delta_c f) &= m(r,\Delta_c f) + N(r,\Delta_c f) \\ &= m\left(r,\frac{f\Delta_c f}{f}\right) + N(r,\Delta_c f) \\ &\leq m\left(r,\frac{\Delta_c f}{f}\right) + m(r,f) + N(r,f) + N\big(r,f(z+c)\big). \end{aligned}$$

It follows from Lemma 2.2 and Lemma 2.5 that

$$T(r,\Delta_c f) \leq T(r,f) + N(r,f) + S(r,f).$$

As $\delta(\infty,f) = 1$, so

$$\limsup_{r\to\infty} \frac{T(r,\Delta_c f)}{T(r,f)} \leq 1 + \limsup_{r\to\infty} \frac{N(r,f)}{T(r,f)} = 1.$$

Therefore

$$\lim_{r\to+\infty} \frac{T(r,\Delta_c f)}{T(r,f)} = 1. \tag{16}$$

Since $\delta(0,f) = 1$ and $\delta(\infty,f) = 1$, then

$$N\left(r,\frac{1}{f}\right) = S(r,f), N(r,f) = S(r,f). \tag{17}$$

By (17) and Lemma 2.6, we have

$$\begin{aligned} T(r,f) &\leq N\left(r,\frac{1}{(\Delta_c f) - z}\right) + S(r,f) \\ &\leq T\left(r,\frac{1}{(\Delta_c f) - z}\right) + S(r,f) \\ &\leq T(r,\Delta_c f) + S(r,f). \end{aligned} \tag{18}$$

Combining (16) and (18) implies

$$T(r,\Delta_c f) \sim T(r,f) \sim N\left(r,\frac{1}{(\Delta_c f) - z}\right),$$

as $r \to \infty$.

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**Authors' contributions**

All authors drafted the manuscript, read and approved the final manuscript. All authors contributed equally.

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