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Optical soliton solutions to the $(2+1)$-dimensional Chaffee–Infante equation and the dimensionless form of the Zakharov equation

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Abstract

The $(2+1)$-dimensional Chaffee–Infante equation and the dimensionless form of the Zakharov equation have widespread scopes of function in science and engineering fields, such as in nonlinear fiber optics, the waves of electromagnetic field, plasma physics, the signal processing through optical fibers, fluid dynamics, coastal engineering and remarkable to model of the ion-acoustic waves in plasma, the sound waves. In this article, the first integral method has been assigned to search closed form solitary wave solutions to the previously proposed nonlinear evolution equations (NLEEs). We have constructed abundant soliton solutions and discussed the physical significance of the obtained solutions of its definite values of the included parameters through depicting figures and interpreted the physical phenomena. It has been shown that the first integral method is powerful, convenient, straightforward and provides further general wave solutions to diverse NLEEs in mathematical physics.

Keywords: The $(2+1)$-dimensional Chaffee–Infante equation; The dimensionless form of the Zakharov equation; The first integral method

1 Introduction

Nonlinearity is a fascinating element of nature and many scientists consider nonlinear science as the most important frontier for the fundamental understanding of nature. The mathematical modeling of intricate phenomena that change with time depends intensely on the investigation of diverse class of nonlinear ordinary and partial differential equations. These models are developed in highly dissimilar fields of study, ranging from the physical and natural sciences, infectious disease epidemiology, neural networks, population ecology to economics, optical fibers, elasticity, plasma physics, solid state physics, and fluid mechanics. Therefore, an exciting and incredibly dynamic field of research for the previous few decades has been the investigation of soliton solutions of the earlier stated phenomena and the related issue is the development of closed form wave solutions to a broad class of nonlinear evolution equations. Closed form solitary wave solutions provide better internal information about those phenomena. Therefore, considerable efforts have been made by many mathematicians and physical scientists to obtain closed form wave solutions of such NLEEs and a number of powerful and efficient methods, such as the Bäcklund transformation method [1], the first integral method [2], the modified simple equation method [3, 4], the Exp-function method [5], the (G'/G)-expansion method [6], the sine-cosine method [7], the modified Kudryashov method [8], the homogeneous balance method [9], the F-expansion method [10], the variational iteration method [11], the tanh-function method [12], the Adomian decomposition method [13], the projective Riccati equation method [14], the homotopy analysis method [15], and the $(G'/G, 1/G)$-expansion method [16] have been developed.

Among these methods, the first integral method is one of the most important, direct and effective algebraic methods for finding exact solutions to NLEEs. This method was first proposed by Feng [17] in solving the Burgers–KdV equation which is based on the ring theory of commutative algebra. Recently, this useful method has been widely used by many researchers, as for instance, Raslan [18] examined the exact traveling wave solutions to the Fisher equation by applying the first integral method. Taghizadeh et al. [19] determined the exact wave solutions to the modified KdV–KP equation and the Burgers–KP equation by using this method. Abbasbandy and Shirzadi [20] investigated the modified BBM equation by using the first integral method. Also Moosaei et al. [21] examined the perturbed nonlinear Schrodinger equation with Kerr law nonlinearity by using this method.

Motivated by the ongoing research, in this article, we have examined the (2 + 1)-dimensional Chaffee–Infante equation and the dimensionless form of the Zakharov equation (ZE) through the first integral method to extract closed form solitary wave solutions and solitons.

2 The first integral method

In this section, to facilitate further analysis, we initiate our study by briefly reviewing the procedure.

Step 1: Let us consider a general NLEE of the form

$$P(u, u_t, u_x, u_{xx}, u_{tt}, u_{xt}, u_{xxx}, \dots) = 0, \tag{1}$$

where $u = u(x,t)$ is the solution of the NLEE (1). In order to investigate Eq. (1) by means of the first integral method, we use the following transformation:

$$u(x,t) = f(\xi), \tag{2}$$

where $\xi = x - ct$, is the wave variable. This enables us to use the following changes:

$$\frac{\partial}{\partial t}(\cdot) = -c\frac{\partial}{\partial \xi}(\cdot), \qquad \frac{\partial}{\partial x}(\cdot) = \frac{\partial}{\partial \xi}(\cdot), \qquad \frac{\partial^2}{\partial x^2}(\cdot) = \frac{\partial^2}{\partial \xi^2}(\cdot), \qquad \dots \tag{3}$$

Use Eq. (3) to change the NLEE (1) to nonlinear ordinary differential equation

$$G\left(f(\xi), \frac{\partial f(\xi)}{\partial \xi}, \frac{\partial^2 f(\xi)}{\partial \xi^2}, \dots\right) = 0. \tag{4}$$

Step 2: We introduce the new independent variables

$$X(\xi) = f(\xi), \qquad Y(\xi) = \frac{\partial f(\xi)}{\partial \xi}. \tag{5}$$

Step 3: Under the conditions set out in Step 2, Eq. (4) can be converted into a system of nonlinear ordinary differential equations as follows:

$$\frac{\partial X(\xi)}{\partial \xi} = Y(\xi), \qquad \frac{\partial Y(\xi)}{\partial \xi} = F\big(X(\xi), Y(\xi)\big). \tag{6}$$

Step 4: As indicated in the qualitative principle of ordinary differential equations [22], if we can uncover the first integrals of Eq. (6) under the same conditions, then the general solutions of Eq. (6) can be found immediately. Yet, in general, it is certainly problematic to understand this even for one first integral, because, for a given plane autonomous system, there is no methodological hypothesis that can inform us as to how to get its first integrals, nor is there a rational way to tell us what these first integrals are. We implement the reputed division algorithm to get one first integral of Eq. (6) which transforms Eq. (4) to a first order integrable ordinary differential equation. The outcome of the first order integrals provides the solution of Eq. (1). At this time, let us review the division algorithm.

Division Theorem (Feng [2]) *Assume that $R(w,z)$, $S(w,z)$ are two polynomials in the complex domain $C[w,z]$ such that $R(w,z)$ is irreducible in $C[w,z]$. If $S(w,z)$ vanishes at all zero points of $R(w,z)$, then there exists a polynomial $F(w,z)$ in $C[w,z]$ such that*

$$S(w,z) = R(w,z)F(w,z).$$

3 Formulation of the solutions

In this section, we will analyze two NLEEs, namely, the (2 + 1)-dimensional Chaffee–Infante equation and the dimensionless form of the Zakharov equation and establish useful solutions by using the first integral method described in Sect. 2.

3.1 The (2 + 1)-dimensional Chaffee–Infante equation

We first consider the (2 + 1)-dimensional Chaffee–Infante equation (Sakthivel and Chun [23]) in the following form:

$$u_{xt} + \left(-u_{xx} + \alpha u^3 - \alpha u\right)_x + \sigma u_{yy} = 0, \tag{7}$$

where α is the coefficient of diffusion and σ are degradation coefficient. The diffusion of a gas in a homogeneous medium is an important phenomenon in physical context and the Chaffee–Infante equation provides a useful model to study such phenomena. The (2 + 1)-dimensional Chaffee–Infante equation is a notable reaction Duffing equation arising in the physical sciences (Constantin [23, 24]).

In order to investigate Eq. (7) by using the first integral method, we use the following transformation:

$$u(x,y,t) = f(\xi), \quad \xi = x + y - ct, \tag{8}$$

where c is the wave velocity.

Substituting (8) into (7), we reach into the ordinary differential equation

$$-cf'' + \left(-f'' + \alpha f^3 - \alpha f\right)' + \sigma f'' = 0. \tag{9}$$

Integrating Eq. (9) with respect to ξ and neglecting the constant of integration, yields

$$-f'' + \sigma f' - cf' + \alpha f^3 - \alpha f = 0. \tag{10}$$

We are searching for solitary wave solutions and solitary waves are localized, so that they decay as $\xi \to \pm\infty$. Therefore, we have used the boundary conditions $u(\xi) \to 0$, $u'(\xi) \to 0$, $u''^{(\xi)} \to 0, \dots$ as $\xi \to \pm\infty$ and these boundary conditions yield zero constant [25].

Equation (10) can be rewritten as follows:

$$f'' = \sigma f' - cf' + \alpha f^3 - \alpha f, \tag{11}$$

where a prime means differentiation with respect to ξ.

Using (5) and (6) from (11), we get

$$\dot{X}(\xi) = Y(\xi), \tag{12a}$$

$$\dot{Y}(\xi) = \sigma Y(\xi) - cY(\xi) + \alpha\left(X(\xi)\right)^3 - \alpha X(\xi). \tag{12b}$$

Following the first integral method, we suppose that $X(\xi)$ and $Y(\xi)$ are nontrivial solutions of (12a)–(12b) and

$$q(X, Y) = \sum_{i=0}^{m} a_i(X)Y^i = 0,$$

is an irreducible polynomial in the complex domain $C[X, Y]$ such that

$$q\left(X(\xi), Y(\xi)\right) = \sum_{i=0}^{m} a_i\left(X(\xi)\right)Y^i(\xi) = 0, \tag{13}$$

where $a_i(X)$ $(i = 0, 1, \dots, m)$, are polynomials of X such that $a_m(X) \neq 0$. Eq. (13) is called the first integral of (12a)–(12b). As indicated in the division hypothesis, there exists a polynomial $g(X) + h(X)Y$ in the complex domain $C[X, Y]$ such that

$$\frac{dq}{d\xi} = \frac{\partial q}{\partial X}\frac{\partial X}{\partial \xi} + \frac{\partial q}{\partial Y}\frac{\partial Y}{\partial \xi} = \left(g(X) + h(X)Y\right)\sum_{i=0}^{m} a_i(X)Y^i. \tag{14}$$

For the Chaffee–Infante equation, we have discussed two different cases, $m = 1$ and $m = 2$ in Eq. (13), and other values of m are ignored, since the algebraic complexity arises rapidly, however, no viable solution can be found in this proportion.

Case 1: First suppose $m = 1$, then by comparing the coefficients of Y^i $(i = 2, 1, 0)$ on both sides of (14), we obtain

$$\dot{a}_1(X) = h(X)a_1(X), \tag{15a}$$

$$\dot{a_0}(X) = -a_1(X)\sigma + a_1(X)c + g(X)a_1(X) + h(X)a_0(X), \tag{15b}$$

$$a_1(X)\left(\alpha\left(X(\xi)\right)^3 - \alpha X(\xi)\right) = g(X)a_0(X). \tag{15c}$$

Since $a_i(X)$ $(i = 0, 1)$ are polynomials, from (15a), we presume that $a_1(X)$ is constant and $h(X) = 0$. For simplicity, we assume $a_1(X) = 1$, and hence (15a)–(15c) can be written as

$$a_1(X) = 1, \tag{16a}$$

$$\dot{a_0}(X) = -\sigma + c + g(X), \tag{16b}$$

$$\alpha\left(X(\xi)\right)^3 - \alpha X(\xi) = g(X)a_0(X). \tag{16c}$$

Balancing the degree of $g(X)$ and $a_0(X)$, we conclude that $\deg g(X) = 1$ only. Suppose that $g(X) = A_1X + B_0$, therefore we find $a_0(X)$ as follows:

Substituting $g(X) = A_1X + B_0$ in (16b) and integrating with respect to X, we attain

$$a_0(X) = -\sigma X + cX + \frac{A_1X^2}{2} + B_0X + A_0, \tag{17}$$

where A_0 is the integral constant.

Substituting $g(X)$ and $a_0(X)$ into (16c) and setting all the coefficients of powers of X to zero, we thus obtain a system of algebraic equations and by solving them with the aid of Maple, provides

$$A_1 = \pm\sqrt{2\alpha}, \qquad A_0 = \mp\frac{1}{2}\sqrt{2\alpha}, \qquad B_0 = 0, \qquad c = \sigma, \tag{18a}$$

$$A_1 = \pm\sqrt{2\alpha}, \qquad A_0 = 0, \qquad B_0 = \pm\sqrt{2\alpha}, \qquad c = \sigma \mp \frac{3}{2}\sqrt{2\alpha}. \tag{18b}$$

Family I: By means of the values assembled in (18a), from (13) we obtain

$$Y(\xi) = \mp\frac{1}{2}\sqrt{2\alpha}X^2 \pm \frac{1}{2}\sqrt{2\alpha}. \tag{19}$$

Combining (19) and (12a), we obtain the closed form solution to Eq. (11) as follows:

$$f(\xi) = \pm\tanh\left(\frac{1}{2}\sqrt{2\alpha}\xi + \xi_0\right), \tag{20}$$

where ξ_0 is an integral constant.

When $\alpha > 0$, Eq. (20) can be resolved as

$$f(\xi) = \pm\frac{1 - \xi_0 e^{-\sqrt{2\alpha}\xi}}{1 + \xi_0 e^{-\sqrt{2\alpha}\xi}}, \tag{21}$$

where ξ_0 is a constant of integration. Therefore the closed form wave solution of (7) can be found in the following shape:

$$u(x, y, t) = \pm\frac{1 - \xi_0 e^{-\sqrt{2\alpha}(x+y-\sigma t)}}{1 + \xi_0 e^{-\sqrt{2\alpha}(x+y-\sigma t)}}. \tag{22}$$

Since ξ_0 is an unknown constant its value can be selected as desired. Now, if we select $\xi_0 = 1$, from (22) we obtain the following wave solution:

$$u(x,y,t) = \pm\tanh\left(\frac{1}{2}\sqrt{2\alpha}(x+y-\sigma t)\right). \tag{23}$$

Furthermore, if we select $\xi_0 = -1$, then from (22) we obtain the under mentioned wave solution

$$u(x,y,t) = \pm\coth\left(\frac{1}{2}\sqrt{2\alpha}(x+y-\sigma t)\right). \tag{24}$$

When $\alpha < 0$, we set $\alpha = -\beta$, $\beta > 0$, then Eq. (20) can be reached to

$$f(\xi) = \pm\frac{1-\xi_0 e^{-i\sqrt{2\beta}\xi}}{1+\xi_0 e^{-i\sqrt{2\beta}\xi}}, \tag{25}$$

where ξ_0 is an unspecified constant. Thus the exact solution of (7) can be written as

$$u(x,y,t) = \pm\frac{1-\xi_0 e^{-i\sqrt{2\beta}(x+y-\sigma t)}}{1+\xi_0 e^{-i\sqrt{2\beta}(x+y-\sigma t)}}. \tag{26}$$

As ξ_0 is an unspecified constant, we might set $\xi_0 = 1$ into (26), we therefore obtain the subsequent wave solution:

$$u(x,y,t) = \pm i\tan\left(\frac{1}{2}\sqrt{-2\alpha}(x+y-\sigma t)\right). \tag{27}$$

However, if we set $\xi_0 = -1$ into (26), we obtain the wave solution as

$$u(x,y,t) = \pm i\cot\left(\frac{1}{2}\sqrt{-2\alpha}(x+y-\sigma t)\right). \tag{28}$$

Family II: Now, using the values of the constants arranged in (18b), from (13) we obtain

$$Y(\xi) = \mp\frac{1}{2}\sqrt{2\alpha}X^2 \pm \frac{1}{2}\sqrt{2\alpha}X. \tag{29}$$

Combining (29) with (12a), we obtain the closed form wave solution to Eq. (11) as follows:

$$f(\xi) = \frac{1}{2} \pm \frac{1}{2}\tanh\left(\frac{1}{4}\sqrt{2\alpha}\xi + \xi_0\right), \tag{30}$$

where ξ_0 is a constant of integration.

When $\alpha > 0$, Eq. (30) can be resolved into

$$f(\xi) = \frac{1}{2} \pm \frac{1}{2}\frac{1-\xi_0 e^{-\frac{1}{2}\sqrt{2\alpha}\xi}}{1+\xi_0 e^{-\frac{1}{2}\sqrt{2\alpha}\xi}}, \tag{31}$$

where ξ_0 is an arbitrary constant. Therefore the exact solution to (7) can be derived as

$$u(x,y,t) = \frac{1}{2} \pm \frac{1}{2}\frac{1-\xi_0 e^{\frac{-1}{2}\sqrt{2\alpha}(x+y-(\sigma\mp\frac{3}{2}\sqrt{2\alpha})t)}}{1+\xi_0 e^{\frac{-1}{2}\sqrt{2\alpha}(x+y-(\sigma\mp\frac{3}{2}\sqrt{2\alpha})t)}}. \tag{32}$$

In particular, if we opt $\xi_0 = 1$ into (32), we obtain the solitary wave solution

$$u(x,y,t) = \frac{1}{2} \pm \frac{1}{2}\tanh\left(\frac{1}{4}\sqrt{2\alpha}\left(x + y - \left(\sigma \mp \frac{3}{2}\sqrt{2\alpha}\right)t\right)\right). \tag{33}$$

Alternatively, if we opt $\xi_0 = -1$ into (32), we obtain the solitary wave solution

$$u(x,y,t) = \frac{1}{2} \pm \frac{1}{2}\coth\left(\frac{1}{4}\sqrt{2\alpha}\left(x + y - \left(\sigma \mp \frac{3}{2}\sqrt{2\alpha}\right)t\right)\right). \tag{34}$$

When $\alpha < 0$, putting $\alpha = -\lambda$, $\lambda > 0$, Eq. (30) can be rebuild as

$$f(\xi) = \frac{1}{2} \pm \frac{1}{2}\frac{1 - \xi_0 e^{\frac{-i}{2}\sqrt{2\lambda}\xi}}{1 + \xi_0 e^{-\frac{-i}{2}\sqrt{2\lambda}\xi}}, \tag{35}$$

where ξ_0 is an arbitrary constant. Consequently the exact solution to (7) can be obtained:

$$u(x,y,t) = \frac{1}{2} \pm \frac{1}{2}\frac{1 - \xi_0 e^{\frac{-i}{2}\sqrt{2\lambda}(x+y-(\sigma\mp\frac{3i}{2}\sqrt{2\lambda})t)}}{1 + \xi_0 e^{\frac{-i}{2}\sqrt{2\lambda}(x+y-(\sigma\mp\frac{3i}{2}\sqrt{2\lambda})t)}}. \tag{36}$$

Particularly, if we choose $\xi_0 = 1$ into (36), we obtain the solitary wave solution

$$u(x,y,t) = \frac{1}{2} \pm \frac{i}{2}\tan\left(\frac{1}{4}\sqrt{-2\alpha}\left(x + y - \left(\sigma \mp \frac{3}{2}\sqrt{2\alpha}\right)t\right)\right). \tag{37}$$

On the other hand, if we take $\xi_0 = -1$ into (36), we obtain the solitary wave solution

$$u(x,y,t) = \frac{1}{2} \mp \frac{i}{2}\cot\left(\frac{1}{4}\sqrt{-2\alpha}\left(x + y - \left(\sigma \mp \frac{3}{2}\sqrt{2\alpha}\right)t\right)\right). \tag{38}$$

Case 2: Suppose that $m = 2$, then by equating the coefficients of Y^i $(i = 3, 2, 1, 0)$ on both sides of (14), we obtain

$$\dot{a}_2(X) = h(X)a_2(X), \tag{39a}$$

$$\dot{a}_1(X) = -2a_2(X)\sigma + 2a_2(X)c + g(X)a_2(X) + a_1(X)h(X), \tag{39b}$$

$$\begin{aligned}\dot{a}_0(X) = {} & -a_1(X)\sigma + a_1(X)c - 2a_2(X)\left(\alpha\left(X(\xi)\right)^3 - \alpha X(\xi)\right) \\ & + g(X)a_1(X) + h(X)a_0(X),\end{aligned} \tag{39c}$$

$$a_1(X)\left(\alpha\left(X(\xi)\right)^3 - \alpha X(\xi)\right) = g(X)a_0(X). \tag{39d}$$

Since $a_i(X)$ $(i = 0, 1, 2)$ are polynomials, from (39a), we deduce that $a_2(X)$ is a constant and $h(X) = 0$. For simplicity, we have taken $a_2(X) = 1$, and hence (39a)–(39d) can be written as

$$a_2(X) = 1, \tag{40a}$$

$$\dot{a}_1(X) = -2\sigma + 2c + g(X), \tag{40b}$$

$$\dot{a_0}(X) = -a_1(X)\sigma + a_1(X)c - 2\alpha\big(X(\xi)\big)^3 + 2\alpha X(\xi) + g(X)a_1(X), \tag{40c}$$

$$a_1(X)\big(\alpha\big(X(\xi)\big)^3 - \alpha X(\xi)\big) = g(X)a_0(X). \tag{40d}$$

Balancing the degree of $g(X)$, $a_1(X)$ and $a_0(X)$, yield the $\deg(g(X)) = 1$. Therefore, we assume that $g(X) = A_1X + B_0$, and thus we find $a_1(X)$ and $a_0(X)$ as follows:

$$a_1(X) = \frac{A_1X^2}{2} + B_0X + A_0 + 2cX - 2\sigma X, \tag{41}$$

$$\begin{aligned} a_0(X) = {} & -\left(\frac{1}{6}A_1X^3 + \frac{1}{2}B_0X^2 + A_0X + cX^2 - \sigma X^2\right)\sigma - \frac{1}{2}\alpha X^4 + \alpha X^2 + \frac{1}{8}A_1^2X^4 \\ & + \frac{1}{3}\left((-2\sigma + B_0 + 2c)A_1 + \frac{1}{2}A_1B_0\right)X^3 + \frac{1}{2}\big(A_0A_1 + (-2\sigma + B_0 + 2c)B_0\big)X^2 \\ & + A_0B_0X + c\left(\frac{1}{6}A_1X^3 + \frac{1}{2}B_0X^2 + A_0X + cX^2 - \sigma X^2\right) + d, \end{aligned} \tag{42}$$

where A_0 and d are the integral constants.

Substituting $g(X)$, $a_1(X)$ and $a_0(X)$ into (40d) and setting each coefficient of similar power of X to zero provides a system of algebraic equations and solving them with the aid of Maple, we obtain

$$A_1 = \pm 2\sqrt{2\alpha}, \qquad A_0 = 0, \qquad c = \mp\frac{3}{2}\sqrt{2\alpha} + \sigma, \qquad B_0 = \pm 2\sqrt{2\alpha}, \qquad d = 0, \tag{43a}$$

$$A_1 = \mp 2\sqrt{2\alpha}, \qquad A_0 = \pm\sqrt{2\alpha}, \qquad c = \sigma, \qquad B_0 = 0, \qquad d = \frac{1}{2}\alpha. \tag{43b}$$

Family III: Setting (43a) into (13), we obtain

$$Y(\xi) = \pm\frac{1}{2}\sqrt{2\alpha}\big(X - X^2\big). \tag{44}$$

Combining (44) and (12a), we obtain the solution to Eq. (11) as follows:

$$f(\xi) = \frac{1}{2} \pm \frac{1}{2}\tanh\left(\frac{1}{4}\sqrt{2\alpha}\xi + \xi_0\right), \tag{45}$$

where ξ_0 is a constant of integration.

When $\alpha > 0$, from Eq. (45) it can be found

$$f(\xi) = \frac{1}{2} \pm \frac{1}{2}\frac{1 - \xi_0 e^{-\frac{1}{2}\sqrt{2\alpha}\xi}}{1 + \xi_0 e^{-\frac{1}{2}\sqrt{2\alpha}\xi}}, \tag{46}$$

where ξ_0 is an integral constant. Accordingly the exact wave solution of (7) can be written as

$$u(x,y,t) = \frac{1}{2} \pm \frac{1}{2}\frac{1 - \xi_0 e^{-\frac{1}{2}\sqrt{2\alpha}(x+y-(\sigma\mp\frac{3}{2}\sqrt{2\alpha})t)}}{1 + \xi_0 e^{-\frac{1}{2}\sqrt{2\alpha}(x+y-(\sigma\mp\frac{3}{2}\sqrt{2\alpha})t)}}. \tag{47}$$

Particularly, if we set $\xi_0 = 1$ into (47), we obtain the subsequent solitary wave solution

$$u(x,y,t) = \frac{1}{2} \pm \frac{1}{2}\tanh\left(\frac{1}{4}\sqrt{2\alpha}\left(x + y - \left(\sigma \mp \frac{3}{2}\sqrt{2\alpha}\right)t\right)\right). \tag{48}$$

However, if we set $\xi_0 = -1$ into (47), we obtain the succeeding solitary wave solution

$$u(x,y,t) = \frac{1}{2} \pm \frac{1}{2}\coth\left(\frac{1}{4}\sqrt{2\alpha}\left(x+y-\left(\sigma \mp \frac{3}{2}\sqrt{2\alpha}\right)t\right)\right). \tag{49}$$

When $\alpha < 0$, we substitute $\alpha = -\nu$, $\nu > 0$, Eq. (45) can be transformed to

$$f(\xi) = \frac{1}{2} \pm \frac{1}{2}\frac{1-\xi_0 e^{\frac{-i}{2}\sqrt{2\nu}\xi}}{1+\xi_0 e^{\frac{-i}{2}\sqrt{2\nu}\xi}}, \tag{50}$$

where ξ_0 is an arbitrary constant. Thus, the closed form solution to (7) can be written as

$$u(x,y,t) = \frac{1}{2} \pm \frac{1}{2}\frac{1-\xi_0 e^{\frac{-i}{2}\sqrt{2\nu}(x+y-(\sigma\mp\frac{3i}{2}\sqrt{2\nu})t)}}{1+\xi_0 e^{\frac{-i}{2}\sqrt{2\nu}(x+y-(\sigma\mp\frac{3i}{2}\sqrt{2\nu})t)}}. \tag{51}$$

Since ξ_0 is an arbitrary constant, one might instinctively pick its value. Accordingly, if we pick $\xi_0 = 1$, from (51) we obtain the solitary wave solution

$$u(x,y,t) = \frac{1}{2} \pm \frac{i}{2}\tan\left(\frac{1}{4}\sqrt{-2\alpha}\left(x+y-\left(\sigma \mp \frac{3i}{2}\sqrt{-2\alpha}\right)t\right)\right). \tag{52}$$

Moreover, if we pick $\xi_0 = -1$, then from (51) we obtain the closed form wave solution

$$u(x,y,t) = \frac{1}{2} \mp \frac{i}{2}\cot\left(\frac{1}{4}\sqrt{-2\alpha}\left(x+y-\left(\sigma \mp \frac{3i}{2}\sqrt{-2\alpha}\right)t\right)\right). \tag{53}$$

Family IV: Again, using (43b) into (13), we obtain

$$Y(\xi) = \pm\frac{1}{2}\sqrt{2\alpha}\left(X^2-1\right). \tag{54}$$

Combining (54) with (12a), we obtain the exact solution to Eq. (11) as follows:

$$f(\xi) = \pm\tanh\left(\frac{1}{2}\sqrt{2\alpha}\xi + \xi_0\right), \tag{55}$$

where ξ_0 is an integral constant.

When $\alpha > 0$, Eq. (55) can be solved as

$$f(\xi) = \pm\frac{1-\xi_0 e^{-\sqrt{2\alpha}\xi}}{1+\xi_0 e^{-\sqrt{2\alpha}\xi}}. \tag{56}$$

Consequently, we extract the closed form wave solution to Eq. (7):

$$u(x,y,t) = \mp\frac{1-\xi_0 e^{-\sqrt{2\alpha}(x+y-\sigma t)}}{1+\xi_0 e^{-\sqrt{2\alpha}(x+y-\sigma t)}}. \tag{57}$$

Inasmuch as ξ_0 is unspecified constant, we can sort the value of ξ_0 randomly. For simplicity and conciseness, we have selected $\xi_0 = 1$ and thus from (57), we attain the following closed form wave solution:

$$u(x,y,t) = \mp\tanh\left(\frac{1}{2}\sqrt{2\alpha}(x+y-\sigma t)\right). \tag{58}$$

Again, if we select $\xi_0 = -1$, then from (57), we achieve the following wave solution:

$$u(x,y,t) = \mp\coth\left(\frac{1}{2}\sqrt{2\alpha}(x+y-\sigma t)\right). \tag{59}$$

When $\alpha < 0$, we use $\alpha = -\mu$, $\mu > 0$, hence from (55) it can be obtained

$$f(\xi) = \mp\frac{1-\xi_0 e^{-i\sqrt{2\mu}\xi}}{1+\xi_0 e^{-i\sqrt{2\mu}\xi}}, \tag{60}$$

where ξ_0 is an unknown constant. Thus, the exact wave solution to (7) can be found as

$$u(x,y,t) = \mp\frac{1-\xi_0 e^{-i\sqrt{2\mu}(x+y-\sigma t)}}{1+\xi_0 e^{-i\sqrt{2\mu}(x+y-\sigma t)}}. \tag{61}$$

Forasmuch as ξ_0 is an unknown constant, for compact form solution, we have first accepted $\xi_0 = 1$, then from (61), we acquire the wave solution

$$u(x,y,t) = \mp i\tan\left(\frac{1}{2}\sqrt{-2\alpha}(x+y-\sigma t)\right). \tag{62}$$

On the other hand, if we accept $\xi_0 = -1$, then from (61), we gain the compact form solution wave solution

$$u(x,y,t) = \pm i\cot\left(\frac{1}{2}\sqrt{-2\alpha}(x+y-\sigma t)\right). \tag{63}$$

It is remarkable to observe that the obtained closed form wave solutions to the (2 + 1)-dimensional Chaffee–Infante equation are significant and practically well suited. The solutions are obtained in compact form and thus might be useful to analyze the gas diffusion in a homogeneous medium.

3.2 The dimensionless form of the Zakharov equation (ZE)

In order to interpret the interaction between low-frequency ion-acoustic waves and the high-frequency Langmuir waves, Zakharov [26, 27] first derived the dimensionless form of the ZE. In this subsection, we will consider the dimensionless form of the ZE [26–28] in the following form:

$$iq_t + q_{xx} + bF\left(|q|^2\right)q = qr, \tag{64}$$

$$r_{tt} - r_{xx} = \left(|q|^{2\gamma}\right)_{xx}. \tag{65}$$

Here the potential function $r(x,t)$ is the plasma density determined from its equilibrium state and the complex potential $q(x,t)$ is envelope of the high-frequency electric field [26]. The ZE can be deduced from the hydrodynamic explanation of the plasma and a simplified model of strong Langmuir turbulence [29]. Therefore, the ZE was summed up by considering more components. The generalized Zakharov equation (GZE) is a set of coupled equations and might be written as [26]:

$$iq_t + q_{xx} = rq + b_1|q|^{2\gamma} q + b_2|q|^{4\gamma} q, \tag{66}$$

$$r_{tt} - r_{xx} = \left(|q|^{2\gamma}\right)_{xx}, \tag{67}$$

where $b_1, b_2, \gamma > 0$ are real parameters and the above process can be transformed to the original Zakharov equations of plasma physics by setting $b_1 = 0$, $b_2 = 0$ and $\gamma = 1$. Since it is very difficult to examine the closed form wave solutions to the GZE due to strong nonlinearity and if we set $b_1 = -b$, $b_2 = 0$, $F(|q|^2) = |q|^{2\gamma}$ and $\gamma = 1$, the GZE is converted into the dimensionless form of the Zakharov equation, in this article, we have studied the dimensionless form of the ZE (64) and (65).

In (64), F is a real-valued algebraic nonlinear function and it is necessary to have the smoothness of the complex function $F(|q|^2)q : C \mapsto C$. Considering the complex plane C as a two-dimensional linear space R^2, the function $F(|q|^2)q$ is k times continuously differentiable, so that (Kohl [30])

$$F\left(|q|^2\right)q \in \bigcup_{m,n=1}^{\infty} C^k\left((-n,n) \times (-m,m); R^2\right). \tag{68}$$

Therefore, Eqs. (64) and (65) can be rewritten in the following form:

$$\left.\begin{array}{l} iq_t + q_{xx} + b(|q|^2)q = qr, \\ r_{tt} - r_{xx} = (|q|^2)_{xx}. \end{array}\right\} \tag{69}$$

Here, $F(|q|^2) = |q|^2$ and $\gamma = 1$.

In order to investigate Eq. (69) by using the first integral method, we use the following transformation:

$$q(x,t) = f(\xi)e^{i\phi}, \quad \xi = x - ct, \phi = -kx + \omega t + \theta \quad \text{and} \quad r(x,t) = h(\xi), \tag{70}$$

where $f(\xi)$ represents the shape of the pulse and c is the velocity of the soliton, k is the wave number, while ω is the frequency and θ is the phase constant.

Substituting (70) into (69), we obtain the subsequent ordinary differential equations:

$$\left.\begin{array}{l} (-icf' - \omega f + f'' - 2ikf' - k^2 f + bf^3 - fh)e^{i\phi} = 0, \\ c^2h'' - h'' - (f^2)'' = 0. \end{array}\right\} \tag{71}$$

Equation (71) can be rewritten as follows:

$$\left.\begin{array}{l} -i(2k + c)f' + f'' - (\omega + k^2)f + bf^3 - fh = 0, \quad e^{i\phi} \neq 0, \\ c^2h'' - h'' - (f^2)'' = 0. \end{array}\right\} \tag{72}$$

In actuality, we might set $2k + c = 0$ in the first equation of (72) and integrating the second equation of (72) twice with respect to ξ, considering the constant of integration being zero, yields

$$\left.\begin{array}{l} f'' - (\omega + k^2)f + bf^3 - fh = 0, \\ c^2h - h - f^2 = 0. \end{array}\right\} \tag{73}$$

From the second equation of (73), we obtain the affinity

$$h = \frac{f^2}{c^2 - 1}. \tag{74}$$

Now, using the first equation of (73) and the affinity (74), we obtain to the following ordinary differential equation:

$$f'' - (\omega + k^2)f + \left(b - \frac{1}{c^2 - 1}\right)f^3 = 0. \tag{75}$$

Using (5) and (6), we get

$$\dot{X}(\xi) = Y(\xi), \tag{76a}$$

$$\dot{Y}(\xi) = \left((\omega + k^2)X(\xi) - \left(b - \frac{1}{c^2 - 1}\right)\bigl(X(\xi)\bigr)^3\right). \tag{76b}$$

According to the first integral method, we suppose that $X(\xi)$ and $Y(\xi)$ are nontrivial solutions of (76a)–(76b), and

$$q(X, Y) = \sum_{i=0}^{m} a_i(X)Y^i = 0,$$

is an irreducible polynomial in the complex domain $C[X, Y]$ such that

$$q\bigl(X(\xi), Y(\xi)\bigr) = \sum_{i=0}^{m} a_i\bigl(X(\xi)\bigr)Y^i(\xi) = 0, \tag{77}$$

where $a_i(X)$ $(i = 0, 1, \dots, m)$, are polynomials of X and $a_m(X) \neq 0$. Eq. (77) is called the first integral of (76a)–(76b). As a result of the division hypothesis, there exists a polynomial $g(X) + h(X)Y$, in the complex domain $C[x, Y]$ such that

$$\frac{dq}{d\xi} = \frac{\partial q}{\partial X}\frac{\partial X}{\partial \xi} + \frac{\partial q}{\partial Y}\frac{\partial Y}{\partial \xi} = \bigl(g(X) + h(X)Y\bigr)\sum_{i=0}^{m} a_i(X)Y^i. \tag{78}$$

Suppose that $m = 1$ and by comparing with the coefficients of Y^i $(i = 2, 1, 0)$ on both sides of (78), we obtain

$$\dot{a}_1(X) = h(X)a_1(X), \tag{79a}$$

$$\dot{a}_0(X) = g(X)a_1(X) + h(X)a_0(X), \tag{79b}$$

$$a_1(X)\left\{(\omega + k^2)X(\xi) - \left(b - \frac{1}{c^2 - 1}\right)\bigl(X(\xi)\bigr)^3\right\} = g(X)a_0(X). \tag{79c}$$

Since $a_i(X)$ $(i = 0, 1)$ are polynomials, thus from (79a), we deduce that $a_1(X)$ is constant and $h(X) = 0$. For simplicity, we assume $a_1(X) = 1$, and hence (79a)–(79c) can be written as

$$a_1(X) = 1, \tag{80a}$$

$$\dot{a}_0(X) = g(X), \tag{80b}$$

$$(\omega + k^2)X(\xi) - \left(b - \frac{1}{c^2 - 1}\right)\left(X(\xi)\right)^3 = g(X)a_0(X). \tag{80c}$$

Balancing the degree of $g(X)$ and $a_0(X)$, we conclude that $\deg g(X) = 1$ only. Suppose that $g(X) = A_1X + B_0$, therefore we find $a_0(X)$ as follows:

$$a_0(X) = \frac{A_1X^2}{2} + B_0X + A_0, \tag{81}$$

where A_0 is the integral constant.

Substituting $g(X)$ and $a_0(X)$ into (80c) and setting all the coefficients of powers of X to zero, we obtain a system of nonlinear algebraic equations and by solving it with the aid of Maple, we obtain

$$c = \pm\sqrt{\frac{A_1^2 + 2b + 2}{2b + A_1^2}}, \qquad A_0 = \frac{\omega + k^2}{A_1}, \qquad B_0 = 0. \tag{82}$$

Now, using (82) into (77), we obtain

$$Y = -\frac{1}{2}A_1X^2 - \frac{\omega + k^2}{A_1}. \tag{83}$$

Combining (83) and (76a), we obtain the exact solution to Eq. (75) as follows:

$$f(\xi) = \frac{-1}{A_1}\sqrt{2(\omega + k^2)}\tan\left(\frac{1}{2}\sqrt{2(\omega + k^2)}\xi + \xi_0\right), \tag{84}$$

where ξ_0 is an integral constant.

When $\omega + k^2 > 0$, Eq. (84) can be written as

$$f(\xi) = \frac{-1}{A_1}\sqrt{2(\omega + k^2)}\tan\left(\frac{1}{2}\sqrt{2(\omega + k^2)}\xi\right). \tag{85}$$

Thus, the exact solution of (69) reduces to

$$q(x,t) = \frac{-1}{A_1}\sqrt{2(\omega + k^2)}\tan\left(\frac{1}{2}\sqrt{2(\omega + k^2)}\left(x \mp \sqrt{\frac{A_1^2 + 2b + 2}{2b + A_1^2}}t\right)\right)e^{i(-kx+\omega t+\theta)} \tag{86}$$

and

$$r(x,t) = \frac{(\omega + k^2)(2b + A_1^2)}{A_1^2}\tan^2\left(\frac{1}{2}\sqrt{2(\omega + k^2)}\left(x \mp \sqrt{\frac{A_1^2 + 2b + 2}{2b + A_1^2}}t\right)\right). \tag{87}$$

When $\omega + k^2 < 0$, then from Eq. (84), we obtain

$$f(\xi) = \frac{1}{A_1}\sqrt{-2(\omega + k^2)}\left(\frac{1 - \xi_0 e^{-\sqrt{-2(\omega+k^2)}\xi}}{1 + \xi_0 e^{-e^{-\sqrt{-2(\omega+k^2)}\xi}}}\right), \tag{88}$$

where ξ_0 is an arbitrary constant.

Particularly, if we set $\xi_0 = 1$ into (88), we obtain the solitary wave solution

$$f(\xi) = \frac{1}{A_1}\sqrt{-2(\omega + k^2)}\tanh\left(\frac{1}{2}\sqrt{-2(\omega + k^2)}\xi\right). \tag{89}$$

Thus, the exact solution to (69) transformed to:

$$q(x,t) = \frac{1}{A_1}\sqrt{-2(\omega + k^2)}\tanh\left(\frac{1}{2}\sqrt{-2(\omega + k^2)}\left(x \mp \sqrt{\frac{A_1^2 + 2b + 2}{2b + A_1^2}}t\right)\right)e^{i(-kx+\omega t+\theta)} \tag{90}$$

and

$$r(x,t) = -\frac{(\omega + k^2)(2b + A_1^2)}{A_1^2}\tanh^2\left(\frac{1}{2}\sqrt{-2(\omega + k^2)}\left(x \mp \sqrt{\frac{A_1^2 + 2b + 2}{2b + A_1^2}}t\right)\right). \tag{91}$$

On the other hand, if we set $\xi_0 = -1$ into (88), we obtain the solitary wave solution

$$f(\xi) = \frac{1}{A_1}\sqrt{-2(\omega + k^2)}\coth\left(\frac{1}{2}\sqrt{-2(\omega + k^2)}\xi\right). \tag{92}$$

Thus, the exact solution to (69) can be written as

$$q(x,t) = \frac{1}{A_1}\sqrt{-2(\omega + k^2)}\coth\left(\frac{1}{2}\sqrt{-2(\omega + k^2)}\left(x \mp \sqrt{\frac{A_1^2 + 2b + 2}{2b + A_1^2}}t\right)\right)e^{i(-kx+\omega t+\theta)} \tag{93}$$

and

$$r(x,t) = \frac{-(\omega + k^2)(2b + A_1^2)}{A_1^2 a}\coth^2\left(\frac{1}{2}\sqrt{-2(\omega + k^2)}\left(x \mp \sqrt{\frac{A_1^2 + 2b + 2}{2b + A_1^2}}t\right)\right). \tag{94}$$

The obtained solutions are convenient to search the nature the wave profile of the ion-acoustic waves in plasma, the waves of electromagnetic field, the sound waves, the signal processing waves through optical fibers, material science etc.

4 Graphical representations

In this section, we have recapitulated the graphical representations and physical significance of the obtained solutions for the definite values of the included parameters through depicting 3D figures by means of symbolic computational software like Mathematica which are given now.

4.1 The (2 + 1)-dimensional Chaffee–Infante equation

In this subsection, we have depicted the shape of figures of the obtained solutions to the (2 + 1)-dimensional Chaffee–Infante equation which is given below.

The figures of the obtained solutions (23), (33), (48) and (58) are kink shape soliton. For conciseness, we have plotted only the figure of the obtained solution (33) for the definite values of the parameters $\alpha = 3$, $\sigma = 2.75$, $t = 1.40$ (Fig. 1).

The figures of the attained solutions (24), (34), (49) and (59) are singular kink shape solitons. For brevity, we have plotted only the figure of the attained solution (24) for the definite values of the parameters $\alpha = 1.35$, $\sigma = -0.1$, $t = 0.75$ (Fig. 2).

Figure 1 3D plot of kink shape soliton of solution (33) within the interval $-10 \leq x,y \leq 10$

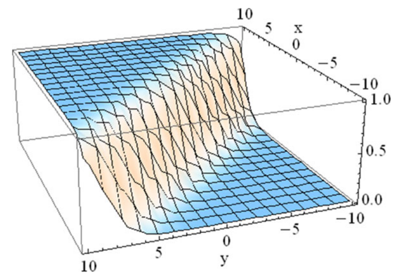


Figure 2 3D plot of singular kink shape soliton of solution (24) within the interval $-10 \leq x,y \leq 10$

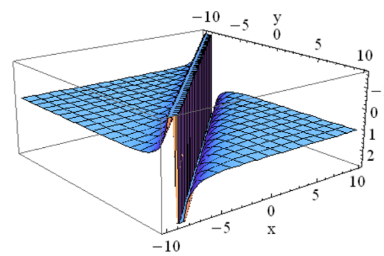


Figure 3 3D modulus plot of solution (27) which is anti-bell shape soliton within the interval $-10 \leq x,y \leq 10$

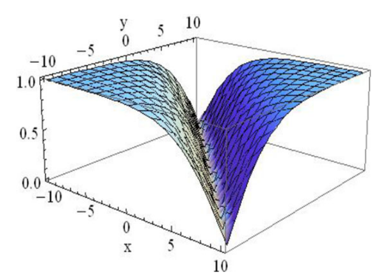
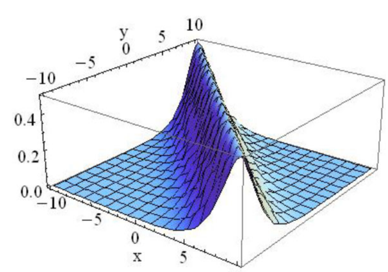


Figure 4 3D modulus plot of solution (37) which is bell shape soliton within the interval $-10 \leq x,y \leq 10$



The figure of the acquired solutions (27) and (58) are anti-bell shape soliton. For minimalism, we have sketched merely the figure of the acquired solution (27) for the definite values $\alpha = -0.08$, $\sigma = 1$, $t = 0.30$ of the parameters and given in Fig. 3.

The figure of the obtained solutions (37) and (52) are bell shape soliton. For compactness, we have depicted only the figure of the obtained solution (37) for the definite values $\alpha = -3$, $\sigma = 2.50$, $t = 0$ of the parameters and given in Fig. 4.

The figure of the achieved solutions (28) and (63) are of a singular bell shape soliton. For simplicity, we have outlined only the figure of the achieved solution (28) for the definite values $\alpha = -2.15$, $\sigma = 1.50$, $t = 0.1$ of the parameters (Fig. 5).

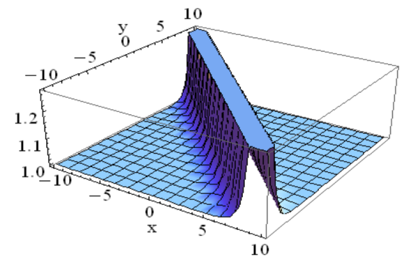


Figure 5 3D modulus plot of solution (28) which is singular bell shape soliton within the interval $-10 \le x, y \le 10$

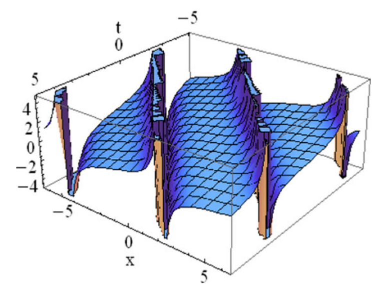


Figure 6 3D plot of singular periodic wave of solution (86) within the interval $-5 \le x, t \le 5$

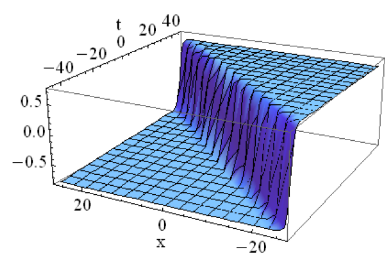


Figure 7 3D plot of kink wave of solution (90) within the interval $-20 \le x \le 20$, $-40 \le t \le 40$

4.2 The dimensionless form of the Zakharov equation (ZE)

In this subsection, we have depicted the shape of figures of the obtained solutions to the dimensionless form of the ZE:

The shape of the solutions (86) and (87) are singular periodic soliton. We have portrayed the figure of the solution (86) only for the definite values $b = 0.45$, $k = 0.65$, $\theta = -0.45$, $\omega = 0.10$, $A_1 = -1.20$ of the parameters (Fig. 6).

The shape of the solution (90) is kink-type soliton. In Fig. 7 it has been interpreted for the definite values of the parameters $b = 0.46$, $k = 1$, $\theta = 2$, $\omega = -2$, $A_1 = -2$.

The shape of the solutions (91) and (94) is a singular bell shape soliton. We have portrayed the solution (91) only to shorten the article for the definite values $b = 1$, $k = 1$, $\omega = -2$, $A_1 = 1$ of the parameters (Fig. 8).

The solution (93) has been traced for the definite values $b = 0.60$, $k = 1$, $\theta = 2$, $\omega = -2$, $A_1 = 1$ of the parameters (Fig. 9).

From the graphical representations clssified above of the attained closed form wave solutions for their definite values of the parameters of the $(2 + 1)$-dimensional Chaffee–Infante equation and the dimensionless form of the Zakharov equation by using the first integral method, we assert that the solutions might be useful to analyze the physical phenomena. It is noteworthy to observe that we have found different well-known shapes of wave solutions, like, kink-type wave solutions, singular kink-type wave solutions, bell shaped wave

Figure 8 3D plot of solution (91) which is singular bell shape soliton within the interval $-10 \le x, t \le 10$

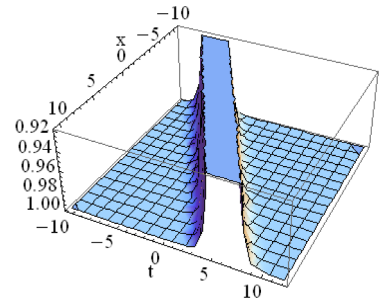
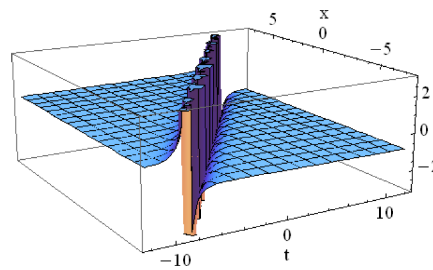


Figure 9 3D plot of singular kink wave of solution (93) within the interval $-8 \le x \le 8$, $-10 \le t \le 10$



solutions, singular bell shaped wave solutions, anti-bell shaped wave solutions, and singular periodic-type wave solutions for suitable intervals.

5 Conclusion

In this article, we have successfully extracted new, useful and further general exact soliton solutions to the (2 + 1)-dimensional Chaffee–Infante equation and the dimensionless form of the ZE through the first integral method. The soliton solutions attained in this study might be useful to analyze the signal through optical fibers, the waves of electromagnetic field and plasma physics. It is remarkable to discern that the solutions are formulated subject to the hyperbolic, trigonometric and exponential functions. The results show that the first integral method is powerful, accurate and effective. Most of the extracted closed form solitary wave solutions may be useful for describing certain nonlinear physical phenomena. Computational software, namely Maple, has been used for computations and programming in this paper. The obtained solutions were verified to check the correctness by putting them back into the original equation and they were found to be correct.

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Authors' contributions
This work was done in coordinated effort among the authors. All authors have made a significant contribution to the planning of the study, and to complete the investigation of this work. All authors read and endorsed the final manuscript.

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