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Modelling and nonlinear boundary stabilization of the modified generalized Korteweg–de Vries–Burgers equation

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Abstract

In this paper, we study the modelling and nonlinear boundary stabilization problem of the modified generalized Korteweg–de Vries–Burgers equation (MGKdVB) when the spatial domain is [0, 1]. First, the MGKdVB equation is derived using the long-wave approximation and perturbation method. Then, two nonlinear boundary controllers are proposed for this equation and the L^2-global exponential stability of the solution is shown. Numerical simulations are given to illustrate the efficiency of the developed control schemes.

Keywords: Modified generalized Korteweg–de Vries–Burgers equation; Exponential stability; Boundary control

1 Introduction

The following modified generalized Korteweg–de Vries–Burgers (MGKdVB) equation is considered:

$$\frac{\partial u}{\partial t} + \gamma_1 u^\alpha \frac{\partial u}{\partial x} - \nu \frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial^3 u}{\partial x^3} + \gamma_2 \frac{\partial^4 u}{\partial x^4} = 0, \quad x \in (0,1), t > 0, \tag{1}$$

subject to the following boundary conditions:

$$u(0,t) = 0, \quad t > 0, \tag{2}$$

$$\frac{\partial^2 u}{\partial x^2}(0,t) = 0, \quad t > 0, \tag{3}$$

$$\frac{\partial u}{\partial x}(1,t) = w_1(t), \quad t > 0, \tag{4}$$

$$\frac{\partial^2 u}{\partial x^2}(1,t) = w_2(t), \quad t > 0, \tag{5}$$

and the following initial condition:

$$u(x,0) = u_0(x), \quad x \in (0,1), \tag{6}$$

where $w_1(t)$ and $w_2(t)$ are nonlinear boundary controls.

In Eq. (1), α is a positive integer and all the parameters γ_1, ν, μ and γ_2 are nonzero known positive real constants. The MGKdVB equation is a nonlinear partial differential equation that is of first order in time and of fourth order in space. It exhibits the standard elements of any nonlinear process that involves wave evolution. The terms $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^4 u}{\partial x^4}$ in Eq. (1) show the features of dissipation and the terms $\frac{\partial^3 u}{\partial x^3}$ and $u^\alpha \frac{\partial u}{\partial x}$ show the features of dispersion and nonlinearity, respectively.

The MGKdVB equation is of significant importance when it comes to describing the physical processes in motion of turbulence and other chaotic process systems. Different physical systems can be modeled using this equation depending on the values of γ_1, ν, μ and γ_2. In fact, if $\alpha = 1, \gamma_2 = \gamma_1 = 1, \mu = 0$ and ν is negative in Eq. (1), then the MGKdVB equation reduces to the Kuramoto–Sivashinsky (KS) equation. Note that Kuramoto [1] derived the KS equation independently to model reaction-diffusion systems, and it was also derived by Sivashinsky [2] to model flame front propagation in turbulent flows. The KS equation is considered as a fourth order nonlinear equation and has been the subject of many research studies [3–13].

When $\alpha = \gamma_1 = 1$ and ν is negative, the MGKdVB equation becomes the Generalized Kuramoto–Sivashinsky (GKS) equation [14–18]. If $\nu = \gamma_2 = 0$, and $\alpha = \gamma_1 = 1$ in Eq. (1), the MGKdVB equation becomes the Korteweg–de Vries (KdV) equation which was derived by Korteweg and de Vries to model the translation of water waves observed by Russell [19]. The KdV equation was used to describe several phenomena such as waves in a rotating atmosphere or ion-acoustic waves in plasma [20].

Also, the Burgers equation can be obtained from Eq. (1) by setting $\alpha = \gamma_1 = 1$ and $\mu = \gamma_2 = 0$. This equation was first derived by Burgers [21] as a prototype model for turbulent liquid flow. Many scientists have intensively studied this equation [22–24]. Furthermore, when $\alpha = \gamma_1 = 1$ and $\gamma_2 = 0$, the MGKdVB equation reduces to the Korteweg–de Vries–Burgers (KdVB) equation [25–30].

When $\gamma_1 = 1$ and $\gamma_2 = 0$ in Eq. (1), the MGKdVB equation gives the generalized Korteweg–de Vries–Burgers (GKdVB) equation; the non-adaptive and adaptive control problems of this equation were studied by Smaoui and Jamal [30] and Smaoui et al. [31–33].

Since the nonlinear stabilization problem of the MGKdVB equation has not been investigated elsewhere, we study this equation analytically as well as numerically and show the L^2-global exponential stability of its solutions.

The existence and uniqueness of solutions of the MGKdVB equation have been investigated by Smaoui et al. [34]. The work in this paper is build upon assuming the existence of a unique solution $u(x,t)$ of this equation in the following space: $L^\infty(0,T;H_0^1(0,1)\cap H^4(0,1))\cap C(0,T;H_0^1(0,1)\cap H^2(0,1))$.

This paper is arranged as follows. In Sect. 2, the MGKdVB equation is derived based on the long-wave approximation and perturbation method when $\alpha = 3$. In Sects. 3 and 4, two nonlinear boundary controllers are proposed for the MGKdVB equation when the parameters ν, μ, γ_1 and γ_2 are known and positive real constants, and when α is a positive integer. A qualitative and numerical study shows the global exponential stability of the solutions in $L^2(0,1)$. Section 5 presents a numerical simulation of the uncontrolled MGKdVB equation. In Sect. 6, the rates of convergence for the solutions of the two designed controllers are compared with the solution obtained without control. Finally, some concluding remarks are presented in Sect. 7.

2 The derivation of the modified generalized Korteweg–de Vries–Burgers equation

In this section, we use the long-wave approximation and the perturbation method [35, 36] to derive the MGKdVB equation when $\alpha = 3$. It should be noted that the asymptotic constructions presented here are formal and only have the goal to derive the MGKdVB equation (1).

In the same spirit of the work of Demiray [35], we assume that blood behaves like an incompressible Newtonian fluid [37], then the conservation of mass and linear momentum equations governing the motion of prestressed thick elastic tube filled with a viscous fluid can be derived in cylindrical polar coordinates as follows:

$$\frac{\partial U_r}{\partial r} + \frac{U_r}{r} + \frac{\partial U_z}{\partial z} = 0, \tag{7}$$

$$\frac{\partial U_r}{\partial t} + U_r\frac{\partial U_r}{\partial r} + U_z\frac{\partial U_r}{\partial z} + \frac{1}{\rho}\frac{\partial P}{\partial r} - \overline{\nu}\left(\frac{\partial^2 U_r}{\partial r^2} + \frac{1}{r}\frac{\partial U_r}{\partial r} - \frac{U_r}{r^2} + \frac{\partial^2 U_r}{\partial z^2}\right) = 0, \tag{8}$$

$$\frac{\partial U_z}{\partial t} + U_r\frac{\partial U_z}{\partial r} + U_z\frac{\partial U_z}{\partial z} + \frac{1}{\rho}\frac{\partial P}{\partial z} - \overline{\nu}\left(\frac{\partial^2 U_z}{\partial r^2} + \frac{1}{r}\frac{\partial U_z}{\partial r} + \frac{\partial^2 U_z}{\partial z^2}\right) = 0, \tag{9}$$

where U_r is the radial fluid velocity component, and U_z is the axial fluid velocity component. ρ is the mass density, P is the fluid pressure function, and $\overline{\nu}$ is the kinematic viscosity of the fluid.

In 2003, Demiray [35] applied an averaging procedure to Eqs. (7)–(9), where he derived the following dimensionless equations to show the propagation of small but finite amplitude wave in a prestressed thick viscoelastic tube that was filled with a viscous fluid (blood):

$$2\frac{\partial u}{\partial t} + (1+u)\frac{\partial w}{\partial x} + 2w\frac{\partial u}{\partial x} = 0, \tag{10}$$

$$\frac{\partial w}{\partial t} + w\frac{\partial w}{\partial x} + \frac{\partial p}{\partial x} - \nu\left(-\frac{8w}{(1+u)^2} + \frac{\partial^2 w}{\partial x^2}\right) = 0, \tag{11}$$

where the dimensionless pressure equation can be represented by

$$p = \beta_1 u + \beta_2\frac{\partial^2 u}{\partial x^2} + \beta_3\frac{\partial^2 u}{\partial t^2} + \beta_4\frac{\partial u}{\partial t} + \beta_5\frac{\partial^3 u}{\partial x^2\partial t} + \beta_6 u^2 + \beta_7 u^3 + \cdots, \tag{12}$$

and where u, w and ν characterize the dimensionless dynamical radial displacement, the averaged axial fluid velocity divided by the Moens–Korteweg speed and the kinematic viscosity, respectively. The coefficients $\beta_1, \beta_2, \beta_6$ and β_7 are the elastic effects, β_4 and β_5 are the viscous effects, and β_3 shows the inertial effect.

Next introduce the following coordinate's transformation:

$$\xi = \varepsilon^{\delta}(x - gt) \tag{13}$$

and

$$\tau = \varepsilon^{\delta+\gamma} gt, \tag{14}$$

where the parameter ε measures the size of the nonlinearity, dispersion and dissipation; δ and γ are positive constants, and the parameter g scales the speed of a linearized wave. By applying the above transformation to Eqs. (10)–(12) one will get the following:

$$2g\left(-\frac{\partial u}{\partial \xi}+\varepsilon^{\gamma}\frac{\partial u}{\partial \tau}\right)+(1+u)\frac{\partial w}{\partial \xi}+2w\frac{\partial u}{\partial \xi}=0, \tag{15}$$

$$g\left(-\frac{\partial w}{\partial \xi}+\varepsilon^{\gamma}\frac{\partial w}{\partial \tau}\right)+w\frac{\partial w}{\partial \xi}+\frac{\partial p}{\partial \xi}-\nu\varepsilon^{-\delta}\left(-\frac{8w}{(1+u)^2}+\varepsilon^{2\delta}\frac{\partial^2 w}{\partial \xi^2}\right)=0, \tag{16}$$

and

$$\begin{aligned}p={}&\beta_1 u+\varepsilon^{2\delta}\beta_2\frac{\partial^2 u}{\partial \xi^2}+\varepsilon^{2\delta}g^2\beta_3\left(\frac{\partial^2 u}{\partial \xi^2}-2\varepsilon^{\gamma}\frac{\partial^2 u}{\partial \xi\partial \tau}+\varepsilon^{2\delta}\frac{\partial^2 u}{\partial \tau^2}\right)\\&+\varepsilon^{(\delta+\beta)}g\beta_4\left(-\frac{\partial u}{\partial \xi}+\varepsilon^{\delta}\frac{\partial u}{\partial \tau}\right)+\varepsilon^{(3\delta+\beta)}g\beta_5\left(-\frac{\partial^3 u}{\partial \xi^3}+\varepsilon^{\delta}\frac{\partial^3 u}{\partial \xi^2\partial \tau}\right)+\beta_6 u^2+\beta_7 u^3.\end{aligned} \tag{17}$$

It should be noted that, since the derivation when $\gamma = 3$ was not treated in [35], we will only consider here this case, and we refer the reader to Demiray [35] for the cases $\gamma = 1$ and $\gamma = 2$.

Assuming that the viscoelastic coefficients are of order ε^{β}, and by using the power series representation of the variables u, w and p in ε as:

$$u=\sum_{n=1}^{\infty}\varepsilon^n u_n(\xi,\tau),\qquad w=\sum_{n=1}^{\infty}\varepsilon^n w_n(\xi,\tau),\qquad p=\sum_{n=1}^{\infty}\varepsilon^n p_n(\xi,\tau), \tag{18}$$

into Eq. (15) and taking $\gamma = 3$, we obtain the following asymptotic expansion in ε:

$$\begin{aligned}&-2g\left(\varepsilon\frac{\partial u_1}{\partial \xi}+\varepsilon^2\frac{\partial u_2}{\partial \xi}+\varepsilon^3\frac{\partial u_3}{\partial \xi}+\varepsilon^4\frac{\partial u_4}{\partial \xi}+\cdots\right)\\&\quad+2g\varepsilon^3\left(\varepsilon\frac{\partial u_1}{\partial \tau}\ldots\right)+\left(\varepsilon\frac{\partial w_1}{\partial \xi}+\varepsilon^2\frac{\partial w_2}{\partial \xi}+\varepsilon^3\frac{\partial w_3}{\partial \xi}+\varepsilon^4\frac{\partial w_4}{\partial \xi}+\cdots\right)\\&\quad+\left(\varepsilon u_1+\varepsilon^2 u_2+\varepsilon^3 u_3+\varepsilon^4 u_4+\cdots\right)\left(\varepsilon\frac{\partial w_1}{\partial \xi}+\varepsilon^2\frac{\partial w_2}{\partial \xi}+\varepsilon^3\frac{\partial w_3}{\partial \xi}+\varepsilon^4\frac{\partial w_4}{\partial \xi}+\cdots\right)\\&\quad+2\left(\varepsilon w_1+\varepsilon^2 w_2+\varepsilon^3 w_3+\varepsilon^4 w_4+\cdots\right)\left(\varepsilon\frac{\partial u_1}{\partial \xi}+\varepsilon^2\frac{\partial u_2}{\partial \xi}+\varepsilon^3\frac{\partial u_3}{\partial \xi}+\varepsilon^4\frac{\partial u_4}{\partial \xi}+\cdots\right)=0,\end{aligned}$$

or

$$\begin{aligned}&\varepsilon\left(-2g\frac{\partial u_1}{\partial \xi}+\frac{\partial w_1}{\partial \xi}\right)+\varepsilon^2\left(-2g\frac{\partial u_2}{\partial \xi}+\frac{\partial w_2}{\partial \xi}+u_1\frac{\partial w_1}{\partial \xi}+2w_1\frac{\partial u_1}{\partial \xi}\right)\\&\quad+\varepsilon^3\left(-2g\frac{\partial u_3}{\partial \xi}+\frac{\partial w_3}{\partial \xi}+u_1\frac{\partial w_2}{\partial \xi}+u_2\frac{\partial w_1}{\partial \xi}+2w_1\frac{\partial u_2}{\partial \xi}+2w_2\frac{\partial u_1}{\partial \xi}\right)\\&\quad+\varepsilon^4\left(-2g\frac{\partial u_4}{\partial \xi}+2g\frac{\partial u_1}{\partial \tau}+\frac{\partial w_4}{\partial \xi}+u_1\frac{\partial w_3}{\partial \xi}+u_2\frac{\partial w_2}{\partial \xi}+u_3\frac{\partial w_1}{\partial \xi}\right.\\&\quad\left.+2w_1\frac{\partial u_3}{\partial \xi}+2w_2\frac{\partial u_2}{\partial \xi}+2w_3\frac{\partial u_1}{\partial \xi}\right)=0.\end{aligned} \tag{19}$$

Similarly, Eq. (16) can be written as

$$
\begin{aligned}
&-g\left(\varepsilon\frac{\partial w_1}{\partial\xi}+\varepsilon^2\frac{\partial w_2}{\partial\xi}+\varepsilon^3\frac{\partial w_3}{\partial\xi}+\varepsilon^4\frac{\partial w_4}{\partial\xi}+\cdots\right)+g\varepsilon^3\left(\varepsilon\frac{\partial w_1}{\partial\tau}+\cdots\right)\\
&\quad+\left(\varepsilon w_1+\varepsilon^2 w_2+\varepsilon^3 w_3+\cdots\right)\left(\varepsilon\frac{\partial w_1}{\partial\xi}+\varepsilon^2\frac{\partial w_2}{\partial\xi}+\varepsilon^3\frac{\partial w_3}{\partial\xi}+\cdots\right)\\
&\quad+\left(\varepsilon\frac{\partial p_1}{\partial\xi}+\varepsilon^2\frac{\partial p_2}{\partial\xi}+\varepsilon^3\frac{\partial p_3}{\partial\xi}+\varepsilon^4\frac{\partial p_4}{\partial\xi}+\cdots\right)\\
&\quad-\nu\varepsilon^{-\delta}\left[-8\frac{\sum_{n=1}^{\infty}\varepsilon^n w_n}{(1+\sum_{n=1}^{\infty}\varepsilon^n u_n)^2}+\varepsilon^{2\delta}\left(\varepsilon\frac{\partial^2 w_1}{\partial\xi^2}+\cdots\right)\right]=0,
\end{aligned}
$$

or

$$
\begin{aligned}
&\varepsilon\left(-g\frac{\partial w_1}{\partial\xi}+\frac{\partial p_1}{\partial\xi}\right)+\varepsilon^2\left(-g\frac{\partial w_2}{\partial\xi}+w_1\frac{\partial w_1}{\partial\xi}+\frac{\partial p_2}{\partial\xi}\right)\\
&\quad+\varepsilon^3\left(-g\frac{\partial w_3}{\partial\xi}+w_1\frac{\partial w_2}{\partial\xi}+w_2\frac{\partial w_1}{\partial\xi}+\frac{\partial p_3}{\partial\xi}\right)\\
&\quad+\varepsilon^4\left(-g\frac{\partial w_4}{\partial\xi}+g\frac{\partial w_1}{\partial\tau}+w_1\frac{\partial w_3}{\partial\xi}+w_2\frac{\partial w_2}{\partial\xi}+w_3\frac{\partial w_1}{\partial\xi}+\frac{\partial p_4}{\partial\xi}+8\nu\varepsilon^{-(\delta+3)}w_1\right)=0. \qquad (20)
\end{aligned}
$$

Also, Eq. (17) can be expanded to yield

$$
\begin{aligned}
&\varepsilon p_1+\varepsilon^2 p_2+\varepsilon^3 p_3+\varepsilon^4 p_4+\cdots\\
&\quad=\left(\varepsilon\beta_1 u_1+\varepsilon^2\beta_1 u_2+\varepsilon^3\beta_1 u_3+\varepsilon^4\beta_1 u_4+\cdots\right)+\varepsilon^{2\delta}\beta_2\left(\varepsilon\frac{\partial^2 u_1}{\partial\xi^2}+\varepsilon^2\frac{\partial^2 u_2}{\partial\xi^2}+\cdots\right)\\
&\qquad+\beta_3\varepsilon^{2\delta}g^2\left(\varepsilon\frac{\partial^2 u_1}{\partial\xi^2}+\varepsilon^2\frac{\partial^2 u_2}{\partial\xi^2}+\cdots\right)-2\beta_3\varepsilon^{2\delta+3}g^2\left(\varepsilon\frac{\partial^2 u_1}{\partial\xi\partial\tau}+\cdots\right)\\
&\qquad+\beta_3\varepsilon^{2\delta}g^2\varepsilon^6\left(\varepsilon\frac{\partial^2 u_1}{\partial\tau^2}+\cdots\right)-\beta_4\varepsilon^{\delta+\beta}g\left(\varepsilon\frac{\partial u_1}{\partial\xi}+\cdots\right)\\
&\qquad+\beta_4\varepsilon^{\delta+\beta}g\varepsilon^3\left(\varepsilon\frac{\partial u_1}{\partial\tau}+\cdots\right)-\beta_5\varepsilon^{3\delta+\beta}g\left(\varepsilon\frac{\partial^3 u_1}{\partial\xi^3}+\varepsilon^2\frac{\partial^3 u_2}{\partial\xi^3}+\cdots\right)\\
&\qquad+\beta_5\varepsilon^{3\delta+\beta}g\varepsilon^3\left(\varepsilon\frac{\partial^3 u_1}{\partial\xi^2\partial\tau}+\cdots\right)+\beta_6\left(\varepsilon u_1+\varepsilon^2 u_2+\varepsilon^3 u_3\right)\left(\varepsilon u_1+\varepsilon^2 u_2+\varepsilon^3 u_3\right)\\
&\qquad+\beta_7\left(\varepsilon u_1+\varepsilon^2 u_2+\varepsilon^3 u_3\right)\left(\varepsilon u_1+\varepsilon^2 u_2+\varepsilon^3 u_3\right)\left(\varepsilon u_1+\varepsilon^2 u_2+\varepsilon^3 u_3\right),
\end{aligned}
$$

that is,

$$
\begin{aligned}
&\varepsilon p_1+\varepsilon^2 p_2+\varepsilon^3 p_3+\varepsilon^4 p_4\\
&\quad=\varepsilon(\beta_1 u_1)+\varepsilon^2\left(\beta_1 u_2+\beta_6 u_1^2\right)+\varepsilon^3\left(\beta_1 u_3+2\beta_6 u_1 u_2+\beta_7 u_1^3\right)\\
&\qquad+\varepsilon^4\left(\beta_1 u_4+\varepsilon^{(2\delta-3)}\beta_2\frac{\partial^2 u_1}{\partial\xi^2}+\beta_3\varepsilon^{(2\delta-3)}g^2\frac{\partial^2 u_1}{\partial\xi^2}-\beta_4\varepsilon^{\delta+\beta-3}g\frac{\partial u_1}{\partial\xi}\right.\\
&\qquad\left.-\beta_5\varepsilon^{3\delta+\beta-3}g\frac{\partial^3 u_1}{\partial\xi^3}+2\beta_6 u_1 u_3+\beta_6 u_2^2+3\beta_7 u_1^2 u_2\right). \qquad (21)
\end{aligned}
$$

By setting the like powers in Eqs. (19)–(21) to zero, we will get the following set of differential equations.

The $O(\varepsilon)$ equations:

$$\begin{aligned}
&-2g\frac{\partial u_1}{\partial \xi}+\frac{\partial w_1}{\partial \xi}=0,\\
&-g\frac{\partial w_1}{\partial \xi}+\frac{\partial p_1}{\partial \xi}=0,\\
&p_1=\beta_1 u_1.
\end{aligned}\tag{22}$$

The $O(\varepsilon^2)$ equations:

$$\begin{aligned}
&-2g\frac{\partial u_2}{\partial \xi}+\frac{\partial w_2}{\partial \xi}+u_1\frac{\partial w_1}{\partial \xi}+2w_1\frac{\partial u_1}{\partial \xi}=0,\\
&-g\frac{\partial w_2}{\partial \xi}+w_1\frac{\partial w_1}{\partial \xi}+\frac{\partial p_2}{\partial \xi}=0,\\
&p_2=\beta_1 u_2+\beta_6 u_1^2.
\end{aligned}\tag{23}$$

The $O(\varepsilon^3)$ equations:

$$\begin{aligned}
&-2g\frac{\partial u_3}{\partial \xi}+\frac{\partial w_3}{\partial \xi}+u_1\frac{\partial w_2}{\partial \xi}+u_2\frac{\partial w_1}{\partial \xi}+2w_1\frac{\partial u_2}{\partial \xi}+2w_2\frac{\partial u_1}{\partial \xi}=0,\\
&-g\frac{\partial w_3}{\partial \xi}+w_1\frac{\partial w_2}{\partial \xi}+w_2\frac{\partial w_1}{\partial \xi}+\frac{\partial p_3}{\partial \xi}=0,\\
&p_3=\beta_1 u_3+2\beta_6 u_1 u_2+\beta_7 u_1^3.
\end{aligned}\tag{24}$$

The $O(\varepsilon^4)$ equations:

$$\begin{aligned}
&-2g\frac{\partial u_4}{\partial \xi}+2g\frac{\partial u_1}{\partial \tau}+\frac{\partial w_4}{\partial \xi}+u_1\frac{\partial w_3}{\partial \xi}+u_2\frac{\partial w_2}{\partial \xi}+u_3\frac{\partial w_1}{\partial \xi}\\
&\quad+2w_1\frac{\partial u_3}{\partial \xi}+2w_2\frac{\partial u_2}{\partial \xi}+2w_3\frac{\partial u_1}{\partial \xi}=0,\\
&-g\frac{\partial w_4}{\partial \xi}+g\frac{\partial w_1}{\partial \tau}+w_1\frac{\partial w_3}{\partial \xi}+w_2\frac{\partial w_2}{\partial \xi}+w_3\frac{\partial w_1}{\partial \xi}+\frac{\partial p_4}{\partial \xi}+8\nu\varepsilon^{-(\delta+3)}w_1=0,\\
&p_4=\beta_1 u_4+\varepsilon^{2\delta-3}\left(\beta_2+g^2\beta_3\right)\frac{\partial^2 u_1}{\partial \xi^2}-\beta_4\varepsilon^{\delta+\beta-3}g\frac{\partial u_1}{\partial \xi}-\beta_5\varepsilon^{3\delta+\beta-3}g\frac{\partial^3 u_1}{\partial \xi^3}\\
&\quad+2\beta_6 u_1 u_3+\beta_6 u_2^2+3\beta_7 u_1^2 u_2.
\end{aligned}\tag{25}$$

For the solution of set (22), we set

$$u_1=U(\xi,t),\qquad w_1=2gU(\xi,t),\qquad g^2=\frac{\beta_1}{2}\quad\text{and}\quad p_1=2g^2U(\xi,t),\tag{26}$$

where $U(\xi,\tau)$ is unknown.

Inserting Eqs. (26) into (23), and noting that $5\beta_1+2\beta_6=0$, we get

$$w_2=2gu_2-3gU^2,\qquad p_2=2g^2u_2-5g^2U^2.\tag{27}$$

Introducing (26)–(27) into the first equation in (24), we get

$$-2g\frac{\partial u_3}{\partial \xi}+\frac{\partial w_3}{\partial \xi}+U(\xi,t)\frac{\partial}{\partial \xi}\left(2gu_2-3gU^2\right)$$
$$+u_2\frac{\partial}{\partial \xi}(2gU)+2(2gU)\frac{\partial u_2}{\partial \xi}+2\left(2gu_2-3gU^2\right)\frac{\partial U}{\partial \xi}=0.$$

The previous equation leads to

$$-2g\frac{\partial u_3}{\partial \xi}+\frac{\partial w_3}{\partial \xi}+U(\xi,t)2g\frac{\partial u_2}{\partial \xi}-3gU\frac{\partial (U^2)}{\partial \xi}+2u_2g\frac{\partial U}{\partial \xi}$$
$$+4gU\frac{\partial u_2}{\partial \xi}+\left(4gu_2-6gU^2\right)\frac{\partial U}{\partial \xi}=0.$$

Hence,

$$\frac{\partial w_3}{\partial \xi}=2g\frac{\partial u_3}{\partial \xi}-6gU(\xi,t)\frac{\partial u_2}{\partial \xi}+6gU^2\frac{\partial U}{\partial \xi}-6gu_2\frac{\partial U}{\partial \xi}+3gU\frac{\partial (U^2)}{\partial \xi}.$$

This implies that

$$\frac{\partial w_3}{\partial \xi}=2g\frac{\partial u_3}{\partial \xi}-6g\frac{\partial (u_2U)}{\partial \xi}+12gU^2\frac{\partial U}{\partial \xi}.$$

Therefore,

$$w_3=2gu_3-6gu_2U+4gU^3$$

and

$$p_3=\beta_1u_3+2\beta_6Uu_2+\beta_7U^3. \tag{28}$$

Introducing the results in (26)–(28) into the first equation in (25) gives the following:

$$-2g\frac{\partial u_4}{\partial \xi}+2g\frac{\partial U}{\partial \tau}+\frac{\partial w_4}{\partial \xi}+U\frac{\partial}{\partial \xi}\left(2gu_3-6gu_2U+4gU^3\right)$$
$$+u_2\frac{\partial}{\partial \xi}\left(2gu_2-3gU^2\right)+u_3\frac{\partial}{\partial \xi}(2gU)$$
$$+2(2gU)\frac{\partial u_3}{\partial \xi}+2\left(2gu_2-3gU^2\right)\frac{\partial u_2}{\partial \xi}+2\left(2gu_3-6gu_2U+4gU^3\right)\frac{\partial U}{\partial \xi}=0,$$

that is,

$$-2g\frac{\partial u_4}{\partial \xi}+2g\frac{\partial U}{\partial \tau}+\frac{\partial w_4}{\partial \xi}+2gU\frac{\partial u_3}{\partial \xi}-6gU\frac{\partial}{\partial \xi}(u_2U)$$
$$+4gU\frac{\partial U^3}{\partial \xi}+2gu_2\frac{\partial u_2}{\partial \xi}-3gu_2\frac{\partial U^2}{\partial \xi}+2u_3g\frac{\partial U}{\partial \xi}$$
$$+4gU\frac{\partial u_3}{\partial \xi}+4gu_3\frac{\partial U}{\partial \xi}-12gu_2U\frac{\partial U}{\partial \xi}+8gU^3\frac{\partial U}{\partial \xi}+4gu_2\frac{\partial u_2}{\partial \xi}-6gU^2\frac{\partial u_2}{\partial \xi}=0.$$

This simplifies to

$$\begin{aligned}
&-2g\frac{\partial u_4}{\partial\xi}+2g\frac{\partial U}{\partial\tau}+\frac{\partial w_4}{\partial\xi}+6gU\frac{\partial u_3}{\partial\xi}-6gU\frac{\partial}{\partial\xi}(u_2U)+6gu_2\frac{\partial u_2}{\partial\xi}\\
&\quad-18gu_2U\frac{\partial U}{\partial\xi}+6gu_3\frac{\partial U}{\partial\xi}+20gU^3\frac{\partial U}{\partial\xi}\\
&\quad-6gU^2\frac{\partial u_2}{\partial\xi}=0,
\end{aligned}$$

that is,

$$\begin{aligned}
&-2g\frac{\partial u_4}{\partial\xi}+2g\frac{\partial U}{\partial\tau}+\frac{\partial w_4}{\partial\xi}+6g\frac{\partial(u_3U)}{\partial\xi}-6gU\frac{\partial}{\partial\xi}(u_2U)+6gu_2\frac{\partial u_2}{\partial\xi}-18gu_2U\frac{\partial U}{\partial\xi}\\
&\quad+20gU^3\frac{\partial U}{\partial\xi}-6gU^2\frac{\partial u_2}{\partial\xi}=0.
\end{aligned}\tag{29}$$

Doing the same for the second equation in the set (25) yields

$$\begin{aligned}
&-g\frac{\partial w_4}{\partial\xi}+g\frac{\partial}{\partial\tau}(2gU)+(2gU)\frac{\partial}{\partial\xi}\left(2gu_3-6gu_2U+4gU^3\right)\\
&\quad+\left(2gu_2-3gU^2\right)\frac{\partial}{\partial\xi}\left(2gu_2-3gU^2\right)\\
&\quad+\left(2gu_3-6gu_2U+4gU^3\right)\frac{\partial}{\partial\xi}(2gU)+\frac{\partial p_4}{\partial\xi}+8\nu\varepsilon^{-(\delta+3)}(2gU)=0,
\end{aligned}$$

that is,

$$\begin{aligned}
&-g\frac{\partial w_4}{\partial\xi}+g\frac{\partial}{\partial\tau}(2gU)+4g^2u_2\frac{\partial u_2}{\partial\xi}-6g^2u_2\frac{\partial U^2}{\partial\xi}-6g^2U^2\frac{\partial u_2}{\partial\xi}\\
&\quad+9g^2U^2\frac{\partial U^2}{\partial\xi}+4g^2u_3\frac{\partial U}{\partial\xi}-12g^2u_2U\frac{\partial U}{\partial\xi}\\
&\quad+8g^2U^3\frac{\partial U}{\partial\xi}+4g^2U\frac{\partial u_3}{\partial\xi}-12g^2U\frac{\partial(u_2U)}{\partial\xi}\\
&\quad+8g^2U\frac{\partial U^3}{\partial\xi}+\frac{\partial p_4}{\partial\xi}+16g\nu\varepsilon^{-(\delta+3)}U=0.
\end{aligned}\tag{30}$$

Similarly, the third equation in (25) can be written as

$$\begin{aligned}
p_4={}&\beta_1u_4+\varepsilon^{2\delta-3}\left(\beta_2+g^2\beta_3\right)\frac{\partial^2U}{\partial\xi^2}-\beta_4\varepsilon^{\delta+\beta-3}g\frac{\partial U}{\partial\xi}-\beta_5\varepsilon^{3\delta+\beta-3}g\frac{\partial^3U}{\partial\xi^3}\\
&+2\beta_6Uu_3+\beta_6u_2^2+3\beta_7U^2u_2.
\end{aligned}\tag{31}$$

Multiplying (29) by g and then adding it to (30) to eliminate u_4, w_4 and p_4 from Eq. (29)–(31) gives the following equation:

$$\begin{aligned}
&4g^2\frac{\partial U}{\partial\tau}+\left(10g^2+2\beta_6\right)\frac{\partial}{\partial\xi}(u_3U)+\left(10g^2+2\beta_6\right)u_2\frac{\partial u_2}{\partial\xi}-\beta_4\varepsilon^{\delta+\beta-3}g\frac{\partial^2U}{\partial\xi^2}\\
&\quad-\beta_5\varepsilon^{3\delta+\beta-3}g\frac{\partial^4U}{\partial\xi^4}+\varepsilon^{2\delta-3}\left(\beta_2+g^2\beta_3\right)\frac{\partial^3U}{\partial\xi^3}+70g^2U^3\frac{\partial U}{\partial\xi}
\end{aligned}$$

$$+ 16g\nu\varepsilon^{-(\delta+3)}U + \left(-30g^2 + 3\beta_7\right)U^2\frac{\partial u_2}{\partial \xi} + \left(-60g^2 + 6\beta_7\right)Uu_2\frac{\partial U}{\partial \xi} = 0. \tag{32}$$

Dividing Eq. (32) by $4g^2$ and noting that $10g^2 + 2\beta_6 = 0, \beta_1 = 2g^2$ and assuming that $-10g^2 + \beta_7 = 0$, we get

$$\begin{aligned}&\frac{\partial U}{\partial \tau} - \frac{\beta_4\varepsilon^{\delta+\beta-3}}{4g}\frac{\partial^2 U}{\partial \xi^2} - \frac{\beta_5}{4g}\varepsilon^{3\delta+\beta-3}\frac{\partial^4 U}{\partial \xi^4} + \varepsilon^{2\delta-3}\left(\frac{\beta_2}{2\beta_1} + \frac{\beta_3}{4}\right)\frac{\partial^3 U}{\partial \xi^3} + \frac{70}{4}U^3\frac{\partial U}{\partial \xi}\\&+ \frac{4}{g}\nu\varepsilon^{-(\delta+3)}U = 0.\end{aligned} \tag{33}$$

Set the coefficients $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and α_5 in Eq. (33) as follows:

$$\begin{aligned}&\alpha_1 = \frac{70}{4}, \qquad \alpha_2 = \frac{\beta_4\varepsilon^{\delta+\beta-3}}{4g}, \qquad \alpha_3 = \left(\frac{\beta_2}{2\beta_1} + \frac{\beta_3}{4}\right)\varepsilon^{2\delta-3},\\&\alpha_4 = \frac{4}{g}\nu\varepsilon^{-(\delta+3)} \quad \text{and} \quad \alpha_5 = \frac{\beta_5}{4g}\varepsilon^{3\delta+\beta-3}.\end{aligned} \tag{34}$$

We get the following master equation:

$$\frac{\partial U}{\partial \tau} + \alpha_1 U^3\frac{\partial U}{\partial \xi} - \alpha_2\frac{\partial^2 U}{\partial \xi^2} + \alpha_3\frac{\partial^3 U}{\partial \xi^3} + \alpha_4 U - \alpha_5\frac{\partial^4 U}{\partial \xi^4} = 0. \tag{35}$$

Setting $\delta = 0, \beta = 3$ and $\nu = O(\varepsilon^4)$, and assuming that $(\frac{\beta_2}{2\beta_1} + \frac{\beta_3}{4})$ is of O (ε^3), Eq. (35) reduces to the following equation:

$$\frac{\partial U}{\partial \tau} + \alpha_1 U^3\frac{\partial U}{\partial \xi} - \alpha_2\frac{\partial^2 U}{\partial \xi^2} + \alpha_3\frac{\partial^3 U}{\partial \xi^3} - \alpha_5\frac{\partial^4 U}{\partial \xi^4} = 0. \tag{36}$$

Now, if we let

$$\alpha_1 = \gamma_1, \qquad \alpha_2 = \nu, \qquad \alpha_3 = \mu, \qquad \alpha_5 = -\gamma_2, \qquad u = U, \qquad t = \tau, \qquad x = \xi,$$

Eq. (36) reduces to the MGKdVB equation with nonlinearity of order 3 (i.e. $\alpha = 3$ in Eq. (1)):

$$\frac{\partial u}{\partial t} + \gamma_1 u^3\frac{\partial u}{\partial x} - \nu\frac{\partial^2 u}{\partial x^2} + \mu\frac{\partial^3 u}{\partial x^3} + \gamma_2\frac{\partial^4 u}{\partial x^4} = 0. \tag{37}$$

Remarks It should be noted that one can obtain the following equations from the master equation (35):

(i) When $\delta = 3, \beta = 0, \nu = O(\varepsilon^7)$, Eq. (35) reduces to the Burgers' equation:

$$\frac{\partial U}{\partial \tau} + \alpha_1 U^3\frac{\partial U}{\partial \xi} - \alpha_2\frac{\partial^2 U}{\partial \xi^2} = 0. \tag{38}$$

(ii) When $\delta = 3, \beta = 0, \nu = O(\varepsilon^6)$, Eq. (35) becomes the perturbed Burgers' equation:

$$\frac{\partial U}{\partial \tau} + \alpha_1 U^3\frac{\partial U}{\partial \xi} - \alpha_2\frac{\partial^2 U}{\partial \xi^2} + \alpha_4 U = 0. \tag{39}$$

(iii) When $\delta = \frac{3}{2}, \beta = 2, \nu = O(\varepsilon^6)$, Eq. (35) reduces to the well-known KdV equation:

$$\frac{\partial U}{\partial \tau} + \alpha_1 U^3 \frac{\partial U}{\partial \xi} + \alpha_3 \frac{\partial^3 U}{\partial \xi^3} = 0. \tag{40}$$

(iv) If $\delta = \frac{3}{2}, \beta = 2, \nu = O(\varepsilon^{9/2})$, the master equation produces the perturbed KdV equation:

$$\frac{\partial U}{\partial \tau} + \alpha_1 U^3 \frac{\partial U}{\partial \xi} + \alpha_3 \frac{\partial^3 U}{\partial \xi^3} + \alpha_4 U = 0. \tag{41}$$

3 The first nonlinear boundary control law for the MGKDVB equation

This section describes the first nonlinear boundary control law for the MGKdVB equation. The following theorem presents the first result of our nonlinear boundary control law.

3.1 Design of the first controller

Theorem 1 *Let α be a positive integer and all the parameters γ_1, ν, μ and γ_2 are nonzero known positive real constants. The modified generalized Korteweg–de Vries–Burgers* (*MGKdVB*) *equation given by Eq.* (1) *subject to the boundary conditions given by Eqs.* (2)–(5) *and with the initial condition $u_0(x) \in L^2(0,1)$ is globally exponentially stable in the $L^2(0,1)$-sense, by applying the following nonlinear control law*:

$$w_1(t) = \frac{-2\nu}{\mu} u(1,t), \tag{42}$$

$$w_2(t) = \frac{-\gamma_1 \mu}{(\alpha+2)(2\gamma_2\nu + \mu^2)} u^{\alpha+1}(1,t) - \frac{\gamma_2 \mu}{2\gamma_2 \nu + \mu^2} \frac{\partial^3 u}{\partial x^3}(1,t). \tag{43}$$

Proof Consider the Lyapunov function candidate:

$$V(t) = \frac{1}{2} \int_0^1 u^2(x,t)\,dx.$$

Note that $V > 0$ when $u \neq 0$, and $V = 0$ iff $u = 0$.

Taking the derivative of $V(t)$ with respect to time and using Eq. (1), we obtain

$$\begin{aligned} \dot{V}(t) &= \int_0^1 u(x,t) \frac{\partial u}{\partial t}(x,t)\,dx \\ &= \int_0^1 u \left[\nu \frac{\partial^2 u}{\partial x^2} - \mu \frac{\partial^3 u}{\partial x^3} - \gamma_1 u^\alpha \frac{\partial u}{\partial x} - \gamma_2 \frac{\partial^4 u}{\partial x^4} \right] dx. \end{aligned} \tag{44}$$

That is,

$$\begin{aligned} &\int_0^1 u(x,t) \frac{\partial u}{\partial t}(x,t)\,dx - \nu \int_0^1 u(x,t) \frac{\partial^2 u}{\partial x^2}(x,t)\,dx + \mu \int_0^1 u(x,t) \frac{\partial^3 u}{\partial x^3}(x,t)\,dx \\ &\quad + \gamma_1 \int_0^1 u(x,t) u^\alpha(x,t) \frac{\partial u}{\partial x}(x,t)\,dx + \gamma_2 \int_0^1 u(x,t) \frac{\partial^4 u}{\partial x^4}(x,t)\,dx = 0. \end{aligned} \tag{45}$$

Integrating by parts, Eq. (45) yields

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\|u(x,t)\right\|^2 \\
&\quad = -\nu\left\|\frac{\partial u}{\partial x}(x,t)\right\|^2 + \nu u(1,t)\frac{\partial u}{\partial x}(1,t) - \nu u(0,t)\frac{\partial u}{\partial x}(0,t) - \mu u(1,t)\frac{\partial^2 u}{\partial x^2}(1,t) \\
&\qquad + \mu u(0,t)\frac{\partial^2 u}{\partial x^2}(0,t) + \frac{\mu}{2}\left(\frac{\partial u}{\partial x}\right)^2(1,t) - \frac{\mu}{2}\left(\frac{\partial u}{\partial x}\right)^2(0,t) - \frac{\gamma_1}{\alpha+2}u^{\alpha+2}(1,t) \\
&\qquad + \frac{\gamma_1}{\alpha+2}u^{\alpha+2}(0,t) - \gamma_2 u(1,t)\frac{\partial^3 u}{\partial x^3}(1,t) + \gamma_2 u(0,t)\frac{\partial^3 u}{\partial x^3}(0,t) + \gamma_2\frac{\partial u}{\partial x}(1,t)\frac{\partial^2 u}{\partial x^2}(1,t) \\
&\qquad - \gamma_2\frac{\partial u}{\partial x}(0,t)\frac{\partial^2 u}{\partial x^2}(0,t) - \gamma_2\left\|\frac{\partial^2 u}{\partial x^2}(x,t)\right\|^2.
\end{aligned} \tag{46}$$

Using the boundary conditions (2)–(5), and noting that $\frac{-\mu}{2}(\frac{\partial u}{\partial x})^2(0,t) \le 0$, and that $-\gamma_2\|\frac{\partial^2 u}{\partial x^2}(x,t)\|^2 \le 0$, Eq. (46) becomes

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\left\|u(x,t)\right\|^2 \le{}& -\nu\left\|\frac{\partial u}{\partial x}(x,t)\right\|^2 + w_1(t)\left(\nu u(1,t) + \frac{\mu}{2}w_1(t)\right) \\
& - \frac{\gamma_1}{\alpha+2}u^{\alpha+2}(1,t) - \gamma_2 u(1,t)\frac{\partial^3 u}{\partial x^3}(1,t) \\
& + w_2(t)\bigl(\gamma_2 w_1(t) - \mu u(1,t)\bigr).
\end{aligned} \tag{47}$$

Applying the first nonlinear control law (i.e. (42)–(43)), we obtain

$$\frac{1}{2}\frac{d}{dt}\left\|u(x,t)\right\|^2 \le -\nu\left\|\frac{\partial u}{\partial x}(x,t)\right\|^2. \tag{48}$$

Using Poincaré inequality leads to

$$\frac{d}{dt}\left\|u(x,t)\right\|^2 \le -2\nu\left\|\frac{\partial u}{\partial x}(x,t)\right\|^2 \le -2\nu\left\|u(x,t)\right\|^2.$$

Thus,

$$\frac{d}{dt}\left\|u(x,t)\right\|^2 \le -2\nu\left\|u(x,t)\right\|^2. \tag{49}$$

Integrating inequality (49) with respect to time, we obtain

$$\left\|u(x,t)\right\| \le e^{-\nu t}\left\|u_0(x)\right\|. \tag{50}$$

Since $u_0(x) \in L^2(0,1)$, one can conclude from inequality (50) that $\|u(x,t)\|$ converges exponentially to zero as $t \longrightarrow \infty$. This proves that the equation is exponentially stable under the first nonlinear non-adaptive control law. □

In the next subsection, we present the dynamical behavior of the MGKdVB equation numerically when applying the nonlinear boundary control law presented in Eqs. (42)–(43).

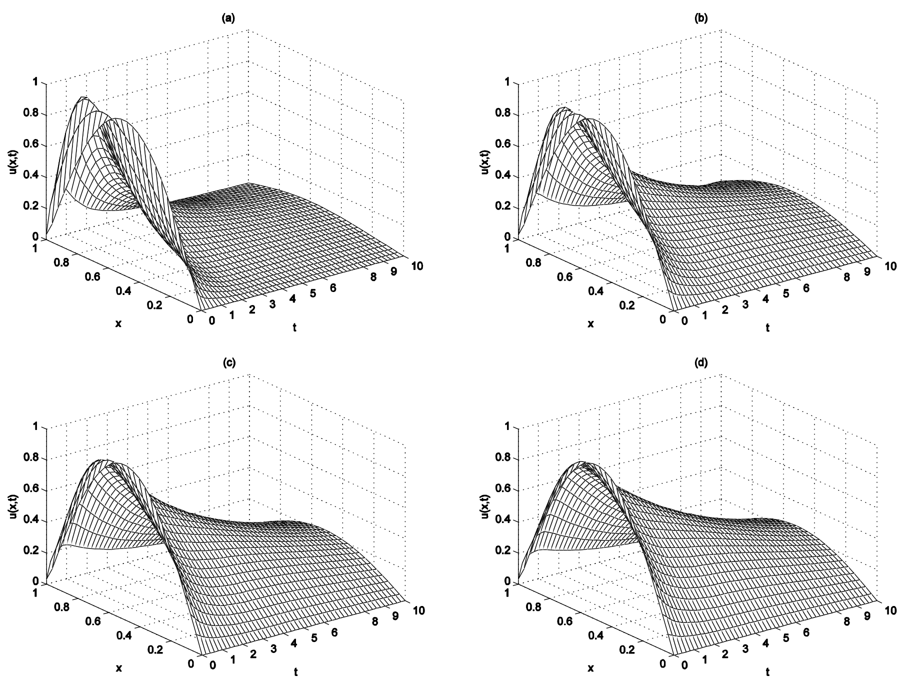

Figure 1 A 3-d landscape of the dynamics of the MGKdVB equation when using the first nonlinear control law; $v = 0.01, \mu = 0.001, \gamma_1 = 1, \gamma_2 = 0.0005$ and $u_0(x) = \sin(\pi x)$; (**a**) $\alpha = 1$; (**b**) $\alpha = 2$; (**c**) $\alpha = 3$; (**d**) $\alpha = 4$

3.2 Numerical solutions of the MGKdVB equation using the first nonlinear control law

The dynamical behavior of the MGKDVB equation, subject to the controllers $w_1(t)$ and $w_2(t)$ given by Eqs. (42)–(43), is simulated using the COMSOL Multiphysics software which is based on the Finite Element Method (FEM) . The simulations are shown for several values of α, namely, $\alpha = 1, 2, 3$, and 4.

Different initial conditions $u_0(x)$ were considered in our study. In the numerical simulations reported in this section, we set the kinematic viscosity ν to be 0.01, while the dynamic viscosity μ is chosen to be 0.001, and the parameters γ_1 and γ_2 are set to be 1 and 0.0005, respectively. Figures 1(a)–(d) depict the numerical results obtained when $u_0(x) = \sin(\pi x)$. Moreover, the L^2-norm of $u(x,t)$, $\|u(x,t)\|$, versus time and the natural logarithm of the L^2-norm of $u(x,t)$, $\ln(\|u(x,t)\|)$ versus time are presented in Fig. 2 and Fig. 3, respectively. A careful look at Fig. 3 shows that the curves of $\ln(\|u(x,t)\|)$ after approximately $t = 4$ seconds are presented by parallel lines with a negative slope less than $-\nu = -0.01$, and this is in accordance with the analytical results given by inequality (50). Therefore, one can conclude from Figs. 1–3 that the L^2-norm, $\|u(x,t)\|$, converges exponentially to zero as t tends to infinity. In addition, Figs. 1–3 indicate that, as α increases from 1 to 4, the solutions of the MGKdVB equation takes longer time to reach the steady state solution. This is due to the effect of the nonlinear term $u^\alpha \frac{\partial u}{\partial x}$ over the diffusion term $\frac{\partial^2 u}{\partial x^2}$, and the dispersion term $\frac{\partial^3 u}{\partial x^3}$.

Figures 4(a)–4(d) depict the solution of the MGKdVB equation when the initial condition $u_0(x) = \sin(2\pi x)$. Figures 5 and 6 present the L^2-norm of $u(x,t)$, $\|u(x,t)\|$, versus time and the natural logarithm of the L^2-norm of $u(x,t)$, $\ln(\|u(x,t)\|)$ versus time, respectively. Again, a careful look at Fig. 6 shows that the curves of $\ln(\|u(x,t)\|)$ after approximately

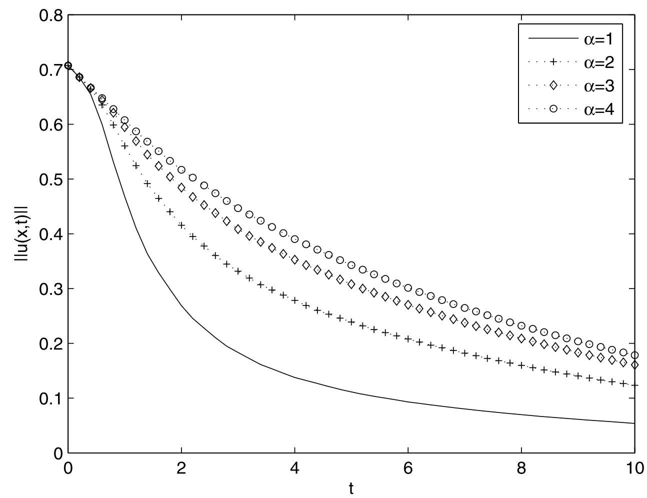

Figure 2 The L^2-norm of $u(x,t)$, $\|u(x,t)\|$, versus time for various values of α when using the first nonlinear non-adaptive control law; $\nu = 0.01$, $\mu = 0.001$, $\gamma_1 = 1$, $\gamma_2 = 0.0005$ and $u_0(x) = \sin(\pi x)$

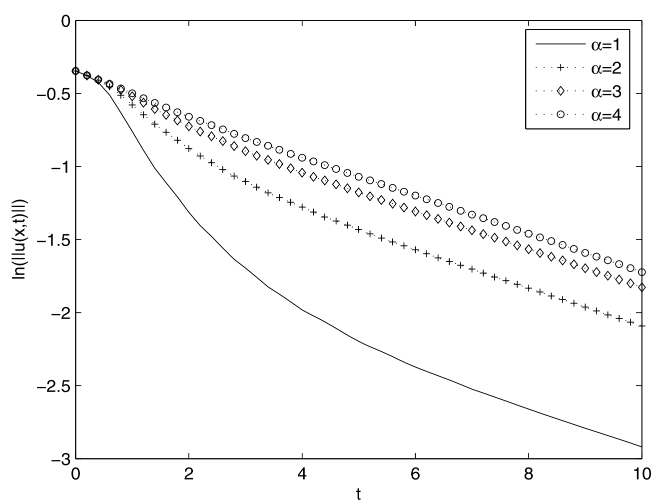

Figure 3 The natural logarithm of the L^2-norm of $u(x,t)$, $\ln(\|u(x,t)\|)$, versus time for various values of α when using the first nonlinear non-adaptive control law; $\nu = 0.01$, $\mu = 0.001$, $\gamma_1 = 1$, $\gamma_2 = 0.0005$ and $u_0(x) = \sin(\pi x)$

$t = 5$ seconds are presented by parallel lines with a negative slope less than $-\nu = -0.01$, and this is in accordance with the analytical results given by inequality (50). Therefore, one can conclude from Figs. 4–6 that the L^2-norm, $\|u(x,t)\|$, converges exponentially to zero as t tends to infinity. However, it should be noted that depending whether α is odd or even as it increases, the solutions of the MGKdVB equation takes longer time to reach the steady state solution.

The numerical simulations presented are in good agreement with the analytical work presented previously in this section. In the next section, another nonlinear control law is proposed to speed up the convergence of the solution to the steady solution.

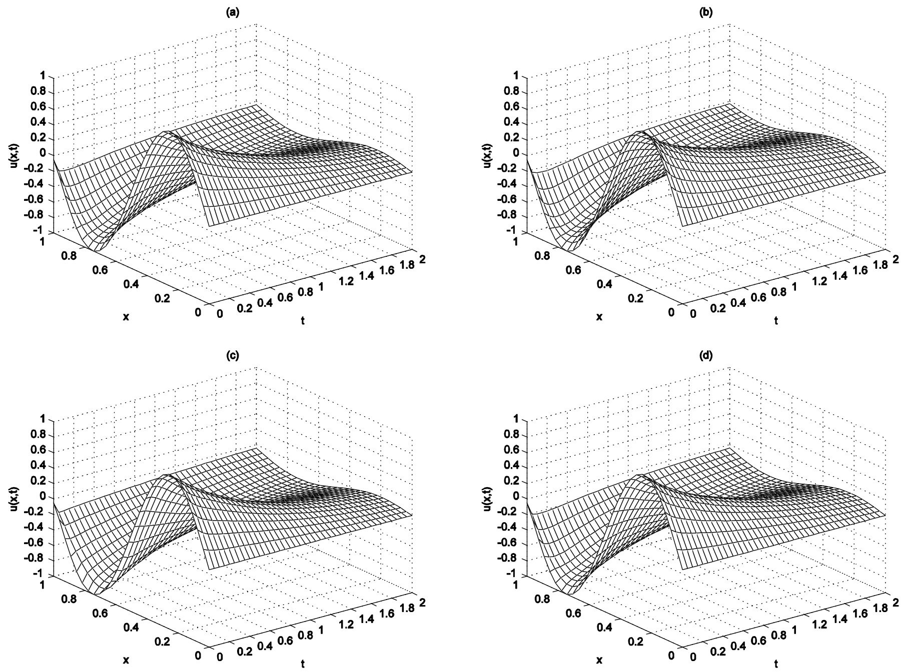

Figure 4 A 3-d landscape of the dynamics of the MGKdVB equation when using the first nonlinear non-adaptive control law; $v = 0.01$, $\mu = 0.001$, $\gamma_1 = 1$, $\gamma_2 = 0.0005$ and $u_0(x) = \sin(2\pi x)$; (**a**) $\alpha = 1$; (**b**) $\alpha = 2$; (**c**) $\alpha = 3$; (**d**) $\alpha = 4$

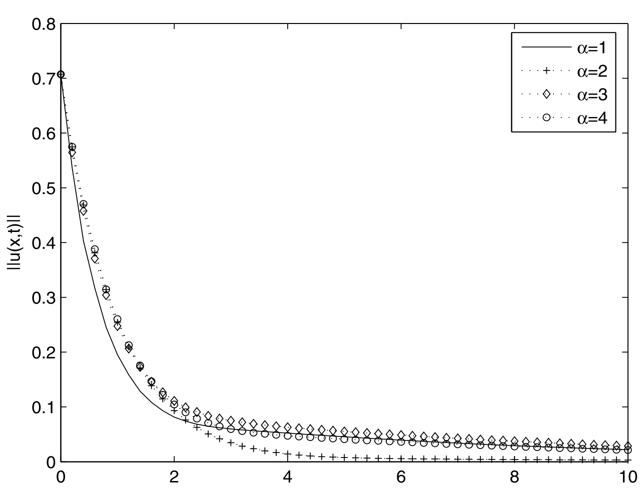

Figure 5 The L^2-norm of $u(x,t)$, $\|u(x,t)\|$, versus time for various values of α when using the first nonlinear non-adaptive control law; $v = 0.01$, $\mu = 0.001$, $\gamma_1, \gamma_2 = 0.0005$ and $u_0(x) = \sin(2\pi x)$

4 The second nonlinear boundary control law for the MGKdVB equation

In this section, the second nonlinear non-adaptive controller for the modified generalized Korteweg–de Vries–Burgers (MGKdVB) equation will be presented. In this control law, a positive control gain c_1 is introduced to speed up the convergence of the solution to

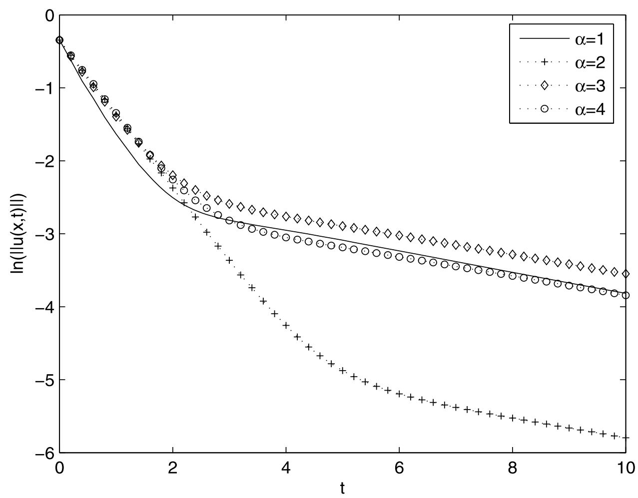

Figure 6 The natural logarithm of the L^2-norm of $u(x,t)$, $\ln(\|u(x,t)\|)$, versus time for various values of α when using the first nonlinear non-adaptive control law; $\nu = 0.01, \mu = 0.001, \gamma_1, \gamma_2 = 0.0005$ and $u_0(x) = \sin(2\pi x)$

the steady state solution. The following theorem gives the results of our second nonlinear non-adaptive boundary control law.

4.1 Design of the second controller

Theorem 2 *Let α be a positive integer and all the parameters γ_1, ν, μ and γ_2 are nonzero known positive real constants. The modified generalized Korteweg–de Vries–Burgers (MGKdVB) equation given by Eq.* (1) *subject to the boundary conditions given by Eqs.* (2)–(5) *and with the initial condition $u_0(x) \in L^2(0,1)$ is globally exponentially stable in the $L^2(0,1)$-sense, by applying the following nonlinear control law*:

$$w_1(t) = \beta_1 u(1,t), \tag{51}$$

$$w_2(t) = \beta_2 u(1,t) - \beta_3 \frac{\partial^3 u}{\partial x^3}(1,t) + \beta_4 u^{\alpha+1}(1,t), \tag{52}$$

where

$$\beta_1 = \left(\frac{-2\nu}{\mu} - \frac{c_1}{\nu}\right), \qquad \beta_2 = \frac{\mu c_1^2 + 4c_1\nu^2}{\frac{4\nu^3\gamma_2}{\mu} + 2\nu\gamma_2 c_1 + 2\nu^2\mu},$$

$$\beta_3 = \frac{\nu\mu\gamma_2}{(2\gamma_2\nu^2 + c_1\gamma_2\mu + \mu^2\nu)}, \qquad \beta_4 = \frac{\nu\mu\gamma_1}{(\alpha+2)(-2\gamma_2\nu^2 - c_1\gamma_2\mu - \mu^2\nu)},$$

and $c_1 > 0$.

Proof Consider the Lyapunov function candidate:

$$V(t) = \frac{1}{2}\int_0^1 u^2(x,t)\,dx.$$

Note that $V > 0$ when $u \neq 0$, and $V = 0$ iff $u = 0$.

Taking the derivative of $V(t)$ with respect to time and using Eq. (1), we obtain

$$\dot{V}(t) = \int_0^1 u(x,t)\frac{\partial u}{\partial t}(x,t)\,dx = \int_0^1 u\left[\nu\frac{\partial^2 u}{\partial x^2} - \mu\frac{\partial^3 u}{\partial x^3} - \gamma_1 u^\alpha\frac{\partial u}{\partial x} - \gamma_2\frac{\partial^4 u}{\partial x^4}\right]dx. \tag{53}$$

That is,

$$\begin{aligned}&\int_0^1 u(x,t)\frac{\partial u}{\partial t}(x,t)\,dx - \nu\int_0^1 u(x,t)\frac{\partial^2 u}{\partial x^2}(x,t)\,dx + \mu\int_0^1 u(x,t)\frac{\partial^3 u}{\partial x^3}(x,t)\,dx\\ &\quad + \gamma_1\int_0^1 u(x,t)u^\alpha(x,t)\frac{\partial u}{\partial x}(x,t)\,dx + \gamma_2\int_0^1 u(x,t)\frac{\partial^4 u}{\partial x^4}(x,t)\,dx = 0.\end{aligned} \tag{54}$$

Integrating by parts, Eq. (54) yields

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|u(x,t)\|^2\\ &\quad = -\nu\left\|\frac{\partial u}{\partial x}(x,t)\right\|^2 + \nu u(1,t)\frac{\partial u}{\partial x}(1,t) - \nu u(0,t)\frac{\partial u}{\partial x}(0,t) - \mu u(1,t)\frac{\partial^2 u}{\partial x^2}(1,t)\\ &\qquad + \mu u(0,t)\frac{\partial^2 u}{\partial x^2}(0,t) + \frac{\mu}{2}\left(\frac{\partial u}{\partial x}\right)^2(1,t) - \frac{\mu}{2}\left(\frac{\partial u}{\partial x}\right)^2(0,t) - \frac{\gamma_1}{\alpha+2}u^{\alpha+2}(1,t)\\ &\qquad + \frac{\gamma_1}{\alpha+2}u^{\alpha+2}(0,t) - \gamma_2 u(1,t)\frac{\partial^3 u}{\partial x^3}(1,t) + \gamma_2 u(0,t)\frac{\partial^3 u}{\partial x^3}(0,t) + \gamma_2\frac{\partial u}{\partial x}(1,t)\frac{\partial^2 u}{\partial x^2}(1,t)\\ &\qquad - \gamma_2\frac{\partial u}{\partial x}(0,t)\frac{\partial^2 u}{\partial x^2}(0,t) - \gamma_2\left\|\frac{\partial^2 u}{\partial x^2}(x,t)\right\|^2.\end{aligned} \tag{55}$$

Using the boundary conditions (2)–(5), and using the fact that μ and γ_2 are positive, Eq. (55) becomes

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u(x,t)\|^2 \leq &-\nu\left\|\frac{\partial u}{\partial x}(x,t)\right\|^2 + w_1(t)\left(\nu u(1,t) + \frac{\mu}{2}w_1(t)\right)\\ &- \frac{\gamma_1}{\alpha+2}u^{\alpha+2}(1,t) - \gamma_2 u(1,t)\frac{\partial^3 u}{\partial x^3}(1,t)\\ &+ w_2(t)\big(\gamma_2 w_1(t) - \mu u(1,t)\big).\end{aligned} \tag{56}$$

Applying the second control law given by Eqs. (51)–(52), we obtain the following:

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|u(x,t)\|^2\\ &\quad \leq -\nu\left\|\frac{\partial u}{\partial x}(x,t)\right\|^2 + \beta_1 u(1,t)\left(\nu u(1,t) + \frac{\mu}{2}\beta_1 u(1,t)\right)\\ &\qquad - \frac{\gamma_1}{\alpha+2}u^{\alpha+2}(1,t) - \gamma_2 u(1,t)\frac{\partial^3 u}{\partial x^3}(1,t)\\ &\qquad + \left(\beta_2 u(1,t) - \beta_3\frac{\partial^3 u}{\partial x^3}(1,t) + \beta_4 u^{\alpha+1}(1,t)\right)\big(\gamma_2\beta_1 u(1,t) - \mu u(1,t)\big).\end{aligned} \tag{57}$$

Or,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u(x,t)\|^2 \\
&\quad\leq -\nu\left\|\frac{\partial u}{\partial x}(x,t)\right\|^2 + \left(\nu\beta_1 + \frac{\mu}{2}\beta_1^2 + \gamma_2\beta_1\beta_2 - \mu\beta_2\right)u^2(1,t) \\
&\qquad + \left(-\frac{\gamma_1}{\alpha+2} + \gamma_2\beta_1\beta_4 - \beta_4\mu\right)u^{\alpha+2}(1,t) \\
&\qquad + (-\gamma_2\beta_1\beta_3 - \gamma_2 + \beta_3\mu)u(1,t)\frac{\partial^3 u}{\partial x^3}(1,t).
\end{aligned} \tag{58}$$

Next, letting $\beta_1 = (\frac{-2\nu}{\mu} - \frac{c_1}{\nu})$, where c_1 is a positive control gain, inequality (58) reduces to

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u(x,t)\|^2 \leq{}& -\nu\left\|\frac{\partial u}{\partial x}(x,t)\right\|^2 - c_1 u^2(1,t) \\
&+ \left(-\frac{2\nu^2}{\mu} + \frac{\mu}{2}\left(\frac{-2\nu}{\mu} - \frac{c_1}{\nu}\right)^2 + \gamma_2\left(\frac{-2\nu}{\mu} - \frac{c_1}{\nu}\right)\beta_2 - \mu\beta_2\right)u^2(1,t) \\
&+ \left(-\frac{\gamma_1}{\alpha+2} + \gamma_2\left(\frac{-2\nu}{\mu} - \frac{c_1}{\nu}\right)\beta_4 - \beta_4\mu\right)u^{\alpha+2}(1,t) \\
&+ \left(-\gamma_2\left(\frac{-2\nu}{\mu} - \frac{c_1}{\nu}\right)\beta_3 - \gamma_2 + \beta_3\mu\right)u(1,t)\frac{\partial^3 u}{\partial x^3}(1,t).
\end{aligned} \tag{59}$$

Now, choosing $\beta_2 = \frac{\mu c_1^2 + 4c_1\nu^2}{\frac{4\nu^3\gamma_2}{\mu} + 2\nu\gamma_2 c_1 + 2\nu^2\mu}$, $\beta_3 = \frac{\nu\mu\gamma_2}{(2\gamma_2\nu^2 + c_1\gamma_2\mu + \mu^2\nu)}$ and $\beta_4 = \frac{\nu\mu\gamma_1}{(\alpha+2)(-2\gamma_2\nu^2 - c_1\gamma_2\mu - \mu^2\nu)}$, inequality (59) reduces to

$$\frac{1}{2}\frac{d}{dt}\|u(x,t)\|^2 \leq -\nu\left\|\frac{\partial u}{\partial x}(x,t)\right\|^2 - c_1 u^2(1,t). \tag{60}$$

Since $c_1 > 0$, (60) becomes

$$\frac{1}{2}\frac{d}{dt}\|u(x,t)\|^2 \leq -\nu\left\|\frac{\partial u}{\partial x}(x,t)\right\|^2. \tag{61}$$

Utilizing the Poincaré inequality leads to

$$\frac{d}{dt}\|u(x,t)\|^2 \leq -2\nu\|u(x,t)\|^2. \tag{62}$$

Integrating inequality (62) with respect to time, we obtain

$$\|u(x,t)\| \leq e^{-\nu t}\|u_0(x)\|. \tag{63}$$

Since $u_0(x) \in L^2(0,1)$, $\|u(x,t)\|$ converges to zero exponentially as $t \longrightarrow \infty$. This proves that the equation is exponentially stable when utilizing the second nonlinear controller. □

In the next subsection, the dynamical behavior of the MGKdVB equation when applying the second nonlinear control law presented by Eqs. (52)–(53) will be shown numerically.

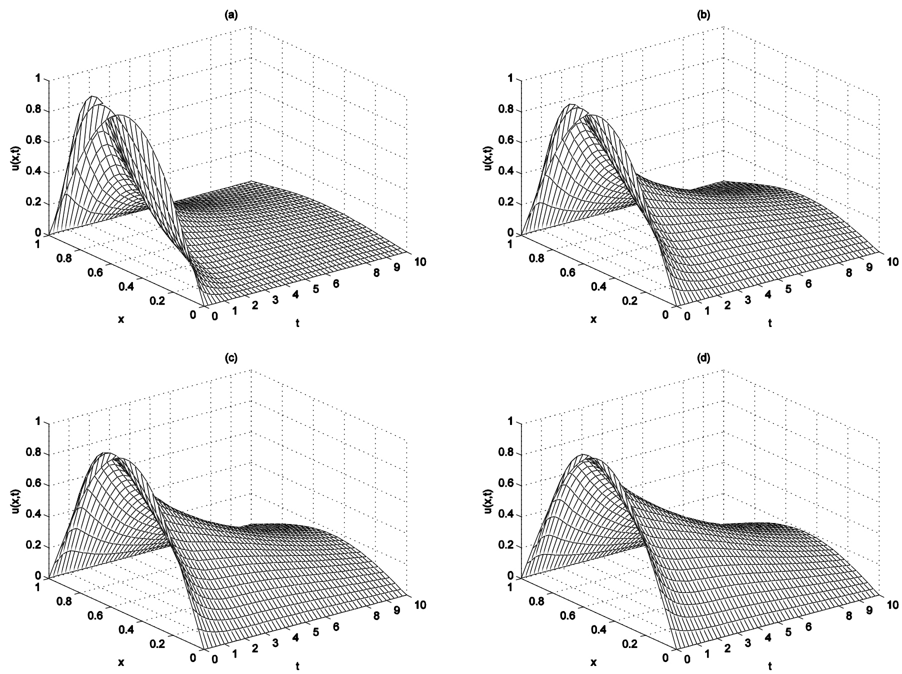

Figure 7 A 3-d landscape of the dynamics of the MGKdVB equation when using the second nonlinear non-adaptive control law; $v = 0.01$, $\mu = 0.001$, $\gamma_1 = 1$, $\gamma_2 = 0.0005$ and $u_0(x) = \sin(\pi x)$; (**a**) $\alpha = 1$; (**b**) $\alpha = 2$; (**c**) $\alpha = 3$; (**d**) $\alpha = 4$

4.2 Numerical solutions of the MGKdVB equation using the second nonlinear control law

Numerical solutions for the modified generalized Korteweg–de Vries–Burgers (MGKdVB) equation (i.e. Eq. (1)–(6)) with the controllers $w_1(t)$ and $w_2(t)$ as presented by Eqs. (51)–(52) were simulated using COMSOL Multiphysics software. The solutions are carried out for several values for α. These values are 1, 2, 3 and 4.

In the numerical simulations reported in this section, we set the kinematic viscosity ν to be 0.01, while the dynamic viscosity μ is chosen to be 0.001, and the parameters γ_1 and γ_2 are set to be 1 and 0.0005, respectively. Figures 7(a)–(d) present a 3-d landscape of the numerical results obtained when $u_0(x) = \sin(\pi x)$. Moreover, the L^2-norm of $u(x,t)$, $\|u(x,t)\|$, versus time and the natural logarithm of the L^2-norm of $u(x,t)$, $\ln(\|u(x,t)\|)$ versus time are presented in Fig. 8 and Fig. 9, respectively. A careful look at Fig. 9 shows that after approximately $t = 4$ seconds the curves of $\ln(\|u(x,t)\|)$ for different values of α are presented by parallel lines with a negative slope less than $-\nu = -0.01$, and this is in accordance with the analytical results given by inequality (50). Therefore, one can conclude from Figs. 7–9 that the L^2-norm, $\|u(x,t)\|$, converges exponentially to zero as t tends to infinity. In addition, Figs. 7–9 indicate that, as α increases from 1 to 4, the solutions of the MGKdVB equation converge slowly to the steady state solution.

Figures 10(a)–(d) depict the solution of the MGKdVB equation when the initial condition $u_0(x) = \sin(2\pi x)$. Figures 11 and 12 present the L^2-norm of $u(x,t)$, $\|u(x,t)\|$, versus time and the natural logarithm of the L^2-norm of $u(x,t)$, $\ln(\|u(x,t)\|)$ versus time, respectively. Again, a careful look at Fig. 12 shows that after approximately $t = 4$ seconds the curves of $\ln(\|u(x,t)\|)$ are presented by parallel lines with a negative slope less than

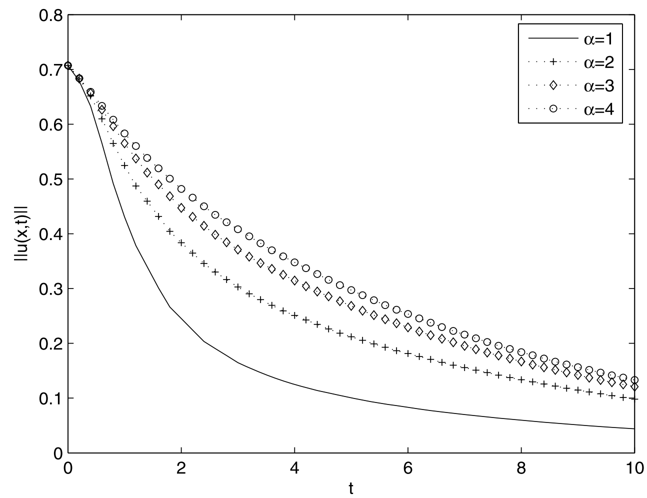

Figure 8 The L^2-norm of $u(x,t)$, $\|u(x,t)\|$, versus time for various values of α when using the second nonlinear non-adaptive control law; $v = 0.01$, $\mu = 0.001$, $\gamma_1 = 1$, $\gamma_2 = 0.0005$ and $u_0(x) = \sin(\pi x)$

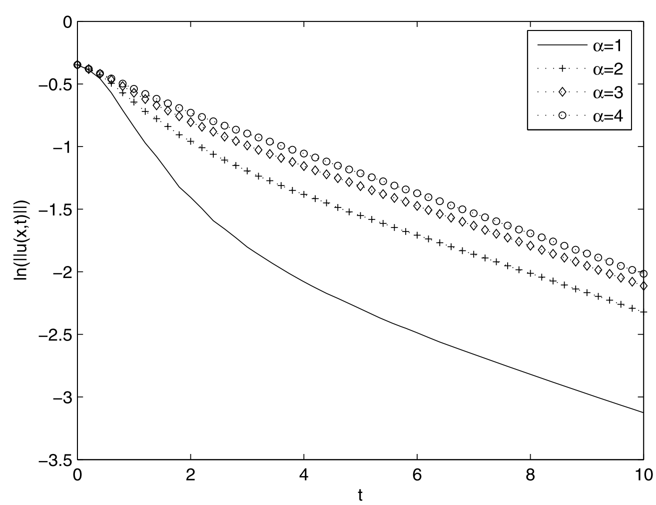

Figure 9 The natural logarithm of the L^2-norm of $u(x,t)$, $\ln(\|u(x,t)\|)$, versus time for various values of α when using the second nonlinear non-adaptive control law; $v = 0.01$, $\mu = 0.001$, $\gamma_1 = 1$, $\gamma_2 = 0.0005$ and $u_0(x) = \sin(\pi x)$

$-\nu = -0.01$, and this is in accordance with the analytical results given by inequality (50). Therefore, one can conclude from Figs. 4–6 that the L^2-norm, $\|u(x,t)\|$, converges exponentially to zero as t tends to infinity.

In Sect. 5, numerical solutions of the MGKdVB equation without control are presented, and a comparison between the performances of the two proposed nonlinear controllers will be discussed in Sect. 6 for each value of α. Moreover, the performances of these control laws will be compared to the behavior of the solutions without applying any control.

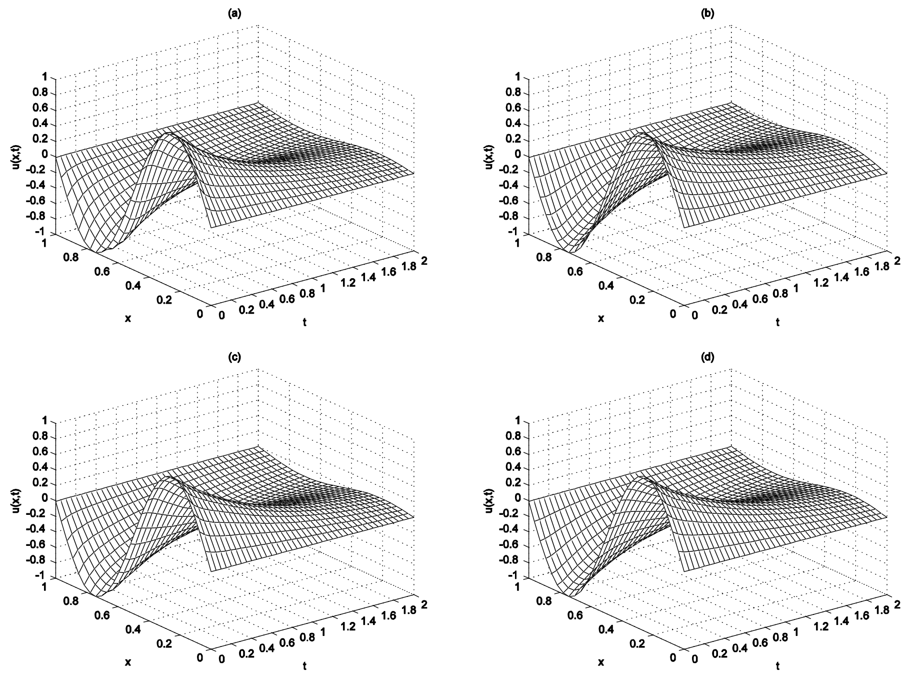

Figure 10 A 3-d landscape of the dynamics of the MGKdVB equation when using the second nonlinear non-adaptive control law; $v = 0.01$, $\mu = 0.001$, $\gamma_1 = 1$, $\gamma_2 = 0.0005$ and $u_0(x) = \sin(2\pi x)$; (**a**) $\alpha = 1$; (**b**) $\alpha = 2$; (**c**) $\alpha = 3$; (**d**) $\alpha = 4$

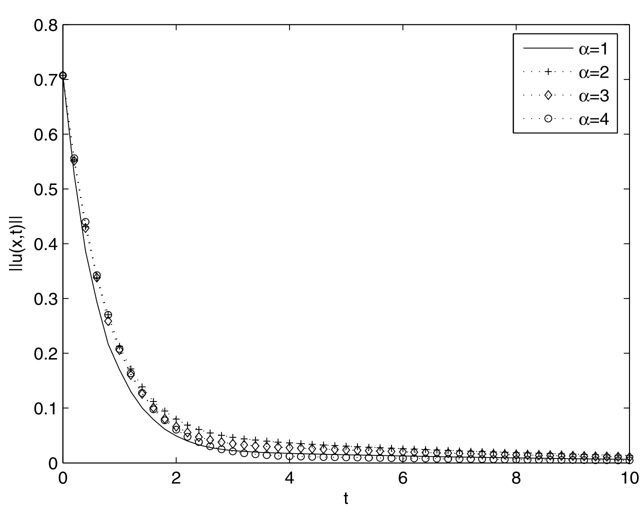

Figure 11 The L^2-norm of $u(x,t)$, $\|u(x,t)\|$, versus time for various values of α when using the second nonlinear non-adaptive control law; $v = 0.01$, $\mu = 0.001$, $\gamma_1, \gamma_2 = 0.0005$ and $u_0(x) = \sin(2\pi x)$

5 Numerical solutions of the MGKdVB equation without control

The COMSOL Multiphysics software is used to simulate the numerical solution of the MGKdVB equation Eqs. (1)–(6) subject to the homogeneous boundary condition (i.e., $w_1(t)$ and $w_2(t)$ are set to be zero in Eqs. (4)–(5)). The simulations were tackled for α having the values: 1, 2, 3 and 4.

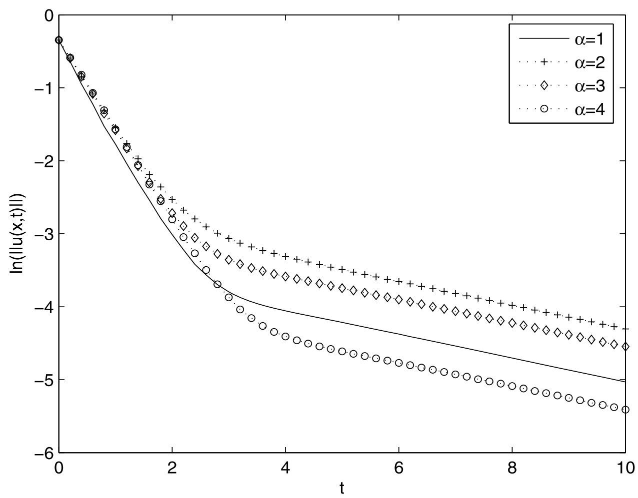

Figure 12 The natural logarithm of the L^2-norm of $u(x,t)$, $\ln(\|u(x,t)\|)$, versus time for various values of α when using the second nonlinear non-adaptive control law; $v = 0.01, \mu = 0.001, \gamma_1, \gamma_2 = 0.0005$ and $u_0(x) = \sin(2\pi x)$

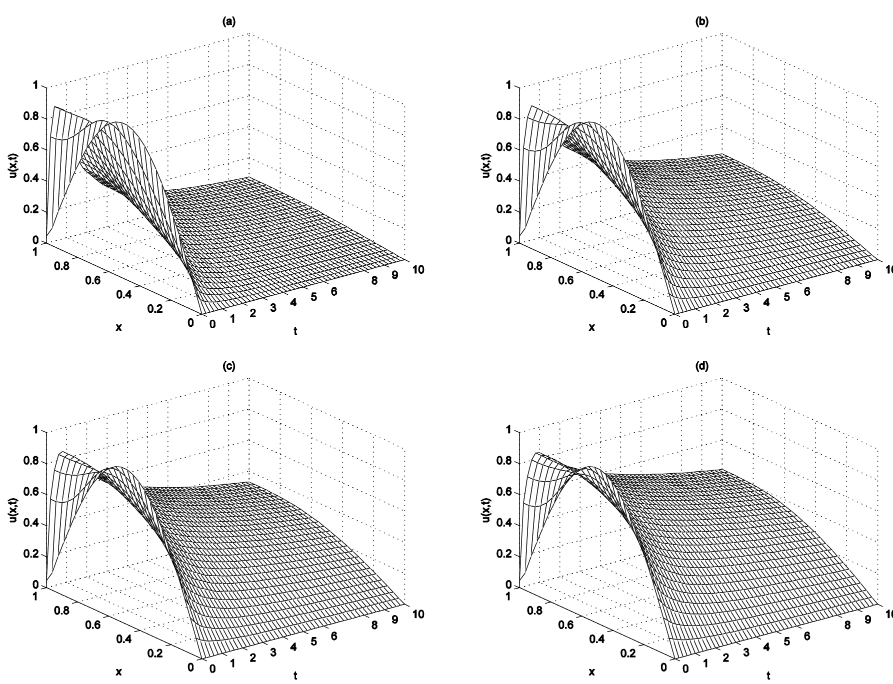

Figure 13 A 3-d landscape of the dynamics of the MGKdVB equation without control; $v = 0.01, \mu = 0.001, \gamma_1 = 1, \gamma_2 = 0.0005$ and $u_0(x) = \sin(\pi x)$; (**a**) $\alpha = 1$; (**b**) $\alpha = 2$; (**c**) $\alpha = 3$; (**d**) $\alpha = 4$

Different initial conditions $u_0(x)$ were considered in our study. In the simulations, we set the parameters v, μ, γ_1 and γ_2 to be 0.01, 0.001, 1, and 0.0005, respectively. Figures 13(a)–(d) depict a 3-d landscape of the behavior of the solution $u(x,t)$ as it evolves in time when $u_0(x) = \sin(\pi x)$. The L^2-norm of $u(x,t)$, $\|u(x,t)\|$, versus time and the natural logarithm of the L^2-norm of $u(x,t)$, $\ln(\|u(x,t)\|)$, versus time are presented in Fig. 14, and Fig. 15,

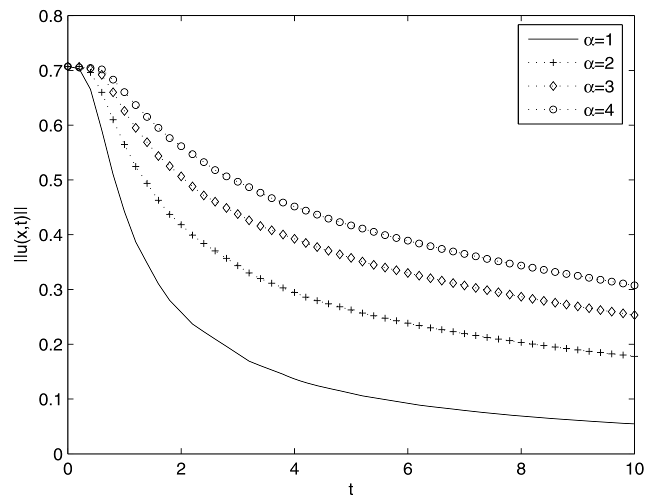

Figure 14 The L^2-norm of $u(x,t)$, $\|u(x,t)\|$, versus time for various values of α without control; $v = 0.01, \mu = 0.001, \gamma_1 = 1, \gamma_2 = 0.0005$ and $u_0(x) = \sin(\pi x)$

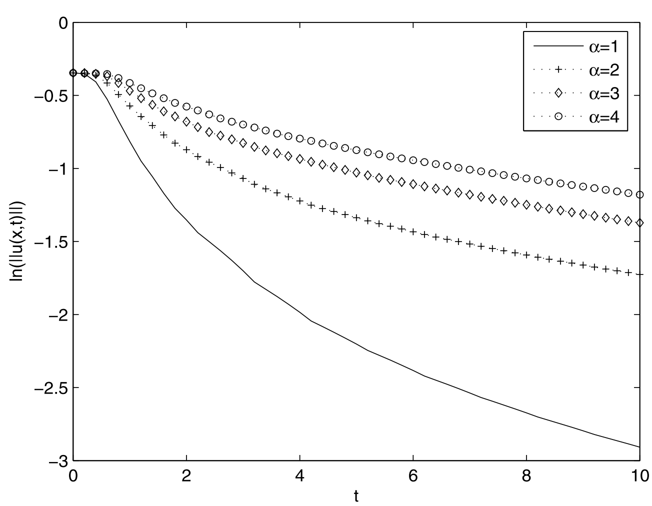

Figure 15 The natural logarithm of the L^2-norm of $u(x,t)$, $\ln(\|u(x,t)\|)$, versus time for various values of α without control; $v = 0.01, \mu = 0.001, \gamma_1 = 1, \gamma_2 = 0.0005$ and $u_0(x) = \sin(\pi x)$

respectively. This figures show that $\|u(x,t)\|$ takes a longer time to approach the steady state solution. It can also be seen from Figs. 13–15 that the time taken to approach the steady state solution increases as α increases.

Figures 16(a)–(d) present the solution of the MGKdVB equation when the initial condition $u_0(x) = \sin(2\pi x)$. In the simulations, we set the parameters v, μ, γ_1 and γ_2 to be 0.01, 0.001, 1, and 0.0005, respectively. The L^2-norm of $u(x,t)$, $\|u(x,t)\|$, versus time and the natural logarithm of the L^2-norm of $u(x,t)$, $\ln(\|u(x,t)\|)$, versus time are presented in Fig. 17 and Fig. 18, respectively. Looking carefully at Figs. 17 and 18 one can see that when $\alpha = 1$; 3, the solution $u(x,t)$ does not converge to the steady state solution; whereas, when $\alpha = 2$; 4, the solution converges very slowly to the steady solution.

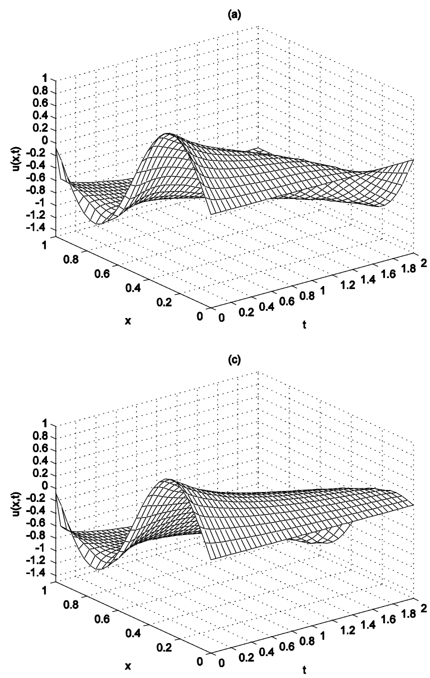

Figure 16 A 3-d landscape of the dynamics of the MGKdVB equation without control; $v = 0.01, \mu = 0.001, \gamma_1 = 1, \gamma_2 = 0.0005$ and $u_0(x) = \sin(2\pi x)$; (**a**) $\alpha = 1$; (**b**) $\alpha = 2$; (**c**) $\alpha = 3$; (**d**) $\alpha = 4$

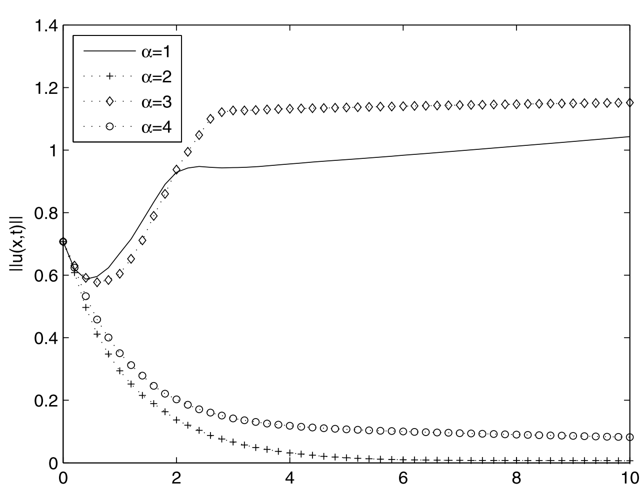

Figure 17 The L^2-norm of $u(x,t)$, $\|u(x,t)\|$, versus time for various values of α without control; $v = 0.01, \mu = 0.001, \gamma_1 = 1, \gamma_2 = 0.0005$ and $u_0(x) = \sin(2\pi x)$

6 Comparison of the performances of the nonlinear controllers proposed in Theorems 1–2 with the one without control

In this section, a comparison between the performances of the nonlinear non-adaptive designed controllers presented in Theorems 1 and 2 is given numerically for different values of α. Moreover, a comparison between the behavior of the uncontrolled system and

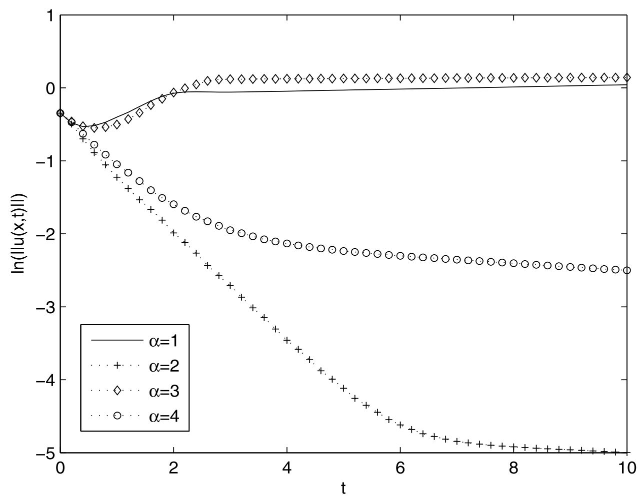

Figure 18 The natural logarithm of the L^2-norm of $u(x,t)$, $\ln(\|u(x,t)\|)$, versus time for various values of α without control; $v = 0.01$, $\mu = 0.001$, $\gamma_1 = 1$, $\gamma_2 = 0.0005$ and $u_0(x) = \sin(2\pi x)$

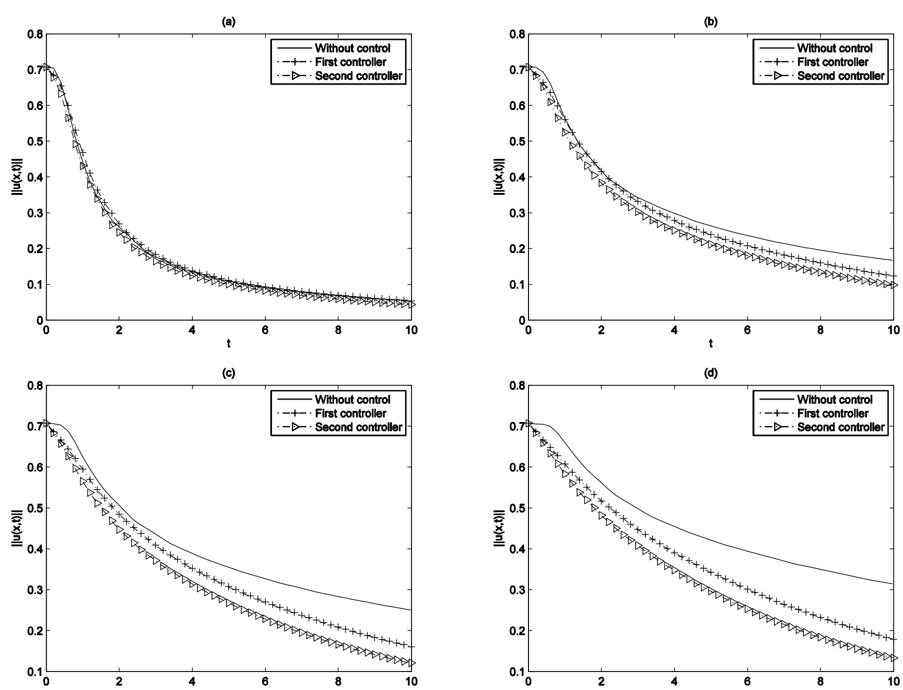

Figure 19 The L^2-norm of $u(x,t)$, $\|u(x,t)\|$, versus time for different values of α; comparison between the behavior of the equation with and without control when $v = 0.01$, $\mu = 0.001$, $\gamma_1 = 1$, $\gamma_2 = 0.0005$ and $u_0(x) = \sin(\pi x)$; (**a**) $\alpha = 1$; (**b**) $\alpha = 2$; (**c**) $\alpha = 3$; (**d**) $\alpha = 4$

the system after applying the two nonlinear controllers proposed previously will be also presented.

The L^2-norm of the solutions $u(x,t)$ of the MGKdVB equation is used for comparison. Figures 19(a)–(d) show the L^2-norm of $u(x,t)$ versus time for different values of α when $u_0(x) = \sin(\pi x)$. These figures show that the solution of the MGKdVB equation obtained

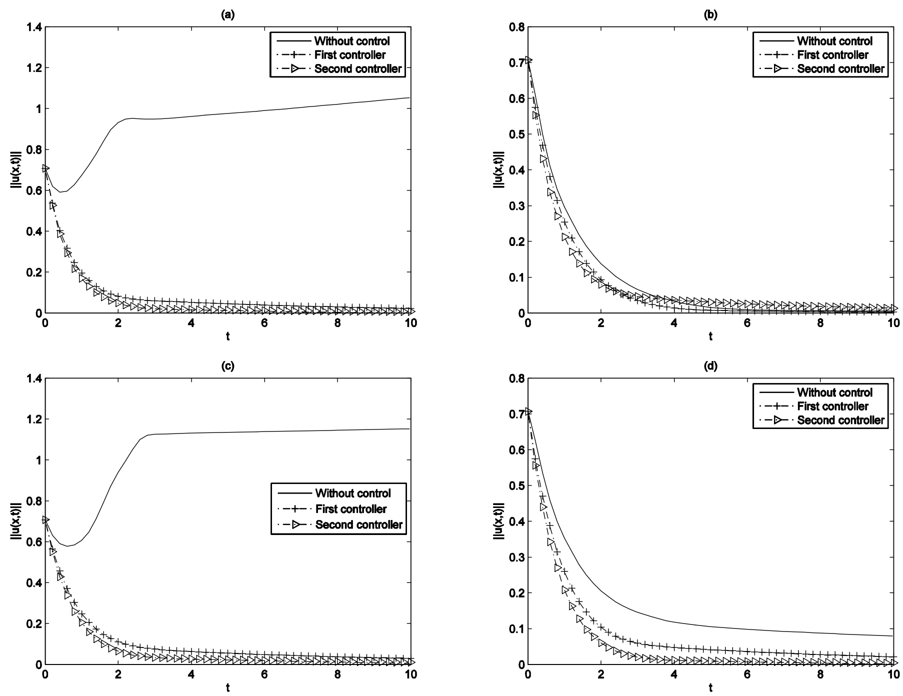

Figure 20 The L^2-norm of $u(x,t)$, $\|u(x,t)\|$, versus time for various values of α; comparison between the behavior of the equation with and without control when $v = 0.01$, $\mu = 0.001$, $\gamma_1 = 1$, $\gamma_2 = 0.0005$ and $u_0(x) = \sin(2\pi x)$; (**a**) $\alpha = 1$; (**b**) $\alpha = 2$; (**c**) $\alpha = 3$; (**d**) $\alpha = 4$

using the second controller outperforms the solution obtained using the first controller for $\alpha = 1, 2, 3$ and 4. A careful look at the figures also demonstrates that, for $\alpha = 2, 3$ and 4, the two controllers give better results than the solutions obtained without applying any control. On the other hand, for $\alpha = 1$, one can notice that solutions of the MGKdVB equation obtained using the two control laws seem to have a similar decay rate to the case when no control is applied.

Figures 20(a)–(d) show the L^2-norm of $u(x,t)$ versus time when $u_0(x) = \sin(2\pi x)$ for $\alpha = 1, 2, 3, 4$. A thorough observation of the figures demonstrates that the first and the second nonlinear controllers force the solutions to converge to the trivial solution faster than the case when having no control. One can also notice from Figs. 20(a) and 20(c) the significant effect of the first and the second controllers in speeding up the convergence to the steady state solution when α is odd. Also, it can be clearly seen that the solution of the MGKdVB equation obtained using the second controller outperforms the solutions obtained when applying the first controller, for all values of α.

7 Concluding remarks

The boundary stabilization of the MGKdVB equation was considered in this paper. First, the derivation of the MGKdVB equation for the case when $\alpha = 3$ is obtained. Then, two different control laws were designed for this equation when the physical parameters of the MGKdVB equation are known and positive. The global exponential stability of the solution in $L^2(0, 1)$ was presented analytically as well as numerically. Also, a comparison between the convergence rates of the two presented control laws was shown.

The adaptive control of the MGKdVB equation will be the subject of future research studies.

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Competing interests
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Authors' contributions
NS suggested and derived the model, analyzed the results, performed the numerical simulations and wrote the manuscript. BC revised and edited the manuscript. AA derived the model and performed the numerical simulations. All authors read and approved the final manuscript.

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