EL-Kalaawy et al. *Advances in Difference Equations* (2019) 2019:445
https://doi.org/10.1186/s13662-019-2375-y

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH **Open Access**

Conservation laws, analytical solutions and stability analysis for the time-fractional Schamel–Zakharov–Kuznetsov–Burgers equation

O.H. EL-Kalaawy[1*], S.M. Moawad[1], M.M. Tharwat[1] and Rasha B. Al-Denari[1]

*Correspondence:
ohkalaawy7@gmail.com
[1]Department of Mathematics and Computer Science, Faculty of Science, Beni-Suef University, Beni-Suef, Egypt

Abstract

In this paper, we consider the (3 + 1)-dimensional time-fractional Schamel–Zakharov–Kuznetsov–Burgers (SZKB) equation. With the help of the Riemann–Liouville derivatives, the Lie point symmetries of the (3 + 1)-dimensional time-fractional SZKB equation are derived. By applying the Lie point symmetry method as well as Erdélyi–Kober fractional operator, we get the similarity reductions of the time-fractional SZKB equation. Conservation laws of the time-fractional SZKB are constructed. Moreover, we obtain its power series solutions with the convergence analysis. In addition, the analytical solution is obtained by modified trial equation method. Finally, stability is analyzed graphically in different planes.

Keywords: Lie point symmetries; Conservation laws; Time-fractional Schamel–Zakharov–Kuznetsov–Burgers equation; Riemann–Liouville derivatives; Similarity reduction; Explicit power series; Modified trial equation

1 Introduction

Partial differential equations (PDEs) are frequently used to describe most of the phenomena that arise in engineering fields, mathematical physics, plasma physics, solid state physics, quantum mechanics, fluid mechanics, ecology, optical fibers, biology, chemical kinematics, geochemistry, meteorology, electricity and so on. Therefore, investigating analytical solutions (traveling wave solutions or soliton solutions) is very interesting. As a result, many new techniques have been successfully proposed, developed and extended by groups of researchers to find exact or analytical solutions for PDEs, such as the (G'/G)-expansion method [1–3], the Kudryashov method [4, 5], the functional variable method [5, 6], the first integral method [7], the $\exp(-\phi(\xi))$-expansion method [8], the sine–cosine function method and Bernoulli's equation approach [5, 9], the trial solution method [5, 10], Hirota's bilinear method [3] etc.

Fractional calculus was developed as one of the best tools for studying various models in each discipline [11, 12]. Using fractional models to describe the special characteristic phenomenon of fractional order, it can well reveal the essence of the nature of phenomena and their behavior. Fractional calculus is a generalization of integer calculus, integer

Springer

© The Author(s) 2019. This article is distributed under the terms of the Creative Commons Attribution 4.0 International License (http://creativecommons.org/licenses/by/4.0/), which permits unrestricted use, distribution, and reproduction in any medium, provided you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made.

calculus is a special case of fractional calculus. It has universal significance to study the fractional calculus system. Compared with the integer order model in describing the actual physical process, the physical meaning of the fractional order model is clearer and the expression is more concise.

On the other hand, in recent years, fractional ordinary differential equations (FODEs) and fractional partial differential equations (FPDEs) have been widely used to describe many various physical effects and many complex nonlinear phenomena. This is due to their accurate description of nonlinear phenomena in fluid mechanics, viscoelasticity, electrical chemistry, quantum biology, physics, engineering mechanics [13–19] and other scientific fields. So the study of FODEs and FPDEs has attracted much attention. In other words, by using the theory of derivatives and integrals of fractional order, many physical phenomena can be accurately modeled.

The exact solutions of most of the FPDEs cannot be found easily, so analytical and numerical methods [20, 21] must be used. The solutions of the FPDEs are investigated by many authors using powerful analytical methods. Several numerical methods such as the homotopy perturbation method [22, 23], the Adomian decomposition method [24], the variational iteration method [25], the differential transform method [26], the fractional Riccati expansion method [27], the fractional sub-equation method [28–34] have been suggested for solving FDEs. However, solutions obtained through all these methods are of a local nature and it is important to explore other techniques in order to find exact analytical solutions of FDEs.

Lie point symmetry plays a very important role in various fields of science, especially in integrable systems where infinitely many symmetries exist. So Lie symmetry analysis is considered to be one of the efficient approaches for obtaining analytical solutions of nonlinear partial differential equations (NLPDEs). A large number of studies are devoted to the theory of Lie point symmetry and their applications to DEs. The symmetry analysis of FDEs and the fractional derivatives are proposed by Gazizov and his collaborators [35]. They proposed a prolongation formula for two basic fractional derivatives: Riemann–Liouville and Caputo. This method has been used to study many of the FDEs [36–38]. Furthermore Lie point symmetry is used to construct conservation laws which play an important role in the study of nonlinear physical phenomena [12]. Conservation laws are a mathematical formulation which statement that the total amount of a certain physical quantity stays the same during the evolution of a physical system [12]. Conservation laws are also used in the development of numerical methods to establish the existence and uniqueness of solution. There are many methods of constructing conservation laws for differential equations (DEs) [39–45]. The well-known Noether theorem [46] establishes a connection between Lie point symmetries and conservation laws of DEs, provided that the equations are Euler–Lagrange equations. Ibragimov [47] suggested a new conservation law theorem. In [39], conservation laws for time-fractional sub-diffusion and diffusion-wave equations were obtained based on the new conservation laws theorem [47]. Also there are researches that discussed conservation laws for time FPDEs [12, 48, 49].

In this work, we focus on the time-fractional SZKB equation of the form

$$\frac{\partial^\alpha u}{\partial t^\alpha} + a\sqrt{u}u_x + bu_{xxx} + c(u_{xyy} + u_{xzz}) + du_{xx} = 0, \tag{1.1}$$

where ∂_t^α is the fractional derivative of order α, with $0 < \alpha < 1$ and $u(x, y, z, t)$ is the potential function of space x, y, z and time t. If $\alpha = 1$, Eq. (1.1) is reduced to the classical SZKB equation, which describes the nonlinear plasma-dust ion acoustic waves DIAWs in a magnetized dusty plasma and it is derived using the standard reductive perturbation technique in small amplitude. The coefficients of dispersion b, non-linearity a, mixed derivative c and dissipation d are given in [50, 51]. *Note that Sahoo and Ray* [52] *have studied the* $(3+1)$-*dimensional time-fractional mKdV-ZK equation without the Schamel and Burgers terms*. The main purpose of this paper is to obtain the Lie point symmetries, conservation laws and analytical solutions of the time-fractional SZKB equation.

The paper is organized as follows: The introduction is presented in Sect. 1. In Sect. 2, some definitions and description of Lie symmetry analysis for fractional partial differential equations (FPDEs) are briefly presented. Lie point symmetries and similarity reduction of the Eq. (1.1) are obtained In Sect. 3. In Sect. 4, the conservation laws of the Eq. (1.1) are obtained. In Sect. 5, which is based on the power series, the analytical solution of the Eq. (1.1) is constructed with convergence analysis. We construct the analytical solution of Eq. (1.1) by using a modified trial equation method In Sect. 6. Finally, the discussions and conclusions of this paper are presented in Sect. 7.

2 Preliminaries

Here in this section, we focus on some of the concepts that revolve around the subject of our article

Definition 1 Let $\alpha > 0$. The operator I^α defined by

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds, \tag{2.1}$$

is called the Riemann–Liouville (R-L) fractional integral operator of order α, and $\Gamma(\cdot)$ denotes the gamma function.

Definition 2 Let $\alpha > 0$. The operator D_t^α defined by

$$D_t^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} f(s)\, ds & \text{if } n-1 < \alpha < n, n \in N, \\ \frac{d^n f(t)}{dt^n} & \text{if } \alpha = n, n \in N, \end{cases} \tag{2.2}$$

is called the R-L fractional partial derivative [16, 53–56].

2.1 Description of Lie point symmetries

Let us consider the symmetry analysis for a FPDE of the form

$$D_t^\alpha u(x,y,z,t) = G(x,y,z,t,u,u_x,u_t,u_y,u_z,u_{xx},u_{xy},\ldots), \quad 0 < \alpha < 1. \tag{2.3}$$

Now, let Eq. (2.3) be invariant under the following one-parameter Lie group of point transformations acting on both the dependent and the independent variables:

$$\begin{aligned} \bar{x} &= x + \varepsilon\xi(x,y,z,t) + O(\varepsilon^2), \\ \bar{y} &= y + \varepsilon\zeta(x,y,z,t) + O(\varepsilon^2), \\ \bar{z} &= z + \varepsilon\nu(x,y,z,t) + O(\varepsilon^2), \end{aligned}$$

$$
\begin{aligned}
&\bar{t} = t + \varepsilon\tau(x,y,z,t) + O(\varepsilon^2),\\
&\bar{u} = u + \varepsilon\eta(x,y,z,t) + O(\varepsilon^2),\\
&D_{\bar{t}}^{\alpha}\bar{u} = D_t^{\alpha}u + \varepsilon\eta_{\alpha}^{0}(x,y,z,t) + O(\varepsilon^2),\\
&\frac{\partial\bar{u}}{\partial\bar{x}} = \frac{\partial u}{\partial x} + \varepsilon\eta^{x}(x,y,z,t) + O(\varepsilon^2),\\
&\frac{\partial^2\bar{u}}{\partial\bar{x}^2} = \frac{\partial^2 u}{\partial x^2} + \varepsilon\eta^{xx}(x,y,z,t) + O(\varepsilon^2),\\
&\frac{\partial^3\bar{u}}{\partial\bar{x}^3} = \frac{\partial^3 u}{\partial x^3} + \varepsilon\eta^{xxx}(x,y,z,t) + O(\varepsilon^2),\\
&\frac{\partial^3\bar{u}}{\partial\bar{x}\,\partial\bar{y}^2} = \frac{\partial^3 u}{\partial x\,\partial y^2} + \varepsilon\eta^{xyy}(x,y,z,t) + O(\varepsilon^2),\\
&\frac{\partial^3\bar{u}}{\partial\bar{x}\,\partial\bar{z}^2} = \frac{\partial^3 u}{\partial x\,\partial z^2} + \varepsilon\eta^{xzz}(x,y,z,t) + O(\varepsilon^2),
\end{aligned}
\tag{2.4}
$$

where $\varepsilon \ll 1$ is the Lie group parameter and ξ, ζ, ν, τ, η are the infinitesimals of the transformations for dependent and independent variables, respectively. The explicit expressions of η^x, η^{xx}, η^{xxx}, η^{xyy}, η^{xzz} are given by

$$
\begin{aligned}
&\eta^{x} = D_x(\eta) - u_x D_x(\xi) - u_y D_x(\zeta) - u_z D_x(\nu) - u_t D_x(\tau),\\
&\eta^{xx} = D_x(\eta^{x}) - u_{xx} D_x(\xi) - u_{xy} D_x(\zeta) - u_{xz} D_x(\nu) - u_{xt} D_x(\tau),\\
&\eta^{xxx} = D_x(\eta^{xx}) - u_{xxx} D_x(\xi) - u_{xxy} D_x(\zeta) - u_{xxz} D_x(\nu) - u_{xxt} D_x(\tau),\\
&\eta^{xyy} = D_x(\eta^{yy}) - u_{xxy} D_x(\xi) - u_{xyy} D_x(\zeta) - u_{xyz} D_x(\nu) - u_{xyt} D_x(\tau),\\
&\eta^{xzz} = D_x(\eta^{zz}) - u_{xxz} D_x(\xi) - u_{xyz} D_x(\zeta) - u_{xzz} D_x(\nu) - u_{xzt} D_x(\tau),
\end{aligned}
\tag{2.5}
$$

where D_x, D_y, D_z, and D_t are the total derivatives with respect to x, y, z, and t, respectively, that are defined for $x^1 = x$, $x^2 = y$, $x^3 = z$ as

$$
D_{xj} = \frac{\partial}{\partial x^j} + u_j\frac{\partial}{\partial u} + u_{jk}\frac{\partial}{\partial u_k} + \cdots, \quad j,k = 1,2,3,\ldots,
$$

where $u_j = \frac{\partial u}{\partial x^j}$, $u_{jk} = \frac{\partial^2 u}{\partial x^j \partial x^k}$ and so on.

The corresponding Lie algebra of symmetries consists of a set of vector fields of the form

$$
V = \xi\frac{\partial}{\partial x} + \zeta\frac{\partial}{\partial y} + \nu\frac{\partial}{\partial z} + \tau\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial u}.
$$

The invariance condition of Eq. (2.3) under the infinitesimal transformations is given as

$$
\mathrm{Pr}^{(n)}\, V(\Delta)\big|_{\Delta=0} = 0, \quad n = 1,2,3,\ldots,
$$

where

$$
\Delta := D_t^{\alpha}u(x,y,z,t) - G(x,y,z,t,u,u_x,u_t,u_y,u_z,u_{xx},u_{xy},\ldots).
$$

Also, the invariance condition gives

$$\tau(x,y,z,t,u)|_{t=0}=0. \tag{2.6}$$

The αth extended infinitesimal related to RL fractional time derivative with Eq. (2.6) can be represented as follows:

$$\begin{aligned}\eta_\alpha^0 &= D_t^\alpha(\eta)+\xi D_t^\alpha(u_x)-D_t^\alpha(\xi u_x)+\zeta D_t^\alpha(u_y)-D_t^\alpha(\zeta u_y)+\nu D_t^\alpha(u_z)\\ &\quad -D_t^\alpha(\nu u_z)+D_t^\alpha\bigl(D_t(\tau)u\bigr)-D_t^{\alpha+1}(\tau u)+\tau D_t^{\alpha+1}(u),\end{aligned} \tag{2.7}$$

where D_t^α is the total fractional derivative operator and by using the generalized Leibnitz rule

$$D_t^\alpha\bigl(f(t)g(t)\bigr)=\sum_{n=0}^{\infty}\binom{\alpha}{n}D_t^{\alpha-n}f(t)D_t^n g(t),\qquad \binom{\alpha}{n}=\frac{(-1)^{n-1}\alpha\Gamma(n-\alpha)}{\Gamma(1-\alpha)\Gamma(n+1)}.$$

By applying the Leibnitz rule, Eq. (2.7) becomes

$$\begin{aligned}\eta_\alpha^0 &= D_t^\alpha(\eta)-\alpha D_t^\alpha(\tau)\frac{\partial^\alpha u}{\partial t^\alpha}-\sum_{n=1}^{\infty}\binom{\alpha}{n}D_t^n(\xi)D_t^{\alpha-n}u_x-\sum_{n=1}^{\infty}\binom{\alpha}{n}D_t^n(\zeta)D_t^{\alpha-n}u_y\\ &\quad -\sum_{n=1}^{\infty}\binom{\alpha}{n}D_t^n(\nu)D_t^{\alpha-n}u_z-\sum_{n=1}^{\infty}\binom{\alpha}{n+1}D_t^{n+1}(\xi)D_t^{\alpha-n}u.\end{aligned} \tag{2.8}$$

Now the chain rule for the compound function is defined as follows:

$$\frac{d^n\phi(h(t))}{dt^n}=\sum_{k=0}^{n}\sum_{r=0}^{k}\binom{k}{r}\frac{1}{k!}\bigl[-h(t)\bigr]^r\frac{d^n}{dt^n}\bigl[-h(t)^{k-r}\bigr]\times\frac{d^k\phi(h)}{dh^k}.$$

By applying this rule and the generalized Leibnitz rule with $f(t)=1$, we have

$$D_t^\alpha(\eta)=\frac{\partial^\alpha\eta}{\partial t^\alpha}+\eta_u\frac{\partial^\alpha u}{\partial t^\alpha}-u\frac{\partial^\alpha\eta_u}{\partial t^\alpha}+\sum_{n=1}^{\infty}\binom{\alpha}{n}\frac{\partial^n\eta_u}{\partial t^n}D_t^{\alpha-n}(u)+\mu,$$

where

$$\mu=\sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\alpha}{n}\binom{n}{m}\binom{k}{r}\frac{1}{k!}\times\frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)}[-u]^r\frac{\partial^m}{\partial t^m}\bigl[u^{k-r}\bigr]\frac{\partial^{n-m+k}\eta}{\partial t^{n-m}\partial u^k}.$$

Therefore, Eq. (2.8) yields

$$\begin{aligned}\eta_\alpha^0 &= \frac{\partial^\alpha\eta}{\partial t^\alpha}+\bigl(\eta_u-\alpha D_t^\alpha(\tau)\bigr)\frac{\partial^\alpha u}{\partial t^\alpha}-u\frac{\partial^\alpha\eta_u}{\partial t^\alpha}+\mu\\ &\quad +\sum_{n=1}^{\infty}\left[\binom{\alpha}{n}\frac{\partial^\alpha\eta_u}{\partial t^\alpha}-\binom{\alpha}{n+1}D_t^{n+1}(\tau)\right]D_t^{\alpha-n}(u)+\sum_{n=1}^{\infty}(\alpha)D_t^n(\xi)D_t^{\alpha-n}u_x\\ &\quad -\sum_{n=1}^{\infty}\binom{\alpha}{n}D_t^n(\zeta)D_t^{\alpha-n}u_y-\sum_{n=1}^{\infty}\binom{\alpha}{n}D_t^n(\nu)D_t^{\alpha-n}u_z.\end{aligned} \tag{2.9}$$

Definition 3 The function $u = \theta(x,y,z,t)$ is an invariant solution of Eq. (2.3) associated with the vector field W, such that

1. $u = \theta(x,y,z,t)$ is an invariant surface of Eq. (2.3), this means

$$V\theta = 0 \quad \Leftrightarrow \quad \left(\xi\frac{\partial}{\partial x} + \zeta\frac{\partial}{\partial y} + \nu\frac{\partial}{\partial z} + \tau\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial u}\right)\theta = 0,$$

2. $u = \theta(x,y,z,t)$ satisfies Eq. (2.3).

3 Lie point symmetries and similarity reduction for Eq. (1.1)

In this section, we implemented Lie group method for Eq. (1.1) and then, used these symmetries to reduce Eq. (1.1) to be a FODE as shown in the next two subsections

3.1 Lie point symmetries for Eq. (1.1)

Let us consider Eq. (1.1) as an invariant under Eq. (2.4), we get

$$\frac{\partial^\alpha \bar{u}}{\partial \bar{t}^\alpha} + a\sqrt{\bar{u}}\bar{u}_x + b\bar{u}_{xxx} + c(\bar{u}_{xyy} + \bar{u}_{xzz}) + d\bar{u}_{xx} = 0, \tag{3.1}$$

such that $u = u(x,y,z,t)$ satisfies Eq. (1.1), then using the point transformations Eq. (2.4) in Eq. (3.1), we get the invariant equation

$$\eta_\alpha^0 + a\sqrt{u}\eta^x + \frac{a}{2\sqrt{u}}\eta u_x + b\eta^{xxx} + c\left(\eta^{xyy} + \eta^{xzz}\right) + d\eta^{xx} = 0, \tag{3.2}$$

By substituting Eq. (2.5) and Eq. (2.9) into Eq. (3.2), we acquire

$$\begin{aligned}
&\frac{\partial^\alpha \eta}{\partial t^\alpha} + \left(\eta_u - \alpha D_t^\alpha(\tau)\right)\frac{\partial^\alpha u}{\partial t^\alpha} - u\frac{\partial^\alpha \eta_u}{\partial t^\alpha} + \mu \\
&+ \sum_{n=1}^{\infty}\left[\binom{\alpha}{n}\frac{\partial^\alpha \eta_u}{\partial t^\alpha} - \binom{\alpha}{n+1}D_t^{n+1}(\tau)\right]D_t^{\alpha-n}(u) + \sum_{n=1}^{\infty}(\alpha)D_t^n(\xi)D_t^{\alpha-n}u_x \\
&- \sum_{n=1}^{\infty}\binom{\alpha}{n}D_t^n(\zeta)D_t^{\alpha-n}u_y - \sum_{n=1}^{\infty}\binom{\alpha}{n}D_t^n(\nu)D_t^{\alpha-n}u_z \\
&+ a\sqrt{u}\big(\eta_x + (\eta_u - \xi_x)u_x - \xi_u u_x^2 - (\zeta_x + \zeta_u u_x)u_y - (\nu_x + \nu_x u_x)u_z - (\tau_x + \tau_u u_x)u_t\big) \\
&+ \frac{a}{2\sqrt{u}}\eta u_x + b\big(\eta_{xxx} + (\eta_{xxu} - \xi_{xxx})u_x + (3\eta_{xu} - 3\xi_{xx})u_{xx} + (\eta_u - 3\xi_x)u_{xxx} \\
&+ (3\eta_{xuu} - 3\xi_{xxu})u_x^2 + (\eta_{uuu} - \xi_{xuu})u_x^3 - \xi_{uuu}u_x^4 + (3\eta_{uu} - 9\xi_{xu})u_x u_{xx} \\
&- 3\xi_{uu}u_x^2 u_{xx} - 3\xi_u u_{xx}^2 - 4\xi_u u_x u_{xxx} - 3(\zeta_x + \zeta_u u_x)u_{xxy} - 3(\nu_x + \nu_x u_x)u_{xxz} \\
&- 3(\tau_x + \tau_u u_x)u_{xxt} - 3\big(\zeta_{xx} + 2\zeta_{xu}u_x + \zeta_{uu}u_x^2 + \zeta_u u_{xx}\big)u_{xy} \\
&- 3\big(\nu_{xx} + 2\nu_{xu}u_x + \nu_{uu}u_x^2 + \nu_u u_{xx}\big)u_{xz} - 3\big(\tau_{xx} + 2\tau_{xu}u_x + \tau_{uu}u_x^2 + \tau_u u_{xx}\big)u_{xt} \\
&- \big(\zeta_{xxx} + 3\zeta_{xxu}u_x + 3\zeta_{xuu}u_x^2 + \zeta_{uuu}u_x^3 + 3\zeta_{uu}u_x u_{xx} + 3\zeta_{xu}u_{xx} + \zeta_u u_{xxx}\big)u_y \\
&- \big(\nu_{xxx} + 3\nu_{xxu}u_x + 3\nu_{xuu}u_x^2 + \nu_{uuu}u_x^3 + 3\nu_{uu}u_x u_{xx} + 3\nu_{xu}u_{xx} + \nu_u u_{xxx}\big)u_z \\
&- \big(\tau_{xxx} + 3\tau_{xxu}u_x + 3\tau_{xuu}u_x^2 + \tau_{uuu}u_x^3 + 3\tau_{uu}u_x u_{xx} + 3\tau_{xu}u_{xx} + \tau_u u_{xxx}\big)u_t\big) \\
&+ c(\eta_{xyy} + (\eta_{uyy} - \xi_{xyy})u_x + (2\eta_{uy} - \zeta_{yy} - 2\xi_{xy})u_{xy}
\end{aligned}$$

$$
\begin{aligned}
&+ (2\eta_{xuy} + 2\eta_{uuy}u_x - \zeta_{xyy} - \zeta_{uyy}u_x)u_y + (\eta_u - 2\zeta_y - \xi_x - \zeta_u u_x)u_{xyy} \\
&+ 2(\eta_{uu} - 2\zeta_{uy})u_y u_{xy} + (\eta_{xu} + \eta_{uu}u_x - 2\zeta_{xy} - 2\zeta_{uy}u_x)u_{yy} - 3\zeta_{uu}u_y^2 u_{xy} \\
&+ (3\eta_{uux} + \eta_{uuu}u_x - 2\zeta_{xuy} - 2\zeta_{uuy}u_x)u_y^2 - 3\zeta_u u_{xy}u_{yy} - (\zeta_{uux} + \zeta_{uuu}u_x)u_y^3 \\
&- 3\zeta_u u_y u_{xyy} - 3(\zeta_{xu} + \zeta_{uu}u_x)u_y u_{yy} - (2\xi_y + 2\xi_u u_y + \xi_x + \xi_u u_x)u_{xxy} \\
&- 2(\xi_{uy}u_x + \xi_{xu}u_y + \xi_{uu}u_x u_y + \xi_u u_{xy})u_{xy} - (2\nu_y + 2\nu_u u_y + \nu_x + \nu_u u_x)u_{xyz} \\
&- 2(\nu_{uy}u_x + \nu_{xu}u_y + \nu_{uu}u_x u_y + \nu_u u_{xy})u_{zy}(2\tau_y + 2\tau_u u_y + \tau_x + \tau_u u_x)u_{xyt} \\
&- 2(\tau_{uy}u_x + \tau_{xu}u_y + \tau_{uu}u_x u_y + \tau_u u_{xy})u_{yt} - \left(\xi_{yy}u_x + 2\xi_{uy}u_y + \xi_{uu}u_y^2 + \xi_u u_{yy}\right)u_{xx} \\
&- \left(\nu_{yy} + 2\nu_{uy}u_y + \nu_{uu}u_y^2 + \nu_u u_{yy}\right)u_{xz} - \left(\tau_{yy} + 2\tau_{uy}u_y + \tau_{uu}u_y^2 + \tau_u u_{yy}\right)u_{xt} \\
&- \left(\xi_{xyy} + \xi_{uyy}u_x + 2\xi_{xyu}u_y + 2\xi_{uuy}u_x u_y + 2\xi_{uy}u_{xy} + \xi_{xuu}u_y^2 + \xi_{uuu}u_x u_y^2 + 2\xi_{uu}u_x u_{xy}\right. \\
&\left.+ \xi_{xu}u_{yy} + \xi_{xu}u_x u_{yy} + \xi_u u_{xyy}\right)u_x - \left(\nu_{xyy} + \nu_{uyy}u_x + 2\nu_{xyu}u_y + 2\nu_{uuy}u_x u_y + 2\nu_{uy}u_{xy}\right. \\
&\left.+ \nu_{xuu}u_y^2 + \nu_{uuu}u_x u_y^2 + 2\nu_{uu}u_x u_{xy} + \nu_{xu}u_{yy} + \nu_{xu}u_x u_{yy} + \nu_u u_{xyy}\right)u_z \\
&- \left(\tau_{xyy} + \tau_{uyy}u_x + 2\tau_{xyu}u_y + 2\tau_{uuy}u_x u_y + 2\tau_{uy}u_{xy} + \tau_{xuu}u_y^2 + \tau_{uuu}u_x u_y^2 + 2\tau_{uu}u_x u_{xy}\right. \\
&\left.+ \tau_{xu}u_{yy} + \tau_{xu}u_x u_{yy} + \tau_u u_{xyy}\right)u_t + \eta_{xzz} + (\eta_{uzz} - \xi_{xzz})u_x + (2\eta_{uy} - \zeta_{yy} - 2\xi_{xy})u_{xz} \\
&+ (2\eta_{xuz} + 2\eta_{uuz}u_x - \nu_{xzz} - \nu_{uzz}u_x)u_z + (\eta_u - 2\nu_z - \xi_x - \nu_u u_x)u_{xzz} \\
&+ 2(\eta_{uu} - 2\nu_{uz})u_z u_{xz} + (\eta_{xu} + \eta_{uu}u_x - 2\nu_{xz} - 2\nu_{uz}u_x)u_{zz} - 3\nu_{uu}u_z^2 u_{xz} \\
&+ (3\eta_{uux} + \eta_{uuu}u_x - 2\nu_{xuz} - 2\nu_{uuz}u_x)u_z^2 - 3\nu_u u_{xz}u_{zz} - (\nu_{uux} + \nu_{uuu}u_x)u_z^3 - 3\nu_u u_z u_{xzz} \\
&- 3(\nu_{xu} + \nu_{uu}u_x)u_z u_{zz} - (2\xi_z + 2\xi_u u_z + \xi_x + \xi_u u_x)u_{xxz} \\
&- 2(\xi_{uz}u_x + \xi_{xu}u_z + \xi_{uu}u_x u_z + \xi_u u_{xz})u_{xz} - (2\zeta_z + 2\zeta_u u_z + \zeta_x + \zeta_u u_x)u_{xyz} \\
&- 2(\zeta_{uz}u_x + \zeta_{xu}u_z + \zeta_{uu}u_x u_z + \zeta_u u_{xz})u_{zy} - (2\tau_z + 2\tau_u u_z + \tau_x + \tau_u u_x)u_{xzt} \\
&- 2(\tau_{uz}u_x + \tau_{xu}u_z + \tau_{uu}u_x u_z + \tau_u u_{xz})u_{zt} - \left(\xi_{zz}u_x + 2\xi_{uz}u_z + \xi_{uu}u_z^2 + \xi_u u_{zz}\right)u_{xx} \\
&- \left(\zeta_{zz} + 2\zeta_{uz}u_y + \zeta_{uu}u_z^2 + \zeta_u u_{zz}\right)u_{xy} - \left(\tau_{zz} + 2\tau_{uz}u_y + \tau_{uu}u_z^2 + \tau_u u_{zz}\right)u_{xt} \\
&- \left(\xi_{xzz} + \xi_{uzz}u_x + 2\xi_{xzu}u_z + 2\xi_{uuz}u_x u_z + 2\xi_{uz}u_{xz} + \xi_{xuu}u_z^2 + \xi_{uuu}u_x u_z^2 + 2\xi_{uu}u_x u_{xz}\right. \\
&\left.+ \xi_{xu}u_{zz} + \xi_{xu}u_x u_{zz} + \xi_u u_{xzz}\right)u_x - \left(\zeta_{xzz} + \zeta_{uzz}u_x + 2\zeta_{xzu}u_z + 2\zeta_{uuz}u_x u_z + 2\zeta_{uz}u_{xz}\right. \\
&\left.+ \zeta_{xuu}u_z^2 + \zeta_{uuu}u_x u_z^2 + 2\zeta_{uu}u_x u_{xz} + \zeta_{xu}u_{zz} + \zeta_{xu}u_x u_{zz} + \zeta_u u_{xzz}\right)u_y \\
&- \left(\tau_{xzz} + \tau_{uzz}u_x + 2\tau_{xzu}u_z + 2\tau_{uuz}u_x u_z + 2\tau_{uz}u_{xz} + \tau_{xuu}u_z^2\right. \\
&\left.+ \tau_{uuu}u_x u_z^2 + 2\tau_{uu}u_x u_{xz} + \tau_{xu}u_{zz} + \tau_{xu}u_x u_{zz} + \tau_u u_{xzz}\right)u_t) \\
&+ d\left(\eta_{xx} + (2\eta_{xu} - \xi_{xx})u_x + (\eta_u - 2\xi_x)u_{xx} + (\eta_{uu} - 2\xi_{xu})u_x^2 - \xi_{uu}u_x^3 - 3\xi_u u_x u_{xx}\right. \\
&- 2(\zeta_x + \zeta_u u_x)u_{xy} - 2(\nu_x + \nu_u u_x)u_{xz} - 2(\tau_x + \tau_u u_x)u_{xt} \\
&- \left(\zeta_{xx} + 2\zeta_{xu}u_x + \zeta uuu_x^2 + \zeta_u u_{xx}\right)u_y - \left(\nu_{xx} + 2\nu_{xu}u_x + \nu uuu_x^2 + \nu_u u_{xx}\right)u_z \\
&\left.- \left(\tau_{xx} + 2\tau_{xu}u_x + \tau uuu_x^2 + \tau_u u_{xx}\right)u_t\right) = 0,
\end{aligned}
\tag{3.3}
$$

now, for Eq. (3.3) by making the whole powers of derivatives of u to zero, we get the following system of equations:

$$\begin{aligned}
&\binom{\alpha}{n}\frac{\partial^\alpha \eta_u}{\partial t^\alpha}-\binom{\alpha}{n+1}D_t^{n+1}(\tau)=0, \quad n=1,2,3,\ldots,\\
&\frac{a}{2\sqrt{u}}\eta+a\sqrt{u}(\eta_u-\xi_x)+b(3\eta_{xxu}-\xi_{xxx})+c(\eta_{yyu}-\xi_{xyy}+\eta_{zzu}-\xi_{xzz})+d(2\xi_{xu}-\xi_{xx})=0,\\
&a\sqrt{u}\eta_x+b\eta_{xxx}+c(\eta_{xyy}+\eta_{xzz})+d\eta_{xx}+\mu=0,\\
&\tau_u=\xi_u=\nu_u=\tau_x=\zeta_u=\nu_u=0,\\
&\tau_{uu}=\xi_{uu}=\nu_{uu}=\zeta_{uu}=\nu_{uu}=0,\\
&D_t^n(\xi)=0, \quad n=1,2,3,\ldots,\\
&\eta_u-\alpha D_t(\tau)=0,\\
&\eta_u-2\zeta_y-\xi_x=0,\\
&\eta_u-3\xi_x=0,\\
&\eta_u-2\nu_z-\xi_x=0.
\end{aligned}$$

On solving the previous system of equations, we get the following infinitesimals:

$$\xi=c_1x+c_2, \qquad \tau=\frac{3}{\alpha}c_1t+c_3, \qquad \zeta=c_1y+c_4, \qquad \nu=c_1z+c_5, \qquad \eta=-4c_1u, \tag{3.4}$$

where c_i, $i=1,2,3,4,5$ are arbitrary constants. So, the associated vector fields are given by

$$\begin{aligned}
&V_1=\frac{\partial}{\partial x}, \qquad V_2=\frac{\partial}{\partial t}, \qquad V_3=\frac{\partial}{\partial y}, \qquad V_4=\frac{\partial}{\partial z},\\
&V_5=x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}+\frac{3}{\alpha}t\frac{\partial}{\partial t}-4u\frac{\partial}{\partial u},
\end{aligned}\tag{3.5}$$

and then, all of the infinitesimal generators of Eq. (1.1) can be expressed as follows:

$$V=c_1V_1+c_2V_2+c_3V_3+c_4V_4+c_5V_5.$$

3.2 The similarity reduction for Eq. (1.1)

In order to reduce Eq. (1.1) to become a FODE, we used the infinitesimal generator V_5 defined in Eq. (3.5) to form the following characteristic equation:

$$\frac{dx}{x}=\frac{dy}{y}=\frac{dz}{z}=\frac{dt}{\frac{3}{\alpha}t}=\frac{du}{-4u}.$$

Solving the above equation yields the following similarity function:

$$u=t^{-\frac{4\alpha}{3}}f(\xi), \qquad \xi=(x+y+z)t^{-\frac{\alpha}{3}}. \tag{3.6}$$

By means of the similarity transformation $u=t^{-\frac{4\alpha}{3}}f(\xi)$, the time-fractional SZKB Eq. (1.1) can be reduced to a nonlinear FODE with a new independent variable $\xi=(x+y+z)t^{-\frac{\alpha}{3}}$. Consequently, one can get the following theorem.

Theorem 1 *The transformation equation* (3.6) *reduces the time-fractional SZKB Eq.* (1.1) *to the following nonlinear FODE*:

$$\left(P_{\frac{3}{\alpha}}^{1-\frac{7\alpha}{3},\alpha}f\right)(\xi)+a\sqrt{f}f_{\xi}+(b+2c)f_{\xi\xi\xi}+dt^{\frac{\alpha}{3}}f_{\xi\xi}=0, \tag{3.7}$$

with Erdélyi–Kober (*EK*) *fractional differential operator* $(P_{\beta}^{\tau,\alpha}f)(\xi)$, *which is defined as*

$$\left(P_{\beta}^{\tau,\alpha}f\right)(\xi)=\prod_{j=0}^{n-1}\left(\tau+j-\frac{1}{\beta}\frac{d}{d\tau}\right)\left(K_{\beta}^{\tau+\alpha,n-\alpha}f\right)(\xi),\quad n=\begin{cases}|\alpha|+1, & n\notin N,\\ \alpha, & n\in N,\end{cases} \tag{3.8}$$

where

$$\left(K_{\beta}^{\tau+\alpha,n-\alpha}f\right)(\xi)=\begin{cases}\frac{1}{\Gamma(\alpha)}\int_{1}^{\infty}(u-1)^{\alpha-1}u^{-(\tau+\alpha)}f(\xi u^{\frac{1}{\beta}})\,du, & \alpha>0,\\ f(\xi), & \alpha=0,\end{cases} \tag{3.9}$$

is the EK fractional integral operator.

Proof Let $n-1<\alpha<1$, $n=1,2,3,\ldots$, we obtain the similarity transformation based on Riemann–Liouville fractional derivatives as follows:

$$\frac{\partial^{\alpha}u}{\partial t^{\alpha}}=\frac{\partial^{n}}{\partial t^{n}}\left[\frac{1}{\Gamma(n-\alpha)}\int_{1}^{t}(t-s)^{n-\alpha-1}s^{\frac{-4}{3}\alpha}f\left((x+y+z)s^{-\frac{\alpha}{3}}\right)ds\right]. \tag{3.10}$$

Let $\nu=\frac{t}{s}$, one can get $ds=-\frac{t}{\nu^{2}}$, thus Eq. (3.10) can be written as

$$\frac{\partial^{\alpha}u}{\partial t^{\alpha}}=\frac{\partial^{n}}{\partial t^{n}}\left[\frac{t^{n-\frac{7}{3}\alpha}}{\Gamma(n-\alpha)}\int_{1}^{\infty}(\nu-1)^{n-\alpha-1}\nu^{-(n+1-\frac{7}{3}\alpha)}f\left(\xi\nu^{\frac{\alpha}{3}}\right)d\nu\right], \tag{3.11}$$

by using the definition of EK fractional differential operator, then Eq. (3.11) becomes

$$\frac{\partial^{\alpha}u}{\partial t^{\alpha}}=\frac{\partial^{n}}{\partial t^{n}}\left[t^{n-\frac{7}{3}\alpha}\left(K_{\frac{3}{\alpha}}^{1-\frac{\alpha}{3},n-\alpha}f\right)(\xi)\right], \tag{3.12}$$

and now it is appropriate to simplify the right hand side of Eq. (3.11). Consider $\xi=\chi t^{-\frac{\alpha}{3}}$, $\varphi\in(0,\infty)$ we get

$$t\frac{\partial}{\partial t}\varphi(\xi)=t\chi\left(-\frac{\alpha}{3}\right)t^{-\frac{\alpha}{3}-1}\varphi'(\xi)=-\frac{\alpha}{3}\xi\frac{d}{d\xi}\varphi(\xi).$$

From this,

$$\begin{aligned}\frac{\partial^{n}}{\partial t^{n}}\left[t^{n-\frac{7}{3}\alpha}\left(K_{\frac{3}{\alpha}}^{1-\frac{\alpha}{3},n-\alpha}f\right)(\xi)\right]&=\frac{\partial^{n-1}}{\partial t^{n-1}}\left[\frac{\partial}{\partial t}\left(t^{n-\frac{7}{3}\alpha}\left(K_{\frac{3}{\alpha}}^{1-\frac{\alpha}{3},n-\alpha}f\right)(\xi)\right)\right]\\&=\frac{\partial^{n-1}}{\partial t^{n-1}}\left[t^{n-\frac{7}{3}\alpha-1}\left(n-\frac{7}{3}\alpha-\frac{\alpha}{3}\xi\frac{\partial}{\partial\xi}\right)\left(K_{\frac{3}{\alpha}}^{1-\frac{\alpha}{3},n-\alpha}f\right)(\xi)\right],\end{aligned}$$

and thus repeating we have

$$\begin{aligned}\frac{\partial^n}{\partial t^n}\left[t^{n-\frac{7}{3}\alpha}\left(K_{\frac{3}{\alpha}}^{1-\frac{\alpha}{3},n-\alpha}f\right)(\xi)\right] &= \frac{\partial^{n-1}}{\partial t^{n-1}}\left[\frac{\partial}{\partial t}\left(t^{n-\frac{7}{3}\alpha}\left(K_{\frac{3}{\alpha}}^{1-\frac{\alpha}{3},n-\alpha}f\right)(\xi)\right)\right]\\ &= \frac{\partial^{n-1}}{\partial t^{n-1}}\left[t^{n-\frac{7}{3}\alpha-1}\left(n-\frac{7}{3}\alpha-\frac{\alpha}{3}\xi\frac{d}{d\xi}\right)\left(K_{\frac{3}{\alpha}}^{1-\frac{\alpha}{3},n-\alpha}f\right)(\xi)\right]\\ &\vdots\\ &= t^{-\frac{7}{3}\alpha}\prod_{j=0}^{n-1}\left[\left(1-\frac{7}{3}\alpha+j-\frac{\alpha}{3}\xi\frac{d}{d\varsigma}\right)\left(K_{\frac{3}{\alpha}}^{1-\frac{\alpha}{3},n-\alpha}f\right)(\xi)\right].\end{aligned}$$

On using the definition of the EK fractional differential operator Eq. (3.9), we get

$$\frac{\partial^n}{\partial t^n}\left[t^{n-\frac{7}{3}\alpha}\left(K_{\frac{3}{\alpha}}^{1-\frac{\alpha}{3},n-\alpha}f\right)(\xi)\right] = t^{-\frac{7}{3}\alpha}\left(P_{\frac{3}{\alpha}}^{1-\frac{7}{3}\alpha,\alpha}f\right)(\xi). \tag{3.13}$$

By substituting Eq. (3.13) into Eq. (3.12), we have

$$\frac{\partial^\alpha u}{\partial t^\alpha} = t^{-\frac{7}{3}\alpha}\left(P_{\frac{3}{\alpha}}^{1-\frac{7}{3}\alpha,\alpha}f\right)(\xi). \tag{3.14}$$

Thus, Eq. (3.13) can be reduced into a fractional ordinary differential equation

$$\left(P_{\frac{3}{\alpha}}^{1-\frac{7}{3}\alpha,\alpha}f\right)(\xi) + a\sqrt{f}f_\xi + (b+2c)f_{\xi\xi\xi} + dt^{\frac{\alpha}{3}}f_{\xi\xi} = 0. \tag{3.15}$$

The proof of the theorem is completed. □

4 Conservation laws for Eq. (1.1)

In this section, the conservation laws of the time-fractional SZKB equation (1.1) are derived, based on the formal lagrangian and Lie point symmetries as described in the following explanation:

Consider a vector $C = (C^t, C^x, C^y, C^z)$ admitting the following conservation equation:

$$\left[D_t\left(C^t\right) + D_x\left(C^x\right) + D_y\left(C^y\right) + D_z\left(C^z\right)\right]_{\text{Eq. (1.1)}} = 0, \tag{4.1}$$

where $C^t = C^t(x,y,z,t,u,\ldots)$, $C^x = C^x(x,y,z,t,u,\ldots)$, $C^y = C^y(x,y,z,t,u,\ldots)$, and $C^z = C^z(x,y,z,t,u,\ldots)$ are called the conserved vectors for Eq. (1.1). According to the new conservation theorem due to Ibragimov [47], the formal Lagrangian for Eq. (1.1) can be given by

$$L = \omega(x,y,z,t)\left[\frac{\partial^\alpha u}{\partial t^\alpha} + a\sqrt{u}u_x + bu_{xxx} + c(u_{xyy} + u_{xzz}) + du_{xx}\right] = 0, \tag{4.2}$$

here $\omega(x,y,z,t)$ is a new dependent variable. Depending on the definition of the Lagrangian, we get an action integral as follows:

$$\int_0^t \int_{\Omega_1}\int_{\Omega_2}\int_{\Omega_3} L\left(x,y,z,t,u,\omega,D_t^\alpha,u_x,u_{xxx},u_{xyy},u_{xzz},u_{xx}\right)dx\,dy\,dz\,dt.$$

The Euler–Lagrange operator is defined as

$$\frac{\delta}{\delta u}=\frac{\partial}{\partial u}+\left(D_t^\alpha\right)^*\frac{\partial}{\partial D_t^\alpha u}-D_x\frac{\partial}{\partial u_x}+D_x^2\frac{\partial}{\partial u_{xx}}-D_x^3\frac{\partial}{\partial u_{xxx}}-D_xD_y^2\frac{\partial}{\partial u_{xyy}}-D_xD_z^2\frac{\partial}{\partial u_{xzz}},$$

where $(D_t^\alpha)^*$ denotes to the adjoint operator of D_t^α, and the adjoint equation to the nonlinear by means of the Euler–Lagrange equation given by

$$\frac{\delta L}{\delta u}=0.$$

The adjoint operator $(D_t^\alpha)^*$ for R-L is defined by

$$\left(D_t^\alpha\right)^*=(-1)^n I_T^{n-\alpha}\left(D_t^n\right)\equiv {}_t^C D_T^\alpha,$$

where $I_T^{n-\alpha}$ is the right-sided operator of fractional integration of order $n-\alpha$ that is defined by

$$I_T^{n-\alpha}f(t,x)=\frac{1}{\Gamma(n-\alpha)}\int_t^T(\tau-t)^{n-\alpha-1}f(\tau,x)\,d\tau.$$

Considering the case of one dependent variable $u(x,y,z,t)$ with four independent variables x, y, z, t, we get

$$\bar{X}+D_t(\tau)I+D_x(\xi)I+D_y(\zeta)I+D_z(\nu)I=W\frac{\delta}{\delta u}+D_t\left(C^t\right)+D_x\left(C^x\right)+D_y\left(C^y\right)+D_z\left(C^z\right),$$

where $\bar{X}$ is defined by

$$\begin{aligned}\bar{X}&=\tau\frac{\partial}{\partial t}+\xi\frac{\partial}{\partial x}+\zeta\frac{\partial}{\partial y}+\nu\frac{\partial}{\partial z}+\eta\frac{\partial}{\partial u}+\eta_\alpha^0\frac{\partial}{\partial D_t^\alpha u}\\&\quad+\eta^x\frac{\partial}{\partial u_x}+\eta^{xx}\frac{\partial}{\partial u_{xx}}+\eta^{xxx}\frac{\partial}{\partial u_{xxx}}+\eta^{xyy}\frac{\partial}{\partial u_{xyy}}+\eta^{xzz}\frac{\partial}{\partial u_{xzz}},\end{aligned}$$

and the Lie characteristic function is defined by

$$\begin{aligned}&W=\eta-\tau u_t-\xi u_x-\zeta u_y-\nu u_z,\quad\text{and then}\\&W_1=-u_x,\qquad W_2=-u_t,\qquad W_3=-u_y,\qquad W_t=-u_z.\end{aligned}\tag{4.3}$$

For the R-L time-fractional derivative, the density component C^t of conservation law is defined by

$$C^t=\tau L+\sum_{k=0}^{n-1}(-1)^k{}_0D_t^{\alpha-1-k}(W_m)D_t^k\frac{\partial L}{\partial_0 D_t^\alpha u}-(-1)^n J\left(W_m,D_t^n\frac{\partial L}{\partial_0 D_t^\alpha u}\right),\tag{4.4}$$

where the operator $J(\cdot)$ is defined by

$$J(f,g)=\frac{1}{\Gamma(n-\alpha)}\int_0^t\int_t^T\frac{f(\tau,x,y,z)g(\mu,x,y,z)}{(\mu-\tau)^{\alpha+1-n}}\,d\mu\,d\tau,$$

and the other (flux) components are defined as

$$C^i = \xi^i L + W_m\left[\frac{\partial L}{\partial u_i^m} - D_j\left(\frac{\partial L}{\partial u_{ij}^m}\right) + D_j D_k\left(\frac{\partial L}{\partial u_{ijk}^m} - \cdots\right)\right] + D_j(W_m)\left[\frac{\partial L}{\partial u_{ij}^m} - D_k\left(\frac{\partial L}{\partial u_{ijk}^m}\right) + \cdots\right] + D_j D_k(W_m)\left(\frac{\partial L}{\partial u_{ijk}^m} - \cdots\right) + \cdots, \tag{4.5}$$

where $\xi^1 = \xi$, $\xi^2 = \zeta$, $\xi^3 = \nu$ and $m = 1, 2, \ldots, 5$.

Now by using Eq. (4.3) with the help of Eqs. (4.4) and (4.5), we obtain the components of the conservation laws for the time-fractional SZKB equation as follows.

Case 1: $W_1 = -u_x$ where $\xi^x = 1$, $\xi^t = 0$, $\xi^y = 0$, $\xi^z = 0$ and $\eta = 0$ we get the following conserved vectors:

$$\begin{aligned}
C_1^t &= \omega_0 D_t^{\alpha-1}(-u_x)\frac{\partial L}{\partial_0 D_t^\alpha u} - J\left(-u_x, D_t^n \frac{\partial L}{\partial_0 D_t^\alpha u}\right) = -\omega D_t^\alpha(u_x) - u_x D_t^\alpha(\omega),\\
C_1^x &= \omega\left[D_t^\alpha u + a\sqrt{u}u_x + \frac{c}{3}(u_{yxy} + u_{yyx} + u_{zxz} + u_{zzx})\right]\\
&\quad - u_x\left[-d\omega_x + b\omega_{xx} + \frac{c}{3}(\omega_{yy} + \omega_{zz})\right] + b\omega_{xx}\omega_x + \frac{c}{3}(u_{xy}\omega_y + u_{xz}\omega_z),\\
C_1^y &= -\frac{c}{3}u_x[\omega_{xy} + \omega_{yx}] + \frac{c}{3}(u_{xx}\omega_y + u_{xy}\omega_x) - \frac{c}{3}\omega(u_{xyx} + u_{yxx}),\\
C_1^z &= -\frac{c}{3}u_x[\omega_{xz} + \omega_{zx}] + \frac{c}{3}(u_{xx}\omega_z + u_{xz}\omega_x) - \frac{c}{3}\omega(u_{xzx} + u_{zxx}).
\end{aligned}$$

Case 2: $W_2 = -u_t$ where $\xi^x = 0$, $\xi^t = 1$, $\xi^y = 0$, $\xi^z = 0$ and $\eta = 0$ we acquire the following conserved vectors:

$$\begin{aligned}
C_2^t &= \omega L\omega_0 D_t^{\alpha-1}(-u_t)\frac{\partial L}{\partial_0 D_t^\alpha u} - J\left(-u_t, D_t^n \frac{\partial L}{\partial_0 D_t^\alpha u}\right)\\
&= \omega\left[D_t^\alpha u + a\sqrt{u}u_x + \frac{c}{3}(u_{yxy} + u_{yyx} + u_{zxz} + u_{zzx}) + du_{xx}\right] - \omega D_t^\alpha(u_t) - u_t D_t^\alpha(\omega),\\
C_2^x &= -u_t\left[a\omega\sqrt{u} - d\omega_x + b\omega_{xx} + \frac{c}{3}(\omega_{yy} + \omega_{zz})\right]\\
&\quad - u_{xt}[d\omega_x - b\omega_x] + \frac{c}{3}(u_{yt}\omega_y + u_{zt}\omega_z) - b\omega u_{xxt} - \frac{c}{3}\omega(u_{yyt} + u_{zzt}),\\
C_2^y &= -\frac{c}{3}u_t[\omega_{xy} + \omega_{yx}] + \frac{c}{3}(u_{xt}\omega_y + u_{yt}\omega_x) - \frac{c}{3}\omega(u_{xyt} + u_{yxt}),\\
C_2^z &= -\frac{c}{3}u_t[\omega_{xz} + \omega_{zx}] + \frac{c}{3}(u_{xt}\omega_z + u_{zt}\omega_x) - \frac{c}{3}\omega(u_{xzt} + u_{zxt}).
\end{aligned}$$

Case 3: $W_3 = -u_y$ where $\xi^x = 0$, $\xi^t = 0$, $\xi^y = 1$, $\xi^z = 0$ and $\eta = 0$ we obtain the following conserved vectors:

$$\begin{aligned}
C_3^t &= \omega_0 D_t^{\alpha-1}(-u_y)\frac{\partial L}{\partial_0 D_t^\alpha u} - J\left(-u_y, D_t^n \frac{\partial L}{\partial_0 D_t^\alpha u}\right) = -\omega D_t^\alpha(u_y) - u_y D_t^\alpha(\omega),\\
C_3^x &= -u_y\left[a\omega\sqrt{u} - d\omega_x + b\omega_{xx} + \frac{c}{3}(\omega_{yy} + \omega_{zz})\right]\\
&\quad - u_{xy}[d\omega_x - b\omega_x] + \frac{c}{3}(u_{yy}\omega_y + u_{yz}\omega_z) - b\omega u_{xxy} - \frac{c}{3}\omega(u_{yyy} + u_{zzy}),
\end{aligned}$$

$$C_3^y = \omega\left[D_t^\alpha u + a\sqrt{u}u_x + \frac{c}{3}(u_{yxy} + u_{yyx} + u_{zxz} + u_{zzx}) + du_{xx}\right]$$
$$- \frac{c}{3}u_y[\omega_{xy} + \omega_{yx}] + \frac{c}{3}(u_{xy}\omega_y + u_{yy}\omega_x) - \frac{c}{3}\omega(u_{xyy} + u_{yxy}),$$
$$C_3^z = -\frac{c}{3}u_y[\omega_{xz} + \omega_{zx}] + \frac{c}{3}(u_{xy}\omega_z + u_{yz}\omega_x) - \frac{c}{3}\omega(u_{xzy} + u_{zxy}).$$

Case 4: $W_4 = -u_z$ where $\xi^x = 0$, $\xi^t = 0$, $\xi^y = 0$, $\xi^z = 0$ and $\eta = 0$ we obtain the following conserved vectors:

$$C_4^t = \omega_0 D_t^{\alpha-1}(-u_z)\frac{\partial L}{\partial_0 D_t^\alpha u} - J\left(-u_z, D_t^n \frac{\partial L}{\partial_0 D_t^\alpha u}\right) = -\omega D_t^\alpha(u_z) - u_z D_t^\alpha(\omega),$$
$$C_4^x = -u_z\left[a\omega\sqrt{u} - d\omega_x + b\omega_{xx} + \frac{c}{3}(\omega_{yy} + \omega_{zz})\right] - u_{xy}[d\omega_x - b\omega_x]$$
$$+ \frac{c}{3}(u_{yy}\omega_y + u_{yz}\omega_z) - b\omega u_{xxy} - \frac{c}{3}\omega(u_{yyy} + u_{zzy}),$$
$$C_4^y = \omega\left[D_t^\alpha u + a\sqrt{u}u_x + \frac{c}{3}(u_{yxy} + u_{yyx} + u_{zxz} + u_{zzx}) + du_{xx}\right]$$
$$- \frac{c}{3}u_y[\omega_{xy} + \omega_{yx}] + \frac{c}{3}(u_{xy}\omega_y + u_{yy}\omega_x) - \frac{c}{3}\omega(u_{xyy} + u_{yxy}),$$
$$C_4^z = -\frac{c}{3}u_y[\omega_{xz} + \omega_{zx}] + \frac{c}{3}(u_{xy}\omega_z + u_{yz}\omega_x) - \frac{c}{3}\omega(u_{xzy} + u_{zxy}).$$

Remark We have verified that all the solutions satisfy the original equation.

5 Explicit power series and convergence analysis for Eq. (1.1)

In this section, we derived the analytic solution for Eq. (1.1) via the power series method [57, 58] and proved the convergence of the power series solution as demonstrated in the next two subsections.

5.1 Power series and analytical solutions for Eq. (1.1)

Considering Eq. (1.1) and by using $u(x,y,z,t) = \psi^2(x,y,z,t)$, Eq. (1.1) becomes

$$\frac{\partial^\alpha \psi^2}{\partial t^\alpha} + a\psi\left(\psi^2\right)_x + b\left(\psi^2\right)_{xxx} + c\left(\left(\psi^2\right)_{xyy} + \left(\psi^2\right)_{xzz}\right) + d\left(\psi^2\right)_{xx} = 0. \tag{5.1}$$

Let us introduce the important transformation

$$\psi(x,y,z,t) = \psi(\xi), \quad \xi = px + qy + rz - \frac{kt^\alpha}{\Gamma(1+\alpha)}, \tag{5.2}$$

where p, q, r and k are constants to be determined later. The substitution of (5.2) into (5.1) leads to the following nonlinear ODE:

$$-\frac{k}{2}\left(\psi^2\right)' + \frac{a}{3}\left(\psi^3\right)' + \frac{1}{2}p\left(bp^2 + c\left(q^2 + r^2\right)\right)\left(\psi^2\right)''' + dp^2\left(\psi^2\right)'' = 0. \tag{5.3}$$

By integrating Eq. (5.3) with respect to ξ, we get

$$-\frac{k}{2}\left(\psi^2\right) + \frac{a}{3}\left(\psi^3\right) + \frac{1}{2}p\left(bp^2 + c\left(q^2 + r^2\right)\right)\left(\psi^2\right)'' + dp^2\left(\psi^2\right)' + \lambda = 0, \tag{5.4}$$

where λ is the constant of integration. Let us assume that the solution of Eq. (5.4) has the following form:

$$\psi(\xi) = \sum_{n=0}^{\infty} a_n \xi^n, \tag{5.5}$$

where a_n are constants to be determined later. On substituting (5.5) into (5.4), we have the following relation:

$$\begin{aligned}
&-\frac{k}{2}\sum_{n=0}^{\infty}\sum_{m=0}^{n}(a_m a_{n-m})\xi^n + \frac{a}{3}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{l=0}^{n-m}(a_l a_{m-l} a_{n-m})\xi^n + p\bigl(bp^2 + c\bigl(q^2+r^2\bigr)\bigr)\\
&\quad\times\left(\sum_{n=0}^{\infty}\sum_{m=0}^{n}(n-m+1)(n-m+2)a_m a_{n-m+2}\right.\\
&\quad\left.+\sum_{n=0}^{\infty}\sum_{m=0}^{n}(n-m+1)(m+1)a_{m+1}a_{n-m+1}\right)\xi^n\\
&\quad+2dp^2\sum_{n=0}^{\infty}\sum_{m=0}^{n}\bigl((n-m+1)a_m a_{n-m+1}\bigr)\xi^n + \lambda = 0.
\end{aligned} \tag{5.6}$$

From Eq. (5.6), by comparing the coefficients for $n = 0$ one can get

$$a_2 = -\frac{2aa_0^3 - 3ka_0^2 + 12dp^2 a_0 a_1 + 6(\lambda + bp^2 + c(q^2+r^2))a_1^2}{12p(bp^2 + c(q^2+r^2))a_0}, \tag{5.7}$$

where a_0 and a_1 are arbitrary constants where p and $a_0 \neq 0$. Generally, for $n \geq 1$ we can get

$$\begin{aligned}
a_{n+2} &= \frac{1}{6\rho(n+1)(n+2)a_0}\left(3k\sum_{m=0}^{n}(a_m a_{n-m}) - 2a\sum_{m=0}^{n}\sum_{l=0}^{n-m}(a_l a_{m-l}a_{n-m}) - 6\rho\right.\\
&\quad\times\left(\sum_{m=0}^{n}(n-m+1)(n-m+2)a_m a_{n-m+2} + \sum_{m=0}^{n}(n+1)(n-m+1)a_{m+1}a_{n-m+1}\right)\\
&\quad\left.-12dp^2\sum_{m=0}^{n}\bigl((n-m+1)a_m a_{n-m+1}\bigr) - 6\lambda\right),
\end{aligned} \tag{5.8}$$

where $\rho = p(bp^2 + c(q^2+r^2))$. Now, from (5.8) and (5.7) we can get all the coefficients a_{n+2}, $n \geq 1$ of the power series (5.5) in which a_0, p, q, r, and k are the arbitrary constants. Thus, the power series solution of Eq. (5.4) could be written as follows:

$$\begin{aligned}
u(\xi) &= \left(a_0 + a_1\xi + a_2\xi^2 + \sum\nolimits_{n=1}^{\infty} a_{n+2}\xi^{n+2}\right)^2\\
&= (a_0 + a_1\xi + \left(-\frac{6\lambda - 3ka_0^2 + 2aa_0^3 + 6p(2dp + bp^2 + c(q^2+r^2))a_0 a_1}{12p(bp^2 + c(q^2+r^2))a_0}\right)\xi^2\\
&\quad+\sum_{n=1}^{\infty}\left(\frac{1}{6\rho(n+1)(n+2)a_0}\left(3k\sum_{m=0}^{n}(a_n a_{n-m}) - 2a\sum_{m=0}^{n}\sum_{l=0}^{n-m}(a_l a_{m-l}a_{n-m}) - 6\rho\right.\right.
\end{aligned}$$

$$\times \left(\sum_{m=0}^{n} (n-m+1)(n-m+2) a_m a_{n-m+2} + \sum_{m=0}^{n} (n+1)(n-m+1) a_{m+1} a_{n-m+1} \right)$$
$$- 12dp^2 \sum_{m=0}^{n} \bigl((n-m+1) a_m a_{n-m+1} \bigr) - 6\lambda \Biggr) \xi^{n+2} \Biggr)^2. \tag{5.9}$$

Thus, we obtain the exact power series solution for Eq. (3.15) as follows:

$$\begin{aligned} u(x,y,z,t) = {}& \Biggl(a_0 + a_1 \left(px + qy + rz - \frac{kt^\alpha}{\Gamma(1+\alpha)} \right) \\ & + \left(- \frac{6\lambda - 3ka_0^2 + 2aa_0^3 + 6p(2dp + bp^2 + c(q^2 + r^2)) a_0 a_1}{12p(bp^2 + c(q^2 + r^2)) a_0} \right) \\ & \times \left(px + qy + rz - \frac{kt^\alpha}{\Gamma(1+\alpha)} \right)^2 \\ & + \sum_{n=1}^{\infty} \Biggl(\frac{1}{6\rho(n+1)(n+2)a_0} \Biggl(3k \sum_{m=0}^{n} (a_m a_{n-m}) - 2a \sum_{m=0}^{n} \sum_{l=0}^{n-m} (a_l a_{m-l} a_{n-m}) \\ & - 6\rho \Biggl(\sum_{m=0}^{n} (n-m+1)(n-m+2) a_m a_{n-m+2} \\ & + \sum_{m=0}^{n} (n+1)(n-m+1) a_{m+1} a_{n-m+1} \Biggr) \\ & - 12dp^2 \sum_{m=0}^{n} \bigl((n-m+1) a_m a_{n-m+1} \bigr) - 6\lambda \Biggr) \Biggr) \\ & \times \left(px + qy + rz - \frac{kt^\alpha}{\Gamma(1+\alpha)} \right)^{n+2} \Biggr)^2. \end{aligned} \tag{5.10}$$

We represented the solution defined by Eq. (5.10) using the 2D plot, see Fig. 1b, and the 3D plots, see Fig. 1a and Fig. 2. The result in Eq. (5.10) shows that there is an analytical power series solution for Eq. (5.4). It is important to find new solutions, because either new exact solutions or numerical approximate solutions and analytical solutions may deepen our understanding of the physical phenomena.

5.2 Convergence analysis

In this subsection, to complete the analysis of Eq. (1.1), we study the convergence of the power series solution Eq. (5.5) with the coefficients denoted by Eq. (5.7) and (5.8), where Eq. (5.8) can be enlarged to the following form:

$$\begin{aligned} |a_{n+2}| \le M \Biggl[& \sum_{m=0}^{n} |a_m||a_{n-m}| - \sum_{m=0}^{n} \sum_{l=0}^{n-m} |a_l||a_{m-l}||a_{n-m}| - \sum_{m=0}^{n} |a_m||a_{n-m+2}| \\ & - \sum_{m=0}^{n} |a_{m+1}||a_{n-m+1}| - \sum_{m=0}^{n} |a_m||a_{n-m+1}| \Biggr], \end{aligned} \tag{5.11}$$

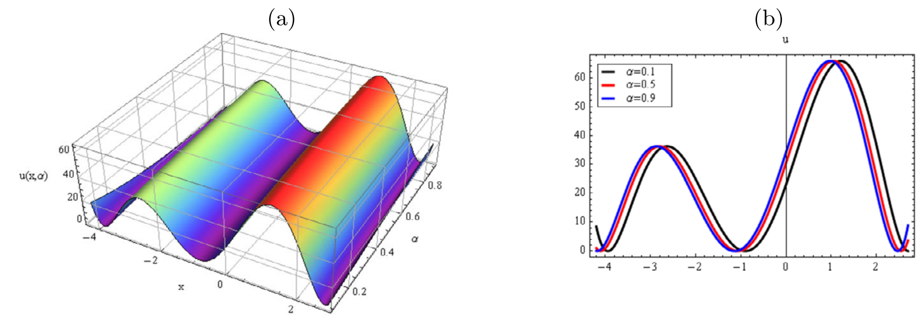

Figure 1 (**a**) The effect of α on the 3D solitary wave solution of equation (5.10) with a fixed time t and the parameters $k = 0.3$, $p = 0.2$, $q = 0.7$, $r = 0.3$, $a_0 = 6$, $a_1 = 1.7$. (**b**) The corresponding 2D plot of (**a**), which shows that the envelope of the wave decays far away from the peak quickly when the value of α increases meaning that the solitary wave solution behaves as a dynamical system

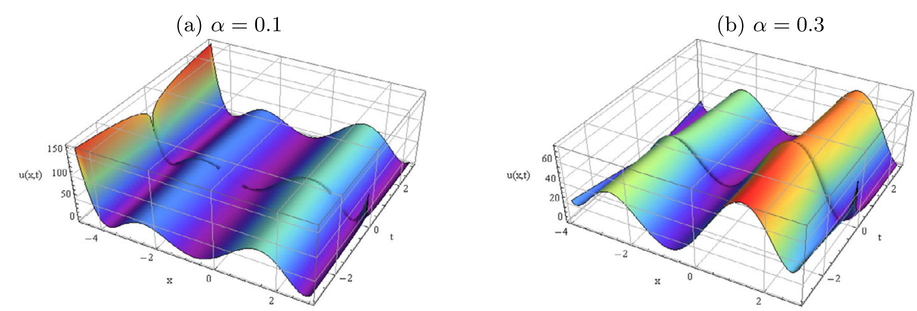

Figure 2 The effect of α on the 3D solitary wave solution of equation (5.10) at different values of $\alpha = 0.1, 0.3$ with the suitable parameters $k = 0.3$, $p = 0.2$, $q = 0.7$, $r = 0.3$, $a_0 = 6$, $a_1 = 1.7$

where $M = \max\{\frac{3k}{\rho}, \frac{2a}{\rho}, 12dp^2\}$, $\rho = bp^2 + c(q^2 + r^2)$. Now, consider another power series,

$$S(\xi) = \sum_{n=0}^{\infty} s_n \xi^n, \tag{5.12}$$

with $s_i = |a_i|$, $i = 0, 1$. Then we can get

$$\begin{aligned} s_{n+2} = M\Bigg[&\sum_{m=0}^{n} |s_m||s_{n-m}| \\ &- \sum_{m=0}^{n}\sum_{l=0}^{n-m} |s_l||s_{m-l}||s_{n-m}| - |s_m||s_{n-m+2}| - |s_{m+1}||s_{n-m+1}| - |s_m||s_{n-m+1}|\Bigg], \end{aligned} \tag{5.13}$$

where $n = 0, 1, 2, \ldots$ thus, it is easily noted that

$$|a_{n+2}| \le s_{n+2} \quad \text{this leads to } |a_n| \le s_n,$$

depending on this result, we could say that the series defined by Eq. (5.12) is a majorant series of Eq. (5.5). Next, we have shown that the series $S = S(\xi)$ has positive radius of convergence. Then, we write this series in the following form:

$$\begin{aligned}
S(\xi) &= s_0 + s_1\xi + \sum_{n=2}^{\infty} s_n \xi^n \\
&= s_0 + s_1\xi + M\Biggl[\sum_{m=0}^{n} |s_m||s_{n-m}| \\
&\quad - \sum_{m=0}^{n}\sum_{l=0}^{n-m} |s_l||s_{m-l}||s_{n-m}| - |s_m||s_{n-m+2}| - |s_{m+1}||s_{n-m+1}| - |s_m||s_{n-m+1}|\Biggr]\xi^{n+2} \\
&= s_0 + s_1\xi + M\bigl[S^2(\xi)\xi^2 - S^3(\xi)\xi^2 - 2(S-s_0)(S-s_0)\bigr].
\end{aligned}$$

Consider an implicit functional system with respect to the independent variable ξ as follows:

$$\vartheta(\xi,S) = S - s_0 - s_1\xi - M\bigl[S^2(\xi)\xi^2 - S^3(\xi)\xi^2 - 2(S-s_0)(S-s_0)\bigr], \tag{5.14}$$

since ϑ is an analytic in a neighborhood of (o, s_0), where $\vartheta(o,s_0) = 0$ and, $\frac{\partial\vartheta}{\partial S}(o,s_0) \neq 0$.

Theorem ([59]) *Let f be a $\wp'$-mapping of an open set $E \subset s^{n+m}$ into S^n, such that $f(a,b) = 0$ for some point $(a,b) \in E$. Assume that $A = f'(a,b)$ and A_x is invertible. Then the following properties hold in the open sets $U \subset s^{n+m}$ and $W \subset S^m$ with $(a,b) \in U$ and $b \in W$.*

(i) *For each $y \in W$, there exists a unique x such that $(x,y) \in U$ and $f(x,y) = 0$.*

(ii) *If x is defined to be $g(y)$, then*

$$\begin{aligned}
&g(b) = a, \\
&f\bigl(g(y),y\bigr) = 0 \quad (y \in W), \\
&g'(b) = -(A_x)^{-1}A_y,
\end{aligned}$$

where g is a $\wp'$-mapping of W into S^n.

(iii) *The function g is implicitly defined by* (ii).

One can see that $S = S(\xi)$ is analytical in a neighborhood of the point (o,s_0) and has positive radius. It shows that the power series Eq. (5.5) *is convergent in a neighborhood of the point* (o,s_0).

6 The modified trial equation method for Eq. (1.1)

In this section, we gave a brief description of the modified trial method that was proposed by Liu [60, 61]; some authors [62–64] improved this method. Now, let us consider the time FPDEs, say in the four variables x, y, z and t. In the following we give the main steps of the modified trial method and we obtain some traveling wave solutions of Eq. (1.1):

Step 1: Consider the time FPDE defined by Eq. (2.3) and take the wave transformation as Eq. (5.2). Under this transformation, we were permitted to reduce Eq. (2.3) to ODE.

Step 2: Take the trial equation as follows:

$$\begin{aligned} \psi' &= \frac{F(\psi)}{G(\psi)} = \frac{\sum_{i=0}^{n} A_i \psi^i}{\sum_{j=0}^{l} B_j \psi^j} = \frac{A_0 + a_1\psi + A_2\psi^2 + \cdots + A_n\psi^n}{B_0 + B_1\psi + B_2\psi^2 + \cdots + B_l\psi^l}, \\ \psi'' &= \frac{F(\psi)(F'(\psi)G(\psi) - F(\psi)G'(\psi))}{G^3(\psi)}, \end{aligned} \tag{6.1}$$

where F and G are polynomials in ψ. On substituting (6.1) into (5.4) yields an equation of $\Omega(\psi)$ of polynomial in ψ

$$\Omega(\psi) = p_s\psi^s + \cdots + p_1\psi + p_0,$$

we found a relation between n and l according to the balance principle and determined some values of them.

Step 3: Setting the coefficients of $\Omega(\psi)$ to zero yields a system of algebraic equations:

$$p_i = 0, \quad i = 0, 1, \ldots, s.$$

By solving the obtained system of algebraic equations, we can determine the values of $A_0, \ldots, A_n$ and $B_0, \ldots, B_l$.

Step 4: Rewrite Eq. (5.11) by the following integral form:

$$\xi - \xi_0 = \int \frac{F(\psi)}{G(\psi)}\, d\psi, \tag{6.2}$$

where ξ_0 is the constant of integration. When we classify the roots of $F(\psi)$ using the complete discrimination system, we obtain the analytical solutions of Eq. (1.1).

Now, applying the modified trial equation method for the (3 + 1)-dimensional time-fractional SZKB equation with the wave transformation defined by Eq. (5.2), by employing (6.1), (6.2), and using the balance principle yields $n = l + 2$. This resolution procedure was applied and we obtained results as follows.

Case 1: If we take $l = 0$, then $n = 2$, and

$$\begin{aligned} \psi' &= \frac{A_0 + A_1\psi + A_2\psi^2}{B_0}, \\ \psi'' &= \frac{(A_1 + 2A_2\psi)(A_0 + A_1\psi + A_2\psi^2)}{B_0^2}, \end{aligned} \tag{6.3}$$

thus, we have a system of algebraic equations from the coefficients of polynomial of u. Solving this system, we get

$$A_0 = 0, \qquad A_2 = -\frac{aB_0}{6dp}, \qquad r = \pm\sqrt{-\frac{bp^2 + cq^2}{c}}, \qquad k = -\frac{4dp^2A_1}{B_0}.$$

Substituting from the above coefficients into (6.2) and integrating, we get the solutions to (5.4), as follows:

$$\xi - \xi_0 = \int \frac{B_0}{A_2\psi^2 + a_1\psi}\, d\psi, \tag{6.4}$$

this implies

$$\psi(\xi) = \frac{3kB_0}{2p(6dP\exp(\pm\frac{k}{4dp^2B_0}(\xi - \xi_0)) - aB_0)}, \tag{6.5}$$

thus,

$$u(x,y,z,t) = \left(\frac{3kB_0}{2p(6dP\exp(\pm\frac{k}{4dp^2B_0}(px + qy + rz - \frac{kt^\alpha}{\Gamma(1+\alpha)} - \xi_0)) - aB_0)}\right)^2. \tag{6.6}$$

For simplicity, take $\xi_0 = 0, A_1 = A_2 = 1$ then the solution Eq. (6.6) is reduced to the following solution:

$$u(x,y,z,t) = \left(\frac{1}{\exp(\gamma(px + qy + rz - \frac{kt^\alpha}{\Gamma(\alpha+1)})) - 1}\right)^2, \tag{6.7}$$

where $\gamma = \pm\frac{A_1}{B_0}$. We represent the solution defined by Eq. (6.7) using the 2D plot, see Fig. 3b, and the 3D plots, see Fig. 3a and Fig. 4.

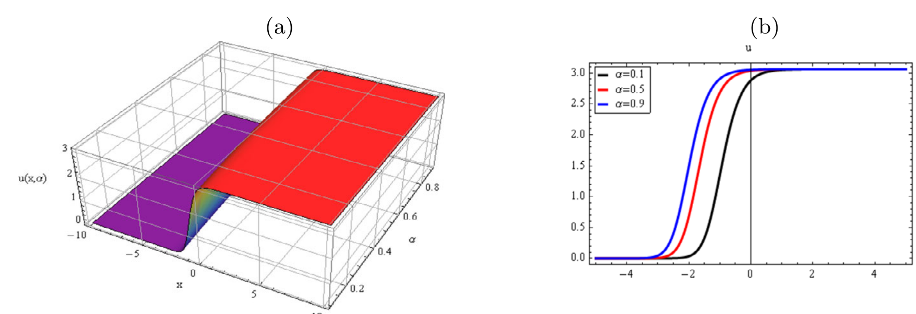

Figure 3 (**a**) The effect of α on the 3D double-layer solution of equation (6.7) with a fixed time t and the parameters $k = 0.3$, $p = 0.2$, $q = 0.2$. (**b**) The corresponding 2D plot of (**a**) with the same parameters

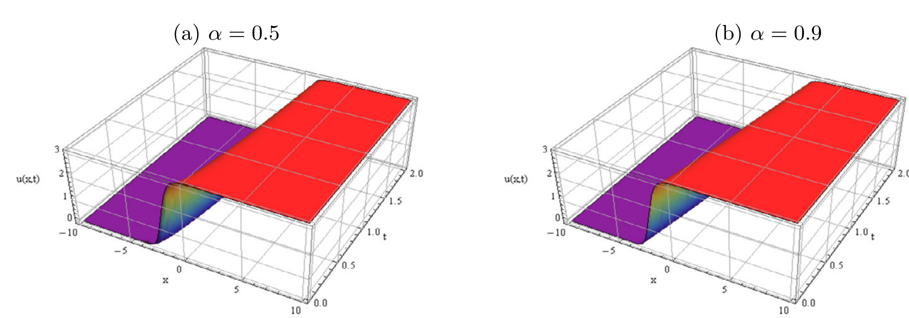

Figure 4 The 3D double-layer solution of equation (6.7) at different values of $\alpha = 0.5, 0.9$ with the suitable parameters $k = 0.3$, $p = 0.2$, $q = 0.2$ which shows that when the value of α decreases the singularity becomes clearer

Case 2: If we take $l = 1$, then $n = 3$, and

$$\begin{aligned}\psi' &= \frac{A_0 + A_1\psi + A_2\psi^2 + A_3\psi^3}{B_0 + B_1\psi},\\ \psi'' &= \Big(\big(A_0 + A_1\psi + A_2\psi^2 + A_3\psi^3\big)\big(\big(B_0 + B_1\psi\big)\big(A_1 + 2A_2\psi + 3A_3\psi^2\big)\big)\\ &\quad - B_1\big(A_0 + A_1\psi + A_2\psi^2 + A_3\psi^3\big)\Big)\\ &\quad /(B_0 + B_1\psi)^3,\end{aligned} \tag{6.8}$$

thus, we have a system of algebraic equations from the coefficients of polynomial of u. Solving this system, we get

$$\begin{aligned}&A_0 = 0, \qquad A_2 = \frac{6dpA_1B_1 - aB_0^2}{6dpB_0}, \qquad A_3 = -\frac{aB_1}{6dp},\\ &k = \frac{4dp^2A_1}{B_0}, \qquad r = \pm\sqrt{-\frac{bp^2 + cq^2}{c}}.\end{aligned}$$

Substituting from the above coefficients into Eq. (6.8) and integrating, we get the solutions to Eq. (5.4), as follows:

$$\xi - \xi_0 = \int \frac{B_0 + B_1 u}{A_3\psi^3 + A_2\psi^2 + a_1\psi}\, d\psi, \tag{6.9}$$

this implies

$$\begin{aligned}\xi - \xi_0 = -\frac{1}{2A_1}\bigg(&B_0 \ln\big(A_1 + \psi(A_2 + A_3\psi)\big) - 2B_0\ln(\psi)\\ &+ \frac{1}{\sigma}\bigg(2(A_2B_0 - 2A_1B_1)tan^{-1}\bigg(\frac{A_2 + 2A_3\psi}{\sigma}\bigg)\bigg)\bigg),\end{aligned} \tag{6.10}$$

where $\sigma = \sqrt{4A_1A_3 - 2A_2^2}$. Substituting the values of A_i, $i = 1, 2, 3$, thus

$$\psi(\xi) = \left(\frac{3k}{2ap + \exp(\frac{k(\xi_0 - \xi)}{4dp^2})}\right), \tag{6.11}$$

where $\xi = px + qy + rz - \frac{kt^\alpha}{\Gamma(\alpha+1)}$.

For simplicity, take $\xi_0 = 0$ then the solution Eq. (6.11) is reduced to the following solution:

$$u(x, y, z, t) = \left(\frac{3k}{2ap + \exp(\frac{-k(px+qy+rz-\frac{kt^\alpha}{\Gamma(\alpha+1)})}{4dp^2})}\right)^2. \tag{6.12}$$

We represent the solution defined by Eq. (6.12) using the 2D plot, see Fig. 5b, and the 3D plots, see Fig. 5a and Fig. 6.

7 Conclusion

In this paper, we considered a (3 + 1)-dimensional time-fraction Schamel–Zakharov–Kuznetsov–Burgers equation, which described the nonlinear plasma-dust ion acoustic

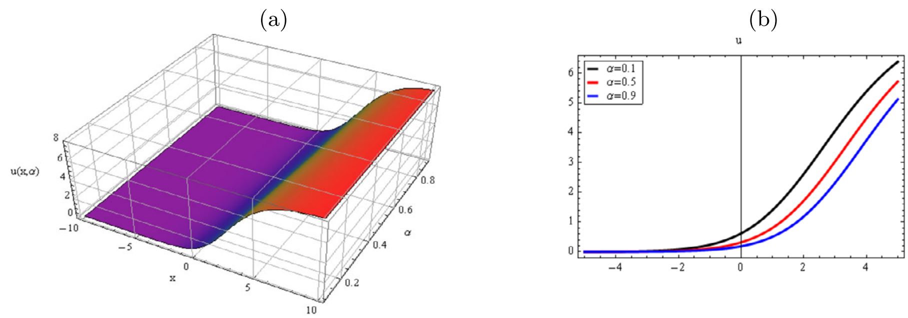

Figure 5 (**a**) The effect of α on the 3D double-layer solution of equation (6.12) with a fixed time t and the parameters $k = 0.3$, $p = 0.2$, $q = 0.2$. (**b**) The corresponding 2D plot of (**a**) which shows that when the value of α increases the interval of the stability of the solution also increases

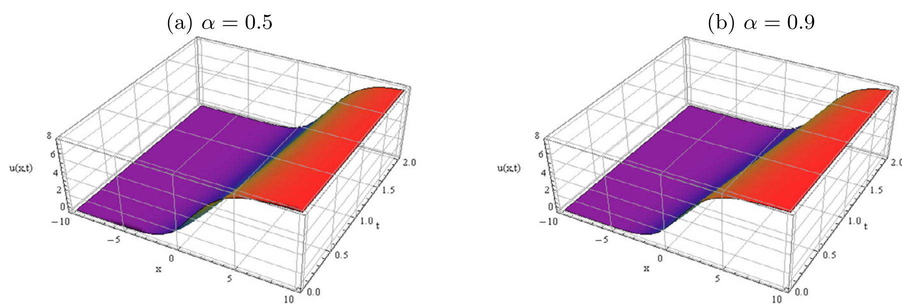

Figure 6 The 3D double-layer solution of equation (6.12) at different values of $\alpha = 0.5, 0.9$ with the suitable parameters $k = 0.3$, $p = 0.2$, $q = 0.2$

waves (DIAWs) in a magnetized dusty plasma. The Lie point symmetries were applied successfully to the study of the (3 + 1)-dimensional time-fraction Schamel–Zakharov–Kuznetsov–Burgers equation. Based on the Riemann–Liouville derivatives, we deduced the corresponding vector fields, which helped us to construct the symmetry reductions of the time-fractional SZKB equation. Furthermore, we obtained four kinds of conservation laws with independent variables laying the foundation of Lie point symmetries. Also, we constructed a new set of analytical solutions via two powerful methods, which are the explicit power series method and the modified trial equation method. Moreover, to introduce a better understanding of the dynamics of these solutions, we provided their graphic analysis of Eq. (1.1). As shown in Fig. 3, It was noticed that the solution $u(x, y, z, t)$ defined by Eq. (6.7) remained stable for $-4.35 < x < 2.75$ and has a singularity outside this interval. It can easily be observed in Figs. 4 and 6 that the solutions are in line with the values of α, it means that the singularity becomes clearer when the value of α decreases. Thus, we expect that the obtained results might serve as the explanation of the physical meaning of the time-fractional SZKB equation with more accuracy. However, we can extend the symmetry analysis to the time-space fractional (3 + 1)-dimensional Schamel–Zakharov–Kuznetsov–Burgers equation where that work may be presented later elsewhere.

Acknowledgements
The authors would like to thank the reviewers for insightful comments and constructive suggestions leading to improvements into the current form of the paper. The authors expresses sincere thanks to the editor.

Funding
This research work is not supported by any funding agencies.

Competing interests
The authors declare that no competing interests exist.

Authors' contributions
The authors declare that the work was realized in collaboration with the same responsibility. All authors read and approved the final paper.

Publisher's Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 12 July 2019 Accepted: 9 October 2019 Published online: 22 October 2019

References
1. Biswas, A., Mirzazadeh, M., Eslami, M.: Dispersive dark optical soliton with Schödinger–Hirota equation (G'/G)-expansion approach in power law medium. Optik **125**(16), 4215–4218 (2014)
2. Eslami, M., Ebrahimi, M.: Explicit solutions of nonlinear $(2+1)$-dimensional dispersive long wave equation. J. King Saud Univ., Sci. **24**(1), 69–71 (2012)
3. Liu, J., Eslami, M., Rezazadeh, H., Mirzazadeh, M.: Rational solutions and lump solutions to a non-isospectral and generalized variable-coefficient Kadomtsev–Petviashvili equation. Nonlinear Dyn. **95**(2), 1027–1033 (2019)
4. Mirzazadeh, M., Eslami, M., Biswas, A.: Dispersive optical solitons by Kudryashov's method. Optik, Int. J. Light Electron Opt. **125**(23), 6874–6880 (2014)
5. Mirzazadeh, M., Eslami, M., Zerrad, E., Mahmood, M., Biswas, A., Belic, M.: Optical solitons in nonlinear directional couplers by sine–cosine function method and Bernoulli's equation approach. Nonlinear Dyn. **81**(4), 1933–1949 (2015)
6. Nazarzadeh, A., Eslami, M., Mirzazadeh, M.: Exact solutions of some nonlinear partial differential equations using functional variable method. Pramana **81**(2), 225–236 (2013)
7. Mirzazadeh, M., Eslami, M.: Exact solutions of the Kudryashov–Sinelshchikov equation and nonlinear telegraph equation via the first integral method. Nonlinear Anal., Model. Control **17**(4), 481–488 (2012)
8. Eslami, M., Neirameh, A.: New exact solutions for higher order nonlinear Schrödinger equation in optical fibers. Opt. Quantum Electron. **50**(1), Article ID 47 (2018)
9. Biswas, A., Mirzazadeh Eslami, M.: Optical solitons with Biswas–Milovic equation for power law and dual-power law nonlinearities. Nonlinear Dyn. **83**(1–2), 731–738 (2016)
10. Mirzazadeh, M., Eslami, M., Biswas, A.: Trial solution technique to chiral nonlinear Schrodinger's equation in $(1+2)$-dimensions. Nonlinear Dyn. **85**(2), 813–816 (2016)
11. Song, F., Yu, Z., Yang, H.: Modeling and analysis of fractional neutral disturbance waves in arterial vessels. Math. Model. Nat. Phenom. **14**, Article ID 301 (2019)
12. Yang, H., Guo, M., He, H.: Conservation laws of space-time fractional mZK equation for Rossby solitary waves with complete Coriolis force. Int. J. Nonlinear Sci. Numer. Simul. **20**, 17–32 (2019)
13. Oldham, K.B., Spanier, J.: The Fractional Calculus, pp. 1–84. Academic Press, New York (1974)
14. Samko, S., Kilbas, A.A., Marichev, O.: Fractional Integrals and Derivatives: Theory and Applications. Gordon & Breach, Yverdon (1993)
15. Miller, K.S., Ross, B.: An Introduction to the Fractional Calculus and Fractional Deferential Equations. Wiley, New York (1993)
16. Kiryakova, V.: Generalised Fractional Calculus and Applications. Pitman Research Notes in Mathematics, vol. 301. Longman, London (1994)
17. Ahmad, B., Ntouyas, S.K., Alsaedi, A.: On a coupled system of fractional differential equations with coupled nonlocal and integral boundary conditions. Chaos Solitons Fractals **83**, 234–241 (2016)
18. Biala, T.A., Jator, S.N.: Block implicit Adams methods for fractional differential equations. Chaos Solitons Fractals **81**, 365–377 (2015)
19. Shah, K., Khalil, H., Khan, R.A.: Investigation of positive solution to a coupled system of impulsive boundary value problems for nonlinear fractional order differential equations. Chaos Solitons Fractals **77**, 240–246 (2015)
20. Ren, Y., Tao, M., Dong, H., Yang, H.: Analytical research of $(3+1)$-dimensional Rossby waves with dissipation effect in cylindrical coordinate based on Lie symmetry approach. Adv. Differ. Equ. **2019**, Article ID 13 (2019)
21. Abdelkawy, M.A., Zaky, M.A., Bhrawy, A.H., Baleanu, D.: Numerical simulation of time variable fractional order mobile–immobile advection–dispersion model. Rom. Rep. Phys. **67**, 773–791 (2015)
22. Wang, Q.: Homotopy perturbation method for fractional KdV Burgers equation. Chaos Solitons Fractals **35**, 843–850 (2008)
23. Gupta, P.K., Singh, M.: Homotopy perturbation method for fractional Fornberg–Whitham equation. Comput. Math. Appl. **61**, 250–254 (2011)
24. Ray, S.S.: Analytical solution for the space fractional diffusion equation by two-step Adomian decomposition method. Commun. Nonlinear Sci. Numer. Simul. **14**, 1295–1306 (2009)
25. Nazari, D., Shahmorad, S.: Application of the fractional differential transform method to fractional-order integro differential equations with nonlocal boundary conditions. J. Comput. Appl. Math. **234**, 883–891 (2010)
26. Gupta, P.K.: Approximate analytical solutions of fractional Benney–Lin equation by reduced differential transformmethod and the homotopy perturbation method. Comput. Math. Appl. **61**, 2829–2842 (2011)
27. Abdel-Salam, E.A., Yousif, E.A.: Solution of nonlinear space-time fractional differential equations using the fractional Riccati expansion method. Math. Probl. Eng. **14**, 1–6 (2013)

28. Zhang, S.H., Zhang, Q.: Fractional sub-equation method and its applications to nonlinear fractional PDEs. Phys. Lett. A **375**, 1069–1073 (2011)
29. Abdelkawy, M.A., El-Kalaawy, O.H., Al-Denari, R.B., Biswas, A.: Application of fractional subequation method to nonlinear evolution equations. Nonlinear Anal., Model. Control **23**, 710–723 (2018)
30. Gepreel, K.A., Mohamed, M.S.: Analytical approximate solution for nonlinear space-time fractional Klein–Gordon equation. Chin. Phys. B **22**, 201–211 (2013)
31. Yulita Molliq, R., Batiha, B.: Approximate analytic solutions of fractional Zakharov–Kuznetsov equation by fractional complex transform. Int. J. Eng. Technol. **1**, 1–13 (2012)
32. Asiru, M.A.: Sumudu transform and the solution of integral equations of convolution type. Int. J. Math. Educ. Sci. Technol. **32**, 906–910 (2001)
33. Rana, M.A., Siddiqui, A.M., Ghori, Q.K., Qamar, R.: Application of He's homotopy perturbation method to Sumudu transform. Int. J. Nonlinear Sci. Numer. Simul. **8**, 185–190 (2007)
34. Eltayeb, H., Kylycman, A.: On some applications of a new integral transform. Int. J. Math. Anal. **4**, 123–132 (2010)
35. Gazizov, R.K., Kasatkin, A.A., Lukashchuk, S.Y.: Symmetry properties of fractional diffusion equations. Phys. Scr. **36**, 1–5 (2009)
36. Bluman, G.W., Kumei, S.: Symmetries and Differential Equations, pp. 31–89. Springer, New York (1989)
37. Olver, P.J.: Application of Lie Groups to Differential Equations, pp. 2–24. Springer, New York (1993)
38. Ibragimov, N.H.: Elementary Lie Group Analysis and Ordinary Differential Equations. Wiley, Chichester (1999)
39. Lukashchuk, S.Y.: Conservation laws for time-fractional subdiffusion and diffusion-wave equations. Nonlinear Dyn. **23**, 153–163 (2015)
40. Abdullahi, R.A., Muatjetjeja, B.: Conservation laws and exact solutions for a 2D Zakharov–Kuznetsov equation. Appl. Math. Lett. **1**, 109–117 (2015)
41. El-Kalaawy, O.H.: Variational principle, conservation laws and exact solutions for dust ion acoustic shock waves modeling modified Burger equation. Comput. Math. Appl. **72**, 1031–1041 (2016)
42. El-Kalaawy, O.H.: Modulational instability: conservation laws and bright soliton solution of ion-acoustic waves in electron-positron-ion-dust plasmas. Eur. Phys. J. Plus **133**, Article ID 58 (2018)
43. El-Kalaawy, O.H.: New: variational principle-exact solutions and conservation laws for modified ion-acoustic shock waves and double layers with electron degenerate in plasma. Phys. Plasmas **24**, 308 (2017)
44. El-Kalaawy, O.H., Moawad, S.M., Wael, S.: Stability: conservation laws, Painlevé analysis and exact solutions for S-KP equation in coupled dusty plasma. Results Phys. **7**, 934–946 (2017)
45. Adem, A.R., Khalique, C.M.: Symmetry reductions exact solutions and conservation laws of a new coupled KdV system. Commun. Nonlinear Sci. Numer. Simul. **17**, 3465–3475 (2012)
46. Noether, E.: Invariant variation problems. Transp. Theory Stat. Phys. **1**, 186–207 (1971)
47. Ibragimov, N.H.: A new conservation theorem. J. Math. Anal. Appl. **28**, 311–333 (2007)
48. Lu, C., Xie, L., Yang, H.: Analysis of Lie symmetries with conservation laws and solutions for the generalized $(3+1)$-dimensional time fractional Camassa–Holm–Kadomtsev–Petviashvili equation. Comput. Math. Appl. **77**, 3154–3171 (2019)
49. Yang, H., Sun, J., Fu, C.: Time-fractional Benjamin–Ono equation for algebraic gravity solitary waves in baroclinic atmosphere and exact multi-soliton solution as well as interaction. Commun. Nonlinear Sci. Numer. Simul. **71**, 187–201 (2019)
50. El-Kalaawy, O.H., Engy, A.A.: Shock waves, variational principle and conservation laws of a Schamel–Zakharov–Kuznetsov–Burgers equation in a magnetised dust plasma. Z. Naturforsch. A **73**, 693–704 (2018)
51. Hadjaz, I., Tribeche, M.: Alternative dust-ion acoustic waves in a magnetized charge varying dusty plasma with nonthermal electrons having a vortex-like velocity distribution. Astrophys. Space Sci. **351**, 591–598 (2014)
52. Sahoo, S., Saha Ray, S.: Analysis of Lie symmetries with conservation laws for the $(3+1)$ dimensional time-fractional mKdV-ZK equation in ion-acoustic waves. Nonlinear Dyn. **90**, 1105–1113 (2017)
53. Gazizov, R.K., Kasatkin, A.A., Lukashcuk, S.Y.: Continuous transformation groups of fractional differential equations. Vestn., UGATU **9**, 125–135 (2007)
54. Sahadevan, R., Bakkyaraj, T.: Invariant analysis of time fractional generalized Burgers and Korteweg–de Vries equations. J. Math. Anal. Appl. **393**, 341–347 (2012)
55. Baleanu, D., Diethelm, K., Scalas, E., Trujillo, J.J.: Fractional Calculus: Models and Numerical Methods. World Scientific, Singapore (2012)
56. Uchaikin, V., Sibatov, R.: Fractional kinetics in solids: anomalous charge transport in semiconductors. In: Dielectrics and Nanosystems, vol. 9, pp. 401–402. World Scientific, Singapore (2013)
57. Galaktionov, V.A., Svirshchevskii, S.R.: Exact Solutions and Invariant Subspaces of Nonlinear Partial Differential Equations in Mechanics and Physics. Chapman & Hall/CRC, Boca Raton (2006)
58. Tian, S.F., Zhang, H.Q.: On the integrability of a generalized variable-coefficient Kadomtsev–Petviashvili equation. J. Phys. A **132**, 212–246 (2012)
59. Rudin, W.: Principles of Mathematical Analysis, pp. 223–228. China Machine Press, Beijing (2004)
60. Liu, C.S.: Trial equation method to nonlinear evolution equations with rank inhomogeneous: mathematical discussions and its applications. Commun. Theor. Phys. **45**, 219–223 (2006)
61. Liu, C.S.: New exact envelope traveling wave solutions of high-order dispersive cubic-quintic nonlinear Schrödinger equation. Commun. Theor. Phys. **44**, 799–801 (2005)
62. Jun, C.Y.: Cubic structures applied to ideals of BCI-algebras. Comput. Math. Appl. **62**, 3987–3996 (2011)
63. Pandir, Y., Gurefe, Y., Kadak, U., Misirli, E.: Classification of exact solutions for some nonlinear partial differential equations with generalized evolution. Abstr. Appl. Anal. **2012**, Article ID 478531 (2012)
64. Pandir, Y., Gurefe, Y., Misirli, E.: Classification of exact solutions to the generalized Kadomtsev–Petviashvili equation. Phys. Scr. **87**, 1–12 (2013)