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Robust adaptive synchronization of complex network with bounded disturbances

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Abstract
In this paper, we investigate distributed robust adaptive synchronization for complex networked systems with bounded disturbances. We propose both average synchronization protocol and leader-following synchronization protocol based on adaptive control and variable structure control strategies. The synchronization conditions do not require any global information except a connection assumption under the adaptive control method. Furthermore, the external disturbances are attenuated effectively. Finally, we present numerical simulations to illustrate the theoretical findings.

Keywords: Complex network; Synchronization; Adaptive control; Variable structure control; Disturbance

1 Introduction

Recently, distributed cooperative control for complex networked systems has absorbed a mount of attention due to its widely applications in biological, physical, social, and many engineering sciences. Researches including synchronization [1–4], consensus [5, 6], containment [7, 8], and flocking [9] are intensively investigated.

Among the distributed cooperative control for complex networked systems, synchronization is one of the most fundamental problems, which means that the states of the agents reach an agreement on a common physical quantity of interest by implementing an appropriate consensus protocol based on the information from local neighbors [10]. In the past decades, many different control protocols have been reported for driving the complex network to synchronize, such as adaptive control [11, 12], impulsive control [13, 14], intermittent control [15], and event-triggered control [16, 17]. Furthermore, synchronization behavior is mainly influenced by the dynamics of each node. Synchronization (or consensus) of networked systems with nonlinear dynamics or disturbances is intensively investigated [18]. Synchronization was studied for heterogenous networks with piecewise smooth nonlinear coupling topology [19]. On one hand, external disturbance is a main source of instability and poor performance, which widely exists in real processes. Thereby it is of great significance to investigate distributed coordination for nonlinear multiagent systems with bounded disturbances. A robust consensus algorithm was studied for double integrator multiagent systems with exogenous disturbances by utilizing a nonsmooth back-stepping control technique [20]. The work in [21] investigated the consensus of the multiagent systems with a nonlinear coupling function and external disturbances based on disturbance observer and H_∞ control method. Robust consensus tracking was investigated for a class of second-order multiagent systems with disturbances and unmodeled dynamics [22]. Using sliding-mode control method, the work in [23] investigated the finite consensus and containment of first-order nonlinear multiagent systems with disturbances under directed topology. In [24] a distributed leader-following consensus problem was studied for second-order multiagent systems with bounded disturbances. Leader-following consensus conditions were derived for nonlinear multiagent systems with communication delay and communication noise under switching topology [25]. Adaptive consensus was investigated for uncertain parabolic PDE agents [26]. On the other hand, for reducing the number of controlled nodes, a pinning control is proposed for synchronization control of complex network [27, 28], in which the authors drive the agents to realize synchronization via controlling a part of the nodes. For adjusting the coupling gains, adaptive pinning control protocols were proposed for networked systems [29]. Adaptive pinning impulsive synchronization was investigated for time-delayed complex networks [30]. An adaptive pinning synchronization criterion was obtained for linearly coupled reaction–diffusion neural networks with mixed delays [31].

Motivated by the works mentioned, in this paper, we focus on nonlinear multiagent systems with external disturbances. With the hybrid aid of adaptive control, pinning control, and variable structure control strategy, we propose a fully distributed synchronization protocol, which can guarantee that the consensus condition requires no any global information. The main contribution of this paper is twofold: (a) The disturbances are modeled as a nonlinear function dependent on the relative information between the neighboring agents. This leads to that the subsystems are coupled by an unknown nonlinear function, which exactly improves the complexity of the stability analysis; (b) adaptive control is involved for adjusting the coupling gains for the average synchronization, and adaptive pinning control protocol is designed for the leader-following case; (c) variable structure control strategy is used for attenuating the bounded channel disturbances.

The rest of the paper is organized as follows. In Sect. 2, we state the model considered in the paper and give some basic definitions, lemmas, and assumptions. In Sect. 3, we propose an adaptive average synchronization protocol and give the convergency analysis for the nonlinear complex network with bounded disturbances. In Sect. 4, we investigate adaptive leader-following synchronization. Numerical examples are included to demonstrate the proposed protocol in Sect. 5. Finally, Sect. 6 concludes the paper.

2 Preliminaries and model description

In this section, we introduce some notations and preliminaries. By I_n we denote the $n \times n$ identity matrix. For a matrix A (or a vector x), A^T (or x^T) represents the transpose of A (or x); $\|x\|_1$, $\|x\|_2$, and $\|x\|_\infty$ denote the 1-, 2-, and ∞-norms of a vector x, respectively; $A \otimes B$ denotes the Kronecker product of matrices A and B.

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ be a undirected graph with a nonempty set of nodes $\mathcal{V} = (\upsilon_1, \upsilon_2, \ldots, \upsilon_N)$, a set of edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, and a weighted adjacent matrix $\mathcal{A} = [a_{ij}]$. In a undirected graph, we denote an edge by (υ_i, υ_j), which means that vertices i and j can obtain information from each other; $a_{ij} = a_{ji}$ represents the weight of the edge (υ_j, υ_i), and $a_{ij} > 0 \Longleftrightarrow (\upsilon_j, \upsilon_i) \in \mathcal{E}$; the neighbor sets are defined as $\mathcal{N}_i = \{\upsilon_j | (\upsilon_j, \upsilon_i) \in \mathcal{E}\}$; and the Laplacian matrix $L = [l_{ij}] \in R^{N\times N}$ is defined as $l_{ii} = \sum_{j=1, j\neq i}^{N} a_{ij}$ and $l_{ij} = -a_{ij}$, $i \neq j$ [32].

In this paper, we investigate distributed robust adaptive synchronization for complex networked systems with bounded disturbances. The dynamics of the ith subsystem are described as

$$\begin{aligned}\dot{x}_i(t) &= f(t,x_i) + c_i(t)\sum_{j\in\mathcal{N}_i} a_{ij}(x_j - x_i) \\ &\quad + \sum_{j\in\mathcal{N}_i} a_{ij} g(x_j - x_i) + u_i, \quad i = 1,2,\ldots,N,\end{aligned} \tag{1}$$

where $x_i, u_i \in R^n$ are the state and input vectors of the ith subsystem, respectively, $f : R \times R^n \to R^n$, $i = 1,2,\ldots,k$, are continuous vector-value functions, and $g : R^n \to R^n$ denotes the disturbances dependent on the relative information between nodes i and j.

Remark 1 System (1) can be considered as a class of nonlinear coupled complex networks. Also, it can be used to describe the network with channel disturbances.

The following assumptions and lemmas are necessary for the main results.

Assumption 1 The function $f(t,x)$ satisfies the global Lipschitz condition, that is, there exist a constant $\eta > 0$ such that

$$\left\| f(t,x_1) - f(t,x_2)\right\|_2 \le \eta \|x_1 - x_2\|_2, \quad \forall x_1, x_2 \in R^N.$$

Assumption 2 There exists a constant $\gamma > 0$ such that

$$\left\| g(x_j - x_i)\right\|_\infty \le \frac{\gamma}{N}, \quad i,j = 1,2,\ldots,N.$$

Assumption 3 The topology graph is fixed and connected.

Lemma 1 ([32]) *The Laplacian matrix L has a simple eigenvalue* 0, *and the remaining eigenvalues are positive if and only if the undirected graph is connected.*

Lemma 2 ([10]) *For an undirected connected graph G with Laplacian matrix L and a vector x satisfying $\mathbf{1}^T x = 0$, we have*

$$\min_{x\neq 0}\left\{\frac{x^T L x}{x^T x}\right\} = \lambda_2(L).$$

Lemma 3 ([33]) *If $L = (l_{ij}) \in R^{N\times N}$ is a symmetric irreducible matrix with $l_{ii} = -\sum_{j=1,j\neq i}^{N} l_{ij}$, $l_{ij} = l_{ji} \le 0$ $(i \neq j)$, then L is semipositive definite, and for any matrix $E = \operatorname{diag}(e,0,\ldots,0)$ with $e > 0$, all eigenvalues of the matrix $(L + E)$ are positive.*

Lemma 4 ([34]) *Suppose that a scalar function $V(x,t)$ satisfies the following conditions*:

(a) *$V(x,t)$ is lower bounded*;
(b) *$\dot{V}(x,t)$ is negative semidefinite*;
(c) *$\dot{V}(x,t)$ is uniformly continuous in t.*

Then $\dot{V}(x,t) \to 0$ as $t \to \infty$.

3 Average synchronization of complex network with bounded disturbances

In this section, we investigate distributed robust adaptive average synchronization for complex networked systems with bounded disturbances. The main purpose of this section is to design a distributed consensus protocol for system (1) such that $x_i(t) \to x_j(t) \to \bar{x}(t)$ as $t \to \infty$, where $\bar{x}(t) = \frac{\sum_{i=1}^N x_i(t)}{N}$. The proposed consensus protocol is

$$u_i(t) = c_i(t)\operatorname{sgn}\Biggl(\sum_{j\in\mathcal{N}_i} a_{ij}(x_j - x_i)\Biggr), \quad i = 1, 2, \ldots, N, \tag{2}$$

where the initial values of adaptive parameters $c_i(0) > 0$, $i = 1, 2, \ldots, N$, $c_i(t)$ are decided by

$$\begin{aligned}\dot{c}_i(t) &= \tau_i\Biggl(\sum_{j\in\mathcal{N}_i} a_{ij}(x_j - x_i)\Biggr)^T\Biggl(\sum_{j\in\mathcal{N}_i} a_{ij}(x_j - x_i)\Biggr) \\ &\quad + \tau_i\Biggl\|\sum_{j\in\mathcal{N}_i} a_{ij}(x_j - x_i)\Biggr\|_1 \\ &= \tau_i\Biggl(\sum_{j\in\mathcal{N}_i} l_{ij}x_j\Biggr)^T\Biggl(\sum_{j\in\mathcal{N}_i} l_{ij}x_j\Biggr) + \tau_i\Biggl\|\sum_{j\in\mathcal{N}_i} l_{ij}x_j\Biggr\|_1,\end{aligned} \tag{3}$$

$\tau_i > 0$ is the weight of $c_i(t)$, and $\operatorname{sgn}(\cdot)$ is defined as $\operatorname{sgn}(x) = 1$ for $x > 0$, $\operatorname{sgn}(x) = -1$ for $x < 0$, and $\operatorname{sgn}(x) = 0$ for $x = 0$.

Remark 2 Adaptive synchronization protocol (2)–(3) is called a node-based adaptive control [11, 12], in which the adaptive parameters are decided by the addition of relative information between the ith node and its neighbor nodes, whereas the adaptive parameter of the edge-based adaptive control is adjusted by any two adjacent nodes. Both adaptive methods can guarantee that the synchronization conditions do not depend on the information of the Laplacian matrix. The disadvantage is that the computing complexity increases. The cost-guaranteed adaptive control will be given in the future work.

Remark 3 According to (3), $\dot{c}_i(t) > 0$. Then we can conclude that $c_i(t) > 0$ for all $t \geq 0$ because $c_i(0) > 0$.

Under the proposed protocol, system (1) can be rewritten as

$$\begin{aligned}\dot{x}_i(t) &= f(t, x_i) + c_i(t)\sum_{j\in\mathcal{N}_i} a_{ij}(x_j - x_i) + \sum_{j\in\mathcal{N}_i} a_{ij}g(x_j - x_i) \\ &\quad + c_i(t)\operatorname{sgn}\Biggl(\sum_{j\in\mathcal{N}_i} a_{ij}(x_j - x_i)\Biggr), \quad i = 1, 2, \ldots, N.\end{aligned} \tag{4}$$

Let $e_i(t) = x_i(t) - \bar{x}(t)$. Then

$$\begin{aligned}\dot{e}_i(t) &= f(t, x_i) + c_i\sum_{j=1}^N a_{ij}(x_j - x_i) + \sum_{j=1}^N a_{ij}g(x_j - x_i) \\ &\quad + c_i\operatorname{sgn}\Biggl(\sum_{j=1}^N a_{ij}(x_j - x_i)\Biggr) - \frac{\sum_{j=1}^N f(t, x_j)}{N}\end{aligned}$$

$$
\begin{aligned}
&-\frac{\sum_{k=1}^{N} c_k \sum_{j=1}^{N} a_{kj}(x_j-x_k)}{N}-\frac{\sum_{k=1}^{N} \sum_{j=1}^{N} a_{kj} g(x_j-x_k)}{N}\\
&-\frac{\sum_{k=1}^{N} c_k \operatorname{sgn}(\sum_{j=1}^{N} a_{kj}(x_j-x_k))}{N}\\
={}& f(t,x_i)-\frac{\sum_{j=1}^{N} f(t,x_j)}{N}-c_i \sum_{j=1}^{N} l_{ij} e_j+\sum_{j=1}^{N} a_{ij} g(x_j-x_i)\\
&-c_i \operatorname{sgn}\left(\sum_{j=1}^{N} l_{ij} e_j\right)+\frac{\sum_{k=1}^{N} c_k \sum_{j=1}^{N} l_{kj} e_j}{N}-\frac{\sum_{k=1}^{N} \sum_{j=1}^{N} a_{kj} g(x_j-x_k)}{N}\\
&+\frac{\sum_{k=1}^{N} c_k \operatorname{sgn}(\sum_{j=1}^{N} l_{kj} e_j)}{N}.
\end{aligned} \tag{5}
$$

Denote $e(t)=(e_1^T(t),\ldots,e_n^T(t))^T$, $F(t,x)=(f^T(t,x_1),\ldots,f^T(t,x_N))^T$, $\bar{f}(t)=\frac{\sum_{j=1}^{N} f(t,x_j)}{N}$, $H=-\frac{\sum_{k=1}^{N}\sum_{j=1}^{N} a_{kj} g(x_j-x_k)}{N}+\frac{\sum_{k=1}^{N} c_k \operatorname{sgn}(\sum_{j=1}^{N} l_{kj} e_j)}{N}$, and $\bar{g}(t)=(g_1^T(t),\ldots,g_n^T(t))^T$, where $g_i(t)=\sum_{j=1}^{N} a_{ij} g(x_j-x_i)$, then

$$
\begin{aligned}
\dot{e}(t)={}& F(t,x)-\mathbf{1}_N \otimes \bar{f}(t,x)-(CL\otimes I_n)e-(C\otimes I_n)\operatorname{sgn}\big((L\otimes I_n)e\big)+\bar{g}(t)\\
&-\left(\frac{1}{N}\mathbf{1}_N \mathbf{1}_N^T CL\otimes I_n\right)e+\mathbf{1}_N\otimes H,
\end{aligned} \tag{6}
$$

where $C=\operatorname{diag}(c_1,\ldots,c_n)$.

Theorem 1 *Consider a networked multiagent system with N following nodes, where each following node has dynamics as in (1). Suppose that Assumptions 1, 2, and 3 hold. Using the consensus protocol (4) with adaptive strategy (5) for (1), average synchronization of system (1) can be achieved. Furthermore, all the following nodes will asymptotically track the average state.*

Proof According to Lemma 1, $\tilde{L}$ is positive definite. We choose the Lyapunov candidate function

$$
V=\frac{1}{2}e^T(L\otimes I_n)e+\sum_{i=1}^{N}\frac{(c_i-c)^2}{2\tau_i}, \tag{7}
$$

where $c>\max\{1,\gamma\}$ is a positive constant.

Differentiating V respect to t along (10), we obtain

$$
\begin{aligned}
\dot{V}={}& e^T(L\otimes I_n)\dot{e}+2\sum_{i=1}^{N}\frac{(c_i-c)^2}{2\tau_i}\dot{c}_i\\
={}& e^T(L\otimes I_n)\Big[F(t,x)-\mathbf{1}_N\otimes\bar{f}(t,x)-(CL\otimes I_n)e\\
&-(C\otimes I_n)\operatorname{sgn}\big((L\otimes I_n)e\big)+\bar{g}(t)-\left(\frac{1}{N}\mathbf{1}_N\mathbf{1}_N^T CL\otimes I_n\right)e+\mathbf{1}_N\otimes H\Big]\\
&+\sum_{i=1}^{N}(c_i-c)\left(\left(\sum_{j\in\mathcal{N}_i} l_{ij}e_j\right)^T\left(\sum_{j\in\mathcal{N}_i} l_{ij}e_j\right)+\left\|\sum_{j\in\mathcal{N}_i} l_{ij}e_j\right\|_1\right).
\end{aligned} \tag{8}
$$

Noting that $L\mathbf{1}_N = 0$, we have

$$\begin{aligned}\dot{V} &= -e^T(LCL \otimes I_n)e - e^T(LC \otimes I_n)\operatorname{sgn}\big((L \otimes I_n)e\big) \\ &\quad + e^T(L \otimes I_n)\big(F(t,x) - \mathbf{1}_N \otimes f(t,\bar{x}) + \bar{g}(t)\big) \\ &\quad + \sum_{i=1}^{N}(c_i - c)\Bigg(\bigg(\sum_{j\in\mathcal{N}_i} l_{ij}e_j\bigg)^T\bigg(\sum_{j\in\mathcal{N}_i} l_{ij}e_j\bigg) + \bigg\|\sum_{j\in\mathcal{N}_i} l_{ij}e_j\bigg\|_1\Bigg).\end{aligned} \tag{9}$$

We have

$$\begin{aligned}&\sum_{i=1}^{N} c_i\bigg(\sum_{j\in\mathcal{N}_i} l_{ij}e_j\bigg)^T\bigg(\sum_{j\in\mathcal{N}_i} l_{ij}e_j\bigg) \\ &\quad = \sum_{i=1}^{N}\bigg(\sum_{j\in\mathcal{N}_i} c_i l_{ij}e_j\bigg)^T\bigg(\sum_{j\in\mathcal{N}_i} l_{ij}e_j\bigg) = e^T(LCL \otimes I_n)e\end{aligned} \tag{10}$$

and

$$\begin{aligned}e^T(LC \otimes I_n)\operatorname{sgn}\big((L \otimes I_n)e\big) &= \sum_{i=1}^{N} c_i\bigg(\sum_{j\in\mathcal{N}_i} l_{ij}e_j\bigg)^T\operatorname{sgn}\bigg(\sum_{j\in\mathcal{N}_i} l_{ij}e_j\bigg) \\ &= \sum_{i=1}^{N}\bigg(c_i\bigg\|\sum_{j\in\mathcal{N}_i} l_{ij}e_j\bigg\|_1\bigg).\end{aligned} \tag{11}$$

Substituting (10) and (11) into (9), we get

$$\begin{aligned}\dot{V} &= -ce^T\big(L^2 \otimes I_n\big)e - \sum_{i=1}^{N}\bigg(cI_n\bigg\|\sum_{j\in\mathcal{N}_i} l_{ij}e_j\bigg\|_1\bigg) \\ &\quad + e^T(L \otimes I_n)\big(F(t,x) - \mathbf{1}_N \otimes f(t,\bar{x}) + \bar{g}(t)\big).\end{aligned} \tag{12}$$

According to Assumption 1, we have

$$\begin{aligned}&e^T(L \otimes I_n)\big(F(t,x) - \mathbf{1}_N \otimes f(t,\bar{x})\big) \\ &\quad = \big((L \otimes I_n)e\big)^T\big(F(t,x) - \mathbf{1}_N \otimes f(t,\bar{x})\big) \\ &\quad \le \big((L \otimes I_n)e\big)^T(L \otimes I_n)e + \big(F(t,x) - \mathbf{1}_N \otimes f(t,\bar{x})\big)^T\big(F(t,x) - \mathbf{1}_N \otimes f(t,\bar{x})\big) \\ &\quad = e^T\big(L^2 \otimes I_n\big)e + \sum_{i=1}^{N}\big(f(t,x_i) - f(t,\bar{x})\big)^T\big(f(t,x_i) - f(t,\bar{x})\big) \\ &\quad \le e^T\big(L^2 \otimes I_n\big)e + \eta^2 e^T e\end{aligned} \tag{13}$$

and

$$e^T(L \otimes I_n)\bar{g}(t) \le \sum_{i=1}^{N}\bigg|\bigg(\sum_{j\in\mathcal{N}_i} l_{ij}e_j\bigg)^T g_i(t)\bigg| \le \sum_{i=1}^{N}\bigg(\gamma\bigg\|\sum_{j\in\mathcal{N}_i} l_{ij}e_j\bigg\|_1\bigg). \tag{14}$$

Substituting (13) and (14) into (12), we can conclude

$$\dot{V} \leq -(c-1)e^T\left(L^2 \otimes I_n\right)e + \eta^2 e^T e - (c-\gamma)\left\|\sum_{j\in\mathcal{N}_i} l_{ij}e_j\right\|_1. \tag{15}$$

Since L is real and symmetric, there is an orthogonal matrix Q such that $L = Q^T \Lambda Q$. Denoting $L^{\frac{1}{2}} = Q^T \Lambda^{\frac{1}{2}} Q$, we have $\mathbf{1}^T L \mathbf{1} = \mathbf{1}^T L^{\frac{1}{2}} L^{\frac{1}{2}} \mathbf{1} = 0$. Therefore $\mathbf{1}^T L^{\frac{1}{2}} = 0$, and thus $\mathbf{1}^T L^{\frac{1}{2}} e = 0$. According to Lemma 2, $e^T(L^2 \otimes I_n)e = ((L^{\frac{1}{2}} \otimes I_n)e)^T (L \otimes I_n)((L^{\frac{1}{2}} \otimes I_n)e) \leq \lambda_2((L^{\frac{1}{2}} \otimes I_n)e)^T((L^{\frac{1}{2}} \otimes I_n)e) = \lambda_2 e^T(L \otimes I_n) \leq \lambda_2^2 e^T e$. Then we have

$$\dot{V} \leq -(c-1)\lambda_2^2 e^T e + \eta^2 e^T e - (c-\gamma)\left\|\sum_{j\in\mathcal{N}_i} l_{ij}e_j\right\|_1. \tag{16}$$

Since $c > \max\{1 + \frac{\eta}{\lambda_2^2}, \gamma\}$, $\dot{V} \leq 0$, and thus V is not increasing and is bounded. Then e_i, c_i are bounded, which means that $\bar{f}(t,x)$ is also bounded. From (12), $\dot{V}$ is bounded. So V is uniformly continuous. By Lemma 4 we conclude that $\dot{V}(e,t) \to 0$ as $t \to \infty$. Denoting $W(e(t)) = ((c-1)\lambda_2^2 - \eta^2)e^T e$, we have $0 \leq W(e(t)) \leq -\dot{V}$. We know that $W(e(t)) \to 0$ as $t \to \infty$, and thus $e_i(t) \to 0$, that is, $\lim_{t\to\infty}(x_i - \bar{x}) = 0$ for all $i = 1,2,\ldots,N$. Theorem 1 is proved. □

Remark 4 In Theorem 1, we see that the synchronization condition does not depend on the eigenvalue of the Laplacian matrix. This is different from the nonadaptive control case, in which the control gain is no less than a threshold value depending on the eigenvalue.

4 Leader-following synchronization of complex network with bounded disturbances

In this section, we investigate distributed robust adaptive leader-following synchronization for complex networked systems with bounded disturbances. Suppose that there exists a leader (or virtual leader) in the network, which is described as

$$\dot{x}_0(t) = f\left(t, x_0(t)\right). \tag{17}$$

The main purpose of this section is designing a distributed consensus protocol for system (1) such that

$$\lim_{t\to\infty} \|x_i - x_0\| = 0.$$

We propose the consensus protocol

$$u_i(t) = c_i(t)h_i(x_0 - x_i) + c_i(t)\left(\sum_{j\in\mathcal{N}_i} a_{ij}(x_j - x_i) + h_i(x_0 - x_i)\right), \quad i = 1,2,\ldots,N. \tag{18}$$

In (4) the adaptive parameters c_i is decided by

$$\begin{aligned}\dot{c}_i(t) = {} & \tau_i\left(\sum_{j\in\mathcal{N}_i} a_{ij}(x_j - x_i) + h_i(x_0 - x_i)\right)^T \left(\sum_{j\in\mathcal{N}_i} a_{ij}(x_j - x_i) + h_i(x_0 - x_i)\right) \\ & + \tau_i\left\|\left(\sum_{j\in\mathcal{N}_i} \tilde{l}_{ij}e_j\right)\right\|_1, \quad i = 1,2,\ldots,N,\end{aligned} \tag{19}$$

where $\tau_i > 0$ is the weight of $c_i(t)$ and $c_i(0) > 0$, $\operatorname{sgn}(\cdot)$ is defined as $\operatorname{sgn}(x) = 1$ for $x > 0$, $\operatorname{sgn}(x) = -1$ for $x < 0$, and $\operatorname{sgn}(x) = 0$ for $x = 0$. If the ith node is pinned by the virtual leader, then $h_i = 1$; otherwise, $h_i = 0$.

Under the proposed protocol, system (1) can be rewritten as

$$\begin{aligned}\dot{x}_i(t) &= f(t,x_i) + c_i(t)\Bigl(\sum_{j\in\mathcal{N}_i} a_{ij}(x_j - x_i) + h_i(x_0 - x_i)\Bigr) + \sum_{j\in\mathcal{N}_i} a_{ij}g_{ij}(x_j - x_i)\\ &\quad + c_i(t)\operatorname{sgn}\Bigl(\sum_{j\in\mathcal{N}_i} a_{ij}(x_j - x_i) + h_i(x_0 - x_i)\Bigr), \quad i = 1,2,\ldots,N. \end{aligned} \tag{20}$$

Let $e_i(t) = x_i(t) - \bar{x}(t)$. Then we have

$$\begin{aligned}\dot{e}_i(t) &= f(t,x_i) - f(t,x_0) + c_i\Bigl(\sum_{j\in\mathcal{N}_i} a_{ij}(x_j - x_i) + h_i(x_0 - x_i)\Bigr)\\ &\quad + c_i\operatorname{sgn}\Bigl(\sum_{j\in\mathcal{N}_i} a_{ij}(x_j - x_i) + h_i(x_0 - x_i)\Bigr) + \sum_{j\in\mathcal{N}_i} a_{ij}g_{ij}(e_j - e_i)\\ &= f(t,x_i) - f(t,x_0) + c_i\Bigl(\sum_{j\in\mathcal{N}_i} a_{ij}(e_j - e_i) - h_ie_i\Bigr)\\ &\quad + c_i\operatorname{sgn}\Bigl(\sum_{j\in\mathcal{N}_i} a_{ij}(e_j - e_i) - h_ie_i\Bigr) + \sum_{j\in\mathcal{N}_i} a_{ij}g_{ij}(e_j - e_i)\\ &= f(t,x_i) - f(t,x_0) - c_i\Bigl(\sum_{j\in\mathcal{N}_i} l_{ij}e_j + h_ie_i\Bigr)\\ &\quad - c_i\operatorname{sgn}\Bigl(\sum_{j\in\mathcal{N}_i} l_{ij}e_j + h_ie_i\Bigr) + \sum_{j\in\mathcal{N}_i} a_{ij}g_{ij}(e_j - e_i). \end{aligned} \tag{21}$$

Let $\tilde{L} = (\tilde{l}_{ij})$, $\tilde{l}_{ii} = l_{ii} + h_i$, $\tilde{l}_{ij} = l_{ij}$, $i \neq j$. Then

$$\begin{aligned}\dot{e}_i(t) &= f(t,x_i) - f(t,x_0) - c_i\sum_{j\in\mathcal{N}_i} \tilde{l}_{ij}e_j - c_i\operatorname{sgn}\Bigl(\sum_{j\in\mathcal{N}_i} l_{ij}e_j\Bigr)\\ &\quad + \sum_{j\in\mathcal{N}_i} a_{ij}g_{ij}(e_j - e_i), \end{aligned} \tag{22}$$

and (19) can be rewritten as

$$\dot{c}_i = \tau_i\Bigl(\sum_{j\in\mathcal{N}_i} \tilde{l}_{ij}e_j\Bigr)^T\Bigl(\sum_{j\in\mathcal{N}_i} \tilde{l}_{ij}e_j\Bigr) + \tau_i\Bigl\|\sum_{j\in\mathcal{N}_i} (\tilde{l}_{ij}e_j)\Bigr\|_1. \tag{23}$$

Denote the error vector $e(t) = (e_1^T(t),\ldots,e_n^T(t))^T$, $\bar{f}(t,x) = ((f(t,x_1) - f(t,x_0))^T,\ldots, (f(t,x_N) - f(t,x_0))^T)^T$, and $\bar{g}(t) = (g_1^T(t),\ldots,g_n^T(t))^T$, where $g_i(t) = \sum_{j=1}^N a_{ij}g(e_j - e_i)$. Then

$$\dot{e}(t) = -(C\tilde{L} \otimes I_n)e - (C \otimes I_n)\operatorname{sgn}\bigl((\tilde{L} \otimes I_n)e\bigr) + \bar{f}(t,x) + \bar{g}(t), \tag{24}$$

where $C = \operatorname{diag}(c_1,\ldots,c_n)$.

Theorem 2 *Consider a networked multiagent system with N following nodes and a virtual leader, where each following node has dynamics as in* (1), *and the virtual leader is described as in* (17). *Suppose that Assumptions* 1, 2, *and* 3 *hold. Using the consensus protocol* (18) *with adaptive strategy* (19) *for* (1), *leader-following synchronization of system* (1) *can be achieved if there exists at least one pinned node. Furthermore, all the following nodes will asymptotically track the virtual leader.*

Proof According to Lemma 1, $\tilde{L}$ is positive definite. We choose the Lyapunov candidate function

$$V = \frac{1}{2}e^T(\tilde{L} \otimes I_n)e + \sum_{i=1}^{N} \frac{(c_i - c)^2}{2\tau_i}, \tag{25}$$

where $c > \max\{1, \gamma\}$ is a constant.

Differentiating V with respect to t along (10), we obtain

$$\begin{aligned} \dot{V} &= e^T(\tilde{L} \otimes I_n)\dot{e} + 2\sum_{i=1}^{N} \frac{(c_i - c)^2}{2\tau_i}\dot{c}_i \\ &= e^T(\tilde{L} \otimes I_n)\bigl[-(C\tilde{L} \otimes I_n)e - (C \otimes I_n)\operatorname{sgn}\bigl((\tilde{L} \otimes I_n)e\bigr) + \bar{f}(t,x) + \bar{g}(t)\bigr] \\ &\quad + \sum_{i=1}^{N}(c_i - c)\biggl(\biggl(\sum_{j\in\mathcal{N}_i}\tilde{l}_{ij}e_j\biggr)^T\biggl(\sum_{j\in\mathcal{N}_i}\tilde{l}_{ij}e_j\biggr) + \biggl\|\sum_{j\in\mathcal{N}_i}\tilde{l}_{ij}e_j\biggr\|_1\biggr) \\ &= -e^T(\tilde{L}C\tilde{L} \otimes I_n)e - e^T(\tilde{L}C \otimes I_n)\operatorname{sgn}\bigl((\tilde{L} \otimes I_n)e\bigr) \\ &\quad + e^T(\tilde{L} \otimes I_n)\bigl(\bar{f}(t,x) + \bar{g}(t)\bigr) \\ &\quad + \sum_{i=1}^{N}(c_i - c)\biggl(\biggl(\sum_{j\in\mathcal{N}_i}\tilde{l}_{ij}e_j\biggr)^T\biggl(\sum_{j\in\mathcal{N}_i}\tilde{l}_{ij}e_j\biggr) + \biggl\|\sum_{j\in\mathcal{N}_i}\tilde{l}_{ij}e_j\biggr\|_1\biggr). \end{aligned} \tag{26}$$

Note that

$$\begin{aligned} &\sum_{i=1}^{N} c_i\biggl(\sum_{j\in\mathcal{N}_i}\tilde{l}_{ij}e_j\biggr)^T\biggl(\sum_{j\in\mathcal{N}_i}\tilde{l}_{ij}e_j\biggr) \\ &\quad = \sum_{i=1}^{N}\biggl(\sum_{j\in\mathcal{N}_i}c_i\tilde{l}_{ij}e_j\biggr)^T\biggl(\sum_{j\in\mathcal{N}_i}\tilde{l}_{ij}e_j\biggr) \\ &\quad = e^T(\tilde{L}C\tilde{L} \otimes I_n)e \end{aligned} \tag{27}$$

and

$$\begin{aligned} &e^T(\tilde{L}C \otimes I_n)\operatorname{sgn}\bigl((\tilde{L} \otimes I_n)e\bigr) \\ &\quad = \sum_{i=1}^{N} c_i\biggl(\sum_{j\in\mathcal{N}_i}\tilde{l}_{ij}e_j\biggr)^T\operatorname{sgn}\biggl(\sum_{j\in\mathcal{N}_i}\tilde{l}_{ij}e_j\biggr) \\ &\quad = \sum_{i=1}^{N}\biggl(c_i\biggl\|\sum_{j\in\mathcal{N}_i}\tilde{l}_{ij}e_j\biggr\|_1\biggr). \end{aligned} \tag{28}$$

Substituting (27) and (28) into (26), we have

$$\dot{V} = -ce^T(\tilde{L}^2 \otimes I_n)e - \sum_{i=1}^{N}\left(cI_n\left\|\sum_{j\in\mathcal{N}_i}\tilde{l}_{ij}e_j\right\|_1\right) + e^T(\tilde{L}\otimes I_n)\big(\bar{f}(t,x) + d(t)\big). \tag{29}$$

According to Assumption 1, we have

$$\begin{aligned}
&e^T(\tilde{L}\otimes I_n)\bar{f}(t,x)\\
&\quad = \big((\tilde{L}\otimes I_n)e\big)^T\bar{f}(t,x) \le \big((\tilde{L}\otimes I_n)e\big)^T(\tilde{L}\otimes I_n)e + \big(\bar{f}(t,x)\big)^T\bar{f}(t,x)\\
&\quad = e^T(\tilde{L}^2\otimes I_n)e + \sum_{i=1}^{N}\big(f(t,x_i) - f(t,x_0)\big)^T\big(f(t,x_i) - f(t,x_0)\big)\\
&\quad \le e^T(\tilde{L}^2\otimes I_n)e + \eta^2 e^T e
\end{aligned} \tag{30}$$

and

$$\begin{aligned}
e^T(\tilde{L}\otimes I_n)d(t) &\le \sum_{i=1}^{N}\left|\left(\sum_{j\in\mathcal{N}_i}\right)^T d_i(t)\right|\\
&\le \gamma\left|e^T(\tilde{L}\otimes I_n)\right|\\
&\le \sum_{i=1}^{N}\left(\gamma\left\|\sum_{j\in\mathcal{N}_i}\tilde{l}_{ij}e_j\right\|_1\right).
\end{aligned} \tag{31}$$

Substituting (30) and (31) into (29), we get

$$\dot{V} \le -(c-1)e^T(\tilde{L}^2\otimes I_n)e - (c-\gamma)\left\|\sum_{j\in\mathcal{N}_i}\tilde{l}_{ij}e_j\right\|_1. \tag{32}$$

Since $c > \max\{1,\gamma\}$, $\dot{V} \le 0$, and thus V is not increasing, and $\lim_{t\to\infty} V(t)$ exists. Denote $V(\infty) = \lim_{t\to\infty} V(t)$. Furthermore, e_i, c_i are bounded, which means that $\bar{f}(t,x)$ is also bounded. According to (8), $\dot{e}_i$ is bounded.

Let

$$W\big(e(t)\big) = (c-1)e^T(\tilde{L}^2\otimes I_n)e. \tag{33}$$

Note that $W(e(t)) \le -\dot{V}(e(t),c_i(t))$, $\int_0^\infty W(t)\,dt \le -\int_0^\infty \dot{V}(t)\,dt = V(0) - V(\infty)$, and $\int_0^\infty W(t)\,dt$ exists and is bounded. Since $e_i(t)$ and $\dot{e}_i$ are bounded, $\dot{W}(e(t)) = -2e^T(\tilde{L}^2\otimes I_n)\dot{e}$ is also bounded. Then $W(e(t))$ is uniformly continuous with respect to t. According to Lemma 4, $W(e(t)) \to 0$ as $t\to\infty$. Then $e_i(t)\to 0$, that is, $\lim_{t\to\infty}(x_i - x_0) = 0$ for all $i = 1,2,\ldots,N$. Theorem 2 is proved. □

5 Simulations

In this section, we give a numerical simulation to illustrate the effectiveness of the obtained results. For convenience, choose a network with eight nodes. The topology of the network

Figure 1 Topology graph with 8 nodes

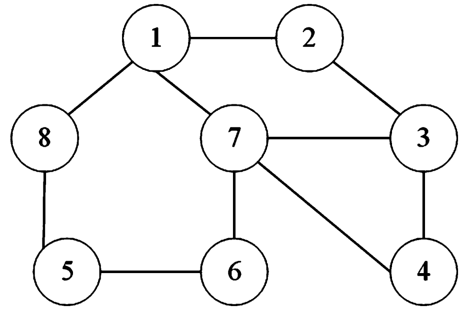


Figure 2 Trajectories of x_{i1} under average synchronization protocol (2)

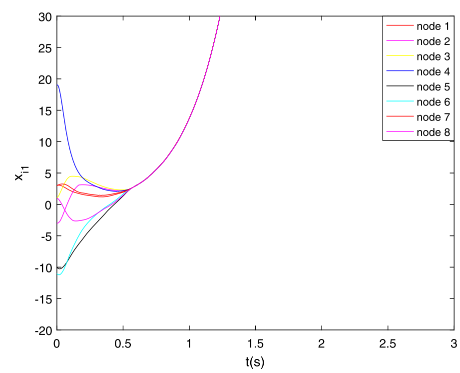
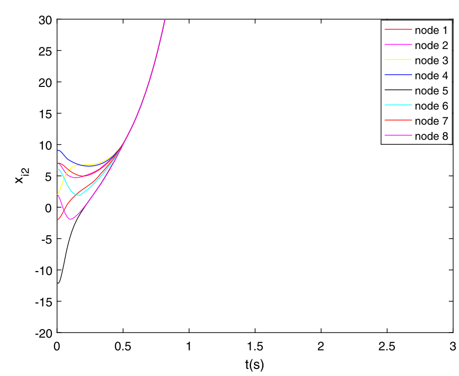


Figure 3 Trajectories of x_{i2} under average synchronization protocol (2)



is described as the graph in Fig. 1. According to the definition of the topology graph, we conclude that the Laplacian matrix is

$$\tilde{L} = \begin{pmatrix} 4 & -1 & 0 & 0 & 0 & 0 & -1 & -1 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 3 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ -1 & 0 & -1 & -1 & 0 & -1 & 4 & 0 \\ -1 & 0 & 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}.$$

The ith node is described as

$$\dot{x}_i(t) = f(t, x_i) + c_i(t) \sum_{j \in \mathcal{N}_i} a_{ij}(x_j - x_i) + \sum_{j \in \mathcal{N}_i} a_{ij} g(x_j - x_i) + u_i,$$

where $f(t, x_i) = 3x_{i1} + 2\sin^2 x_i + \cos x_{i2}$ satisfies the global Lipschitz condition. The nonlinear disturbance function is chosen as $g(x_j - x_i) = \sin 3(x_j - x_i) + \cos(x_j - x_i)$. It is easy to verify that Assumptions 1 and 2 hold.

According to Theorem 1, the average synchronization should be realized under protocol (2) with adaptive strategy (3). The simulation results are shown in Figs. 2–3. The state of each node synchronizes to the average state. We have chosen the initial values as $c_1(0) = 0.4$, $c_2(0) = 0.7$, $c_3(0) = 0.6$, $c_4(0) = 1.2$, $c_5(0) = 0.2$, $c_6(0) = 0.5$, $c_7(0) = 0.9$, $c_8(0) = 0.8$. According to Fig. 4, the trajectories of the adaptive parameters c_i, $i = 1, 2, \ldots, 8$, asymptotically converge to constants.

According to Theorem 2, the leader-following synchronization should be realized under protocol (18) with adaptive strategy (19). The initial values of the leaders are chosen as $(1.2, -1)^T$. Choose the first node as a pinned node. From Figs. 5–6, all the followers can track the leader asymptotically; $c_i(0)$, $i = 1, \ldots, 8$, are chosen as 0.3, 1.2, 0.7, 0.8, 1.6, 0.5, 2.0, 0.4. According to Fig. 7, the trajectories of the adaptive parameters c_i, $i = 1, 2, \ldots, 8$, asymptotically converge to constants.

Figure 4 Trajectories of the adaptive parameters c_i in (3)

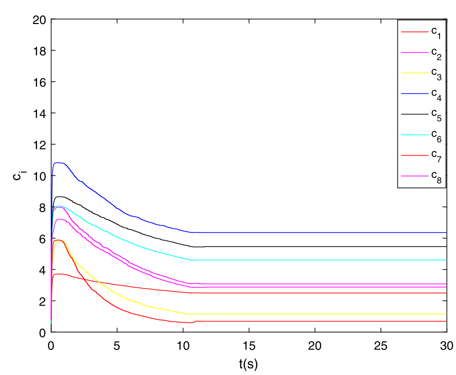
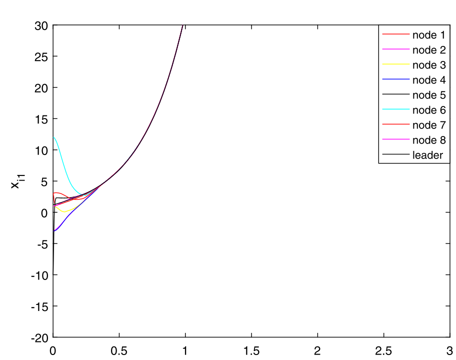


Figure 5 Trajectories of x_{i1} under leader-following synchronization protocol (18)



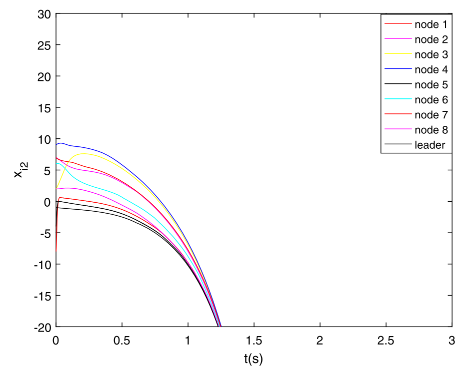


Figure 6 Trajectories of x_{i2} under leader-following synchronization protocol (18)

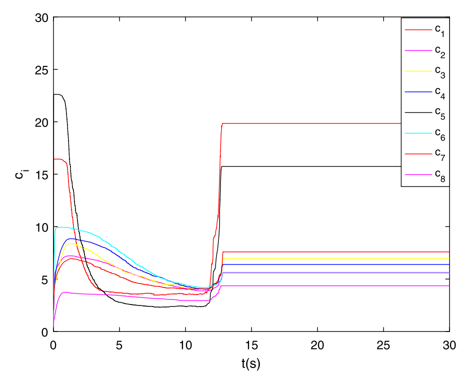


Figure 7 Trajectories of the adaptive parameters in (19)

6 Conclusions

In this paper, we investigated distributed robust adaptive synchronization problem for nonlinear complex networked systems with bounded disturbances. Based on adaptive control and variable control strategies, we proposed both the average synchronization and leader-following synchronization protocols and obtained the synchronization, which can guarantee that the synchronization conditions require no any global information except a connection assumption under the adaptive control method. Finally, we presented numerical simulations to illustrate the theoretical results.

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Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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