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Stability analysis of a nonlinear coupled implicit switched singular fractional differential system with p-Laplacian

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Abstract

This paper deals with existence, uniqueness, and Hyers–Ulam stability of solutions to a nonlinear coupled implicit switched singular fractional differential system involving Laplace operator ϕ_p. The proposed problem consists of two kinds of fractional derivatives, that is, Riemann–Liouville fractional derivative of order β and Caputo fractional derivative of order σ, where $m-1<\beta,\sigma<m$, $m\in\{2,3,\dots\}$. Prior to proceeding to the main results, the system is converted into an equivalent integral form by the help of Green's function. Using Schauder's fixed point theorem and Banach's contraction principle, the existence and uniqueness of solutions are proved. The main results are demonstrated by an example.

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Keywords: Singular fractional differential equation; Riemann–Liouville fractional derivative; Caputo fractional derivative; Schauder's fixed point theorem; Banach contraction principle; Hyers–Ulam stability

1 Introduction

Fractional differential equations (FDEs) arise in different branches of applied mathematics. Recently, it has been evidently realized that the mathematical models of systems and processes involving fractional order derivatives often appear in the fields of physics, chemistry, biology, viscoelasticity, control hypothesis, speculation, fluid dynamics, hydrodynamics, aerodynamics, information processing system networking, notable and picture processing; see the remarkable monographs [22, 24, 37]. The study of fractional order differential models has been associated with the fact that they provide a more accurate description of real phenomena than the counterpart integer order models. The reason behind this intensive interest is that FDEs provide practical tools for the depictions of memory and inherited properties of many materials and processes. As a result, FDEs have experienced significant developments in recent years; see [5, 6, 13, 28, 30, 32, 34, 35, 41, 45] for further details.

One of the most interesting research areas in the field of FDEs, which has attracted great consideration amongst researchers, is dedicated to the existence theory of the solutions of fractional models. The aforesaid part has been extensively explored for integer order differential equations (DEs). However, for arbitrary order DEs, there are still many aspects which need further study and research. Different mathematicians have explored the existence of solutions of FDEs in various directions [1–3, 9, 12, 17, 21, 27]. Another imperative and more remarkable area of research, which has recently attracted more attention, is committed to the stability analysis of DEs of integer and noninteger order. The first effort was initiated by Ulam himself and later was confirmed by Hyers in [19]. That is why this type of stability is referred to as Ulam–Hyers (UH) stability. Further, Rassias introduced the Ulam–Hyers–Rassias (UHR) stability; see some recently reported stability results in the sense of Ulam [4, 7, 20, 29, 33, 38, 42–44, 47–50]. It is to be noted that the above said areas of interest (existence and stability) have been often deliberated within the settings of Riemann–Liouville and Caputo derivatives. The above results can also be studied for Caputo-Fabrizio derivative [10, 11, 14–16, 23].

In the solutions of differential and integral equations, the concept of fixed point theory is very important. Different fixed point theorems, which have numerous applications in the mentioned equations, are presented. Few important fixed point theorems can be found in [18, 39, 46].

For the sake of completeness and comparison, we assemble herein some relevant results. In [26], Liu *et al.* investigated the existence results of fractional Sturm–Liouville boundary value problem:

$$\begin{cases} D_{0^+}^{\sigma}(\Phi(\rho(t)))(D_{0^+}^{\sigma'}u(t)) + f(t,u(t),D_{0^+}^{\sigma'}u(t)) = 0, \quad t\in(0,1), 0<\sigma,\sigma'<1, \\ a_0\lim_{t\to 0} t^{1-\sigma'}u(t) - b_0\lim_{t\to 0}\Phi^{-1}(t^{1-\sigma})\rho(t)D_{0^+}^{\sigma'}u(0) = 0, \\ c_0\lim_{t\to 1}\Phi^{-1}(t^{1-\sigma})\rho(t)D_{0^+}^{\sigma'}u(0) + d_0\lim_{t\to 1} t^{1-\sigma'}u(t) = 0, \end{cases}$$

where $D_{0^+}^{\sigma}$, $D_{0^+}^{\sigma'}$ denote the Riemann–Liouville fractional derivatives of order σ and σ' respectively, $a_0, b_0, c_0, d_0 \in \mathcal{R}$, while $\rho : (0,1) \to \mathcal{R}^+$ is a given continuous function. The function $f : (0,1)\times\mathcal{R}\times\mathcal{R}\to\mathcal{R}$ is a quasi-Carathéodory function which may be singular at the points $t = 0, 1$. The p-Laplacian operator Φ is defined as $\Phi(s) = |s|^{p-2}$ with inverse operator represented by $\Phi^{-1}(s) = |s|^{q-2}$, where $\frac{1}{p}+\frac{1}{q} = 1$. The analysis relies on the well-known Leray–Schauder alternative principle.

In [25], Li studied the existence of a positive solution to the fractional differential equation involving integral boundary conditions with nonlinear p-Laplacian operator of the form:

$$\begin{cases} D^{\alpha}(\phi_p({}^cD^{\sigma}u(t))) + f(t,u(t)) = 0, \quad t\in(0,1), 2<\alpha,\sigma<3, \\ \phi_p({}^cD^{\sigma}u(0)) = [\phi_p({}^cD^{\sigma}u(0))]' = ({}^cD^{\sigma}u(1)) = 0, \\ u''(0) = u'(1) = 0, \\ \lambda u(0) + \zeta u'(0) = \int_0^1 u(t)\varphi(t)\,dt, \end{cases}$$

where D^{α}, ${}^cD^{\sigma}$ denote the Riemann–Liouville and Caputo fractional derivatives of order α and σ, respectively, and $\phi(s) = |s|^{p-2}, p>1$. The function φ satisfies $\varphi : [0,1]\to\mathcal{R}^+$ with $\varphi\in L^1[0,1]$, $\int_0^1\varphi(t)\,dt > 0$ and $\int_0^1 t\varphi(t)\,dt > 0$, $a,b\in\mathcal{R}^+$ with $\int_0^1\varphi(t)\,dt < a$, where $b>a$, and $f : [0,1]\times(0,\infty)\to(0,\infty)$ is continuous. By employing the Avery–Henderson fixed point theorem, new results have been obtained for the above problem.

In [8], Alkhazzan *et al.* studied the existence and stability results for a class of nonlinear fractional differential equations with singularity of the form:

$$\begin{cases} {}^{c}D^{\sigma}[\phi_p D^{\beta}u(t)] + F_1(t)\psi_1(t,u(t)) = 0, \\ ([\phi_p D^{\beta}u(0)])^{(j)} = 0, \quad j = 0,1,\dots,m-1, \\ I^{k-\beta}(u(0)) = 0, \quad k = 2,3,\dots,m, \\ D^{\delta}(u(1)) = 0, \end{cases}$$

where D^{β} and ${}^{c}D^{\sigma}$ respectively represent the Riemann–Liouville and Caputo fractional derivatives of order β and σ, $m-1 < \beta$, $\sigma \le m$, $m \in \{2,3,\dots\}$, $1 < \delta \le 2$. The nonlinear p-Laplacian operator ϕ_p has expression in the form $\phi_p(\theta) = \frac{\theta}{|\theta|^{2-p}}$, $\phi_p(0) = 0$, with inverse ϕ_q, that is, $\phi_q = \phi_p^{-1}$ such that $\frac{1}{p} + \frac{1}{q} = 1$. The nonlinear function $\psi_1 \in \mathcal{C}[0,1]$ is continuous and perhaps singular with respect to t, u. Some classical fixed point theorems are utilized to prove the main results.

The objective of this paper is to use the concepts mentioned in [8] to examine the existence, uniqueness as well as different kinds of Hyers–Ulam stability for the solution of the nonlinear coupled implicit switched singular system of fractional differential equations with singularities of the form:

$$\begin{cases} {}^{c}D^{\sigma}[\phi_p D^{\beta}u(t)] + \mathcal{F}_1(t)\psi_1(t,u(t),{}^{c}D^{\rho}[\phi_p D^{\beta}v(t)]) = 0, \quad t \in \mathrm{J} =]0,1[, \\ {}^{c}D^{\rho}[\phi_p D^{\beta}v(t)] + \mathcal{F}_2(t)\psi_2(t,{}^{c}D^{\sigma}[\phi_p D^{\beta}u(t)],v(t)) = 0, \quad t \in \mathrm{J}, \\ ([\phi_p D^{\beta}u(0)])^{(j)} = 0, \quad j = 0,1,\dots,m-1, \\ ([\phi_p D^{\beta}v(0)])^{(j)} = 0, \quad j = 0,1,\dots,m-1, \\ I^{k-\beta}(u(0)) = I^{k-\beta}(v(0)) = 0, \quad k = 2,3,\dots,m, \\ D^{\delta}(u(1)) = D^{\delta}(v(1)) = 0, \end{cases} \tag{1.1}$$

where D^{β} and ${}^{c}D^{\sigma}$ respectively denote the Riemann–Liouville and Caputo fractional derivatives of order β and σ, $m-1 < \beta$, $\sigma \le m$, $m \in \{2,3,\dots\}$, $1 < \delta \le 2$, and $\mathcal{F}_1(\cdot)$, $\mathcal{F}_2(\cdot)$ are linear and bounded operators on $\mathcal{R}$. Furthermore, the nonlinear p-Laplacian operator ϕ_p has expression in the form $\phi_p(\theta) = \frac{\theta}{|\theta|^{2-p}}$, $\phi_p(0) = 0$, with inverse operator ϕ_q, that is, $\phi_q = \phi_p^{-1}$ such that $\frac{1}{p} + \frac{1}{q} = 1$. The nonlinear functions $\psi_1, \psi_2 \in \mathcal{C}[0,1]$ are continuous and perhaps singular with respect to t, u, v.

The current work is organized as follows: In Sect. 2, we present some basic definitions and assertions that will be used in the subsequent sections. In Sect. 3 we state and prove our main existence results. We discuss the Ulam stability of the proposed problem in Sect. 4. Concrete example is illustrated to demonstrate consistency with the proposed results.

2 Basic definitions and assertions

Here we state some fundamental facts, definitions, and lemmas which will be used throughout this paper.

Let $\mathrm{C}(\mathrm{J},\mathcal{X})$ be the space of all continuous functions of the form $u(t) : \mathrm{J} \to \mathcal{X}$, $t \in \mathrm{J}$. It is obvious that $\mathrm{C}(\mathrm{J},\mathcal{X})$ is a Banach space with norm $\|u\| = \max\{|u(t)|, t \in \mathrm{J}\}$. Further, we understand that $\mathrm{C}(\mathrm{J},\mathcal{X}) \times \mathrm{C}(\mathrm{J},\mathcal{X})$ is a Banach space with norm $\|(u,v)\| = \|u\| + \|v\|$.

Definition 2.1 ([22]) Let $\alpha \in \mathcal{R}^+$. Then the noninteger order integral in the Riemann–Liouville sense for a function $\theta : \mathrm{J} \to \mathcal{R}$ is given as

$$\mathcal{I}^\alpha \theta(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \theta(s)\, ds,$$

such that the integral on the right-hand side is pointwise defined on $\mathcal{R}^+$.

Definition 2.2 ([22]) Let $\alpha \in (n-1, n]$ with $n-1 = [\alpha]$. Then the noninteger order derivative in the Caputo sense of $\theta : [a,b] \to \mathcal{R}$ is stated as

$$\frac{d^\alpha}{dt^\alpha}\theta(t) = \int_a^t \frac{(t-s)^{n-\alpha-1}}{\Gamma(n-\alpha)} \left(\frac{d^n}{ds^n}\theta(s)\right) ds.$$

In particular, if p is defined on the interval $[a,b]$ and $\alpha \in (0,1]$, then

$$\frac{d^\alpha}{dt^\alpha}\theta(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{\theta'(s)}{(t-s)^\alpha}\, ds, \quad \text{where } \theta'(s) = \frac{d\theta(s)}{ds}.$$

It is to be noted that the integral on the right-hand side is pointwise defined on $\mathcal{R}^+$.

Definition 2.3 ([22]) The noninteger order derivative in the Riemann–Liouville sense having order σ for a function $\theta : (0,\infty) \to \mathcal{R}^+$ is defined by

$$D_{1^+}^\sigma \theta(t) = \frac{1}{\Gamma(n-\sigma)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\sigma-1} \theta(s)\, ds, \quad n-1 < \sigma < n = 1 + \lceil \sigma \rceil,$$

where the integral on the right-hand side is pointwise continuous and defined on the interval $(0,\infty)$ and $\lceil \sigma \rceil$ is the integer part of σ.

Lemma 2.4 ([8]) *Let $\sigma \in (m-1, m]$, $\theta \in \mathcal{C}^{m-1}$, and ${}^cD^\sigma$ be the Caputo fractional derivative. Then*

$${}^cI^\sigma {}^cD^\sigma \theta(t) = \theta(t) + b_1 + b_2 t + b_3 t^2 + \cdots + b_m t^{m-1},$$

where $b_i \in \mathcal{R}$, $i = 1, 2, \ldots m$, $m = [\sigma] + 1$.

Lemma 2.5 ([8]) *Let $\sigma \in (m-1, m]$, $\theta \in \mathcal{C}^{m-1}$, and D^σ be the Riemann–Liouville fractional derivative. Then*

$$I^\sigma D^\sigma \theta(t) = \theta(t) + c_1 t^{\sigma-1} + c_2 t^{\sigma-2} + \cdots + c_m t^{\sigma-m},$$

where $c_i \in \mathcal{R}$, $i = 1, 2, \ldots m$, $m = [\sigma] + 1$.

Lemma 2.6 ([36], Arzelä–Ascoli theorem) *An operator $\mathcal{H} : \mathcal{B}_r \cap (\bar{\Omega}_1/\Omega_2) \to \mathcal{B}_r$ is said to be compact if and only if $\mathcal{H}$ is uniformly bounded and discontinuous.*

Lemma 2.7 (Schauder fixed point theorem [39]) *Let $\mathcal{S} \neq \emptyset$ be a convex and closed subset of the Banach space $\mathcal{X}$. Let $\phi : \mathcal{S} \to \mathcal{S}$ be a continuous operator such that $\phi(\mathcal{S})$ is a relatively compact subset of $\mathcal{X}$. Then the operator system ϕ has at least one fixed point in $\mathcal{S}$.*

Lemma 2.8 ([27]) *Let $\phi_p : \mathcal{R} \to \mathcal{R}$ be a nonlinear p-Laplacian operator, that is, $\phi_p(\zeta) = |\zeta|^{p-2}\zeta$, $\zeta \in \mathcal{R}$. Then*

$$\frac{d\phi_p}{d\zeta} = (p-1)|\zeta|^{p-2}.$$

Some basic properties of the operator ϕ_p are as follows:

($\mathcal{A}_1$) If $1 < p \le 2$, $\zeta_1, \zeta_2 > 0$, $0 < \varrho \le |\zeta_1|, |\zeta_2|$, then

$$\left|\phi_p(\zeta_1) - \phi_p(\zeta_2)\right| \le (p-1)\varrho^{p-2}|\zeta_1 - \zeta_2|.$$

($\mathcal{A}_2$) For $p > 2$, $|\zeta_1, \zeta_2| \le \varrho*$, then

$$\left|\phi_p(\zeta_1) - \phi_p(\zeta_2)\right| \le (p-1)\varrho^{p-2}|\zeta_1 - \zeta_2|.$$

Lemma 2.9 ([31]) *Let $\mathcal{X}$ be a Banach space and $\mathcal{B} \subset \mathcal{X}$ be a nonempty, closed, and convex set. If a map $\mathcal{H} : \mathcal{B} \to \mathcal{B}$ is compact, then $\mathcal{H}$ has a fixed point.*

Definition 2.10 (Urs [40], Definition 2) Let $\mathcal{X}$ be a Banach space such that $\Upsilon_1, \Upsilon_2 : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ are two operators. Then the system

$$\begin{cases} y(t) = \Upsilon_1(y,z)(t), \\ z(t) = \Upsilon_2(y,z)(t), \end{cases} \tag{2.1}$$

is said to be Hyers–Ulam stable if there exist constants $F_j (j = 1,2,3,4) > 0$ with $\alpha_j (j = 1,2) > 0$, and for each solution $(y^*, z^*) \in \mathcal{X} \times \mathcal{X}$ of the inequalities

$$\begin{cases} \|y^* - \Phi(y^*,z^*)\| \le \alpha_1, \\ \|z^* - \Psi(y^*,z^*)\| \le \alpha_2, \end{cases} \tag{2.2}$$

there exists a solution $(\widetilde{y}, \widetilde{z}) \in \mathcal{X} \times \mathcal{X}$ of system (2.1), which satisfies

$$\begin{cases} \|y^* - \widetilde{y}\| \le F_1\alpha_1 + F_2\alpha_2, \\ \|z^* - \widetilde{z}\| \le F_3\alpha_1 + F_4\alpha_2. \end{cases} \tag{2.3}$$

Definition 2.11 Let ν_J, where $J = 1,2,\ldots,k$, be the eigenvalues (real or complex) of a matrix $F \in \varpi^{k\times k}$. Then the term spectral radius $\alpha(F)$ of $F \in \varpi^{k\times k}$ is defined as

$$\alpha(F) = \max\left\{|\nu_J| \text{ for } J = 1,2,\ldots,k\right\}.$$

It is well known that the system corresponding to matrix $F \in \varpi^{k\times k}$ will converge to zero if $\alpha(F)$ is less than one.

Theorem 2.12 (Urs [40], Theorem 4) *Consider a Banach space $\mathcal{X}$ and define two operators $\Upsilon_1, \Upsilon_2 : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ such that*

$$\begin{cases} \|\Upsilon_1(\mathbf{y},\mathbf{z}) - \Upsilon_1(\mathbf{y}^*,\mathbf{z}^*)\| \le F_1\|\mathbf{y} - \mathbf{y}^*\| + F_2\|\mathbf{z} - \mathbf{z}^*\|, \\ \|\Upsilon_2(\mathbf{y},\mathbf{z}) - \Upsilon_2(\mathbf{y}^*,\mathbf{z}^*)\| \le F_3\|\mathbf{y} - \mathbf{y}^*\| + F_4\|\mathbf{z} - \mathbf{z}^*\|. \end{cases}$$

If the spectral radius of the matrix

$$F = \begin{pmatrix} F_1 & F_2 \\ F_3 & F_4 \end{pmatrix}$$

is less than one, then the fixed points correlated with operating system (2.1) *are Hyers–Ulam stable.*

3 Existence results

This section is devoted to investigating the existence of solutions. The first result converts the proposed problem into an equivalent integral form by the help of Green's function.

Theorem 3.1 *Let* ψ_1, ψ_2 *be integrable functions and* $u, v \in C(J, \mathcal{X})$ *satisfy* (1.1). *Then, for* $3 < \sigma, \beta, \rho \leq m$, $m \geq 4$, *the solution of the switched coupled system*

$$\begin{cases} {}^cD^\sigma[\phi_p D^\beta u(t)] + \mathcal{F}_1(t)\psi_1(t, u(t), {}^cD^\rho[\phi_p D^\beta v(t)]) = 0, & t \in J, \\ {}^cD^\rho[\phi_p D^\beta v(t)] + \mathcal{F}_2(t)\psi_2(t, {}^cD^\sigma[\phi_p D^\beta u(t)], v(t)) = 0, & t \in J, \\ ([\phi_p D^\beta u(0)])^{(j)} = 0, & j = 0, 1, \ldots, m-1, \\ ([\phi_p D^\beta v(0)])^{(j)} = 0, & j = 0, 1, \ldots, m-1, \\ I^{k-\beta}(u(0)) = I^{k-\beta}(v(0)) = 0, & k = 2, 3, \ldots, m, \\ D^\delta(u(1)) = D^\delta(v(1)) = 0 \end{cases} \tag{3.1}$$

is equivalent to the integral equations

$$u(t) = \int_0^1 \mathcal{G}^\beta(t,s)\phi_q\left(\int_0^s (s-\tau)^{\sigma-1}\mathcal{F}_1(\tau)\psi_1\big(\tau, u(\tau), {}^cD^\sigma\big(\phi_p D^\beta u(\tau)\big)\big)\,d\tau\right) ds$$

and

$$v(t) = \int_0^1 \mathcal{G}^\beta(t,s)\phi_q\left(\int_0^s (s-\tau)^{\sigma-1}\mathcal{F}_2(\tau)\psi_2\big(\tau, {}^cD^\rho\big(\phi_p D^\beta u(\tau)\big), v(\tau)\big)\,d\tau\right) ds,$$

where $\mathcal{G}^\beta(t,s)$ *is Green's function given by*

$$\mathcal{G}^\beta(t,s) = \begin{cases} \frac{-(t-s)^{\beta-1}}{\Gamma(\beta)} - \frac{t^{\beta-1}(1-s)^{\beta-\delta-1}}{\Gamma(\beta)}, & s \leq t \leq 1, t \in J = (0,1), \\ \frac{-t^{\beta-1}(1-s)^{\beta-\delta-1}}{\Gamma(\beta)}, & t \leq s \leq 1. \end{cases} \tag{3.2}$$

Proof Let $u, v \in \mathcal{C}(J, \mathcal{X})$ be the solution of (3.1), then

$$\begin{cases} {}^cD^\sigma[\phi_p D^\beta u(t)] + \mathcal{F}_1(t)\psi_1(t, u(t), {}^cD^\rho[\phi_p D^\beta v(t)]) = 0, & t \in J, \\ {}^cD^\rho[\phi_p D^\beta v(t)] + \mathcal{F}_2(t)\psi_2(t, {}^cD^\sigma[\phi_p D^\beta u(t)], v(t)) = 0, & t \in J, \\ ([\phi_p D^\beta u(0)])^{(j)} = 0, & j = 0, 1, \ldots, m-1, \\ ([\phi_p D^\beta v(0)])^{(j)} = 0, & j = 0, 1, \ldots, m-1, \\ I^{k-\beta}(u(0)) = I^{k-\beta}(v(0)) = 0, & k = 2, 3, \ldots, m, \\ D^\delta(u(1)) = D^\delta(v(1)) = 0. \end{cases}$$

Since

$$ {}^cD^\sigma\left(\phi_p\big(D^\beta u(t)\big)+\mathcal{F}_1(t)\psi_1\big(t,u(t),{}^cD^\rho\big[\phi_pD^\beta v(t)\big]\big)=0,\tag{3.3}$$

where $m-1<\sigma<m$, $m-1<\beta\leq m$, $t\in \mathrm{J}$. Using Lemma 2.4, we have

$$\begin{aligned}\phi_p\big(D^\beta u(t)\big)&=-b_0-b_1t-b_2t^2-\cdots-b_{m-1}t^{m-1}\\&\quad-\frac{1}{\Gamma(\sigma)}\int_0^t(t-s)^{\sigma-1}\mathcal{F}_1(s)\psi_1\big(s,u(s),{}^cD^\rho\big[\phi_pD^\beta v(s)\big]\big).\end{aligned}\tag{3.4}$$

The $(\phi_p(D^\beta u(t)))^{(0)}|_{t=0}=0$ implies that $-b_0=0$ or $b_0=0$. Therefore (3.4) becomes

$$\begin{aligned}\phi_p\big(D^\beta u(t)\big)&=-b_1t-b_2t^2-\cdots-b_{m-1}t^{m-1}\\&\quad-\frac{1}{\Gamma(\sigma)}\int_0^t(t-s)^{\sigma-1}\mathcal{F}_1(s)\psi_1\big(s,u(s),{}^cD^\rho\big[\phi_pD^\beta v(s)\big]\big).\end{aligned}\tag{3.5}$$

Differentiating (3.5) with respect to t, we have

$$\begin{aligned}\big(\phi_p\big(D^\beta u(t)\big)\big)'&=-b_1-2b_2t-\cdots-(m-1)b_{m-1}t^{m-2}\\&\quad-\frac{1}{\Gamma(\sigma-1)}\int_0^t(t-s)^{\sigma-2}\mathcal{F}_1(s)\psi_1\big(s,u(s),{}^cD^\rho\big[\phi_pD^\beta v(s)\big]\big).\end{aligned}$$

Using the condition $(\phi_p(D^\beta u(t)))'|_{t=0}=0$ implies that $b_1=0$. Similarly, by applying the conditions $(\phi_p(D^\beta u(t)))^{(j)}|_{t=0}=0$, we get $b_j=0$, $\forall j=2,3,\dots m$. Therefore, (3.4) becomes

$$\begin{aligned}D^\beta u(t)&=-\phi_p^{-1}\big(I^\sigma\big(\mathcal{F}_1(t)\psi_1\big(t,u(t),{}^cD^\rho\big(\phi_pD^\beta v(t)\big)\big)\big)\\&=-\phi_q\big(I^\sigma\mathcal{F}_1(t)\psi_1\big(t,u(t),{}^cD^\rho\big(\phi_pD^\beta v(t)\big)\big).\end{aligned}$$

Applying I^β to both sides and using Lemma 2.5, we have

$$\begin{aligned}u(t)&=-d_1t^{\beta-1}-d_2t^{\beta-2}-\cdots-d_mt^{\beta-m}\\&\quad-\frac{1}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}\phi_q\big(I^\sigma\mathcal{F}_1(t)\psi_1\big(t,u(t),{}^cD^\rho\big(\phi_pD^\beta v(t)\big)\big).\end{aligned}\tag{3.6}$$

Putting $I^{k-\beta}u(t)|_{t=0}=0$ for $k=2,3,\dots,m$, we obtain $d_2=d_3=\cdots=d_m=0$, and using $D^\delta u(t)|_{t=1}=0$, we get

$$d_1=-\frac{\Gamma(\beta-\delta)}{\Gamma(\beta)}\phi_q\big(I^\sigma\mathcal{F}_1(t)\psi_1\big(t,u(t),{}^cD^\rho\big(\phi_pD^\beta v(t)\big)\big)|_{t=1}.$$

It follows that

$$\begin{aligned}u(t)&=-\frac{1}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}\phi_q\left(\int_0^s(s-\tau)^{\sigma-1}\mathcal{F}_1(\tau)\psi_1\big(\tau,u(\tau),{}^cD^\rho\big(\phi_pD^\beta v(\tau)\big)\big)\,d\tau\right)ds\\&\quad-\frac{t^{\beta-1}}{\Gamma(\beta)}\int_0^1(1-s)^{\beta-\delta-1}\\&\quad\times\phi_q\left(\int_0^s(s-\tau)^{\sigma-1}\mathcal{F}_1(\tau)\psi_1\big(\tau,u(\tau),{}^cD^\rho\big(\phi_pD^\beta v(\tau)\big)\big)\,d\tau\right)ds\end{aligned}$$

$$= \int_0^1 \mathcal{G}^\beta(t,s)\phi_q\left(\int_0^s (s-\tau)^{\sigma-1}\mathcal{F}_1(\tau)\psi_1\left(\tau,u(\tau),{}^cD^\rho\left(\phi_p D^\beta v(\tau)\right)\right)d\tau\right)ds.$$

In a similar manner, we may conclude that

$$v(t) = \int_0^1 \mathcal{G}^\beta(t,s)\phi_q\left(\int_0^s (s-\tau)^{\sigma-1}\mathcal{F}_2(\tau)\psi_2\left(\tau,{}^cD^\rho\left(\phi_p D^\beta u(\tau)\right),v(\tau)\right)d\tau\right)ds, \tag{3.7}$$

where $\mathcal{G}^\beta(t,s)$ is the Green's function defined above. □

Lemma 3.2 ([8]) *The Green's function* $\mathcal{G}^\beta(t,s)$ *satisfies the following properties*:

($\mathcal{P}_1$) $\mathcal{G}^\beta(t,s) > 0$, $\forall 0 < s, t < 1$;

($\mathcal{P}_2$) $\mathcal{G}^\beta(t,s)$ *is a nondecreasing function and* $\max_{t\in(0,1)}\mathcal{G}^\beta(t,s) = \mathcal{G}^\beta(1,s)$;

($\mathcal{P}_3$) $t^{\beta-1}\max_{t\in(0,1)}\mathcal{G}^\beta(t,s) \le \mathcal{G}^\beta(t,s)$ *for* $0 < s, t < 1$.

We introduce the following assumptions:

($\mathbf{H}_1$) The functions $\psi_1,\psi_2 : \mathrm{J}\times\mathcal{X}\times\mathcal{X}\to\mathcal{X}$ are continuous, and $\forall u,v,\overline{u},\overline{v}\in\mathcal{X}$ and $t\in \mathrm{J}$, there exist $\mathcal{M}_{\psi_1},\mathcal{M}_{\psi_2},\mathcal{M}'_{\psi 1},\mathcal{M}'_{\psi_2} > 0$ such that

$$\left\|\psi_1(t,u,v)-\psi_1(t,\overline{u},\overline{v})\right\| \le \mathcal{M}_{\psi_1}\|u-\overline{u}\| + \mathcal{M}'_{\psi_1}\|v-\overline{v}\|$$

and

$$\left\|\psi_2(t,u,v)-\psi_2(t,\overline{u},\overline{v})\right\| \le \mathcal{M}_{\psi_2}\|u-\overline{u}\| + \mathcal{M}'_{\psi_2}\|v-\overline{v}\|.$$

($\mathbf{H}_2$) The functions $\psi_1,\psi_2 : J\times\mathcal{X}\times\mathcal{X}\to\mathcal{X}$ are completely continuous such that, $\forall u,v\in\mathcal{X}$ and $t\in \mathrm{J}$, there exist nondecreasing continuous linear functions $\mu_{\psi_1},\mu_{\psi_2} : \mathcal{R}^+\to\mathcal{R}$ such that

$$\left\|\psi_1(t,u,v)\right\| \le \phi_p\left(\mu_{\psi_1}\|u\| + \mu'_{\psi_1}\|v\|\right)$$

and

$$\left\|\psi_2(t,u,v)\right\| \le \phi_p\left(\mu_{\psi_2}\|u\| + \mu'_{\psi_2}\|v\|\right),$$

where

$$\sup\left\{\mu_{\psi_1}(t), t\in \mathrm{J}\right\} = \mu_{\psi_1}, \qquad \sup\left\{\mu_{\psi_2}(t), t\in \mathrm{J}\right\} = \mu_{\psi_2},$$
$$\sup\left\{\mu'_{\psi_1}(t), t\in \mathrm{J}\right\} = \mu'_{\psi_1}, \qquad \sup\left\{\mu'_{\psi_2}(t), t\in \mathrm{J}\right\} = \mu'_{\psi_2}.$$

($\mathbf{H}_3$) The functions $\mathcal{F}_1,\mathcal{F}_2 : (0,1)\to\mathcal{X}$ are nonzero and continuous with

$$\|\mathcal{F}_1\| = \max_{t\in \mathrm{J}}|\mathcal{F}_1| < \infty^+, \qquad \|\mathcal{F}_2\| = \max_{t\in \mathrm{J}}|\mathcal{F}_2| < \infty^+.$$

Let $\mathcal{B}_r\subset\mathbf{B} = \mathrm{C}(\mathrm{J},\mathcal{X})\times\mathrm{C}(\mathrm{J},\mathcal{X})$ be a cone of nonnegative functions of the form

$$\mathcal{B}_r = \left\{(u,v)\in\mathbf{B}, \min_{t\in \mathrm{J}}\left(u(t)+v(t)\right) \ge t^\beta\left\|(u,v)\right\|\right\},$$

and

$$\Omega(r) = \left\{ \|(u,v)\| < r, \|u\| < \frac{r}{2}, \|v\| < \frac{r}{2} \right\}, \quad \partial\Omega(r) = \left\{ \|(u,v)\| = r \right\}.$$

Consider the operator $\mathcal{H}^* = (\mathcal{H}^*{}_1, \mathcal{H}^*{}_2) : \mathcal{B}_r/(0,0) \to \mathbf{B}$, where $\mathcal{H}^*{}_1$, $\mathcal{H}^*{}_2$ are defined as follows:

$$\begin{cases} \mathcal{H}^*{}_1(u,v)(t) \\ \quad = \int_0^1 \mathcal{G}^\beta(t,s)\phi_q(\frac{1}{\sigma}\int_0^s (s-\tau)^{\sigma-1}\mathcal{F}_1(\tau)\psi_1(\tau,u(\tau),{}^cD^\rho(\phi_p D^\beta v(\tau)))\,d\tau)\,ds, \\ \mathcal{H}^*{}_2(u,v)(t) \\ \quad = \int_0^1 \mathcal{G}^\beta(t,s)\phi_q(\frac{1}{\rho}\int_0^s (s-\tau)^{\rho-1}\mathcal{F}_2(\tau)\psi_2(\tau,{}^cD^\sigma(\phi_p D^\beta u(\tau)),v(\tau))\,d\tau)\,ds. \end{cases} \tag{3.8}$$

Theorem 3.3 *Let assumptions* $(\mathbf{H}_1)$ *to* $(\mathbf{H}_3)$ *hold. Then* (1.1) *has at least one solution.*

Proof For any $(u,v) \in \overline{\Omega(r_2)}/\Omega(r_1)$, and using Lemma 3.2, we have

$$\begin{aligned} &\mathcal{H}^*{}_1(u,v)(t) \\ &\quad = \int_0^1 \mathcal{G}^\beta(t,s)\phi_q\left(\frac{1}{\sigma}\int_0^s (s-\tau)^{\sigma-1}\mathcal{F}_1(\tau)\psi_1\left(\tau,u(\tau),{}^cD^\rho\left(\phi_p D^\beta v(\tau)\right)\right)d\tau\right)ds \\ &\quad \le \int_0^1 \mathcal{G}^\beta(1,s)\phi_q\left(\frac{1}{\sigma}\int_0^s (s-\tau)^{\sigma-1}\mathcal{F}_1(\tau)\psi_1\left(\tau,u(\tau),{}^cD^\rho\left(\phi_p D^\beta v(\tau)\right)\right)d\tau\right)ds \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} &\mathcal{H}^*{}_1(u,v)(t) \\ &\quad = \int_0^1 \mathcal{G}^\beta(t,s)\phi_q\left(\frac{1}{\sigma}\int_0^s (s-\tau)^{\sigma-1}\mathcal{F}_1(\tau)\psi_1\left(\tau,u(\tau),{}^cD^\rho\left(\phi_p D^\beta v(\tau)\right)\right)d\tau\right)ds \\ &\quad \ge t^{\beta-1}\int_0^1 \mathcal{G}^\beta(1,s) \\ &\qquad \times \phi_q\left(\frac{1}{\sigma}\int_0^s (s-\tau)^{\sigma-1}\mathcal{F}_1(\tau)\psi_1\left(\tau,u(\tau),{}^cD^\rho\left(\phi_p D^\beta v(\tau)\right)\right)d\tau\right)ds. \end{aligned} \tag{3.10}$$

By the help of inequalities (3.7) and (3.8), we have

$$\mathcal{H}^*{}_1(u,v)(t) \ge t^{\beta-1}\left\|\mathcal{H}^*{}_1(u,v)(t)\right\|. \tag{3.11}$$

Similarly, we may obtain

$$\mathcal{H}^*{}_2(u,v)(t) \ge t^{\beta-1}\left\|\mathcal{H}^*{}_2(u,v)(t)\right\|. \tag{3.12}$$

Combining (3.11) and (3.12), we get

$$\mathcal{H}^*(u,v)(t) \ge t^{\beta-1}\left\|\mathcal{H}^*(u,v)(t)\right\|.$$

Thus $\mathcal{H}^* : \overline{\Omega(r_2)}/\Omega(r_1) \to \mathbf{B}$ is closed.

For the uniform boundedness of the operator $\mathcal{H}^*$, we consider

$$\begin{aligned}
&\left\|\mathcal{H}^*(u,v)(t)\right\| \\
&\quad= \sup_{t\in J}\left|\mathcal{H}^*(u,v)(t)\right| \\
&\quad= \sup_{t\in J}\left|\left(\mathcal{H}^*{}_1,\mathcal{H}^*{}_2\right)(u,v)(t)\right| \\
&\quad\le \sup_{t\in J}|(\mathcal{H}^*{}_1(u,v)(t)| + \sup_{t\in J}\left|\mathcal{H}^*{}_2(u,v)(t)\right| \\
&\quad\le \sup_{t\in J}\left|\int_0^1 \mathcal{G}^\beta(t,s)\right. \\
&\qquad \left.\times \phi_q\left(\frac{1}{\Gamma(\sigma)}\int_0^s (s-\tau)^{\sigma-1}\mathcal{F}_1(\tau)\psi_1\left(\tau,u(\tau),{}^cD^\rho\left(\phi_p D^\beta v(\tau)\right)\right)d\tau\right)ds\right| \\
&\qquad + \sup_{t\in J}\left|\int_0^1 \mathcal{G}^\beta(t,s)\right. \\
&\qquad \left.\times \phi_q\left(\frac{1}{\Gamma(\rho)}\int_0^s (s-\tau)^{\rho-1}\mathcal{F}_2(\tau)\psi_2\left(\tau,{}^cD^\sigma\left(\phi_p D^\beta u(\tau)\right),v(\tau)\right)d\tau\right)ds\right| \\
&\quad\le \int_0^1 \left|\mathcal{G}^\beta(1,s)\right| \\
&\qquad \times \phi_q\left(\frac{1}{\Gamma(\sigma)}\int_0^s (s-\tau)^{\sigma-1}\left\|\mathcal{F}_1(\tau)\right\|\left\|\psi_1\left(\tau,u(\tau),{}^cD^\rho\left(\phi_p D^\beta v(\tau)\right)\right)\right\|d\tau\right)ds \\
&\qquad + \int_0^1 \left|\mathcal{G}^\beta(1,s)\right| \\
&\qquad \times \phi_q(\frac{1}{\Gamma(\rho)}\int_0^s (s-\tau)^{\rho-1}\left\|\mathcal{F}_2(\tau)\right\|\left\|\psi_2\left(\tau,{}^cD^\sigma\left(\phi_p D^\beta u(\tau),v(\tau)\right)\right)\right\|d\tau)\,ds \\
&\quad\le \left(\frac{1}{\Gamma(\beta+1)}-\frac{1}{\Gamma(\beta-\delta)\Gamma(\beta)}\right)\left[\frac{1}{\Gamma(\sigma+1)}\right]^{q-1} \\
&\qquad \times \|\mathcal{F}_1\|^{q-1}\left(\frac{\mu_{\psi_1}\|u\| + \mu'_{\psi_1}\mu'_{\psi_2}\|\mathcal{F}_2\|\|v\|}{1-\|\mathcal{F}_1\|\|\mathcal{F}_2\|\mu'_{\psi_1}\mu_{\psi_2}}\right) \\
&\qquad + \left(\frac{1}{\Gamma(\beta+1)}-\frac{1}{\Gamma(\beta-\delta)\Gamma(\beta)}\right)\left[\frac{1}{\Gamma(\rho+1)}\right]^{q-1} \\
&\qquad \times \|\mathcal{F}_2\|^{q-1}\left(\frac{\mu_{\psi_1}\mu_{\psi_2}\|\mathcal{F}_1\|\|u\| + \mu_{\psi_2}\|v\|}{1-\|\mathcal{F}_1\|\|\mathcal{F}_2\|\mu'_{\psi_1}\mu_{\psi_2}}\right) \\
&\quad= \left(\frac{1}{\Gamma(\beta+1)}-\frac{1}{\Gamma(\beta-\delta)\Gamma(\beta)}\right)\left[\left(\frac{1}{\Gamma(\sigma+1)}\right)^{q-1}\right. \\
&\qquad \times \|\mathcal{F}_1\|^{q-1}\left(\frac{\mu_{\psi_1}\|u\| + \mu_{\psi_1}\mu'_{\psi_2}\|\mathcal{F}_2\|\|v\|}{1-\|\mathcal{F}_1\|\|\mathcal{F}_2\|\mu'_{\psi_1}\mu_{\psi_2}}\right) \\
&\qquad \left.+ \left(\frac{1}{\Gamma(\rho+1)}\right)^{q-1}\|\mathcal{F}_2\|^{q-1}\left(\frac{\mu_{\psi_1}\mu_{\psi_2}\|\mathcal{F}_1\|\|u\| + \mu'_{\psi_2}\|v\|}{1-\|\mathcal{F}_1\|\|\mathcal{F}_2\|\mu'_{\psi_1}\mu_{\psi_2}}\right)\right] \\
&\quad< \infty.
\end{aligned}$$

Hence $\mathcal{H}^*$ is a uniformly bounded operator.

Now, we show that the operator $\mathcal{H}^*$ is continuous and compact. For this reason, we construct a sequence $\xi_n = (u_n, v_n)$ such that $(u_n, v_n) \to (u, v)$ as $n \to \infty$. Therefore, we have

$$
\begin{aligned}
&\big\| \big(\mathcal{H}^*(u_n, v_n) - \mathcal{H}^*(u, v)\big) \big\| \\
&\quad = \big\| \big(\mathcal{H}^*{}_1, \mathcal{H}^*{}_2\big)(u_n, v_n) - \big(\mathcal{H}^*{}_1, \mathcal{H}^*{}_2\big)(u, v)\big) \big\| \\
&\quad \le \big\| \big(\mathcal{H}^*{}_1(u_n, v_n) - \mathcal{H}^*{}_1(u, v)\big) \big\| + \big\| \big(\mathcal{H}^*{}_2(u_n, v_n) - \mathcal{H}^*{}_2(u, v)\big) \big\| \\
&\quad = \sup_{t\in J} \bigg| \int_0^1 \mathcal{G}^\beta(t,s) \\
&\qquad \times \phi_q\bigg(\frac{1}{\Gamma(\sigma)} \int_0^s (s-\tau)^{\sigma-1} \mathcal{F}_1(\tau)\psi_1\big(\tau, u_n(\tau), {}^cD^\rho\big(\phi_p D^\beta v_n(\tau)\big)\big)\, d\tau \bigg)\, ds \\
&\qquad - \int_0^1 \mathcal{G}^\beta(t,s)\phi_q\bigg(\frac{1}{\Gamma(\sigma)} \int_0^s (s-\tau)^{\sigma-1} \mathcal{F}_1(\tau)\psi_1\big(\tau, u(\tau), {}^cD^\rho\big(\phi_p D^\beta v(\tau)\big)\big)\, d\tau \bigg)\, ds \bigg| \\
&\qquad + \sup_{t\in J} \bigg| \int_0^1 \mathcal{G}^\beta(t,s) \\
&\qquad \times \phi_q\bigg(\frac{1}{\Gamma(\rho)} \int_0^s (s-\tau)^{\rho-1} \mathcal{F}_2(\tau)\psi_2\big(\tau, {}^cD^\sigma\big(\phi_p D^\beta u_n(\tau)\big), v_n(\tau)\big)\, d\tau \bigg)\, ds \\
&\qquad - \int_0^1 \mathcal{G}^\beta(t,s)\phi_q\bigg(\frac{1}{\Gamma(\rho)} \int_0^s (s-\tau)^{\rho-1} \mathcal{F}_2(\tau)\psi_2\big(\tau, {}^cD^\sigma\big(\phi_p D^\beta u(\tau)\big), v(\tau)\big)\, d\tau \bigg)\, ds \bigg| \\
&\quad \le \int_0^1 \big|\mathcal{G}^\beta(t,s)\big| \\
&\qquad \times \bigg\{ \bigg\| \phi_q\bigg(\frac{1}{\Gamma(\sigma)} \int_0^s (s-\tau)^{\sigma-1} \mathcal{F}_1(\tau)\psi_1\big(\tau, u_n(\tau), {}^cD^\rho\big(\phi_p D^\beta v_n(\tau)\big)\big)\, d\tau \bigg)\, ds \\
&\qquad - \phi_q\bigg(\frac{1}{\Gamma(\sigma)} \int_0^s (s-\tau)^{\sigma-1} \mathcal{F}_1(\tau)\psi_1\big(\tau, u(\tau), {}^cD^\rho\big(\phi_p D^\beta v(\tau)\big)\big)\, d\tau \bigg)\, ds \bigg\| \bigg\} \\
&\qquad + \int_0^1 \big|\mathcal{G}^\beta(t,s)\big| \\
&\qquad \times \bigg\{ \bigg\| \phi_q\bigg(\frac{1}{\Gamma(\rho)} \int_0^s (s-\tau)^{\rho-1} \mathcal{F}_2(\tau)\psi_2\big(\tau, {}^cD^\sigma\big(\phi_p D^\beta u_n(\tau)\big), v_n(\tau)\big)\, d\tau \bigg)\, ds \\
&\qquad - \phi_q\bigg(\frac{1}{\Gamma(\rho)} \int_0^s (s-\tau)^{\rho-1} \mathcal{F}_2(\tau)\psi_2\big(\tau, {}^cD^\sigma\big(\phi_p D^\beta u(\tau)\big), v(\tau)\big)\, d\tau \bigg)\, ds \bigg\| \bigg\} \\
&\quad \le (q-1)\varrho^2 \int_0^1 \big|\mathcal{G}^\beta(t,s)\big| \\
&\qquad \times \bigg\{ \frac{1}{\Gamma(\sigma)} \int_0^s (s-\tau)^{\sigma-1} \big\|\mathcal{F}_1(\tau)\big\| \big\| \psi_1\big(\tau, u_n(\tau), {}^cD^\rho\big(\phi_p D^\beta v_n(\tau)\big)\big) \\
&\qquad - \psi_1\big(\tau, u(\tau), {}^cD^\rho\big(\phi_p D^\beta v(\tau)\big)\big\|\, d\tau\big)\, ds \\
&\qquad + \frac{1}{\Gamma(\rho)} \int_0^s (s-\tau)^{\rho-1} \big\|\mathcal{F}_2(\tau)\big\| \big\| \psi_2\big(\tau, {}^cD^\sigma\big(\phi_p D^\beta u_n(\tau)\big), v_n(\tau)\big) \\
&\qquad - \psi_2\big(\tau, {}^c D^\sigma\big(\phi_p D^\beta u(\tau), v(\tau)\big)\big\|\, d\tau\big)\, ds \bigg\} \\
&\quad \le (q-1)\varrho^2 \int_0^1 \big|\mathcal{G}^\beta(t,s)\big| \bigg\{ \frac{\mathcal{M}_{\psi_1}\|\mathcal{F}_1\|\|u_n-u\| + \mathcal{M}'_{\psi_1}\mathcal{M}'_{\psi_2}\|\mathcal{F}_1\|\|\mathcal{F}_2\|\|v_n-v\|}{\Gamma(\sigma+1)}
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \frac{\mathcal{M}_{\psi_1}\mathcal{M}_{\psi_2}\|\mathcal{F}_1\|\mathcal{F}_2\|\|u_n - u\| + \mathcal{M}'_{\psi_2}\|\mathcal{F}_2\|\|v_n - v\|}{\Gamma(\rho+1)}\Bigg\} \\
&\to 0, \quad \text{as } n \to \infty.
\end{aligned}
$$

Therefore, $\|\mathcal{H}^*(u_n, v_n) - \mathcal{H}^*(u,v)\| \to 0$ as $n \to \infty$. Hence $\mathcal{H}^*$ is continuous.

For equicontinuity, take $\upsilon_1, \upsilon_2 \in \mathrm{J}$ with $\upsilon_1 < \upsilon_2$, and for any $(u,v) \in \Omega(r)$, we have

$$
\begin{aligned}
&\big\|\big(\mathcal{H}^*(u,v)(\upsilon_1) - \mathcal{H}^*(u,v)(\upsilon_2)\big)\big\| \\
&\quad \le \big\|\big(\big(\mathcal{H}^*{}_1(u,v)(\upsilon_1) - \mathcal{H}^*{}_1(u,v)(\upsilon_2)\big)\big)\big\| + \big\|\big(\mathcal{H}^*{}_2(u,v)(\upsilon_1) - \mathcal{H}^*{}_2(u,v)(\upsilon_2)\big)\big\| \\
&\quad = \sup_{t\in \mathrm{J}}\bigg|\int_0^1 \mathcal{G}^\beta(\upsilon_1,s)\phi_q\bigg(\frac{1}{\Gamma(\sigma)}\int_0^s (s-\tau)^{\sigma-1}\mathcal{F}_1(\tau)\psi_1\big(\tau, u(\tau), {}^cD^\rho\big(\phi_p D^\beta v(\tau)\big)\big)\,d\tau\bigg)\,ds \\
&\qquad - \int_0^1 \mathcal{G}^\beta(\upsilon_2,s)\phi_q\bigg(\frac{1}{\Gamma(\sigma)}\int_0^s (s-\tau)^{\sigma-1}\mathcal{F}_1(\tau)\psi_1\big(\tau, u(\tau), {}^cD^\rho\big(\phi_p D^\beta v(\tau)\big)\big)\,d\tau\bigg)\,ds\bigg| \\
&\qquad + \sup_{t\in \mathrm{J}}\bigg|\int_0^1 \mathcal{G}^\beta(\tau_1,s) \\
&\qquad \times \phi_q\bigg(\frac{1}{\Gamma(\rho)}\int_0^s (s-\tau)^{\rho-1}\mathcal{F}_2(\tau)\psi_1\big(\tau, {}^cD^\sigma\big(\phi_p D^\beta u(\tau), v(\tau)\big)\big)\,d\tau\bigg)\,ds \\
&\qquad - \int_0^1 \mathcal{G}^\beta(\upsilon_2,s)\phi_q\bigg(\frac{1}{\Gamma(\rho)}\int_0^s (s-\tau)^{\rho-1}\mathcal{F}_2(\tau)\psi_1\big(\tau, {}^cD^\sigma\big(\phi_p D^\beta u(\tau), v(\tau)\big)\big)\,d\tau\bigg)\,ds\bigg| \\
&\quad \le \int_0^1 \big|\mathcal{G}^\beta(\tau_1,s) - \mathcal{G}^\beta(\tau_2,s)\big| \\
&\qquad \times \phi_q\bigg(\frac{1}{\Gamma(\sigma)}\int_0^s (s-\tau)^{\sigma-1}\|\mathcal{F}_1\|\big\|\psi_1\big(\tau, u(\tau), {}^cD^\rho\big(\phi_p D^\beta v(\tau)\big)\big)\big\|\,d\tau\bigg)\,ds \\
&\qquad + \int_0^1 \big|\mathcal{G}^\beta(\tau_1,s) - \mathcal{G}^\beta(\tau_2,s)\big| \\
&\qquad \times \phi_q\bigg(\frac{1}{\Gamma(\rho)}\int_0^s (s-\tau)^{\rho-1}\|\mathcal{F}_2\|\big\|\psi_1\big(\tau, {}^cD^\sigma\big(\phi_p D^\beta u(\tau), v(\tau)\big)\big)\big\|\,d\tau\bigg)\,ds \\
&\quad \le \bigg(\frac{|\upsilon_1^\beta - \upsilon_2^\beta|}{\Gamma(\beta+1)} + \frac{|\upsilon_1^{\beta-1} - \upsilon_2^{\beta-1}|}{\Gamma(\beta-\delta)\Gamma(\beta+1)}\bigg)\bigg[\frac{1}{\Gamma(\sigma+1)}\bigg]^{q-1} \\
&\qquad \times \|\mathcal{F}_1\|^{q-1}\bigg(\frac{\mu_{\psi_1}\|u\| + \mu'_{\psi_1}\mu'_{\psi_2}\|\mathcal{F}_2\|\|v\|}{1 - \|\mathcal{F}_1\|\|\mathcal{F}_2\|\mu'_{\psi_1}\mu_{\psi_2}}\bigg) \\
&\quad \le \bigg(\frac{|\upsilon_1^\beta - \upsilon_2^\beta|}{\Gamma(\beta+1)} + \frac{|\upsilon_1^{\beta-1} - \upsilon_2^{\beta-1}|}{\Gamma(\beta-\delta)\Gamma(\beta+1)}\bigg)\bigg[\frac{1}{\Gamma(\rho+1)}\bigg]^{q-1} \\
&\qquad \times \|\mathcal{F}_2\|^{q-1}\bigg(\frac{\mu_{\psi_1}\mu_{\psi_2}\|u\| + \mu'_{\psi_2}\|v\|}{1 - \|\mathcal{F}_1\|\|\mathcal{F}_2\|\mu'_{\psi_1}\mu_{\psi_2}}\bigg).
\end{aligned}
$$

This implies that $\|\mathcal{H}^*(u,v)(\upsilon_1) - \mathcal{H}^*(u,v)(\upsilon_2)\| \to 0$ as $\upsilon_1 \to \upsilon_2$. Therefore $\mathcal{H}^*$ is relatively compact. By Arzelä–Ascolli theorem, $\mathcal{H}^*$ is compact and hence completely continuous operator.

Now let us define a set

$$
\mathrm{W} = \big\{(u,v) \in \overline{\Omega(r_2)}/\Omega(r_1)) \text{ there exist } \lambda \in [0,1] \text{ such that } (u,v) = \lambda\mathcal{H}(u,v)\big\}.
$$

We will show that W is bounded. Suppose on the contrary that W is unbounded. Let $(u,v) \in \mathrm{W}$ such that $\|(u,v)\| = \mathcal{K} \to \infty$. But

$$\begin{aligned}
\|(u,v)\| &= \|\lambda \mathcal{H}(u,v)\| \\
&\leq \|\mathcal{H}(u,v)\| \\
&\leq \left(\frac{1}{\Gamma(\beta+1)} - \frac{1}{\Gamma(\beta-\delta)\Gamma(\beta)}\right) \\
&\quad \times \left[\left(\frac{1}{\Gamma(\sigma+1)}\right)^{q-1} \|\mathcal{F}_1\|^{q-1} \left(\frac{\mu_{\psi_1}\|u\| + \mu_{\psi_1}\mu'_{\psi_2}\|\mathcal{F}_2\|\|v\|}{1 - \|\mathcal{F}_1\|\|\mathcal{F}_2\|\mu'_{\psi_1}\mu_{\psi_2}}\right)\right. \\
&\quad \left. + \left(\frac{1}{\Gamma(\rho+1)}\right)^{q-1} \|\mathcal{F}_2\|^{q-1} \left(\frac{\mu_{\psi_1}\mu_{\psi_2}\|\mathcal{F}_1\|\|u\| + \mu'_{\psi_2}\|v\|}{1 - \|\mathcal{F}_1\|\|\mathcal{F}_2\|\mu'_{\psi_1}\mu_{\psi_2}}\right)\right].
\end{aligned}$$

This implies that

$$\begin{aligned}
\|(u,v)\| &\leq \left(\frac{1}{\Gamma(\beta+1)} - \frac{1}{\Gamma(\beta-\delta)\Gamma(\beta)}\right) \\
&\quad \times \left[\left(\frac{1}{\Gamma(\sigma+1)}\right)^{q-1} \|\mathcal{F}_1\|^{q-1} \left(\frac{\mu_{\psi_1}\|u\| + \mu_{\psi_1}\mu'_{\psi_2}\|\mathcal{F}_2\|\|v\|}{1 - \|\mathcal{F}_1\|\|\mathcal{F}_2\|\mu'_{\psi_1}\mu_{\psi_2}}\right)\right. \\
&\quad \left. + \left(\frac{1}{\Gamma(\rho+1)}\right)^{q-1} \|\mathcal{F}_2\|^{q-1} \left(\frac{\mu_{\psi_1}\mu_{\psi_2}\|\mathcal{F}_1\|\|u\| + \mu'_{\psi_2}\|v\|}{1 - \|\mathcal{F}_1\|\|\mathcal{F}_2\|\mu'_{\psi_1}\mu_{\psi_2}}\right)\right],
\end{aligned}$$

equivalently

$$\begin{aligned}
1 \leq\ & \frac{1}{\|(u,v)\|}\left(\frac{1}{\Gamma(\beta+1)} - \frac{1}{\Gamma(\beta-\delta)\Gamma(\beta)}\right) \\
& \times \left[\left(\frac{1}{\Gamma(\sigma+1)}\right)^{q-1} \|\mathcal{F}_1\|^{q-1} \left(\frac{\mu_{\psi_1}\|u\| + \mu_{\psi_1}\mu'_{\psi_2}\|\mathcal{F}_2\|\|v\|}{1 - \|\mathcal{F}_1\|\|\mathcal{F}_2\|\mu'_{\psi_1}\mu_{\psi_2}}\right)\right. \\
& \left. + \left(\frac{1}{\Gamma(\rho+1)}\right)^{q-1} \|\mathcal{F}_2\|^{q-1} \left(\frac{\mu_{\psi_1}\mu_{\psi_2}\|\mathcal{F}_1\|\|u\| + \mu'_{\psi_2}\|v\|}{1 - \|\mathcal{F}_1\|\|\mathcal{F}_2\|\mu'_{\psi_1}\mu_{\psi_2}}\right)\right] \\
=\ & \frac{1}{\mathcal{K}}\left(\frac{1}{\Gamma(\beta+1)} - \frac{1}{\Gamma(\beta-\delta)\Gamma(\beta)}\right) \\
& \times \left[\left(\frac{1}{\Gamma(\sigma+1)}\right)^{q-1} \|\mathcal{F}_1\|^{q-1} \left(\frac{\mu_{\psi_1}\|u\| + \mu_{\psi_1}\mu'_{\psi_2}\|\mathcal{F}_2\|\|v\|}{1 - \|\mathcal{F}_1\|\|\mathcal{F}_2\|\mu'_{\psi_1}\mu_{\psi_2}}\right)\right. \\
& \left. + \left(\frac{1}{\Gamma(\rho+1)}\right)^{q-1} \|\mathcal{F}_2\|^{q-1} \left(\frac{\mu_{\psi_1}\mu_{\psi_2}\|\mathcal{F}_1\|\|u\| + \mu'_{\psi_2}\|v\|}{1 - \|\mathcal{F}_1\|\|\mathcal{F}_2\|\mu'_{\psi_1}\mu_{\psi_2}}\right)\right] \\
\to\ & 0 \quad \text{as } \mathcal{K} \to \infty.
\end{aligned}$$

This is a contradiction. Ultimately W is bounded, therefore by Lemma 2.7 the operator $\mathcal{H}$ has at least one fixed point in $\Omega(r_2)/\Omega(r_1)$, which is a solution of coupled system (1.1).

Thus, by Lemma 2.9, (1.1) has at least one solution. □

To control the growth bound of the nonlinearity functions ψ_1, ψ_2 and proceed to the next result, we need the following height functions. Let

$$\begin{cases} \Im_{\max_{t\in J,x>0}}(t,x) = \max\{\{\psi_1,\psi_2\} : t^{\beta-1}x \le (u,v) \le x\}, \\ \Im_{\min_{t\in J,x>0}}(t,x) = \min\{\{\psi_1,\psi_2\} : t^{\beta-1}x \le (u,v) \le x\}. \end{cases} \tag{3.13}$$

Theorem 3.4 *Let assumptions* ($\mathbf{H}_1$) *to* ($\mathbf{H}_3$) *hold, and there exist* $r^*, \hbar \in \mathcal{R}^+$ *such that one of the following conditions is satisfied*:

($\Im_1$)

$$\hbar \le \int_0^1 \mathcal{G}^\beta(1,s)\phi_q\left(\frac{1}{\Gamma(\sigma)}\int_0^s (s-\tau)^{\sigma-1}\mathcal{F}_1(\tau)\Im_{\min}\big(\tau,\hbar,{}^cD^\rho\big(\phi_p D^\beta v(\tau)\big)\big)\,d\tau\right)ds < \infty^+$$

and

$$\int_0^1 \mathcal{G}^\beta(1,s)\phi_q\left(\frac{1}{\Gamma(\sigma)}\int_0^s (s-\tau)^{\sigma-1}\mathcal{F}_1(\tau)\Im_{\max}\big(\tau,r^*,{}^cD^\rho\big(\phi_p D^\beta v(\tau)\big)\big)\,d\tau\right)ds \le r^*$$

($\Im_2$)

$$\int_0^1 \mathcal{G}^\beta(1,s)\phi_q\left(\frac{1}{\Gamma(\sigma)}\int_0^s (s-\tau)^{\sigma-1}\mathcal{F}_1(\tau)\Im_{\max}\big(\tau,\hbar,{}^cD^\rho\big(\phi_p D^\beta v(\tau)\big)\big)\,d\tau\right)ds < \hbar;$$

and

$$r^* \le \int_0^1 \mathcal{G}^\beta(1,s)\phi_q\left(\frac{1}{\Gamma(\sigma)}\int_0^s (s-\tau)^{\sigma-1}\mathcal{F}_1(\tau)\Im_{\min}\big(\tau,r^*,{}^cD^\rho\big(\phi_p D^\beta v(\tau)\big)\big)\,d\tau\right)ds < \infty^+;$$

($\Im_3$)

$$\hbar \le \int_0^1 \mathcal{G}^\beta(1,s)\phi_q\left(\frac{1}{\Gamma(\sigma)}\int_0^s (s-\tau)^{\sigma-1}\mathcal{F}_2(\tau)\Im_{\min}\big(\tau,{}^cD^\rho\big(\phi_p D^\beta u(\tau),\hbar\big)\big)\,d\tau\right)ds < \infty^+;$$

and

$$\int_0^1 \mathcal{G}^\beta(1,s)\phi_q\left(\frac{1}{\Gamma(\sigma)}\int_0^s (s-\tau)^{\sigma-1}\mathcal{F}_2(\tau)\Im_{\max}\big(\tau,{}^cD^\rho\big(\phi_p D^\beta u(\tau)\big),r^*\big)\,d\tau\right)ds \le r^*;$$

($\Im_4$)

$$\int_0^1 \mathcal{G}^\beta(1,s)\phi_q\left(\frac{1}{\Gamma(\sigma)}\int_0^s (s-\tau)^{\sigma-1}\mathcal{F}_2(\tau)\Im_{\max}\big(\tau,{}^cD^\rho\big(\phi_p D^\beta u(\tau)\big),\hbar\big)\,d\tau\right)ds < \hbar;$$

and

$$r^* \le \int_0^1 \mathcal{G}^\beta(1,s)\phi_q\left(\frac{1}{\Gamma(\sigma)}\int_0^s (s-\tau)^{\sigma-1}\mathcal{F}_2(\tau)\Im_{\min}\big(\tau,{}^cD^\rho\big(\phi_p D^\beta u(\tau)\big),r^*\big)\,d\tau\right)ds < \infty^+.$$

Then problem (1.1) *has a nonnegative solution* $(u^*,v^*) \in \mathcal{B}_r \times \mathcal{B}_r$, *so that* $\hbar \le \|(u^*,v^*)\| \le r^*$.

Proof Without loss of generality we take only $(\Im_1)$ and $(\Im_2)$. If $(u,v) \in \partial\Omega(\hbar)$, then $\|(u,v)\| = \hbar$ and $t^{\beta-1}\hbar \le (u,v) \le \hbar$, $t \in \mathrm{J}$. By (3.13) we have

$$\begin{aligned}
&\|\mathcal{H}^*(u,v)(t)\| \\
&\quad = \|(\mathcal{H}_1^*,\mathcal{H}_2^*)(u,v)(t)\| \\
&\quad = \sup_{t\in\mathrm{J}} \int_0^1 \mathcal{G}^\beta(t,s)\phi_q\biggl(\frac{1}{\Gamma(\sigma)}\int_0^s (s-\tau)^{\sigma-1}\mathcal{F}_1(\tau)\psi_1\bigl(\tau,u(\tau),{}^cD^\rho\bigl(\phi_p D^\beta v(\tau)\bigr)\bigr)\,d\tau\biggr)\,ds \\
&\qquad + \sup_{t\in\mathrm{J}} \int_0^1 \mathcal{G}^\beta(t,s)\phi_q\biggl(\frac{1}{\Gamma(\rho)}\int_0^s (s-\tau)^{\rho-1}\mathcal{F}_2(\tau)\psi_2\bigl(\tau,{}^cD^\sigma\bigl(\phi_p D^\beta u(\tau)\bigr),v(\tau)\bigr)\,d\tau\biggr)\,ds \\
&\quad \ge t^{\beta-1}\int_0^1 \mathcal{G}^\beta(1,s)\phi_q\biggl(\frac{1}{\Gamma(\sigma)}\int_0^s (s-\tau)^{\sigma-1}\mathcal{F}_1(\tau)\psi_1\bigl(\tau,u(\tau),{}^cD^\rho\bigl(\phi_p D^\beta v(\tau)\bigr)\bigr)\,d\tau\biggr)\,ds \\
&\qquad + t^{\beta-1}\int_0^1 \mathcal{G}^\beta(1,s) \\
&\qquad \times \phi_q\biggl(\frac{1}{\Gamma(\rho)}\int_0^s (s-\tau)^{\rho-1}\mathcal{F}_2(\tau)\psi_2\bigl(\tau,{}^cD^\sigma\bigl(\phi_p D^\beta u(\tau)\bigr),v(\tau)\bigr)\,d\tau\biggr)\,ds \\
&\quad \ge \int_0^1 \mathcal{G}^\beta(1,s)\phi_q\biggl(\frac{1}{\Gamma(\sigma)}\int_0^s (s-\tau)^{\sigma-1}\bigl[\mathcal{F}_1(\tau)\Im_{\min_{t\in\mathrm{J}}}\bigl(\tau,\hbar,{}^cD^\rho\bigl(\phi_p D^\beta v(\tau)\bigr)\bigr)\bigr]\,d\tau\biggr)\,ds \\
&\qquad + \int_0^1 \mathcal{G}^\beta(1,s)\phi_q\biggl(\frac{1}{\Gamma(\rho)}\int_0^s (s-\tau)^{\rho-1}\bigl[\mathcal{F}_2(\tau)\Im_{\min_{t\in\mathrm{J}}}\bigl(\tau,{}^cD^\sigma\bigl(\phi_p D^\beta u(\tau)\bigr),\hbar\bigr)\bigr]\,d\tau\biggr)\,ds \\
&\quad \ge \frac{\hbar}{2}+\frac{\hbar}{2} = \hbar = \|(u,v)\|.
\end{aligned}$$

Thus

$$\|\mathcal{H}^*(u,v)(t)\| \ge \hbar = \|(u,v)\|.$$

When $(u,v) \in \partial\Omega(r^*)$, then $\|(u,v)\| = r^*$, and by (3.13), $t^{\beta-1}r^* \le (u,v) \le r^*$, we have $\Im_{\max_{t\in\mathrm{J}}} \ge \{\psi_1,\psi_2\}$, therefore

$$\begin{aligned}
&\|\mathcal{H}^*(u,v)(t)\| \\
&\quad = \|(\mathcal{H}_1^*,\mathcal{H}_2^*)(u,v)(t)\| \\
&\quad = \max_{t\in\mathrm{J}} \int_0^1 \mathcal{G}^\beta(t,s)\phi_q\biggl(\frac{1}{\Gamma(\sigma)}\int_0^s (s-\tau)^{\sigma-1}\mathcal{F}_1(\tau)\psi_1\bigl(\tau,u(\tau),{}^cD^\rho\bigl(\phi_p D^\beta v(\tau)\bigr)\bigr)\,d\tau\biggr)\,ds \\
&\qquad + \max_{t\in\mathrm{J}} \int_0^1 \mathcal{G}^\beta(t,s) \\
&\qquad \times \phi_q\biggl(\frac{1}{\Gamma(\rho)}\int_0^s (s-\tau)^{\rho-1}\mathcal{F}_2(\tau)\psi_2\bigl(\tau,{}^cD^\sigma\bigl(\phi_p D^\beta u(\tau)\bigr),v(\tau)\bigr)\,d\tau\biggr)\,ds \\
&\quad \le t^{\beta-1}\int_0^1 \mathcal{G}^\beta(1,s)\phi_q\biggl(\frac{1}{\Gamma(\sigma)}\int_0^s (s-\tau)^{\sigma-1}\mathcal{F}_1(\tau)\psi_1\bigl(\tau,u(\tau),{}^cD^\rho\bigl(\phi_p D^\beta v(\tau)\bigr)\bigr)\,d\tau\biggr)\,ds \\
&\qquad + t^{\beta-1}\int_0^1 \mathcal{G}^\beta(1,s) \\
&\qquad \times \phi_q\biggl(\frac{1}{\Gamma(\rho)}\int_0^s (s-\tau)^{\rho-1}\mathcal{F}_2(\tau)\psi_2\bigl(\tau,{}^cD^\sigma\bigl(\phi_p D^\beta u(\tau)\bigr),v(\tau)\bigr)\,d\tau\biggr)\,ds
\end{aligned}$$

$$
\begin{aligned}
&\leq \int_0^1 \mathcal{G}^\beta(1,s)\phi_q\left(\frac{1}{\Gamma(\sigma)}\int_0^s (s-\tau)^{\sigma-1}\left[\mathcal{F}_1(\tau)\Im_{\max_{t\in \mathrm{J}}}\left(\tau, r^*, {}^cD^\rho\left(\phi_p D^\beta v(\tau)\right)\right)\right]d\tau\right)ds \\
&\quad + \int_0^1 \mathcal{G}^\beta(1,s) \\
&\quad \times \phi_q\left(\frac{1}{\Gamma(\rho)}\int_0^s (s-\tau)^{\rho-1}\left[\mathcal{F}_2(\tau)\Im_{\max_{t\in \mathrm{J}}}\left(\tau, {}^cD^\sigma\left(\phi_p D^\beta u(\tau)\right), r^*\right)\right]d\tau\right)ds \\
&\leq \frac{r^*}{2} + \frac{r^*}{2} = r^* = \left\|(u,v)\right\|.
\end{aligned}
$$

Thus

$$
\left\|\mathcal{H}^*(u,v)(t)\right\| \geq \hbar = \left\|(u,v)\right\|.
$$

Combining these inequalities, we say that $\mathcal{H}^*$ has a fixed point in the interval $[\hbar, r^*]$, say $(u^*,v^*) \in \overline{\Omega(r^*)}/\Omega(\hbar)$, such that $\hbar \leq \|(u^*,v^*)\| \leq r^*$. Next we show that (u^*,v^*) is a nonnegative solution for $t \in \mathrm{J}$ as

$$
\begin{aligned}
u^*(t) &= \int_0^1 \mathcal{G}^\beta(t,s)\phi_q\left(\frac{1}{\Gamma(\sigma)}\int_0^s (s-\tau)^{\sigma-1}\mathcal{F}_1(\tau)\psi_1\left(\tau, u^*(\tau), {}^cD^\rho\left(\phi_p D^\beta v^*(\tau)\right)\right)d\tau\right)ds \\
&\geq t^{\beta-1}\max_{t\in \mathrm{J}}\int_0^1 \mathcal{G}^\beta(1,s) \\
&\qquad \times \phi_q\left(\frac{1}{\Gamma(\sigma)}\int_0^s (s-\tau)^{\sigma-1}\mathcal{F}_1(\tau)\psi_1\left(\tau, u^*(\tau), {}^cD^\rho\left(\phi_p D^\beta v^*(\tau)\right)\right)d\tau\right)ds
\end{aligned}
$$

implies

$$
u^*(t) \geq t^{\beta-1}\left\|u^*\right\| \geq \frac{\hbar}{2}t^{\beta-1} > 0.
$$

Similarly, we get

$$
v^*(t) \geq t^{\beta-1}\left\|v^*\right\| \geq \frac{\hbar}{2}t^{\beta-1} > 0.
$$

With the help of Lemma 3.2 and $(\mathcal{P}_3)$, the solution (u^*,v^*) is nondecreasing for $t \in \mathrm{J}$. □

Theorem 3.5 *Let hypotheses* $(\mathbf{H}_1)$ *to* $(\mathbf{H}_3)$ *be true with* $\Delta = \max\{\Delta_1, \Delta_2\} < 1$, *where*

$$
\begin{aligned}
\Delta_1 &= \frac{(q-1)\varrho^{q-1}(2\beta-\delta)\mathcal{M}_{\psi_1}\|\mathcal{F}_1\|}{(\beta-\delta)\Gamma(\beta+1)}\left[\frac{1}{\Gamma(\sigma+1)} + \frac{\mathcal{M}_{\psi_2}\|\mathcal{F}_2\|}{\Gamma(\rho+1)}\right], \\
\Delta_2 &= \frac{(q-1)\varrho^{q-1}(2\beta-\delta)\mathcal{M}'_{\psi_2}\|\mathcal{F}_2\|}{(\beta-\delta)\Gamma(\beta+1)}\left[\frac{\|\mathcal{F}_1\|\mathcal{M}'_{\psi_1}}{\Gamma(\sigma+1)} + \frac{1}{\Gamma(\rho+1)}\right].
\end{aligned}
$$

Then (1.1) *has a unique solution.*

Proof Define operator $\Phi = (\Phi_1, \Phi_2) : \overline{\Omega(r)}/\Omega(r) \to \mathbf{B}$ by

$$
\Phi(u,v)(t) = \left(\Phi_1(u,v), \Phi_2(u,v)\right)(t), \quad t \in \mathrm{J},
$$

where

$$\begin{aligned}
&\Phi_1(u,v)(t)\\
&\quad=\int_0^1 \mathcal{G}^\beta(t,s)\phi_q\left(\frac{1}{\Gamma(\sigma)}\int_0^s (s-\tau)^{\sigma-1}\mathcal{F}_1(\tau)\psi_1\big(\tau,u(\tau),{}^cD^\rho\big(\phi_pD^\beta v(\tau)\big)\big)\,d\tau\right)ds
\end{aligned}$$

and

$$\begin{aligned}
&\Phi_2(u,v)(t)\\
&\quad=\int_0^1 \mathcal{G}^\beta(t,s)\phi_q\left(\frac{1}{\Gamma(\sigma)}\int_0^s (s-\tau)^{\sigma-1}\mathcal{F}_1(\tau)\psi_1\big(\tau,u(\tau),{}^cD^\rho\big(\phi_pD^\beta v(\tau)\big)\big)\,d\tau\right)ds.
\end{aligned}$$

Now, for any $(u,v),(\bar{u},\bar{v})\in\overline{\Omega(r)}/\Omega(r)$, we have

$$\begin{aligned}
&\big\|\Phi(u,v)-\Phi(\bar{u},\bar{v})\big\|\\
&\quad\leq\sup_{t\in J}\int_0^1\big|\mathcal{G}^\beta(t,s)\big|\\
&\qquad\times\left\{\left|\phi_q\left(\frac{1}{\Gamma(\sigma)}\int_0^s(s-\tau)^{\sigma-1}\mathcal{F}_1(\tau)\psi_1\big(\tau,u(\tau),{}^cD^\rho\big(\phi_pD^\beta v(\tau)\big)\big)\,d\tau\right)ds\right.\right.\\
&\qquad\left.\left.-\phi_q\left(\frac{1}{\Gamma(\sigma)}\int_0^s(s-\tau)^{\sigma-1}\mathcal{F}_1(\tau)\psi_1\big(\tau,\bar{u}(\tau),{}^cD^\rho\big(\phi_pD^\beta \bar{v}(\tau)\big)\big)\,d\tau\right)ds\right|\right\}\\
&\qquad+\sup_{t\in J}\int_0^1\big|\mathcal{G}^\beta(t,s)\big|\\
&\qquad\times\left\{\left|\phi_q\left(\frac{1}{\Gamma(\rho)}\int_0^s(s-\tau)^{\rho-1}\mathcal{F}_2(\tau)\psi_2\big(\tau,{}^cD^\sigma\big(\phi_pD^\beta u(\tau)\big),v(\tau)\big)\,d\tau\right)ds\right.\right.\\
&\qquad\left.\left.-\phi_q\left(\frac{1}{\Gamma(\rho)}\int_0^s(s-\tau)^{\rho-1}\mathcal{F}_2(\tau)\psi_2\big(\tau,{}^cD^\sigma\big(\phi_pD^\beta \bar{u}(\tau)\big),\bar{v}(\tau)\big)\,d\tau\right)ds\right|\right\}\\
&\quad\leq\frac{(q-1)\varrho^{q-1}(2\beta-\delta)}{(\beta-\delta)\Gamma(\beta+1)}\left[\frac{\mathcal{M}_{\psi_1}\|\mathcal{F}_1\|\|u-\bar{u}\|+\mathcal{M}'_{\psi_1}\mathcal{M}'_{\psi_2}\|\mathcal{F}_1\|\|\mathcal{F}_2\|\|v-\bar{v}\|}{\Gamma(\sigma+1)}\right.\\
&\qquad\left.+\frac{\mathcal{M}_{\psi_1}\mathcal{M}_{\psi_2}\|\mathcal{F}_1\|\|\mathcal{F}_2\|\|u-\bar{u}\|+\mathcal{M}'_{\psi_2}\|\mathcal{F}_2\|\|v-\bar{v}\|}{\Gamma(\rho+1)}\right]\\
&\quad\leq\frac{(q-1)\varrho^{q-1}(2\beta-\delta)\mathcal{M}_{\psi_1}\|\mathcal{F}_1\|}{(\beta-\delta)\Gamma(\beta+1)}\left[\frac{1}{\Gamma(\sigma+1)}+\frac{\mathcal{M}_{\psi_2}\|\|\mathcal{F}_2\|}{\Gamma(\rho+1)}\right]\|u-\bar{u}\|\\
&\qquad+\frac{(q-1)\varrho^{q-1}(2\beta-\delta)\mathcal{M}'_{\psi_2}\|\mathcal{F}_2\|}{(\beta-\delta)\Gamma(\beta+1)}\left[\frac{\|\mathcal{F}_1\|\mathcal{M}'_{\psi_1}}{\Gamma(\sigma+1)}+\frac{1}{\Gamma(\rho+1)}\right]\|v-\bar{v}\|\\
&\quad=\Delta_1\|u-\bar{u}\|+\Delta_2\|v-\bar{v}\|\leq\Delta\big\|(u,\bar{u})-(v,\bar{v})\big\|.
\end{aligned}$$

Thus

$$\big\|\Phi(u,v)-\Phi(\bar{u},\bar{v})\big\|\leq\Delta\big\|(u,\bar{u})-(v,\bar{v})\big\|.$$

Hence the assumption $\Delta<1$ implies that the operator Φ is a contraction. Therefore, by Theorem 2.9, (1.1) has a unique fixed point. □

4 Stability analysis

In this section, we analyze Hyers–Ulam stability for the proposed problem.

Theorem 4.1 *Let assumptions* $(\mathbf{H}_1)$–$(\mathbf{H}_3)$ *with* $\Delta < 1$ *hold, along with the condition that the spectral radius of* $\mathcal{Q}$ *is less than one. Then the solution of* (1.1) *is Hyers–Ulam stable.*

Proof Let (u,v) be the exact and $(\bar{u},\bar{v})$ be an approximate solution of the considered problem (1.1), then in view of Theorem 3.5 we have

$$
\begin{aligned}
&\bigl\|\Phi(u,v)(t)-\Phi(\bar{u},\bar{v})(t)\bigr\| \\
&\quad= \bigl\|\bigl(\Phi_1(u,v)(t),\Phi_2(u,v)(t)\bigr)-\bigl(\Phi_1(\bar{u},\bar{v})(t),\Phi_2(\bar{u},\bar{v})(t)\bigr)\bigr\| \\
&\quad\leq \bigl\|\Phi_1(u,v)(t)-\Phi_1(\bar{u},\bar{v})\bigr\| + \bigl\|\Phi_2(u,v)(t)-\Phi_2(\bar{u},\bar{v})\bigr\| \\
&\quad= \sup_{t\in J}\bigl|\Phi_1(u,v)(t)-\Phi_1(\bar{u},\bar{v})(t)\bigr| + \sup_{t\in J}\bigl|\Phi_2(u,v)(t)-\Phi_2(\bar{u},\bar{v})(t)\bigr| \\
&\quad= \sup_{t\in J}\biggl|\int_0^1 \mathcal{G}^\beta(t,s)\phi_q\biggl(\frac{1}{\Gamma(\sigma)}\int_0^s(s-\tau)^{\sigma-1}\mathcal{F}_1(\tau)\psi_1\bigl(\tau,u(\tau),{}^cD^\rho\bigl(\phi_pD^\beta v(\tau)\bigr)\bigr)\,d\tau\biggr)\,ds \\
&\qquad- \int_0^1 \mathcal{G}^\beta(t,s)\phi_q\biggl(\frac{1}{\Gamma(\sigma)}\int_0^s(s-\tau)^{\sigma-1}\mathcal{F}_1(\tau)\psi_1\bigl(\tau,\bar{u}(\tau),{}^cD^\rho\bigl(\phi_pD^\beta \bar{v}(\tau)\bigr)\bigr)\,d\tau\biggr)\,ds\biggr| \\
&\qquad+ \sup_{t\in J}\biggl|\int_0^1 \mathcal{G}^\beta(t,s) \\
&\qquad\times \phi_q\biggl(\frac{1}{\Gamma(\rho)}\int_0^s(s-\tau)^{\rho-1}\mathcal{F}_2(\tau)\psi_2\bigl(\tau,{}^cD^\sigma\bigl(\phi_pD^\beta u(\tau)\bigr),v(\tau)\bigr)\,d\tau\biggr)\,ds \\
&\qquad- \int_0^1 \mathcal{G}^\beta(t,s)\phi_q\biggl(\frac{1}{\Gamma(\rho)}\int_0^s(s-\tau)^{\rho-1}\mathcal{F}_2(\tau)\psi_2\bigl(\tau,{}^cD^\sigma\bigl(\phi_pD^\beta \bar{u}(\tau)\bigr),\bar{v}(\tau)\bigr)\,d\tau\biggr)\,ds\biggr| \\
&\quad\leq (q-1)\varrho^{q-2}\int_0^1\bigl|\mathcal{G}^\beta(t,s)\bigr|\biggl\{\biggl(\frac{1}{\Gamma(\sigma)}\int_0^s(s-\tau)^{\sigma-1}\|\mathcal{F}_1\|\bigl\|\psi_1\bigl(\tau,u(\tau),{}^cD^\rho\bigl(\phi_pD^\beta v(\tau)\bigr)\bigr) \\
&\qquad- \psi_1\bigl(\tau,\bar{u}(\tau),{}^cD^\rho\bigl(\phi_pD^\beta \bar{v}(\tau)\bigr)\bigr)\bigr\|\,d\tau\biggr)\,ds \\
&\qquad+ \frac{1}{\Gamma(\rho)}\int_0^s(s-\tau)^{\rho-1}\|\mathcal{F}_2\|\bigl\|\psi_2\bigl(\tau,{}^cD^\sigma\bigl(\phi_pD^\beta u(\tau),v(\tau)\bigr)\bigr) \\
&\qquad- \psi_2\bigl(\tau,{}^cD^\sigma\bigl(\phi_pD^\beta \bar{u}(\tau),\bar{v}(\tau)\bigr)\bigr)\bigr\|\,d\tau\,ds\biggr\} \\
&\quad\leq \frac{(q-1)\varrho^{q-1}(2\beta-\delta)}{(\beta-\delta)\Gamma(\beta+1)}\biggl[\frac{\mathcal{M}_{\psi_1}\|\mathcal{F}_1\|\|u-\bar{u}\|+\mathcal{M}'_{\psi_1}\mathcal{M}'_{\psi_2}\|\mathcal{F}_1\|\|\mathcal{F}_2\|\|v-\bar{v}\|}{\Gamma(\sigma+1)} \\
&\qquad+ \frac{\mathcal{M}_{\psi_1}\mathcal{M}_{\psi_2}\|\mathcal{F}_1\|\mathcal{F}_2\|\|u-\bar{u}\|+\mathcal{M}'_{\psi_2}\|\mathcal{F}_2\|\|v-\bar{v}\|}{\Gamma(\rho+1)}\biggr] \\
&\quad\leq \frac{(q-1)\varrho^{q-1}(2\beta-\delta)\mathcal{M}_{\psi_1}\|\mathcal{F}_1\|}{(\beta-\delta)\Gamma(\beta+1)\Gamma(\sigma+1)}\|u-\bar{u}\| \\
&\qquad+ \frac{(q-1)\varrho^{q-1}(2\beta-\delta)\mathcal{M}'{}_{\psi_1}\mathcal{M}'{}_{\psi_2}\|\mathcal{F}_1\|\|\mathcal{F}_2\|}{(\beta-\delta)\Gamma(\beta+1)\Gamma(\sigma+1)}\|v-\bar{v}\| \\
&\qquad+ \frac{(q-1)\varrho^{q-1}(2\beta-\delta)\mathcal{M}_{\psi_1}\mathcal{M}_{\psi_2}\|\mathcal{F}_1\|\|\mathcal{F}_2\|}{(\beta-\delta)\Gamma(\beta+1)\Gamma(\sigma+1)}\|u-\bar{u}\|
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{(q-1)\varrho^{q-1}(2\beta-\delta)\mathcal{M}'_{\psi_1}\|\mathcal{F}_1\|}{(\beta-\delta)\Gamma(\beta+1)\Gamma(\sigma+1)}\|v-\bar{v}\| \\
&\leq \big\|(u,v)-(\bar{u},\bar{v})\big\|\mathcal{Q},
\end{aligned}
$$

where $\mathcal{Q}=\left(\begin{smallmatrix}\mathcal{C}_1 & \mathcal{C}_2\\ \mathcal{C}_3 & \mathcal{C}_4\end{smallmatrix}\right)$. Since the spectral radius of $\mathcal{Q}$ is less than one, thus the solution of the considered system (1.1) is Hyers–Ulam stable. Here

$$
\begin{aligned}
\mathcal{C}_1&=\frac{(q-1)\varrho^{q-1}(2\beta-\delta)\mathcal{M}_{\psi_1}\|\mathcal{F}_1\|}{(\beta-\delta)\Gamma(\beta+1)\Gamma(\sigma+1)},\\
\mathcal{C}_2&=\frac{(q-1)\varrho^{q-1}(2\beta-\delta)\mathcal{M}'_{\psi_1}\mathcal{M}'_{\psi_2}\|\mathcal{F}_1\|\|\mathcal{F}_2\|}{(\beta-\delta)\Gamma(\beta+1)\Gamma(\sigma+1)},\\
\mathcal{C}_3&=\frac{(q-1)\varrho^{q-1}(2\beta-\delta)\mathcal{M}_{\psi_1}\mathcal{M}_{\psi_2}\|\mathcal{F}_1\|\|\mathcal{F}_2\|}{(\beta-\delta)\Gamma(\beta+1)\Gamma(\sigma+1)},\\
\mathcal{C}_4&=\frac{(q-1)\varrho^{q-1}(2\beta-\delta)\mathcal{M}'_{\psi_1}\|\mathcal{F}_1\|}{(\beta-\delta)\Gamma(\beta+1)\Gamma(\sigma+1)}.
\end{aligned}
$$

□

The same approach can be followed to obtain results regarding the generalized Hyers–Ulam, Hyers–Ulam–Rassias, and generalized Hyers–Ulam–Rassias stability.

5 An illustrative example

For the support of our theoretical results, an example is presented here.

Example 5.1 Corresponding to (1.1), we consider the system of fractional order differential equations involving p-Laplacian operator ϕ_p as follows:

$$
\begin{cases}
{}^cD^\sigma(\phi_pD^\beta u(t))+\frac{t}{\sqrt{1-t}}\frac{u^2(t)+1+{}^cD^\sigma(\phi_pD^\beta v(t))}{10e^{t^2}+1}=0, \quad t\in[0,1)=\mathrm{J},\\
{}^cD^\sigma(\phi_pD^\beta v(t))+\frac{1}{\sqrt{4-4t^2}}\frac{v(t)+2+{}^cD^\sigma(\phi_pD^\beta u(t))}{20+t^3}=0,\\
([\phi_pD^\beta u(0)])^{(j)}=0, \quad j=0,1,2,3,\ldots,m-1,\\
([\phi_pD^\beta v(0)])^{(j)}=0,\\
I^{k-\beta}(u(0))=I^{k-\beta}(v(0))=0, \quad k=2,3,\ldots,m,\\
D^\delta(u(1))=D^\delta(v(1))=0.
\end{cases}
\tag{5.1}
$$

Set

$$
\psi_1\big(t,u(t),{}^cD^\sigma\big(\phi_pD^\beta v(t)\big)\big)=\frac{u^2(t)+1+{}^cD^\sigma(\phi_pD^\beta u(t))}{10e^{t^2}+1}
$$

and

$$
\psi_2\big(t,{}^cD^\sigma\big(\phi_pD^\beta u(t)\big),v(t)\big)=\frac{v(t)+2+{}^cD^\sigma(\phi_pD^\beta u(t))}{20+t^3}.
$$

Now, for any $u,v,\bar{u},\bar{v}\in\mathcal{X}$, we have

$$
\big|\psi_1\big(t,u(t),v(t)\big)-\psi_1\big(t,\bar{u}(t),\bar{v}(t)\big)\big|\leq\frac{1}{10e^2}\|u-\bar{u}\|+\frac{1}{10e^2}\|v-\bar{v}\|
$$

and

$$\left|\psi_2\big(t,u(t),v(t)\big)-\psi_2\big(t,\bar{u}(t),\bar{v}(t)\big)\right|\leq\frac{1}{20}\|u-\bar{u}\|+\frac{1}{20}\|v-\bar{v}\|.$$

Here, $\mathcal{M}_{\psi_1}=\mathcal{M}'_{\psi_1}=\frac{1}{10e^2}$, $\mathcal{M}_{\psi_2}=\mathcal{M}'_{\psi_2}=\frac{1}{20}$. Take $q=\frac{5}{2}$, $\beta=3$, $\rho=\sigma=\frac{7}{2}$, $\delta=\frac{3}{2}$, $\varrho=1$, then $p=\frac{5}{3}$, and upon calculations we have $\Delta=0.000192<1$, so system (5.1) has a unique solution. Further,

$$\mathcal{H}^*=\begin{pmatrix}0.00002 & 0.00001\\ 0.00001 & 0.00002\end{pmatrix}$$

and if ω_1 and ω_2 are the eigenvalues, then $\omega_1=0.00001$ and $\omega_2=0.00003$. Since the spectral radius of $\mathcal{H}^*$ is less than one, thus system (5.1) is Hyers–Ulam stable.

6 Conclusion

In this paper, we have utilized the Arzelä–Ascoli theorem, Banach's contraction principle, and Schauder's fixed point theorem to establish existence and uniqueness criteria for the solution of the nonlinear coupled implicit switched singular fractional differential system given in (1.1). Furthermore, under some particular assumptions and conditions, we have proved stability results in the sense of Ulam for the solutions of the said problem. We claim that the approach used to prove the main results is powerful, effectual, and suitable for investigating different qualitative properties of the solutions of nonlinear fractional differential equations.

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