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Some representations of the general solution to a difference equation of additive type

Stevo Stević^{1,2,3*}

*Correspondence: sstevic@ptt.rs

¹Mathematical Institute of the Serbian Academy of Sciences, Beograd, Serbia

²Department of Medical Research, China Medical University Hospital, China Medical University, Taichung, Taiwan, Republic of China

Full list of author information is available at the end of the article

Abstract

The general solution to the difference equation

$$x_{n+1} = \frac{ax_n x_{n-1} x_{n-2} + bx_{n-1} x_{n-2} + cx_{n-2} + d}{x_n x_{n-1} x_{n-2}}, \quad n \in \mathbb{N}_0,$$

where $a, b, c \in \mathbb{C}$, $d \in \mathbb{C} \setminus \{0\}$, is presented by using the coefficients, the initial values x_{-j} , $j = \overline{0, 2}$, and the solution to the difference equation

$$y_{n+1} = ay_n + by_{n-1} + cy_{n-2} + dy_{n-3}, \quad n \in \mathbb{N}_0,$$

satisfying the initial conditions $y_{-3} = y_{-2} = y_{-1} = 0$, $y_0 = 1$. The representation complements known ones of the general solutions to the corresponding difference equations of the first and second order. Besides, the general representation formula is investigated in detail and refined by using the roots of the characteristic polynomial

$$P_4(\lambda) = \lambda^4 - a\lambda^3 - b\lambda^2 - c\lambda - d$$

of the linear equation. The following cases are considered separately: (1) all the roots of the polynomial are distinct; (2) there is a unique double root of the polynomial; (3) there is a triple root of the polynomial and one simple; (4) there is a quadruple root of the polynomial; (5) there are two distinct double roots of the polynomial.

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1 Introduction

Regarding some notations which are used in this paper, say that we denote the sets of positive, nonnegative, integer, and complex numbers by \mathbb{N} , \mathbb{N}_0 , \mathbb{Z} , and \mathbb{C} , respectively. If $l_1, l_2 \in \mathbb{Z}$, $l_1 \leq l_2$, then the notation $l = \overline{l_1, l_2}$, which is frequently used throughout the paper, denotes the set of all $l \in \mathbb{Z}$ satisfying the inequalities $l_1 \leq l \leq l_2$. We also use the standard convention $\prod_{j=k}^{k-1} a_j = 1$, $k \in \mathbb{Z}$.

A method for solving the linear difference equation

$$x_{n+k} = a_{k-1}x_{n+k-1} + \cdots + a_0x_n, \quad n \geq l, \quad (1)$$

where $k \in \mathbb{N}$, $l \in \mathbb{Z}$, $a_j \in \mathbb{C}$, $j = \overline{0, k-1}$, $a_0 \neq 0$, has been known to de Moivre yet [1]. See also Euler's book [2] where some old results on solvability can be found. Solvability theory of linear difference equations was developed later, and it became a standard topic of almost any book on difference equations (see, e.g., [3–10]). Equation (1) is practically solvable when $k \leq 4$, whereas if $k > 5$ then it is theoretically solvable. These facts are consequences of a well-known form of the general solution to the equation (see, e.g., [3–10]), as well as of the Abel–Ruffini theorem on solvability of the polynomial equations by radicals [11].

Solvability of equation (1) of the corresponding nonhomogeneous linear difference equation, as well as of the nonhomogeneous linear first-order difference equation whose coefficients can be nonconstant, is frequently used in pure and applied mathematics as well as in other branches of science. For some recently studied classes of nonlinear difference equations and systems of nonlinear difference equations which have been solved by using one of these equations, see, e.g., [12–17]. For some recent applications of their solvability, see, e.g., [18–20]. For example, by using solvability methods in [19] and [20], some classes of difference equations are transformed to “integral” forms, which are usually more suitable for studying the existence of bounded or periodic solutions to difference equations than the equations in their original forms.

Employing various changes of variables, many classes of nonlinear difference equations can be transformed to special cases of equation (1), from which solvability of the nonlinear equations can follow.

One of the first of such nonlinear difference equations which appeared in the literature is the following:

$$x_{n+1} = \frac{\alpha x_n + \beta}{\gamma x_n + \delta}, \quad n \in \mathbb{N}_0, \quad (2)$$

where $\alpha, \beta, \gamma, \delta, z_0 \in \mathbb{C}$ (the bilinear difference equation).

Namely, if $\alpha\delta - \beta\gamma \neq 0$ and $\gamma \neq 0$ (these conditions eliminate two simple cases), then the change of variables

$$x_n = \frac{y_{n+1}}{\gamma y_n} - \frac{\delta}{\gamma}, \quad n \in \mathbb{N}_0, \quad (3)$$

transforms equation (2) to

$$y_{n+2} - (\alpha + \delta)y_{n+1} + (\alpha\delta - \beta\gamma)y_n = 0, \quad n \in \mathbb{N}_0, \quad (4)$$

(see, e.g., [3, 10, 21–25]).

For a method which uses an associated two-dimensional system of difference equations in solving equation (2), see, e.g., [7] or [26]. For some other results on the difference equation and for some applications of the equation, see, e.g., [3, 12, 25, 27–30].

As an explanation for the study in this paper, let us note now that the change of variables

$$\tilde{x}_n = \gamma x_n + \delta, \quad n \in \mathbb{N}_0,$$

transforms equation (2) to

$$\tilde{x}_{n+1} = \tilde{\alpha} + \frac{\tilde{\beta}}{\tilde{x}_n}, \quad n \in \mathbb{N}_0, \quad (5)$$

where

$$\tilde{\alpha} = \alpha + \delta \quad \text{and} \quad \tilde{\beta} = \beta\gamma - \alpha\delta,$$

which further by the change of variables

$$\tilde{x}_n = \frac{y_{n+1}}{y_n}, \quad n \in \mathbb{N}_0, \quad (6)$$

where y_0 is an arbitrary number different from zero, is transformed to equation (4), that is, to

$$y_{n+2} = \tilde{\alpha}y_{n+1} + \tilde{\beta}y_n \quad (7)$$

for $n \in \mathbb{N}_0$.

The method for solving equation (5) by using the change of variables (6) is something which is known to any expert. It is much less known that a transformation which transforms equation (7) to equation (5) has been essentially known for more than two centuries and can be found in book [4] by Vincenzo Brunacci. Namely, in nowadays notation, he employed the following change of variables:

$$y_n = A \prod_{j=0}^n z_j, \quad n \in \mathbb{N}_0, \quad (8)$$

from which after some calculation and cancelation, where is tacitly assumed that $z_n \neq 0$, $n \in \mathbb{N}_0$, equation (7) becomes

$$z_{n+2}z_{n+1} = \tilde{\alpha}z_{n+1} + \tilde{\beta}, \quad n \in \mathbb{N}_0,$$

and consequently,

$$z_{n+2} = \tilde{\alpha} + \frac{\tilde{\beta}}{z_{n+1}}, \quad n \in \mathbb{N}_0.$$

Note that the last difference equation is equation (5) with indices shifted forward for one.

Remark 1 Note that the change of variables

$$y_n = y_0 \prod_{j=0}^{n-1} x_j, \quad n \in \mathbb{N}_0, \quad (9)$$

transforms equation (7) to equation (5) directly, and that the change of variables is quite natural.

Namely, from relation (6), where \tilde{x}_n is replaced by x_n , it immediately follows that

$$y_n = x_{n-1}y_{n-1}, \quad n \in \mathbb{N},$$

which is a simple product-type recurrent relation from which relation (9) is easily obtained.

This should be the main reason why Brunacci tried to solve equation (7) by using the change of variables (8). It is also interesting to note that in this way Brunacci “solved” equation (7) in terms of a product of continuous fractions, which is, in fact, a quite complicated representation of general solution to the difference equation.

Now note that the change of variables (8), where y_n is replaced by x_n , transforms equation (1) to the following:

$$z_{n+k} = a_{k-1} + \frac{a_{k-2}}{z_{n+k-1}} + \cdots + \frac{a_0}{\prod_{j=1}^{k-1} z_{n+j}} \quad (10)$$

for $n \geq l$.

In [25] we explained why the general solutions to some special cases of the difference equations in (2), as well as the general solutions to some special cases of the corresponding two-dimensional systems of difference equations, can be written in terms of the Fibonacci sequence, that is, in terms of the solution to the difference equation

$$x_{n+1} = x_n + x_{n-1}, \quad n \in \mathbb{N},$$

such that

$$x_0 = 0 \quad \text{and} \quad x_1 = 1. \quad (11)$$

(Many facts related to the Fibonacci sequence and its numerous applications in various branches in mathematics can be found, e.g., in books [31] and [32]; see also [23] and [26] for some basic facts and relations concerning the sequence). Moreover, we have shown that the general solution to equation (2), as well as the general solution to the corresponding two-dimensional system of difference equations, can be written in terms of their coefficients, initial values, and the solution to an associated second-order linear difference equation with constant coefficients satisfying the initial conditions in (11).

Our further investigations on solvability of difference equations and systems of difference equations have shown that these solutions, the solutions to the linear difference equations with constant coefficients of the third order such that $x_{-2} = x_{-1} = 0$ and $x_0 = 1$, as well as the solutions to the linear difference equations with constant coefficients of the fourth order such that $x_{-3} = x_{-2} = x_{-1} = 0$ and $x_0 = 1$, naturally appear in representations of general solutions to several other classes of difference equations and systems of difference equations. For example, they appear in papers [25, 28–30] predominately connected to the bilinear difference equations, as well as in the representations of general solutions to the product-type difference equations and systems of difference equations, e.g., in [15, 33–39] (see also the references therein).

For some other results on solvability of difference equations and systems of difference equations, related topics such as invariants for difference equations and systems, and their applications, see, e.g., [3, 5–7, 9, 10, 23, 26, 32, 40–47].

If $k = 3$, equation (10) can be written as follows:

$$x_{n+1} = a + \frac{b}{x_n} + \frac{c}{x_n x_{n-1}} \quad (12)$$

for $n \in \mathbb{N}_0$.

Employing the change of variables

$$x_n = \frac{y_n}{y_{n-1}}, \quad n \geq -1, \quad (13)$$

it is transformed to

$$y_{n+1} = ay_n + by_{n-1} + cy_{n-2}, \quad n \in \mathbb{N}_0, \quad (14)$$

which is clearly solvable.

Motivated, among other things, by some results in [15, 25, 33–39], recently in [48] we have proved the following theorem extending and theoretically explaining a recent result in the literature.

Theorem 1 *Let $a, b \in \mathbb{C}$, $c \in \mathbb{C} \setminus \{0\}$, and let $(t_n)_{n \geq -2}$ be the solution to equation (14) such that*

$$t_{-2} = t_{-1} = 0, \quad t_0 = 1. \quad (15)$$

Then every well-defined solution to equation (12) has the following representation:

$$x_n = \frac{x_0 x_{-1} t_n + x_{-1} (t_{n+1} - a t_n) + c t_{n-1}}{x_0 x_{-1} t_{n-1} + x_{-1} (t_n - a t_{n-1}) + c t_{n-2}} \quad (16)$$

for $n \geq -1$.

It is a natural problem to obtain the corresponding representation of the general solution to the following difference equation:

$$x_{n+1} = a + \frac{b}{x_n} + \frac{c}{x_n x_{n-1}} + \frac{d}{x_n x_{n-1} x_{n-2}}, \quad n \in \mathbb{N}_0, \quad (17)$$

where $a, b, c \in \mathbb{C}$, $d \in \mathbb{C} \setminus \{0\}$.

Our aim here is to solve the problem. Besides, we also present more concrete representations of the general solution to equation (17) by using the roots of the polynomial

$$P_4(\lambda) = \lambda^4 - a\lambda^3 - b\lambda^2 - c\lambda - d, \quad (18)$$

which is associated with a naturally chosen linear difference equation with constant coefficients.

The following cases are considered separately: (1) all the roots of the polynomial are distinct; (2) there is a unique double root of the polynomial; (3) there is a triple root of the polynomial and one simple; (4) there is a quadruple root of the polynomial; (5) there are two distinct double roots of the polynomial.

2 Main results

The main results in this paper are stated and proved in this section.

2.1 A representation of general solution to equation (17)

Our first result concerns the problem of representing the general solution to equation (17) in terms of the parameters a, b, c, d , initial values, and a specially chosen solution to a linear difference equation with constant coefficients of the fourth order associated with the equation. It complements known representations of the general solutions to the corresponding difference equation of the first order (i.e., of equation (5)), as well as of the second order (i.e., of equation (12)).

Theorem 2 *Let $a, b, c \in \mathbb{C}$, $d \in \mathbb{C} \setminus \{0\}$, and $(s_n)_{n \geq -2}$ be the solution to the equation*

$$y_{n+1} = ay_n + by_{n-1} + cy_{n-2} + dy_{n-3}, \quad n \in \mathbb{N}_0, \quad (19)$$

such that

$$s_{-3} = s_{-2} = s_{-1} = 0, \quad s_0 = 1. \quad (20)$$

Then every well-defined solution to equation (17) has the following representation:

$$x_n = \frac{s_n x_0 x_{-1} x_{-2} + (s_{n+1} - as_n) x_{-1} x_{-2} + (cs_{n-1} + ds_{n-2}) x_{-2} + ds_{n-1}}{s_{n-1} x_0 x_{-1} x_{-2} + (s_n - as_{n-1}) x_{-1} x_{-2} + (cs_{n-2} + ds_{n-3}) x_{-2} + ds_{n-2}} \quad (21)$$

for $n \geq -2$.

Proof We consider the following four cases: (1) $a \neq 0$, (2) $a = 0, b \neq 0$, (3) $a = b = 0, c \neq 0$, (4) $a = b = c = 0$, separately.

Case $a \neq 0$. The first part of the proof in this case was essentially presented in [49] (see also [36, 38]). Hence, we will present only essential details for the completeness and benefit of the reader.

First note that, by employing (13) in equation (17), it is transformed to equation (19).

Let

$$\alpha_1 := a, \quad \beta_1 := b, \quad \gamma_1 := c, \quad \delta_1 := d. \quad (22)$$

Employing (19), where n is replaced by $n - 2$, in relation (19), where n is replaced by $n - 1$, and using (22), we have

$$\begin{aligned} y_n &= \alpha_1 (ay_{n-2} + by_{n-3} + cy_{n-4} + dy_{n-5}) + \beta_1 y_{n-2} + \gamma_1 y_{n-3} + \delta_1 y_{n-4} \\ &= \alpha_2 y_{n-2} + \beta_2 y_{n-3} + \gamma_2 y_{n-4} + \delta_2 y_{n-5} \end{aligned} \quad (23)$$

for $n \geq 2$, where

$$\alpha_2 := a\alpha_1 + \beta_1, \quad \beta_2 := b\alpha_1 + \gamma_1, \quad \gamma_2 := c\alpha_1 + \delta_1, \quad \delta_2 := d\alpha_1. \quad (24)$$

Assume that for an $l \in \mathbb{N} \setminus \{1\}$ and all $n \geq l$, we have proved

$$y_n = \alpha_l y_{n-l} + \beta_l y_{n-l-1} + \gamma_l y_{n-l-2} + \delta_l y_{n-l-3} \quad (25)$$

and

$$\alpha_l = a\alpha_{l-1} + \beta_{l-1}, \quad \beta_l = b\alpha_{l-1} + \gamma_{l-1}, \quad \gamma_l = c\alpha_{l-1} + \delta_{l-1}, \quad \delta_l = d\alpha_{l-1}. \quad (26)$$

Employing relation (19) where n is replaced by $n - l - 1$ in (25), it follows that

$$\begin{aligned} y_n &= \alpha_l (ay_{n-l-1} + by_{n-l-2} + cy_{n-l-3} + dy_{n-l-4}) + \beta_l y_{n-l-1} + \gamma_l y_{n-l-2} + \delta_l y_{n-l-3} \\ &= \alpha_{l+1} y_{n-l-1} + \beta_{l+1} y_{n-l-2} + \gamma_{l+1} y_{n-l-3} + \delta_{l+1} y_{n-l-4} \end{aligned} \quad (27)$$

for $n \geq l + 1$, where

$$\alpha_{l+1} := a\alpha_l + \beta_l, \quad \beta_{l+1} := b\alpha_l + \gamma_l, \quad \gamma_{l+1} := c\alpha_l + \delta_l, \quad \delta_{l+1} := d\alpha_l. \quad (28)$$

Relations (23), (24), (27), (28), and the induction imply that (25) and (26) hold $2 \leq l \leq n$.

Since $d \neq 0$, (26) can be used to calculate α_l , β_l , γ_l , and δ_l for $l \leq 0$. Namely, by taking $l = 1, 0, -1, -2$, respectively, it easily follows that (see, e.g., [34, 38] for some details)

$$\begin{aligned} \alpha_0 &= 1, & \alpha_{-1} &= 0, & \alpha_{-2} &= 0, & \alpha_{-3} &= 0, \\ \beta_0 &= 0, & \beta_{-1} &= 1, & \beta_{-2} &= 0, & \beta_{-3} &= 0, \\ \gamma_0 &= 0, & \gamma_{-1} &= 0, & \gamma_{-2} &= 1, & \gamma_{-3} &= 0, \\ \delta_0 &= 0, & \delta_{-1} &= 0, & \delta_{-2} &= 0, & \delta_{-3} &= 1. \end{aligned} \quad (29)$$

The relations in (26) further yield

$$\alpha_n = a\alpha_{n-1} + b\alpha_{n-2} + c\alpha_{n-3} + d\alpha_{n-4}, \quad (30)$$

$$\beta_n = \alpha_{n+1} - a\alpha_n, \quad (31)$$

$$\gamma_n = c\alpha_{n-1} + d\alpha_{n-2}, \quad (32)$$

$$\delta_n = d\alpha_{n-1} \quad (33)$$

for $n \in \mathbb{Z}$ (since $d \neq 0$).

If we take $l = n$ in (25), we obtain

$$y_n = \alpha_n y_0 + \beta_n y_{-1} + \gamma_n y_{-2} + \delta_n y_{-3} \quad (34)$$

for $n \in \mathbb{N}_0$, from which along with (31)–(33), we get

$$y_n = \alpha_n y_0 + (\alpha_{n+1} - a\alpha_n) y_{-1} + (c\alpha_{n-1} + d\alpha_{n-2}) y_{-2} + d\alpha_{n-1} y_{-3} \quad (35)$$

for $n \in \mathbb{N}_0$ (in fact, (35) holds for every $n \in \mathbb{Z}$).

Using (35) in (13), we obtain

$$x_n = \frac{\alpha_n y_0 + (\alpha_{n+1} - a\alpha_n)y_{-1} + (c\alpha_{n-1} + d\alpha_{n-2})y_{-2} + d\alpha_{n-1}y_{-3}}{\alpha_{n-1}y_0 + (\alpha_n - a\alpha_{n-1})y_{-1} + (c\alpha_{n-2} + d\alpha_{n-3})y_{-2} + d\alpha_{n-2}y_{-3}}$$

for $n \geq -2$, from which, along with (29) and (30), as well as the definition of the sequence s_n , it follows that

$$x_n = \frac{s_n y_0 + (s_{n+1} - as_n)y_{-1} + (cs_{n-1} + ds_{n-2})y_{-2} + ds_{n-1}y_{-3}}{s_{n-1}y_0 + (s_n - as_{n-1})y_{-1} + (cs_{n-2} + ds_{n-3})y_{-2} + ds_{n-2}y_{-3}} \quad (36)$$

for $n \geq -2$.

Case $a = 0$, $b \neq 0$. Since $a = 0$, we have

$$y_{n+1} = by_{n-1} + cy_{n-2} + dy_{n-3}, \quad n \in \mathbb{N}_0. \quad (37)$$

Let

$$\tilde{\beta}_1 := b, \quad \tilde{\gamma}_1 := c, \quad \tilde{\delta}_1 := d, \quad \tilde{\eta}_1 := 0. \quad (38)$$

Employing (37), where n is replaced by $n - 3$, in equality (37), where n is replaced by $n - 1$, we obtain

$$\begin{aligned} y_n &= \tilde{\beta}_1(by_{n-4} + cy_{n-5} + dy_{n-6}) + \tilde{\gamma}_1y_{n-3} + \tilde{\delta}_1y_{n-4} + \tilde{\eta}_1y_{n-5} \\ &= \tilde{\gamma}_1y_{n-3} + (b\tilde{\beta}_1 + \tilde{\delta}_1)y_{n-4} + (c\tilde{\beta}_1 + \tilde{\eta}_1)y_{n-5} + d\tilde{\beta}_1y_{n-6} \\ &= \tilde{\beta}_2y_{n-3} + \tilde{\gamma}_2y_{n-4} + \tilde{\delta}_2y_{n-5} + \tilde{\eta}_2y_{n-6} \end{aligned} \quad (39)$$

for $n \geq 3$, where

$$\tilde{\beta}_2 := \tilde{\gamma}_1, \quad \tilde{\gamma}_2 := b\tilde{\beta}_1 + \tilde{\delta}_1, \quad \tilde{\delta}_2 := c\tilde{\beta}_1 + \tilde{\eta}_1, \quad \tilde{\eta}_2 := d\tilde{\beta}_1. \quad (40)$$

Assume that for an $l \in \mathbb{N} \setminus \{1\}$ and all $n \geq l + 1$, we have proved

$$y_n = \tilde{\beta}_ly_{n-l-1} + \tilde{\gamma}_ly_{n-l-2} + \tilde{\delta}_ly_{n-l-3} + \tilde{\eta}_ly_{n-l-4} \quad (41)$$

and

$$\tilde{\beta}_l = \tilde{\gamma}_{l-1}, \quad \tilde{\gamma}_l = b\tilde{\beta}_{l-1} + \tilde{\delta}_{l-1}, \quad \tilde{\delta}_l = c\tilde{\beta}_{l-1} + \tilde{\eta}_{l-1}, \quad \tilde{\eta}_l = d\tilde{\beta}_{l-1}. \quad (42)$$

Then, by using equality (37) where n is replaced by $n - l - 2$ in (41), we have

$$\begin{aligned} y_n &= \tilde{\beta}_l(by_{n-l-3} + cy_{n-l-4} + dy_{n-l-5}) + \tilde{\gamma}_ly_{n-l-2} + \tilde{\delta}_ly_{n-l-3} + \tilde{\eta}_ly_{n-l-4} \\ &= \tilde{\gamma}_ly_{n-l-2} + (b\tilde{\beta}_l + \tilde{\delta}_l)y_{n-l-3} + (c\tilde{\beta}_l + \tilde{\eta}_l)y_{n-l-4} + d\tilde{\beta}_ly_{n-l-5} \\ &= \tilde{\beta}_{l+1}y_{n-l-2} + \tilde{\gamma}_{l+1}y_{n-l-3} + \tilde{\delta}_{l+1}y_{n-l-4} + \tilde{\eta}_{l+1}y_{n-l-5}, \end{aligned} \quad (43)$$

where

$$\tilde{\beta}_{l+1} := \tilde{\gamma}_l, \quad \tilde{\gamma}_{l+1} := b\tilde{\beta}_l + \tilde{\delta}_l, \quad \tilde{\delta}_{l+1} := c\tilde{\beta}_l + \tilde{\eta}_l, \quad \tilde{\eta}_{l+1} := d\tilde{\beta}_l. \quad (44)$$

Relations (39), (40), (43), (44) together with the induction imply that (41) and (42) hold for $l, n \in \mathbb{N}$ such that $2 \leq l \leq n-1$.

Since $d \neq 0$, from (42) it follows that $\tilde{\beta}_l, \tilde{\gamma}_l, \tilde{\delta}_l, \tilde{\eta}_l$ can be calculated for $l \leq 0$. Namely, by taking $l = 1, 0, -1, -2, -3$, respectively, it easily follows that

$$\begin{aligned} \tilde{\beta}_{-1} &= 1, & \tilde{\beta}_{-2} &= 0, & \tilde{\beta}_{-3} &= 0, & \tilde{\beta}_{-4} &= 0, \\ \tilde{\gamma}_{-1} &= 0, & \tilde{\gamma}_{-2} &= 1, & \tilde{\gamma}_{-3} &= 0, & \tilde{\gamma}_{-4} &= 0, \\ \tilde{\delta}_{-1} &= 0, & \tilde{\delta}_{-2} &= 0, & \tilde{\delta}_{-3} &= 1, & \tilde{\delta}_{-4} &= 0, \\ \tilde{\eta}_{-1} &= 0, & \tilde{\eta}_{-2} &= 0, & \tilde{\eta}_{-3} &= 0, & \tilde{\eta}_{-4} &= 1, \end{aligned} \quad (45)$$

(for $l = 1$ are obtained $\tilde{\beta}_0, \tilde{\gamma}_0, \tilde{\delta}_0, \tilde{\eta}_0$, but they are not used further, which is why we have not listed them here).

The relations in (42) further yield

$$\tilde{\beta}_n = b\tilde{\beta}_{n-2} + c\tilde{\beta}_{n-3} + d\tilde{\beta}_{n-4}, \quad (46)$$

$$\tilde{\gamma}_n = \tilde{\beta}_{n+1}, \quad (47)$$

$$\tilde{\delta}_n = c\tilde{\beta}_{n-1} + d\tilde{\beta}_{n-2}, \quad (48)$$

$$\tilde{\eta}_n = d\tilde{\beta}_{n-1} \quad (49)$$

for $n \in \mathbb{Z}$.

From (41) with $l = n-1$, we obtain

$$y_n = \tilde{\beta}_{n-1}y_0 + \tilde{\gamma}_{n-1}y_{-1} + \tilde{\delta}_{n-1}y_{-2} + \tilde{\eta}_{n-1}y_{-3} \quad (50)$$

for $n \in \mathbb{N}_0$.

Employing (47)–(49) in (50), we get

$$y_n = \tilde{\beta}_{n-1}y_0 + \tilde{\beta}_ny_{-1} + (c\tilde{\beta}_{n-2} + d\tilde{\beta}_{n-3})y_{-2} + d\tilde{\beta}_{n-2}y_{-3} \quad (51)$$

for $n \in \mathbb{N}_0$.

From (45) and (46) it follows that $\tilde{\beta}_n$ is the solution to equation (19) with $a = 0$, $b \neq 0$ such that

$$\tilde{\beta}_{n-1} = s_n.$$

Hence, from (13) and (51) it follows that (36) also holds in this case.

Case $a = b = 0 \neq c$. Since $a = b = 0$, equation (19) is

$$y_{n+1} = cy_{n-2} + dy_{n-3} \quad (52)$$

for $n \in \mathbb{N}_0$.

Let

$$\widehat{\gamma}_1 := c, \quad \widehat{\delta}_1 := d, \quad \widehat{\eta}_1 := 0, \quad \widehat{\zeta}_1 := 0. \quad (53)$$

Employing (52), where n is replaced by $n-4$, in equality (52), where n is replaced by $n-1$, we obtain

$$\begin{aligned} y_n &= \widehat{\gamma}_1 y_{n-3} + \widehat{\delta}_1 y_{n-4} + \widehat{\eta}_1 y_{n-5} + \widehat{\zeta}_1 y_{n-6} \\ &= \widehat{\gamma}_1 (c y_{n-6} + d y_{n-7}) + \widehat{\delta}_1 y_{n-4} + \widehat{\eta}_1 y_{n-5} + \widehat{\zeta}_1 y_{n-6} \\ &= \widehat{\delta}_1 y_{n-4} + \widehat{\eta}_1 y_{n-5} + (c \widehat{\gamma}_1 + \widehat{\zeta}_1) y_{n-6} + d \widehat{\gamma}_1 y_{n-7} \\ &= \widehat{\gamma}_2 y_{n-4} + \widehat{\delta}_2 y_{n-5} + \widehat{\eta}_2 y_{n-6} + \widehat{\zeta}_2 y_{n-7} \end{aligned} \quad (54)$$

for $n \geq 4$, where

$$\widehat{\gamma}_2 := \widehat{\delta}_1, \quad \widehat{\delta}_2 := \widehat{\eta}_1, \quad \widehat{\eta}_2 := c \widehat{\gamma}_1 + \widehat{\zeta}_1, \quad \widehat{\zeta}_2 := d \widehat{\gamma}_1. \quad (55)$$

Assume that for an $l \in \mathbb{N} \setminus \{1\}$ and all $n \geq l+2$, we have proved

$$y_n = \widehat{\gamma}_l y_{n-l-2} + \widehat{\delta}_l y_{n-l-3} + \widehat{\eta}_l y_{n-l-4} + \widehat{\zeta}_l y_{n-l-5} \quad (56)$$

and

$$\widehat{\gamma}_l = \widehat{\delta}_{l-1}, \quad \widehat{\delta}_l = \widehat{\eta}_{l-1}, \quad \widehat{\eta}_l = c \widehat{\gamma}_{l-1} + \widehat{\zeta}_{l-1}, \quad \widehat{\zeta}_l = d \widehat{\gamma}_{l-1}. \quad (57)$$

Then, by using equality (52) where n is replaced by $n-l-3$ in (56), we have

$$\begin{aligned} y_n &= \widehat{\gamma}_l y_{n-l-2} + \widehat{\delta}_l y_{n-l-3} + \widehat{\eta}_l y_{n-l-4} + \widehat{\zeta}_l y_{n-l-5} \\ &= \widehat{\gamma}_l (c y_{n-l-5} + d y_{n-l-6}) + \widehat{\delta}_l y_{n-l-3} + \widehat{\eta}_l y_{n-l-4} + \widehat{\zeta}_l y_{n-l-5} \\ &= \widehat{\delta}_l y_{n-l-3} + \widehat{\eta}_l y_{n-l-4} + (c \widehat{\gamma}_l + \widehat{\zeta}_l) y_{n-l-5} + d \widehat{\gamma}_l y_{n-l-6} \\ &= \widehat{\gamma}_{l+1} y_{n-l-3} + \widehat{\delta}_{l+1} y_{n-l-4} + \widehat{\eta}_{l+1} y_{n-l-5} + \widehat{\zeta}_{l+1} y_{n-l-6}, \end{aligned} \quad (58)$$

where

$$\widehat{\gamma}_{l+1} := \widehat{\delta}_l, \quad \widehat{\delta}_{l+1} := \widehat{\eta}_l, \quad \widehat{\eta}_{l+1} := c \widehat{\gamma}_l + \widehat{\zeta}_l, \quad \widehat{\zeta}_{l+1} := d \widehat{\gamma}_l. \quad (59)$$

From (54), (55), (58), (59), and the induction it follows that (56) and (57) hold for $l, n \in \mathbb{N}$ such that $2 \leq l \leq n-2$.

Since $d \neq 0$, employing the recurrent relations (57), we can calculate $\widehat{\gamma}_l, \widehat{\delta}_l, \widehat{\eta}_l, \widehat{\zeta}_l$ for $l \leq 0$ and get

$$\begin{aligned} \widehat{\gamma}_{-2} &= 1, & \widehat{\gamma}_{-3} &= 0, & \widehat{\gamma}_{-4} &= 0, & \widehat{\gamma}_{-5} &= 0, \\ \widehat{\delta}_{-2} &= 0, & \widehat{\delta}_{-3} &= 1, & \widehat{\delta}_{-4} &= 0, & \widehat{\delta}_{-5} &= 0, \\ \widehat{\eta}_{-2} &= 0, & \widehat{\eta}_{-3} &= 0, & \widehat{\eta}_{-4} &= 1, & \widehat{\eta}_{-5} &= 0, \\ \widehat{\zeta}_{-2} &= 0, & \widehat{\zeta}_{-3} &= 0, & \widehat{\zeta}_{-4} &= 0, & \widehat{\zeta}_{-5} &= 1 \end{aligned} \quad (60)$$

(see, e.g., the corresponding calculations in [34, 37]).

The relations in (57) further yield

$$\widehat{\gamma}_n = c\widehat{\gamma}_{n-3} + d\widehat{\gamma}_{n-4}, \quad (61)$$

$$\widehat{\delta}_n = \widehat{\gamma}_{n+1}, \quad (62)$$

$$\widehat{\eta}_n = c\widehat{\gamma}_{n-1} + d\widehat{\gamma}_{n-2}, \quad (63)$$

$$\widehat{\zeta}_n = d\widehat{\gamma}_{n-1}, \quad (64)$$

for $n \in \mathbb{Z}$.

If we choose $l = n - 2$ in (56), we obtain

$$y_n = \widehat{\gamma}_{n-2}y_0 + \widehat{\delta}_{n-2}y_{-1} + \widehat{\eta}_{n-2}y_{-2} + \widehat{\zeta}_{n-2}y_{-3} \quad (65)$$

for $n \in \mathbb{N}_0$.

Employing (62)–(64) in (65), it follows that

$$y_n = \widehat{\gamma}_{n-2}y_0 + \widehat{\gamma}_{n-1}y_{-1} + (c\widehat{\gamma}_{n-3} + d\widehat{\gamma}_{n-4})y_{-2} + d\widehat{\gamma}_{n-3}y_{-3} \quad (66)$$

for $n \in \mathbb{N}_0$.

From (60) and (61) we see that $\widehat{\gamma}_n$ is the solution to equation (19) with $a = b = 0$, $c \neq 0$ such that

$$\widehat{\gamma}_{n-2} = s_n,$$

from which along with (13) and (65) it follows that (36) also holds in this case.

Case $a = b = c = 0 \neq d$. Since $a = b = c = 0$, equation (19) is

$$y_{n+1} = dy_{n-3} \quad (67)$$

for $n \in \mathbb{N}_0$.

From (67) we have

$$s_{4m-i} = d^m s_{-i} \quad (68)$$

for $m \in \mathbb{N}_0$ and $i = \overline{0, 3}$. It is not difficult to check that (36) holds for the sequence defined in (68).

From (36) we have

$$\begin{aligned} x_n &= \frac{s_n y_0 + (s_{n+1} - as_n)y_{-1} + (cs_{n-1} + ds_{n-2})y_{-2} + ds_{n-1}y_{-3}}{s_{n-1}y_0 + (s_n - as_{n-1})y_{-1} + (cs_{n-2} + ds_{n-3})y_{-2} + ds_{n-2}y_{-3}} \\ &= \frac{s_n \frac{y_0}{y_{-3}} + (s_{n+1} - as_n) \frac{y_{-1}}{y_{-3}} + (cs_{n-1} + ds_{n-2}) \frac{y_{-2}}{y_{-3}} + ds_{n-1}}{s_{n-1} \frac{y_0}{y_{-3}} + (s_n - as_{n-1}) \frac{y_{-1}}{y_{-3}} + (cs_{n-2} + ds_{n-3}) \frac{y_{-2}}{y_{-3}} + ds_{n-2}} \\ &= \frac{s_n x_0 x_{-1} x_{-2} + (s_{n+1} - as_n)x_{-1} x_{-2} + (cs_{n-1} + ds_{n-2})x_{-2} + ds_{n-1}}{s_{n-1} x_0 x_{-1} x_{-2} + (s_n - as_{n-1})x_{-1} x_{-2} + (cs_{n-2} + ds_{n-3})x_{-2} + ds_{n-2}} \end{aligned}$$

for $n \in \mathbb{Z}$, as claimed. \square

Remark 2 The case $a \neq 0$ has been essentially treated in [49]. However, the proofs in the other cases, in all of which is $a = 0$, have been omitted there; although, as it is seen in the proof of Theorem 2, they follow the idea of the proof in the case $a \neq 0$. Hence, the proof of Theorem 2 can be regarded as a completion of the proof of Theorem 6 in [49] in the case $a = 0$.

2.2 About the roots of polynomial $P_4(\lambda)$

Here we present the roots of polynomial P_4 and some facts related to them.

First, we formulate a well-known lemma, whose proofs can be found, e.g., in [26, 38, 50].

Lemma 1 Let $t_j, j = \overline{1, k}$ be the roots of the polynomial

$$p(t) = \tilde{a}_k t^k + \tilde{a}_{k-1} t^{k-1} + \cdots + \tilde{a}_1 t + \tilde{a}_0$$

such that $t_i \neq t_j, i \neq j$, then

$$\sum_{j=1}^k \frac{t_j^s}{p'(t_j)} = \begin{cases} 0, & s = \overline{0, k-2}; \\ 1/\tilde{a}_k, & s = k-1. \end{cases}$$

Now we discuss the form of the sequence s_n defined in Theorem 2. Equation $P_4(\lambda) = 0$ can be written in the form

$$\left(\lambda^2 - \frac{a}{2} \lambda + \frac{s}{2} \right)^2 - \left(\left(\frac{a^2}{4} + b + s \right) \lambda^2 - \left(\frac{as}{2} - c \right) \lambda + \frac{s^2}{4} + d \right) = 0. \quad (69)$$

Further, choose s so that

$$(as - 2c)^2 = (a^2 + 4b + 4s)(s^2 + 4d)$$

(see, for example, [50]), that is,

$$s^3 + bs^2 + (ac + 4d)s + a^2d + 4bd - c^2 = 0. \quad (70)$$

Then (69) is

$$\left(\lambda^2 - \frac{a}{2} \lambda + \frac{s}{2} \right)^2 - \left(\frac{\sqrt{a^2 + 4b + 4s}}{2} \lambda - \frac{as - 2c}{2\sqrt{a^2 + 4b + 4s}} \right)^2 = 0, \quad (71)$$

from which it follows that

$$\lambda^2 - \left(\frac{a}{2} + \frac{\sqrt{a^2 + 4b + 4s}}{2} \right) \lambda + \frac{s}{2} + \frac{as - 2c}{2\sqrt{a^2 + 4b + 4s}} = 0 \quad (72)$$

or

$$\lambda^2 - \left(\frac{a}{2} - \frac{\sqrt{a^2 + 4b + 4s}}{2} \right) \lambda + \frac{s}{2} - \frac{as - 2c}{2\sqrt{a^2 + 4b + 4s}} = 0. \quad (73)$$

By using the change of variables

$$s = \widehat{s} - \frac{b}{3} \quad (74)$$

in (70), we have

$$\widehat{s}^3 + p\widehat{s} + q = 0, \quad (75)$$

where

$$p = ac + 4d - \frac{b^2}{3}$$

and

$$q = a^2d + \frac{8bd}{3} - \frac{abc}{3} - c^2 + \frac{2b^3}{27}.$$

A standard procedure for solving cubic equation (75) says that the solutions should be looked for in the form $\widehat{s} = u + v$, where it is assumed that $uv = -p/3$. Hence, $u^3 + v^3 = -q$ and $u^3v^3 = -p^3/27$, implying that u^3 and v^3 are the zeros of $z^2 + qz - p^3/27 = 0$.

Thus

$$\widehat{s} = \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}. \quad (76)$$

Let $p = -\Delta_0/3$ and $q = -\Delta_1/27$, then from (74) and (76) we have

$$s = \frac{1}{3\sqrt[3]{2}} \left(\sqrt[3]{\Delta_1 - \sqrt{\Delta_1^2 - 4\Delta_0^3}} + \sqrt[3]{\Delta_1 + \sqrt{\Delta_1^2 - 4\Delta_0^3}} \right) - \frac{b}{3}. \quad (77)$$

From (72) and (73) and by some calculations, we obtain

$$\lambda_1 = \frac{a}{4} + \frac{\sqrt{a^2 + 4b + 4s}}{4} + \frac{1}{2} \sqrt{\frac{a^2}{2} + b - s - \frac{Q}{2\sqrt{a^2 + 4b + 4s}}}, \quad (78)$$

$$\lambda_2 = \frac{a}{4} + \frac{\sqrt{a^2 + 4b + 4s}}{4} - \frac{1}{2} \sqrt{\frac{a^2}{2} + b - s - \frac{Q}{2\sqrt{a^2 + 4b + 4s}}}, \quad (79)$$

$$\lambda_3 = \frac{a}{4} - \frac{\sqrt{a^2 + 4b + 4s}}{4} + \frac{1}{2} \sqrt{\frac{a^2}{2} - s + \frac{Q}{2\sqrt{a^2 + 4b + 4s}}}, \quad (80)$$

$$\lambda_4 = \frac{a}{4} - \frac{\sqrt{a^2 + 4b + 4s}}{4} - \frac{1}{2} \sqrt{\frac{a^2}{2} + b - s + \frac{Q}{2\sqrt{a^2 + 4b + 4s}}}, \quad (81)$$

where

$$\Delta_0 = -3p = b^2 - 3ac - 12d, \quad (82)$$

$$\Delta_1 = -27q = -2b^3 - 27a^2d - 72bd + 9abc + 27c^2, \quad (83)$$

$$Q = -a^3 - 4ab - 8c. \quad (84)$$

Multiplicity of the zeros of the polynomial P_4 are described in the following lemma (see, e.g., [51]).

Lemma 2 *Let polynomial P_4 be defined in (18), Δ_0 in (82), Δ_1 in (83), Q in (84), and let $\Delta = \frac{1}{27}(4\Delta_0^3 - \Delta_1^2)$, $P = -8b - 3a^2$, and $D = -64d - 16b^2 - 16a^2b - 16ac - 3a^4$.*

Then the following statements hold:

- (a) *If $\Delta \neq 0$, then the zeros of P_4 are distinct.*
- (b) *If $\Delta = 0$ and $P < 0$, $D < 0$, $\Delta_0 \neq 0$, or $D > 0$, or $P > 0$, $D \neq 0$, or $P > 0$, $Q \neq 0$, then exactly two zeros of P_4 are equal.*
- (c) *If $\Delta = 0$, $\Delta_0 = 0$, and $D \neq 0$, then there is a triple zero of P_4 and one simple.*
- (d) *If $\Delta = 0$, $D = 0$, and if $P < 0$, or $P > 0$ and $Q = 0$, then P_4 has two distinct double zeros.*
- (e) *If $\Delta = 0$, $D = 0$, and $\Delta_0 = 0$, then all the zeros of P_4 are equal to $a/4$.*

2.3 Possible forms of the sequence s_n

While formula (21) gives a representation of the general solution to equation (17) in terms of the sequence s_n , it does not tell anything about its possible forms. In this section we will describe the forms in terms of the roots λ_j , $j = \overline{1, 4}$, of polynomial (18).

Case $\Delta \neq 0$. In this case, by Lemma 2(a), we have that the roots λ_j , $j = \overline{1, 4}$, are distinct. By using Lemma 1 and the general solution to equation (19) in this case, we have proved in [38] (see also [36]) that the following formula holds:

$$s_n = \sum_{j=1}^4 \frac{\lambda_j^{n+3}}{P'_4(\lambda_j)} \quad (85)$$

for $n \geq -3$ (in fact, (85) holds for every $n \in \mathbb{Z}$).

If one of the roots of P_4 is one, say λ_1 , then we have

$$P_4(\lambda) = (\lambda - 1)(\lambda^3 + (1 - a)\lambda^2 + (1 - a - b)\lambda + 1 - a - b - c). \quad (86)$$

Employing the change of variables

$$\lambda = s - \frac{1 - a}{3} \quad (87)$$

in the equation

$$\lambda^3 + (1 - a)\lambda^2 + (1 - a - b)\lambda + 1 - a - b - c = 0,$$

it is transformed to

$$s^3 + \widehat{p}s + \widehat{q} = 0, \quad (88)$$

where

$$\widehat{p} = 1 - a - b - \frac{(1 - a)^2}{3} \quad \text{and} \quad \widehat{q} = \frac{2(1 - a)^3}{27} + \frac{(1 - a - b)(2 + a)}{3} - c.$$

We have (see [50]; see also (76))

$$s_j = \sqrt[3]{-\frac{\widehat{q}}{2} - \sqrt{\frac{\widehat{q}^2}{4} + \frac{\widehat{p}^3}{27}}} \varepsilon^{j-2} + \sqrt[3]{-\frac{\widehat{q}}{2} + \sqrt{\frac{\widehat{q}^2}{4} + \frac{\widehat{p}^3}{27}}} \varepsilon^{j-2}$$

for $j = \overline{2, 4}$, where ε is a complex root of the equation $\varepsilon^3 = 1$, from which along with (87) it follows that

$$\lambda_j = \sqrt[3]{-\frac{\widehat{q}}{2} - \sqrt{\frac{\widehat{q}^2}{4} + \frac{\widehat{p}^3}{27}}} \varepsilon^{j-2} + \sqrt[3]{-\frac{\widehat{q}}{2} + \sqrt{\frac{\widehat{q}^2}{4} + \frac{\widehat{p}^3}{27}}} \varepsilon^{j-2} - \frac{1-a}{3} \quad (89)$$

for $j = \overline{2, 4}$.

From the above analysis, we get the following corollary.

Corollary 1 Consider equation (17) with $a, b, c \in \mathbb{C}$ and $d \in \mathbb{C} \setminus \{0\}$. Assume that $\Delta \neq 0$. Then the following statements are true.

- (a) If $a + b + c + d \neq 1$, then the general solution to equation (17) is given by (21), where sequence $(s_n)_{n \geq -3}$ is given by (85), whereas $\lambda_j, j = \overline{1, 4}$, are given by (78)–(81).
- (b) If $a + b + c + d = 1$, then the general solution to equation (17) is given by (21), where sequence $(s_n)_{n \geq -3}$ is given by (85) where $\lambda_1 = 1$, $\lambda_j, j = \overline{2, 4}$, are given by (89).

Remark 3 Note that from (85) we have

$$s_{n+1} - as_n = \sum_{j=1}^4 \frac{\lambda_j - a}{P'_4(\lambda_j)} \lambda_j^{n+3}, \quad (90)$$

$$cs_{n-1} + ds_{n-2} = \sum_{j=1}^4 \frac{c\lambda_j + d}{P'_4(\lambda_j)} \lambda_j^{n+1}, \quad (91)$$

$$ds_{n-1} = \sum_{j=1}^4 \frac{d}{P'_4(\lambda_j)} \lambda_j^{n+2} \quad (92)$$

for $n \in \mathbb{Z}$. By using (85), (90)–(92) in (21), we get the following representation of the general solution to equation (17) in the case $\Delta \neq 0$:

$$x_n = \frac{\sum_{j=1}^4 \frac{\lambda_j^2 x_0 x_{-1} x_{-2} + (\lambda_j^3 - a\lambda_j^2) x_{-1} x_{-2} + (c\lambda_j + d) x_{-2} + d\lambda_j}{P'_4(\lambda_j)} \lambda_j^{n+1}}{\sum_{j=1}^4 \frac{\lambda_j^2 x_0 x_{-1} x_{-2} + (\lambda_j^3 - a\lambda_j^2) x_{-1} x_{-2} + (c\lambda_j + d) x_{-2} + d\lambda_j}{P'_4(\lambda_j)} \lambda_j^n} \quad (93)$$

for $n \in \mathbb{Z}$.

Case $\Delta = 0$ and $P < 0$, $D < 0$, $\Delta_0 \neq 0$, or $D > 0$, or $P > 0$, $D \neq 0$, or $P > 0$, $Q \neq 0$. By Lemma 2(b) in this case P_4 has exactly one double zero. We may assume $\lambda_1 = \lambda_2 \neq \lambda_j$,

$j = 3, 4$. To find the solution satisfying (20), we let $\lambda_1 \rightarrow \lambda_2$ in (85) and get

$$s_n = \frac{\lambda_2^{n+2}((n+3)(\lambda_2 - \lambda_3)(\lambda_2 - \lambda_4) - \lambda_2(2\lambda_2 - \lambda_3 - \lambda_4))}{(\lambda_2 - \lambda_3)^2(\lambda_2 - \lambda_4)^2} + \frac{\lambda_3^{n+3}}{(\lambda_3 - \lambda_2)^2(\lambda_3 - \lambda_4)} + \frac{\lambda_4^{n+3}}{(\lambda_4 - \lambda_2)^2(\lambda_4 - \lambda_3)} \quad (94)$$

for $n \in \mathbb{Z}$ (for details, see [36]).

If $\lambda_1 = \lambda_2 = 1$, then from (94) we immediately get

$$s_n = \frac{n(1 - \lambda_3)(1 - \lambda_4) + 3\lambda_3\lambda_4 - 2\lambda_3 - 2\lambda_4 + 1}{(1 - \lambda_3)^2(1 - \lambda_4)^2} + \frac{\lambda_3^{n+3}}{(\lambda_3 - 1)^2(\lambda_3 - \lambda_4)} + \frac{\lambda_4^{n+3}}{(\lambda_4 - 1)^2(\lambda_4 - \lambda_3)} \quad (95)$$

for $n \in \mathbb{Z}$.

In this case we have $a + b + c + d - 1 = 4 - 3a - 2b - c = 0$, $12 \neq 6a + 2b$, and

$$P_4(\lambda) = (\lambda - 1)^2(\lambda^2 + (2 - a)\lambda + 3 - 2a - b), \quad (96)$$

from which it follows that

$$\lambda_{3,4} = \frac{a - 2 \pm \sqrt{a^2 + 4a + 4b - 8}}{2}. \quad (97)$$

Corollary 2 Consider equation (17) with $a, b, c \in \mathbb{C}$ and $d \in \mathbb{C} \setminus \{0\}$. Assume that $\Delta = 0$ and $P < 0$, $D < 0$, $\Delta_0 \neq 0$, or $D > 0$, or $P > 0$, $D \neq 0$, or $P > 0$, $Q \neq 0$. Then the following statements hold.

- (a) If $3a + 2b + c \neq 4$, then the general solution to equation (17) is given by formula (21), where sequence $(s_n)_{n \geq -3}$ is given by (94).
- (b) If $a + b + c + d - 1 = 4 - 3a - 2b - c = 0$, then the general solution to equation (17) is given by formula (21), where sequence $(s_n)_{n \geq -3}$ is given by (95), whereas $\lambda_{3,4}$ are given by (97).

Case $\Delta = 0$, $D = 0$, and $P < 0$, or $P > 0$ and $Q = 0$. By Lemma 2(d) in this case P_4 has two pairs of double zeros. By letting $\lambda_3 \rightarrow \lambda_4$ in (94), it is obtained

$$s_n = \frac{\lambda_2^{n+2}(n(\lambda_2 - \lambda_4)^2 + \lambda_2^2 - 4\lambda_2\lambda_4 + 3\lambda_4^2)}{(\lambda_2 - \lambda_4)^4} + \frac{\lambda_4^{n+2}(n(\lambda_4 - \lambda_2)^2 + \lambda_4^2 - 4\lambda_2\lambda_4 + 3\lambda_2^2)}{(\lambda_4 - \lambda_2)^4} \quad (98)$$

for $n \geq -3$ (see [36]).

If $a + b + c + d = 1$, then two of the zeros, say λ_1 and λ_2 , are equal to 1, and from (98) we obtain

$$s_n = \frac{\lambda_4^{n+2}(n(\lambda_4 - 1)^2 + \lambda_4^2 - 4\lambda_4 + 3) + n(1 - \lambda_4)^2 + 1 - 4\lambda_4 + 3\lambda_4^2}{(\lambda_4 - 1)^4} \quad (99)$$

for $n \geq -3$. Besides, from (97) we have

$$\lambda_{3,4} = \frac{a-2}{2}. \quad (100)$$

Corollary 3 Consider equation (17) with $a, b, c \in \mathbb{C}$ and $d \in \mathbb{C} \setminus \{0\}$. Assume that $\Delta = 0$, $D = 0$, and $P < 0$, or $P > 0$ and $Q = 0$. Then the following statements hold.

- (a) If $a + b + c + d \neq 1$, then the general solution to equation (17) is given by formula (21), where sequence $(s_n)_{n \geq -3}$ is given by (98).
- (b) If $a + b + c + d = 1$, then the general solution to equation (17) is given by formula (21), where sequence $(s_n)_{n \geq -3}$ is given by (99), whereas $\lambda_{3,4}$ are given by (100).

Case $\Delta = 0$, $\Delta_0 = 0$, and $D \neq 0$. By Lemma 2(c) we have that P_4 has a triple zero and one simple. We may assume that $\lambda_1 = \lambda_2 = \lambda_3$. In [35] we proved that in this case

$$s_n = \left(1 - \frac{\lambda_4^3}{(\lambda_4 - \lambda_1)^3} + \frac{\lambda_1(3\lambda_1 - 5\lambda_4)}{2(\lambda_4 - \lambda_1)^2}n + \frac{\lambda_1}{2(\lambda_1 - \lambda_4)}n^2 \right) \lambda_1^n + \frac{\lambda_4^{n+3}}{(\lambda_4 - \lambda_1)^3} \quad (101)$$

for $n \geq -3$.

If $\lambda_j = 1$, $j = \overline{1, 3}$, then from (101) it follows that

$$s_n = 1 - \frac{\lambda_4^3}{(\lambda_4 - 1)^3} + \frac{(3 - 5\lambda_4)}{2(\lambda_4 - 1)^2}n + \frac{1}{2(1 - \lambda_4)}n^2 + \frac{\lambda_4^{n+3}}{(\lambda_4 - 1)^3} \quad (102)$$

for $n \geq -3$. Besides, from (96) we have

$$\lambda_4 = a - 3. \quad (103)$$

Corollary 4 Consider equation (17) with $a, b, c \in \mathbb{C}$ and $d \in \mathbb{C} \setminus \{0\}$. Assume that $\Delta = 0$, $\Delta_0 = 0$, and $D \neq 0$. Then the following statements hold.

- (a) If $3a + 2b + c \neq 4$, then the general solution to equation (17) is given by formula (21), where sequence $(s_n)_{n \geq -3}$ is given by (101).
- (b) If $a + b + c + d = 1$ and $3a + 2b + c = 4$, then the general solution to equation (17) is given by formula (21), where sequence $(s_n)_{n \geq -3}$ is given by (102), whereas λ_4 is given by (103).

Case $\Delta = 0$, $D = 0$, and $\Delta_0 = 0$. By Lemma 2(e) we have that all the zeros to P_4 are equal to $a/4$.

The general solution in this case is given by

$$a_n = (c_1 + c_2n + c_3n^2 + c_4n^3) \left(\frac{a}{4} \right)^n, \quad n \geq -3. \quad (104)$$

From this and (20) it follows that $s_0 = 1 = c_1$,

$$c_1 - c_2 + c_3 - c_4 = 0,$$

$$c_1 - 2c_2 + 4c_3 - 8c_4 = 0,$$

$$c_1 - 3c_2 + 9c_3 - 27c_4 = 0.$$

Solving the system, we get

$$c_1 = c_3 = 1, \quad c_2 = \frac{11}{6}, \quad \text{and} \quad c_4 = \frac{1}{6}.$$

Using this in (104), after some calculation, we obtain

$$s_n = \frac{(n+1)(n+2)(n+3)}{6} \left(\frac{a}{4}\right)^n \quad (105)$$

for $n \geq -3$.

Corollary 5 Consider equation (17) with $a, b, c \in \mathbb{C}$ and $d \in \mathbb{C} \setminus \{0\}$. Assume that $\Delta = 0$, $D = 0$, and $\Delta_0 = 0$. Then the general solution to equation (17) is given by formula (21), where the sequence $(s_n)_{n \geq -3}$ is given by (105).

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The author declares that he has no competing interests.

Authors' contributions

The author has contributed solely to the writing of this paper. He read and approved the manuscript.

Author details

¹Mathematical Institute of the Serbian Academy of Sciences, Beograd, Serbia. ²Department of Medical Research, China Medical University Hospital, China Medical University, Taichung, Taiwan, Republic of China. ³Department of Computer Science and Information Engineering, Asia University, Taichung, Taiwan, Republic of China.

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