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Oscillation of third-order neutral differential equations with damping and distributed delay

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Abstract
The present paper focuses on the oscillation of the third-order nonlinear neutral differential equations with damping and distributed delay. The oscillation of the third-order damped equations is often discussed by reducing the equations to the second-order ones. However, by applying the Riccati transformation and the integral averaging technique, we give an analytical method for the estimation of Riccati dynamic inequality to establish several oscillation criteria for the discussed equation, which show that any solution either oscillates or converges to zero. The results make significant improvement and extend the earlier works such as (Zhang et al. in Appl. Math. Lett. 25:1514–1519 2012). Finally, some examples are given to demonstrate the effectiveness of the obtained oscillation results.

Keywords: Oscillation; Third-order; Distributed delay; Damping term; Riccati transformation

1 Introduction

Differential equations arise in modeling situations to describe population growth, biology, economics, chemical reactions, neural networks, and so forth; see, e.g., [2–8]. In the present paper, we investigate the oscillatory behavior of a third-order neutral differential equation with damping and distributed delay. The equation is given as follows:

$$\begin{aligned}&\biggl(r(t)\biggl(\alpha(t)\biggl(x(t)+\int_a^b p(t,\mu)x\bigl(\tau(t,\mu)\bigr)\,\mathrm{d}\mu\biggr)'\biggr)'\biggr)'\\&\quad+m(t)\biggl(\alpha(t)\biggl(x(t)+\int_a^b p(t,\mu)x\bigl(\tau(t,\mu)\bigr)\,\mathrm{d}\mu\biggr)'\biggr)'\\&\quad+\int_c^d F\bigl(t,\zeta,x\bigl(g(t,\zeta)\bigr)\bigr)\,\mathrm{d}\zeta=0.\end{aligned}\tag{1.1}$$

Throughout this article, we always make the hypotheses as follows:

(H_1) $r(t)\in C^1([t_0,\infty),(0,\infty))$, $m(t)\in C([t_0,\infty),(0,\infty))$, $\int_{t_0}^{\infty}\frac{1}{r(t)}\exp(-\int_{t_0}^{t}\frac{m(s)}{r(s)}\,\mathrm{d}s)\,\mathrm{d}t=\infty$;

(H_2) $\alpha(t)\in C^1([t_0,\infty),(0,\infty))$, $\int_{t_0}^{\infty}\frac{1}{\alpha(t)}\,\mathrm{d}t=\infty$;

(H_3) $p(t,\mu)\in C([t_0,\infty)\times[a,b],(0,\infty))$, $0\le p(t)=\int_a^b p(t,\mu)\,\mathrm{d}\mu\le p<1$;

(H$_4$) $\tau(t,\mu) \in C([t_0,\infty) \times [a,b],(0,\infty))$ is not a decreasing function with respect to μ and satisfies $\tau(t,\mu) \le t$ and $\lim_{t\to\infty} \inf_{\mu\in[a,b]} \tau(t,\mu) = \infty$;

(H$_5$) $g(t,\zeta) \in C([t_0,\infty) \times [c,d],[\delta,\infty))$ for $\delta > 0$ is not a decreasing function with respect to ζ and satisfies $g(t,\zeta) \le t$ and $\lim_{t\to\infty} \inf_{\zeta\in[c,d]} g(t,\zeta) = \infty$;

(H$_6$) $F(t,\zeta,w) \in C([t_0,\infty) \times [c,d] \times (0,\infty),(0,\infty))$, $q(t,\zeta) \in C([t_0,\infty) \times [c,d],(0,\infty))$, $\frac{F(t,\zeta,w)}{w} \ge q(t,\zeta)$.

Letting

$$y(t) = x(t) + \int_a^b p(t,\mu)x\bigl(\tau(t,\mu)\bigr)\,\mathrm{d}\mu,$$

a function $x(t)$ is the solution of equation (1.1) if $x(t)$ satisfies (1.1) on $[T_x,\infty)$ for every $t \ge T_x \ge t_0$ with $x(t)$, $\alpha(t)y'(t)$ and $r(t)(\alpha(t)y'(t))' \in C^1[T_x,\infty)$. We focus on the solutions satisfying $\sup\{|x(t)| : T \le t < \infty\} > 0, T \ge T_x$. The solution with arbitrarily large zeros on $[T_x,\infty)$ is treated as an oscillatory solution.

More and more scholars pay attention to the oscillatory solution of functional differential equations, especially for the first-order or second-order equations. With the development of the oscillation for the second-order equations, researchers began to study the oscillation for the third-order equations, such as [9–18] for the delay equations, [19–27] for the equations on time scales, [28–37] for the damping equations. For the neutral delay equation

$$\bigl(a(t)\bigl(b(t)\bigl(x(t) + px(t-\tau)\bigr)'\bigr)'\bigr)' + q(t)f\bigl(x(t-\sigma)\bigr) = 0,$$

the oscillation was discussed in [38], and its general cases were discussed in [39–42]. For the distributed neutral delay equations

$$\left[r(t)\left[x(t) + \int_a^b p(t,\mu)x\bigl(\tau(t,\mu)\bigr)\,\mathrm{d}\mu\right]''\right]' + \int_c^d q(t,\zeta)f\bigl(x\bigl[\sigma(t,\zeta)\bigr]\bigr)\,\mathrm{d}\zeta = 0,$$

the Philos-type oscillation criteria were studied by Zhang et al. [1], and the further investigation for the oscillation was given in [43–45] by Riccati transformation and integral averaging technique.

However, our focus is on the oscillation for third-order neutral differential equations with distributed delay and damping term, such as [46]. The research on the damped differential equations of third-order has been developed in recent years. Furthermore, the methods discussed are relatively limited. A general means used in the above mentioned papers [28–37] is reducing the third-order equations to the second-order ones. We notice that in the discussion of oscillation for the differential equations, the key is the inequality estimation techniques. In [46], by the Riccati transformation we give a method for the estimation of Riccati dynamic inequality to get some oscillation criteria. Moreover, the main contribution in this paper is that we provide another method for the inequality estimation to discuss the oscillation of differential equations with damping and distributed delay on the basis of the Riccati transformation and the integral averaging technique. The results obtained continue and extend the analytic works in [1], where the methods using Lemmas 2.3 and 2.4 for the inequality estimation cannot be applied for (1.1).

2 Preliminaries

For the oscillatory solutions of (1.1), we usually talk about the eventually positive solutions. In this section, the following results may play an important role in establishing new oscillation criteria for (1.1).

Lemma 2.1 *Assume that $x(t)$ is the positive solution of* (1.1). *Then there are two cases as follows.*

(I) $y(t) > 0$, $y'(t) > 0$, $(\alpha(t)y'(t))' > 0$;

(II) $y(t) > 0$, $y'(t) < 0$, $(\alpha(t)y'(t))' > 0$

for $t \geq t_1 \geq t_0$ with sufficiently large t_1.

Proof We set that $x(t)$ is the positive solution of (1.1) for $[t_0, \infty)$. Then it follows from (H_4) and (H_5) that $x(\tau(t,\mu)) > 0$ and $x(g(t,\zeta)) > 0$ for $t \geq t_1$ with sufficiently large t_1, respectively. It is easy to get $y(t) > x(t) > 0$.

From (1.1) and (H_6), we get

$$\begin{aligned}\bigl(r(t)\bigl(\alpha(t)y'(t)\bigr)'\bigr)' + m(t)\bigl(\alpha(t)y'(t)\bigr)' &= -\int_c^d F\bigl(t,\zeta,x\bigl(g(t,\zeta)\bigr)\bigr)\,\mathrm{d}\zeta \\ &\leq -\int_c^d q(t,\zeta)x\bigl(g(t,\zeta)\bigr)\,\mathrm{d}\zeta \\ &< 0.\end{aligned}$$

It follows that $\frac{\mathrm{d}}{\mathrm{d}t}[\exp(\int_{t_1}^t \frac{m(s)}{r(s)}\,\mathrm{d}s)r(t)(\alpha(t)y'(t))'] < 0$. Then $\exp(\int_{t_1}^t \frac{m(s)}{r(s)}\,\mathrm{d}s)r(t)(\alpha(t)y'(t))'$ is a decreasing function with one sign eventually. Thus, from (H_1),

$$\bigl(\alpha(t)y'(t)\bigr)' < 0 \quad \text{or} \quad \bigl(\alpha(t)y'(t)\bigr)' > 0$$

for $t \geq t_2 \geq t_1$.

We claim that $(\alpha(t)y'(t))' > 0$. Suppose $(\alpha(t)y'(t))' \leq 0$. According to the monotonicity of $\exp(\int_{t_1}^t \frac{m(s)}{r(s)}\,\mathrm{d}s)r(t)(\alpha(t)y'(t))'$, we have

$$\exp\biggl(\int_{t_1}^t \frac{m(s)}{r(s)}\,\mathrm{d}s\biggr)r(t)\bigl(\alpha(t)y'(t)\bigr)' \leq -M$$

for $M > 0$. Integrate the above inequality on $[t_2, t]$ to get

$$\alpha(t)y'(t) \leq \alpha(t_2)y'(t_2) - M\int_{t_2}^t \frac{1}{r(s)}\exp\biggl(-\int_{t_1}^s \frac{m(\eta)}{r(\eta)}\,\mathrm{d}\eta\biggr)\,\mathrm{d}s.$$

Letting $t \to \infty$, we have $\alpha(t)y'(t) \to -\infty$ by (H_1). Then it follows from $(\alpha(t)y'(t))' \leq 0$ that $\alpha(t)y'(t) \leq \alpha(t_3)y'(t_3) < 0$ for $t \geq t_3 \geq t_2$. Dividing by $\alpha(t)$ and integrating on $[t_3, t]$, we have that

$$y(t) - y(t_3) \leq \alpha(t_3)y'(t_3)\int_{t_3}^t \frac{1}{\alpha(s)}\,\mathrm{d}s.$$

From condition (H_2), we have $y(t) \to -\infty$ as $t \to \infty$. This contradicts $y(t) > 0$, which implies $(\alpha(t)y'(t))' > 0$. We complete the proof. □

Lemma 2.2 *Assume that $x(t)$ is the positive solution of* (1.1) *and $y(t)$ satisfies case* (II). *Suppose*

$$\int_{t_0}^{\infty}\frac{1}{\alpha(v)}\int_{v}^{\infty}\frac{1}{r(u)}\int_{u}^{\infty}q(s)\,\mathrm{d}s\,\mathrm{d}u\,\mathrm{d}v=\infty, \tag{2.1}$$

where $q(t)=\int_c^d q(t,\zeta)\,\mathrm{d}\zeta$. Then $\lim_{t\to\infty}x(t)=0$.

Proof We set that $x(t)$ is the positive solution of (1.1) for $[t_0,\infty)$. Due to the fact that case (II) is valid for $y(t)$, we obtain $\lim_{t\to\infty}y(t)=l\geq 0$. And then we use proof by contradiction to prove $l=0$. Suppose $l>0$. Then it follows that $l+\varepsilon>y(t)>l$ for $\varepsilon>0$ with $t\geq t_1\geq t_0$. Taking ε such that $p\varepsilon<l(1-p)$, from (H_3), (H_4), and property (II), we have

$$\begin{aligned}
x(t)&=y(t)-\int_a^b p(t,\mu)x\bigl(\tau(t,\mu)\bigr)\,\mathrm{d}\mu\\
&\geq l-\int_a^b p(t,\mu)y\bigl(\tau(t,\mu)\bigr)\,\mathrm{d}\mu\\
&\geq l-p(t)y\bigl(\tau(t,a)\bigr)\\
&\geq l-p(l+\varepsilon)\\
&>Ky(t),
\end{aligned}\tag{2.2}$$

where $K=\frac{l(1-p)-p\varepsilon}{l+\varepsilon}>0$. It follows from ($H_5$), ($H_6$), (2.2), and property (II) that

$$\begin{aligned}
\bigl(r(t)\bigl(\alpha(t)y'(t)\bigr)'\bigr)'+m(t)\bigl(\alpha(t)y'(t)\bigr)'&\leq-\int_c^d q(t,\zeta)x\bigl(g(t,\zeta)\bigr)\,\mathrm{d}\zeta\\
&\leq -Ky\bigl(g(t,d)\bigr)q(t).
\end{aligned}$$

Taking $z(t)=\exp(\int_{t_1}^t\frac{m(s)}{r(s)}\,\mathrm{d}s)$, we get

$$\bigl(z(t)r(t)\bigl(\alpha(t)y'(t)\bigr)'\bigr)'\leq -Kz(t)y\bigl(g(t,d)\bigr)q(t).$$

Integrate on $[t,\infty)$ to obtain

$$-z(t)r(t)\bigl(\alpha(t)y'(t)\bigr)'+K\int_t^{\infty}z(s)y\bigl(g(s,d)\bigr)q(s)\,\mathrm{d}s\leq 0.$$

By virtue of $y(g(t,d))>l$ and $z'(t)>0$, we conclude

$$-\bigl(\alpha(t)y'(t)\bigr)'+\frac{Kl}{r(t)}\int_t^{\infty}q(s)\,\mathrm{d}s<0.$$

This yields

$$\alpha(t)y'(t)+Kl\int_t^{\infty}\frac{1}{r(u)}\int_u^{\infty}q(s)\,\mathrm{d}s\,\mathrm{d}u<0$$

by the integration from t to ∞. Further integrate on $[t_1,\infty)$ to get

$$\int_{t_1}^{\infty}\frac{1}{\alpha(v)}\int_{v}^{\infty}\frac{1}{r(u)}\int_{u}^{\infty}q(s)\,\mathrm{d}s\,\mathrm{d}u\,\mathrm{d}v<\frac{y(t_1)}{Kl}.$$

This contradicts (2.1), which leads to $l=0$, and then $\lim_{t\to\infty}x(t)=0$ from $y(t)>x(t)>0$. We complete the proof. □

3 Oscillation results

Based on the lemmas in Section 2, some new oscillation criteria for (1.1) are obtained by applying Riccati transformation, inequality estimation, and integral averaging technique due to Philos [47]. Putting

$$D=\bigl\{(t,s):t_0\le s\le t<\infty\bigr\};\qquad D_0=\bigl\{(t,s):t_0\le s<t<\infty\bigr\},$$

a function $H\in C(D,\mathbb{R})$ is said to belong to X class ($H\in X$) if it satisfies

(i) $H(t,t)=0$, $t\ge t_0$ and $H(t,s)>0$, $(t,s)\in D_0$;

(ii) $H(t,s)$ has a continuous and nonpositive partial derivative on D with respect to the second variable;

(iii) There exists $h(t,s)\in C(D,\mathbb{R})$ such that

$$\frac{\partial H(t,s)}{\partial s}=-h(t,s)\sqrt{H(t,s)}\quad\text{for all }(t,s)\in D.$$

Theorem 3.1 *Assume that* (2.1) *holds and there exist $H\in X$ and $\phi\in C([t_0,\infty),\mathbb{R})$ such that*

$$0<\inf_{s\ge t_0}\left[\liminf_{t\to\infty}\frac{H(t,s)}{H(t,t_0)}\right]\le\infty,\tag{3.1}$$

$$\int_{t_0}^{\infty}\frac{\phi_+^2(t)}{\rho(t)r(t)}\,\mathrm{d}t=\infty,\tag{3.2}$$

and

$$\phi(T)\le\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t\left(H(t,s)P(s)-\frac{k\rho(s)r(s)h^2(t,s)}{4}\right)\mathrm{d}s\tag{3.3}$$

for $t\ge T\ge t_0, k>1, \theta>0$, where

$$P(t)=\rho(t)\frac{(1-p)\theta q(t)}{\alpha(t)},\qquad\rho(t)=\exp\int_{t_0}^{t}\frac{m(s)}{r(s)}\,\mathrm{d}s,\qquad\phi_+(t)=\max\bigl\{\phi(t),0\bigr\}.\tag{3.4}$$

Then any solution $x(t)$ of (1.1) *either oscillates or converges to zero.*

Proof Let $x(t)$ be a nonoscillatory solution of (1.1). Without loss of generality, we assume that $x(t)>0$ on $[t_1,\infty)$. From (H_4) and (H_5), we have $x(\tau(t,\mu))>0$, $(t,\mu)\in[t_1,\infty)\times[a,b]$, $x(g(t,\zeta))>0$, $(t,\zeta)\in[t_1,\infty)\times[c,d]$ for sufficiently large t_1. From Lemma 2.1, $y(t)$ is one case of (I) and (II).

If $y(t)$ satisfies case (I), then

$$\begin{aligned} x(t) &\ge y(t) - \int_a^b p(t,\mu) y\bigl(\tau(t,\mu)\bigr)\,\mathrm{d}\mu \\ &\ge \bigl(1-p(t)\bigr) y(t) \\ &\ge (1-p) y(t) \end{aligned}$$

from (H_3) and (H_4). By (H_5), (H_6) and the above inequality, it is obvious that

$$\begin{aligned} \bigl(r(t)\bigl(\alpha(t)y'(t)\bigr)'\bigr)' + m(t)\bigl(\alpha(t)y'(t)\bigr)' &\le -(1-p)\int_c^d q(t,\zeta) y\bigl(g(t,\zeta)\bigr)\,\mathrm{d}\zeta \\ &\le -(1-p) y\bigl(g(t,c)\bigr) q(t). \end{aligned}$$

Putting that $w(t) = \rho(t)\frac{r(t)(\alpha(t)y'(t))'}{\alpha(t)y'(t)}$, $t \ge t_1$ with $\rho(t)$ given in (3.4), we know

$$\begin{aligned} w'(t) &= \frac{\rho'(t)}{\rho(t)} w(t) + \rho(t)\frac{(r(t)(\alpha(t)y'(t))')'}{\alpha(t)y'(t)} - \frac{w^2(t)}{\rho(t)r(t)} \\ &\le \frac{\rho'(t)}{\rho(t)} w(t) - \rho(t)\left[\frac{(1-p)y(g(t,c))q(t)}{\alpha(t)y'(t)} + \frac{m(t)(\alpha(t)y'(t))'}{\alpha(t)y'(t)}\right] - \frac{w^2(t)}{\rho(t)r(t)} \\ &= -\rho(t)\frac{(1-p)y(g(t,c))q(t)}{\alpha(t)y'(t)} - \frac{w^2(t)}{\rho(t)r(t)}. \end{aligned}$$

By property (I), there exists a limit of $\frac{1}{y'(t)}$ as $t \to \infty$, which is denoted by $\lim_{t\to\infty}\frac{1}{y'(t)} = \eta$. Choosing $\varepsilon = \frac{\eta}{2}$, we obtain $\frac{1}{y'(t)} > \frac{\eta}{2}$ for $t \ge t_2 \ge t_1$. Letting $\theta = \frac{y(\delta)\eta}{2}$, from $g(t,c) \ge \delta$ in (H_5) we have

$$w'(t) \le -P(t) - \frac{w^2(t)}{\rho(t)r(t)},$$

where $P(t)$ is defined in (3.4). Multiply the above inequality by $H(t,s)$ and integrate the inequality from t_2 to t to get

$$\begin{aligned} \int_{t_2}^t H(t,s)P(s)\,\mathrm{d}s &\le H(t,t_2)w(t_2) - \int_{t_2}^t h(t,s)\sqrt{H(t,s)}w(s)\,\mathrm{d}s - \int_{t_2}^t \frac{H(t,s)w^2(s)}{\rho(s)r(s)}\,\mathrm{d}s \\ &= H(t,t_2)w(t_2) - \int_{t_2}^t \left(h(t,s)\frac{\sqrt{k\rho(s)r(s)}}{2} + w(s)\sqrt{\frac{H(t,s)}{k\rho(s)r(s)}}\right)^2\mathrm{d}s \\ &\quad + \int_{t_2}^t \frac{k\rho(s)r(s)h^2(t,s)}{4}\,\mathrm{d}s - \int_{t_2}^t \frac{(k-1)H(t,s)w^2(s)}{k\rho(s)r(s)}\,\mathrm{d}s \end{aligned}$$

from the integral averaging technique. Then

$$\begin{aligned} &\limsup_{t\to\infty}\frac{1}{H(t,t_2)}\int_{t_2}^t \left(H(t,s)P(s) - \frac{k\rho(s)r(s)h^2(t,s)}{4}\right)\mathrm{d}s \\ &\quad\le w(t_2) - \liminf_{t\to\infty}\frac{1}{H(t,t_2)}\int_{t_2}^t \frac{(k-1)H(t,s)w^2(s)}{k\rho(s)r(s)}\,\mathrm{d}s. \end{aligned}$$

Thus, it follows from (3.3) that

$$\phi(t) \le w(t), \quad t \ge t_2,$$

and

$$\liminf_{t\to\infty} \frac{1}{H(t,t_2)} \int_{t_2}^{t} \frac{(k-1)H(t,s)w^2(s)}{k\rho(s)r(s)}\,\mathrm{d}s \le w(t_2) - \phi(t_2) < \infty, \quad t \ge t_2. \tag{3.5}$$

Next we claim

$$\int_{t_2}^{\infty} \frac{w^2(t)}{\rho(t)r(t)}\,\mathrm{d}t < \infty.$$

Suppose $\int_{t_2}^{\infty} \frac{w^2(t)}{\rho(t)r(t)}\,\mathrm{d}t = \infty$. It follows from (3.1) that

$$\inf_{s\ge t_0}\left[\liminf_{t\to\infty} \frac{H(t,s)}{H(t,t_0)}\right] > \mu$$

for $\mu > 0$, and then $\frac{H(t,t_3)}{H(t,t_0)} > \mu$ for $t \ge t_3 \ge t_2$. Thus we have

$$\int_{t_3}^{t} \frac{w^2(t)}{\rho(t)r(t)}\,\mathrm{d}t \ge \frac{M_1}{\mu}$$

for $M_1 > 0$. Then, when $t \ge t_3$, we conclude

$$\begin{aligned}\frac{1}{H(t,t_0)} \int_{t_3}^{t} \frac{H(t,s)w^2(s)}{\rho(s)r(s)}\,\mathrm{d}s &= \frac{1}{H(t,t_0)} \int_{t_3}^{t} -\frac{\partial H(t,s)}{\partial s} \int_{t_3}^{s} \frac{w^2(\eta)}{\rho(\eta)r(\eta)}\,\mathrm{d}\eta\,\mathrm{d}s \\ &\ge \frac{1}{H(t,t_0)} \frac{M_1}{\mu} \int_{t_3}^{t} -\frac{\partial H(t,s)}{\partial s}\,\mathrm{d}s \\ &= \frac{M_1}{\mu} \frac{H(t,t_3)}{H(t,t_0)} \\ &\ge M_1.\end{aligned}$$

This implies

$$\liminf_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_3}^{t} \frac{H(t,s)w^2(s)}{\rho(s)r(s)}\,\mathrm{d}s = \infty.$$

This leads to a contradiction with (3.5). Then we conclude $\int_{t_2}^{\infty} \frac{w^2(t)}{\rho(t)r(t)}\,\mathrm{d}t < \infty$, which contradicts (3.2).

If $y(t)$ satisfies case (II), then $\lim_{t\to\infty} x(t) = 0$ from (2.1) and Lemma 2.2. The proof is complete. □

Theorem 3.2 *Assume that* (2.1) *holds and there exist* $H \in X$ *and* $R(t) \in C([t_0,\infty),\mathbb{R})$ *such that*

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_{T}^{t} \left(H(t,s)D(s) - \frac{k\rho(s)r(s)h^2(t,s)}{4(k-1)}\right)\mathrm{d}s = \infty \tag{3.6}$$

for $k > 1$, $\theta > 0$, *and* $t \geq T \geq t_0$, *where*

$$D(t) = Q(t) - k\rho(t)r(t)R^2(t), \qquad \rho(t) = \exp \int_{t_0}^{t} \frac{m(s)}{r(s)}\,\mathrm{d}s, \tag{3.7}$$

$$Q(t) = \rho(t)\left[r(t)R^2(t) - m(t)R(t) - \bigl(r(t)R(t)\bigr)' + \frac{(1-p)\theta q(t)}{\alpha(t)}\right]. \tag{3.8}$$

Then any solution $x(t)$ *of* (1.1) *either oscillates or converges to zero.*

Proof Let $x(t)$ be a nonoscillatory solution of (1.1). Without loss of generality, we assert that $x(t) > 0$ on $[t_1,\infty)$. It follows from (H_4) and (H_5) that $x(\tau(t,\mu)) > 0$, $(t,\mu) \in [t_1,\infty) \times [a,b]$, $x(g(t,\zeta)) > 0$, $(t,\zeta) \in [t_1,\infty) \times [c,d]$ for sufficiently large t_1. From Lemma 2.1, $y(t)$ is one case of (I) and (II).

If $y(t)$ satisfies case (I), then by letting

$$w(t) = \rho(t)\left[\frac{r(t)(\alpha(t)y'(t))'}{\alpha(t)y'(t)} + r(t)R(t)\right], \quad t \geq t_1,$$

we conclude that

$$\begin{aligned} w'(t) \leq{}& \frac{\rho'(t)}{\rho(t)}w(t) - \rho(t)\left[\frac{(1-p)y(g(t,c))q(t)}{\alpha(t)y'(t)} + \frac{m(t)(\alpha(t)y'(t))'}{\alpha(t)y'(t)} + r(t)\left(\frac{(\alpha(t)y'(t))'}{\alpha(t)y'(t)}\right)^2\right] \\ &+ \rho(t)\bigl(r(t)R(t)\bigr)'. \end{aligned}$$

In the same way as Theorem 3.1, taking $\lim_{t\to\infty} \frac{1}{y'(t)} = \eta$, $\varepsilon = \frac{\eta}{2}$, and $\theta = \frac{y(\delta)\eta}{2}$, we have

$$\begin{aligned} w'(t) \leq{}& \frac{\rho'(t)}{\rho(t)}w(t) - \rho(t)\left[\frac{(1-p)\theta q(t)}{\alpha(t)} + \frac{m(t)w(t)}{\rho(t)r(t)} - m(t)R(t) + r(t)\left(\frac{w(t)}{\rho(t)r(t)} - R(t)\right)^2\right] \\ &+ \rho(t)\bigl(r(t)R(t)\bigr)' \\ ={}& -Q(t) + 2R(t)w(t) - \frac{w^2(t)}{\rho(t)r(t)} \\ ={}& -Q(t) + 2R(t)w(t) - \frac{w^2(t)}{k\rho(t)r(t)} - \frac{(k-1)w^2(t)}{k\rho(t)r(t)}, \end{aligned}$$

where $Q(t)$ is defined by (3.8). Based on $Bu - Au^2 \leq \frac{B^2}{4A}$ for $A > 0$, $u \in \mathbb{R}$, we get

$$w'(t) \leq -D(t) - \frac{(k-1)w^2(t)}{k\rho(t)r(t)},$$

where $D(t)$ is given by (3.7). Multiplying the inequality by $H(t,s)$ and integrating on $[T,t]$, we obtain

$$\begin{aligned} \int_T^t H(t,s)D(s)\,\mathrm{d}s &\leq w(T)H(t,T) - \int_T^t \left(h(t,s)\sqrt{H(t,s)}w(s) + \frac{(k-1)H(t,s)w^2(s)}{k\rho(s)r(s)}\right)\mathrm{d}s \\ &= w(T)H(t,T) - \int_T^t \left(h(t,s)\sqrt{\frac{k\rho(s)r(s)}{4(k-1)}} + w(s)\sqrt{\frac{(k-1)H(t,s)}{k\rho(s)r(s)}}\right)^2 \mathrm{d}s \\ &\quad + \int_T^t \frac{k\rho(s)r(s)h^2(t,s)}{4(k-1)}\,\mathrm{d}s. \end{aligned}$$

Then

$$\frac{1}{H(t,T)}\int_T^t\left(H(t,s)D(s)-\frac{k\rho(s)r(s)h^2(t,s)}{4(k-1)}\right)\mathrm{d}s\le w(T).$$

This contradicts (3.6).

If $y(t)$ satisfies case (II), then $\lim_{t\to\infty}x(t)=0$ from (2.1) and Lemma 2.2. The proof is complete. □

Remark 3.3 The proofs of Theorems 3.1 and 3.2 provide a method for the estimation of Riccati dynamic inequality, which is different from [46] and useful for the oscillation criteria.

4 Examples

Example 4.1 Consider the equation

$$\begin{aligned}&\left(\frac{1}{t}\left(e^{-t}\left(x(t)+\int_1^2\frac{\mu}{6t}x(t-\mu)\,\mathrm{d}\mu\right)'\right)'\right)'+\frac{2}{t^2}\left(e^{-t}\left(x(t)+\int_1^2\frac{\mu}{6t}x(t-\mu)\,\mathrm{d}\mu\right)'\right)'\\&\quad+\int_{\frac{1}{2}}^1 t\zeta x\big((t-3\sqrt{t})\zeta\big)\,\mathrm{d}\zeta=0.\end{aligned}\tag{4.1}$$

By (4.1), we note that $r(t)=\frac{1}{t}$, $\alpha(t)=e^{-t}$, $p(t,\mu)=\frac{\mu}{6t}$, $\tau(t,\mu)=t-\mu$, $m(t)=\frac{2}{t^2}$, $g(t,\zeta)=(t-3\sqrt{t})\zeta$, $a=1$, $b=2$, $c=\frac{1}{2}$, $d=1$, which satisfy conditions (H_1)–(H_6). Furthermore, we choose $H(t,s)=(t-s)^2$, $k=2$, $\theta=2$, $p=\frac{1}{2}$, $q(t,\zeta)=t\zeta$, $\phi(t)=t$, $t_0=1$. By Theorem 3.1, we obtain $h(t,s)=2$, $\rho(t)=t^2-1$, $q(t)=\frac{3}{8}t$, $P(t)=\frac{3}{8}t(t^2-1)e^t$ and

$$\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t\left(H(t,s)P(s)-\frac{k\rho(s)r(s)h^2(t,s)}{4}\right)\mathrm{d}s\ge T=\phi(T).$$

Clearly, it is obvious that other conditions of Theorem 3.1 are valid. Thus, it follows from Theorem 3.1 that any solution of (4.1) is oscillatory or converges to zero as $t\to\infty$.

Example 4.2 We consider the equation

$$\begin{aligned}&\left(\frac{2}{t}\left(\sqrt{t}\left(x(t)+\int_0^1 e^{-t}\mu^2x\left(\frac{1}{3}t\mu\right)\mathrm{d}\mu\right)'\right)'\right)'\\&\quad+\frac{1}{t^2}\left(\sqrt{t}\left(x(t)+\int_0^1 e^{-t}\mu^2x\left(\frac{1}{3}t\mu\right)\mathrm{d}\mu\right)'\right)'\\&\quad+\int_{\frac{1}{2}}^1 t\zeta x(t\zeta)\,\mathrm{d}\zeta=0.\end{aligned}\tag{4.2}$$

From (4.2), we find that $r(t)=\frac{2}{t}$, $\alpha(t)=\sqrt{t}$, $p(t,\mu)=e^{-t}\mu^2$, $\tau(t,\mu)=\frac{1}{3}t\mu$, $m(t)=\frac{1}{t^2}$, $g(t,\zeta)=t\zeta$, $a=0$, $b=1$, $c=\frac{1}{2}$, $d=1$, which satisfy conditions (H_1)–(H_6). Furthermore, we choose $H(t,s)=(t-s)^2$, $k=2$, $\theta=2$, $p=\frac{1}{2}$, $q(t,\zeta)=\frac{\zeta}{\sqrt{t}}$, $R(t)=\frac{1}{t}$, $t_0=1$. By Theorem 3.2, we have $h(t,s)=2$, $\rho(t)=\sqrt{t}-1$, $q(t)=\frac{3}{8}t^{-\frac{1}{2}}$, $D(t)=(\sqrt{t}-1)(\frac{3}{8}t^{-1}+t^{-3})$ and

$$\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t\left(H(t,s)D(s)-\frac{k\rho(s)r(s)h^2(t,s)}{4(k-1)}\right)\mathrm{d}s=\infty.$$

Then by Theorem 3.2 we know that any solution of (4.2) is oscillatory or converges to zero as $t \to \infty$.

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